Mathematical Semantics of Computer Systems, *MSCS* (4810-1168)
Handout for Lecture 12 (2015/01/26)

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1 Today's Agenda

We continue the last lecture and follow slides by Samson Abramsky (Oxford) found at `www.math.helsinki.fi/logic/sellc-2010/course/LectureIII.pdf`. See [2] for further details. There is a big body of literature on the λ-calculus, including [1, 3, 4].

- On conversion in λ-calculus
- Categorical Semantics, as a typed λ-calculus and Cartesian closed categories as examples

In the forthcoming lecture (Feb 9): *Automata Theory Categorically, with Coalgebras.*

2 Cartesian Closed Categories as Models of Typed λ-Calculus

Definition. Type judgment. Type derivation tree.

NB: we use the term calculus *a la Church* (where bound variables have explicit types).

Lemma. *Each derivable type judgment has a unique derivation tree.*

Definition. Cartesian closed category: a category with finite products and exponentials.

Definition. Interpretation $[\![\_]\!]$ of typed λ-calculus. Interpreting types, type derivation trees, type judgments, and terms.

Definition. Substitution lemma: interpretation of $s[t/x]$ is given by composition of arrows.

Definition. Conversion rules, including congruence rules.

Theorem. *Soundness of categorical semantics: if $s =_{\beta\eta} t$, then $[\![s]\!] = [\![t]\!]$.*

If we have time:

- The Curry-Howard correspondence; terms as proofs; conversion as proof normalization
- Subject reduction, strong normalization, recursion and observational equivalence

3 Report Assignments

~~Due: the beginning of the next lecture (2015/2/2)~~ POSTPONED: no report assignment this time; the following question will be asked next time.

1. Prove the substitution lemma.
2. Let $f\colon A \to B$ be an arrow in a Cartesian closed category $\mathbb{C}$. It induces natural transformations: $A \times (\_) \Rightarrow B \times (\_)$, and $(\_)^B \Rightarrow (\_)^A$. Show that the adjunctions $A \times (\_) \dashv (\_)^A$ and $B \times (\_) \dashv (\_)^B$ are "compatible" with those natural transformations.
   - You first have to formalize what "compatibility" means.

References

[1] H.P. Barendregt. *The Lambda Calculus. Its Syntax and Semantics.* North-Holland, Amsterdam, 2^{nd} rev. edn., 1984.

[2] J. Lambek and P.J. Scott. *Introduction to higher order Categorical Logic.* No. 7 in Cambridge Studies in Advanced Mathematics. Cambridge Univ. Press, 1986.

[3] M.H. Sørensen and P. Urzyczyn. *Lectures on the Curry-Howard Isomorphism*, vol. 149 of *Studies in Logic and the Foundations of Mathematics.* Elsevier Science Inc., New York, NY, USA, 2006.

[4] G. Winskel. *The Formal Semantics of Programming Languages.* MIT Press, 1993.