Mathematical Semantics of Computer Systems, *MSCS* (4810-1168)
Handout for Lecture 6 (2014/12/1)

Ichiro Hasuo, Dept. Computer Science, Univ. Tokyo `http://www-mmm.is.s.u-tokyo.ac.jp/~ichiro`

We continue using the handout from the last lecture. The relevant part is repeated below just in case.

1 Today's Goal

Identify the following framework of *abstract interpretation* [1] as an instance of adjunction. (Thanks are due to Kengo Kido for a nice introduction.)

Definition (Galois connection)**.** Let L and $\overline{L}$ be posets; and $\alpha\colon L \to \overline{L}$ and $\gamma\colon \overline{L} \to L$ be monotone functions. The pair (α, γ) is said to be a *Galois connection* if, for any $x \in L$ and $\overline{x} \in \overline{x}$,

$$\alpha(x) \leq_{\overline{L}} \overline{x} \quad \text{if and only if} \quad x \leq_L \gamma(\overline{x}) \ .$$

Example (interval domain)**.** Let

$$L \ := \ \mathcal{P}(\mathbb{N}) \quad \text{and} \quad \overline{L} \ := \ \{\emptyset\} \cup \bigl\{\, [l,r] \bigm| l,r \in \mathbb{N} \cup \{-\infty, \infty\},\, l \leq r \,\bigr\}$$

where each set is ordered by inclusion. Moreover,

$$\alpha(S) \ := \ [\min S, \max X] \quad \text{and} \quad \gamma(\overline{S}) \ := \ \{n \in \mathbb{N} \mid n \in \overline{S}\} \ .$$

Then the pair (α, γ) is a Galois connection.

2 Today's Agenda

2.1 Adjunction

Definition. Homset.

Definition. Adjunction.

Example. Free monoids.

Definition. Unit, counit.

Lemma. *Adjoint transposes by units and counits.*

Proposition. *Characterization of adjuction by: 1) the universality of η (Def. 3.2 of [Lambek & Scott], intuitive for free monoids); 2) the triangular equalities (Def. 3.1 of [Lambek & Scott]).*

Lemma. *1. Adjoint functors determine each other uniquely up-to canonical natural isomorphisms.*

2. Composition of adjoints.

2.2 Limits as Adjoints

Definition. Functor category

Proposition. *A limit gives rise to an adjunction.*

3 Exercises

1. Formulate and prove the following statement.

   A right adjoint preserves limits.

2. Prove the following: in an adjunction $F \dashv G$, G is faithful if and only if every component of the counit ε is an epi. [2, Thm. IV.3.1]

References

[1] P. Cousot and R. Cousot. Abstract interpretation: A unified lattice model for static analysis of programs by construction or approximation of fixpoints. In R.M. Graham, M.A. Harrison and R. Sethi, editors, *Conference Record of the Fourth ACM Symposium on Principles of Programming Languages, Los Angeles, California, USA, January 1977*, pp. 238–252. ACM, 1977.

[2] S. Mac Lane. *Categories for the Working Mathematician*. Springer, Berlin, 2nd edn., 1998.