Mathematical Semantics of Computer Systems, *MSCS* (4810-1168)
Handout for Lecture 7 (2014/12/8)

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Part I: Adjuction (ctn'd)

1 Today's Goals

1.1 Goal I: Abstract Interpretation via Adjunction

Identify the following framework of *abstract interpretation* [3] as an instance of adjunction. (Thanks are due to Kengo Kido for a nice introduction.)

Definition (Galois connection)**.** Let L and $\overline{L}$ be posets; and $\alpha\colon L \to \overline{L}$ and $\gamma\colon \overline{L} \to L$ be monotone functions. The pair (α, γ) is said to be a *Galois connection* if, for any $x \in L$ and $\overline{x} \in \overline{x}$,

$$\alpha(x) \leq_{\overline{L}} \overline{x} \quad \text{if and only if} \quad x \leq_L \gamma(\overline{x}) \ .$$

Example (interval domain)**.** Let

$$L \ := \ \mathcal{P}(\mathbb{N}) \quad \text{and} \quad \overline{L} \ := \ \{\emptyset\} \cup \bigl\{\, [l, r] \bigm| l, r \in \mathbb{N} \cup \{-\infty, \infty\},\, l \leq r \,\bigr\}$$

where each set is ordered by inclusion. Moreover,

$$\alpha(S) \ := \ [\min S, \max X] \quad \text{and} \quad \gamma(\overline{S}) \ := \ \{n \in \mathbb{N} \mid n \in \overline{S}\} \ .$$

Then the pair (α, γ) is a Galois connection.

1.2 Goal II: Quantifiers via Adjunction (à la Lawvere)

Let $f\colon X \to Y$ be a function, and 2^X and 2^Y be the posets of *predicates* over X and Y, respectively, whose orders are the inclusion order.

We think of the posets 2^X and 2^Y as categories. Then we have two adjunctions

$$2^X \ \underset{\forall_f}{\overset{\exists_f}{\rightrightarrows}} \!\!\!\!\!\!\!\! \xleftarrow[\ \perp\ ]{\ \perp\ \ f^{-1}} \ 2^Y \ . \tag{1}$$

2 Today's Agenda

2.1 Adjunction

Proposition. *Characterization of adjuction by: 1) the universality of η (Def. 3.2 of [Lambek & Scott], intuitive for free monoids); 2) the triangular equalities (Def. 3.1 of [Lambek & Scott]).*

Corollary. *Adjoint transposes by units and counits.*

Lemma. *1. Adjoint functors determine each other uniquely up-to canonical natural isomorphisms. (Proof: by the Yoneda lemma)*

2. Composition of adjoints.

2.2 Limits as Adjoints

Definition. Functor category

Proposition. *Limits give rise to an adjunction. So do colimits.*

3 Exercises

1. Formulate and prove the following statement.

   A right adjoint preserves limits.

2. Present concrete definitions of the monotone functions $\exists_f$ and $\forall_f$ in (1).
3. (Challenge! Not required) Prove the following: in an adjunction $F \dashv G$, G is faithful if and only if every component of the counit ε is an epi. [6, Thm. IV.3.1]

   Hint: use the presentation of the adjoint transpose $(\_)^\wedge$ by the counit ε.

Part II: the Yoneda Lemma

Remember: we loosely follow [4], but it hardly serves as an introductory textbook. More beginner-friendly ones include [1, 5]; other classical textbooks include [6, 2]. nLab (`ncatlab.org`) is an excellent online information source.

4 Today's Goal

Familiarize yourself with the *Yoneda lemma*. Identify it as a category theory analogue of the *Cayley representation theorem*:

Theorem (Cayley). *Every group G is isomorphic to a subgroup of $\pi(|G|)$.*

5 Today's Agenda

5.1 Equivalence of Categories

Definition. Subcategory, faithful functor, full functor

Lemma. *Any functor preserves isomorphisms.*
A full and faithful functor reflects isomorphisms.

Definition. Equivalence of categories

Proposition. *Equivalence from a full, faithful and iso-dense functor.*

5.2 The Yoneda Lemma

Definition. Covariance, contravariance

Theorem (Yoneda). *The Yoneda lemma, the Yoneda embedding as a full and faithful functor*

Definition. end, coend

Theorem. *The Yoneda lemma, the (co)end form*

Lemma. *Ends as limits [6, Prop. IX.5.1]*

Lemma. *Homfunctors preserve (co)limits, hence also (co)ends*

6 Exercises

1. Formulate the "naturality" of the Yoneda correspondence

$$\mathrm{Nat}\big(\mathbb{C}(\_\,, X),\, F\big) \;\cong\; FX$$

   and prove it.

Report Assignment

Exercise 3.1, 3.2. (Exercise 3.3 is optional.)
Due: 2014.12.22 (the beginning of the lecture)

References

[1] S. Awodey. *Category Theory.* Oxford Logic Guides. Oxford Univ. Press, 2006.

[2] M. Barr and C. Wells. *Toposes, Triples and Theories.* Springer, Berlin, 1985. Available online.

[3] P. Cousot and R. Cousot. Abstract interpretation: A unified lattice model for static analysis of programs by construction or approximation of fixpoints. In R.M. Graham, M.A. Harrison and R. Sethi, editors, *Conference Record of the Fourth ACM Symposium on Principles of Programming Languages, Los Angeles, California, USA, January 1977*, pp. 238–252. ACM, 1977.

[4] J. Lambek and P.J. Scott. *Introduction to higher order Categorical Logic.* No. 7 in Cambridge Studies in Advanced Mathematics. Cambridge Univ. Press, 1986.

[5] T. Leinster. *Basic Category Theory.* Cambridge Univ. Press, 2014.

[6] S. Mac Lane. *Categories for the Working Mathematician.* Springer, Berlin, 2nd edn., 1998.