[1] KS and IH. **Programming with Infinitesimals: A While-Language for Hybrid System Modeling**. Proc. ICALP 2011, Track B.

[2] IH and KS. **Exercises in Nonstandard Static Analysis of Hybrid Systems**. Proc. CAV 2012.

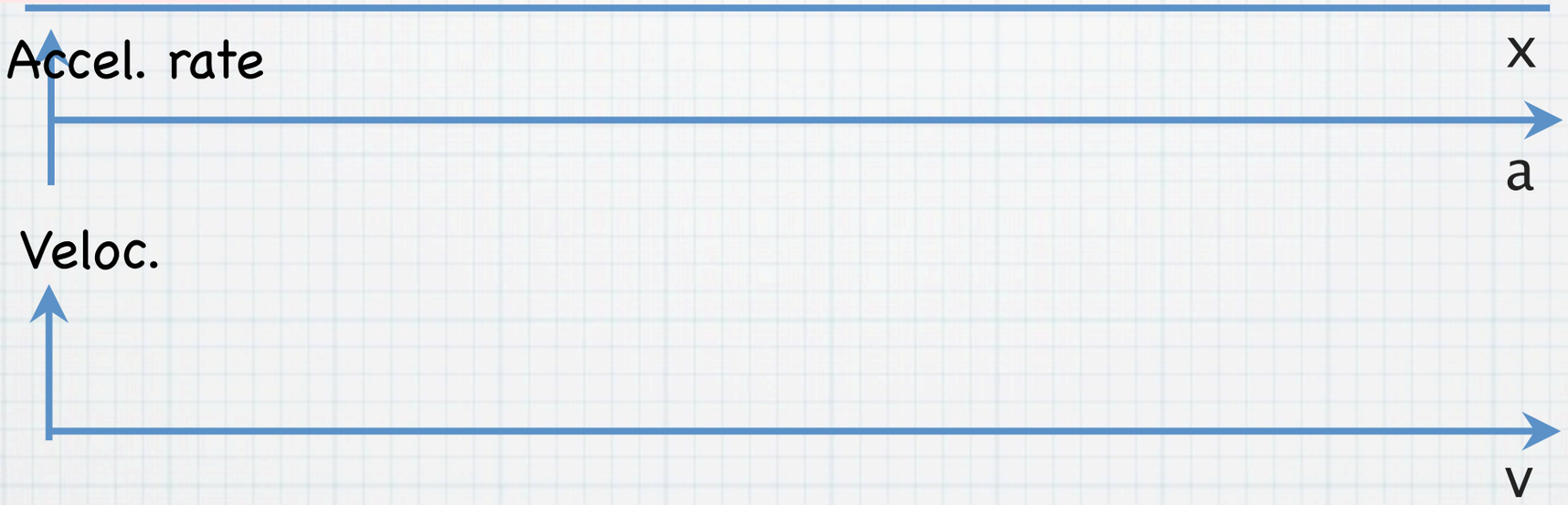
Nonstandard Static Analysis

Transfer Verification to Hybrid Systems

Ichiro Hasuo
University of Tokyo (JP)

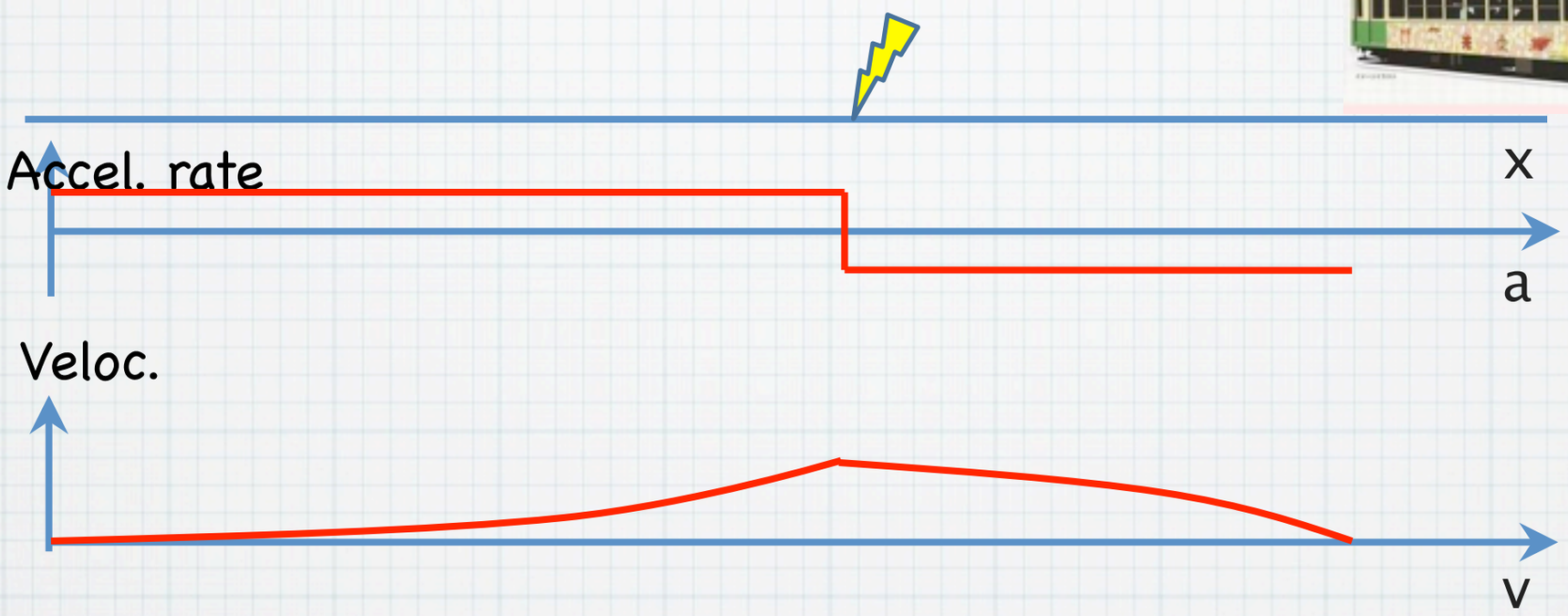
Kohei Suenaga
Kyoto University (JP)

Hybrid System



Hasuo (Tokyo)

Hybrid System



Hasuo (Tokyo)

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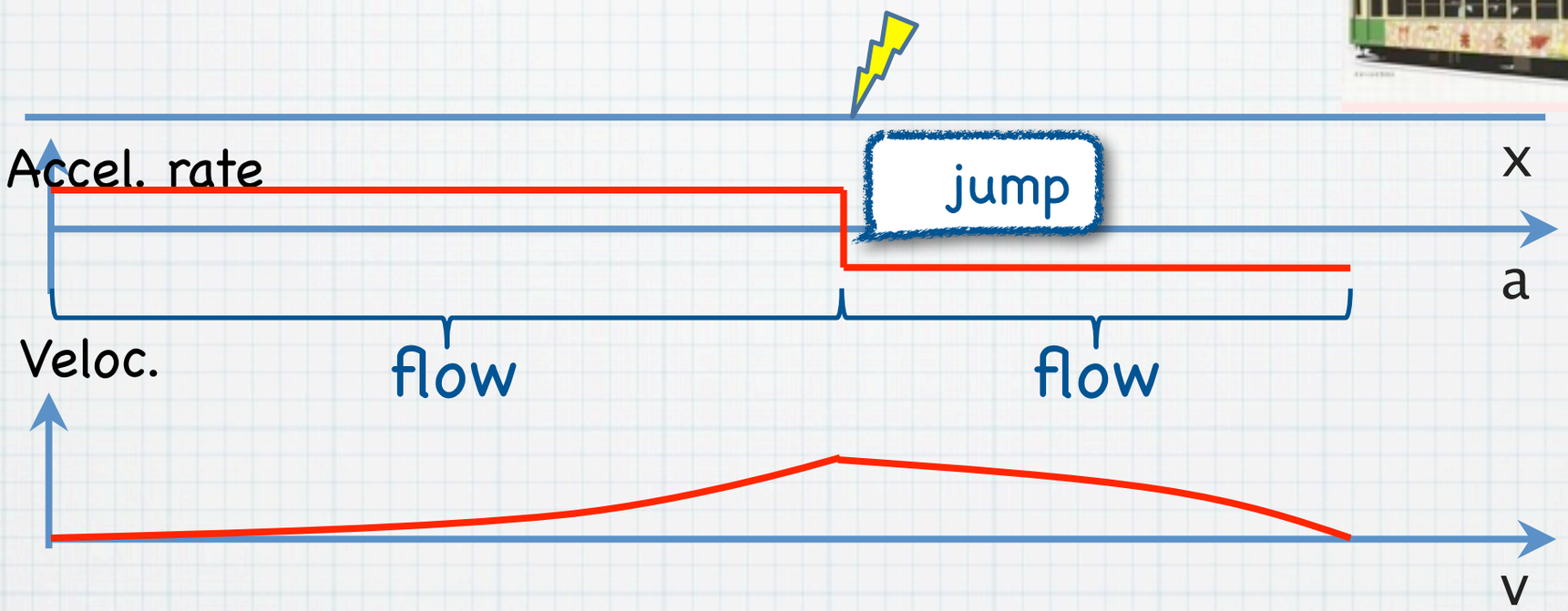
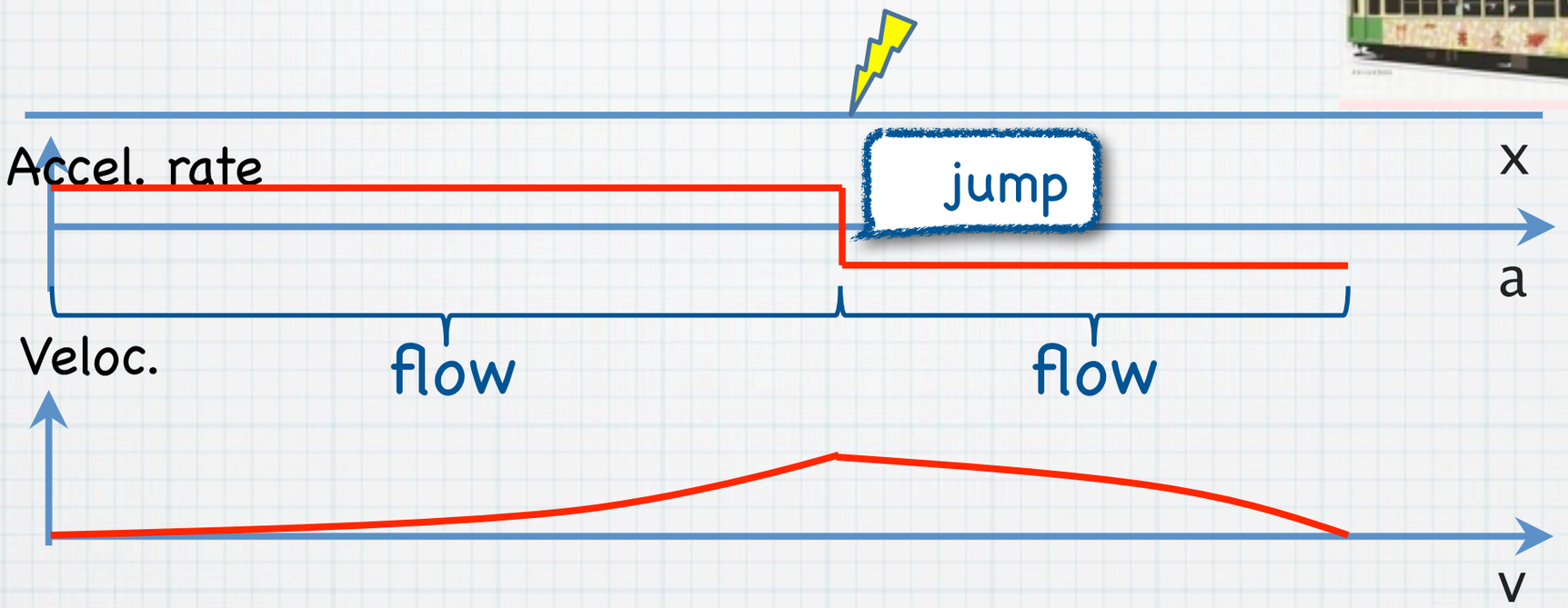


Hasuo (Tokyo)

Hybrid System



* Flow & jump
  * Digital control in a physical environment
  * Component of cyber-physical systems

Hasuo (Tokyo)

Hybrid System

Discrete

"jump"

and

Continuous

"flow"

Hasuo (Tokyo)

Hybrid System

Discrete

"jump"

and

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"flow"

Hasuo (Tokyo)

Hybrid System

Formal verification
(computer science)

Discrete
"**jump**"

and

Continuous
"**flow**"

Control theory
(applied analysis)

Hasuo (Tokyo)

Hybrid System

Formal verification
(computer science)

Hybrid!

Discrete
"**jump**"

and

Continuous
"**flow**"

Hybrid!

Control theory
(applied analysis)

Hasuo (Tokyo)

Hybrid System

Formal verification
(computer science)

Hybrid!

- Flow?
- With minimal cost?

Discrete
"**jump**"

and

Continuous
"**flow**"

Hybrid!

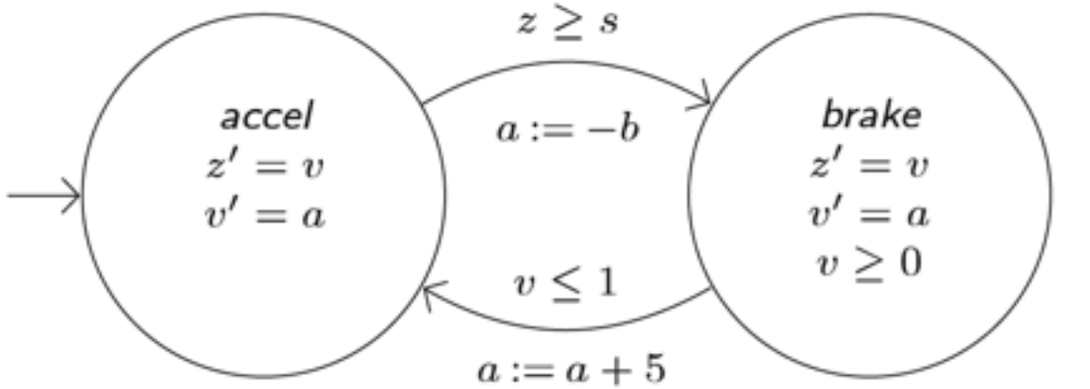
Control theory
(applied analysis)

Hasuo (Tokyo)

Formal Verification Approaches

* Hybrid automata

  [Alur, Henzinger, ...; '90s-]



* Differential dynamic logic

  [Platzer & others, '07-]

$$[\dot{x} = \mathbf{1} \text{ while } x \leq \mathbf{3}]\varphi$$

* Differential equations, explicitly

  ➜ distinction jump vs. flow

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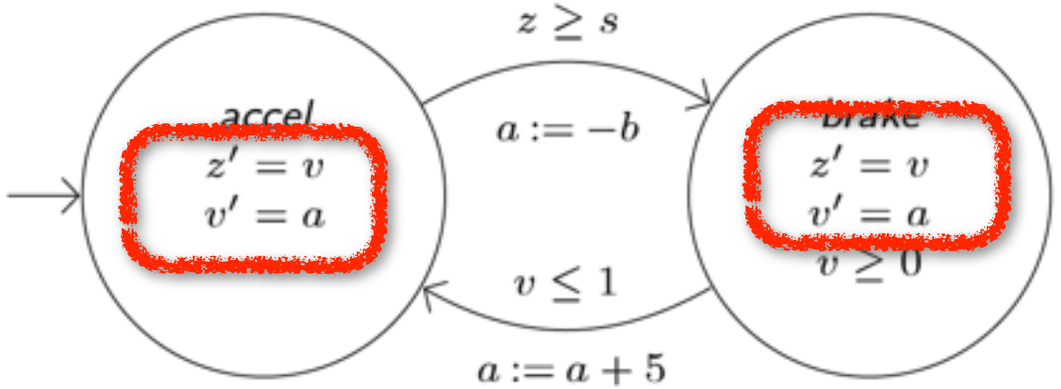

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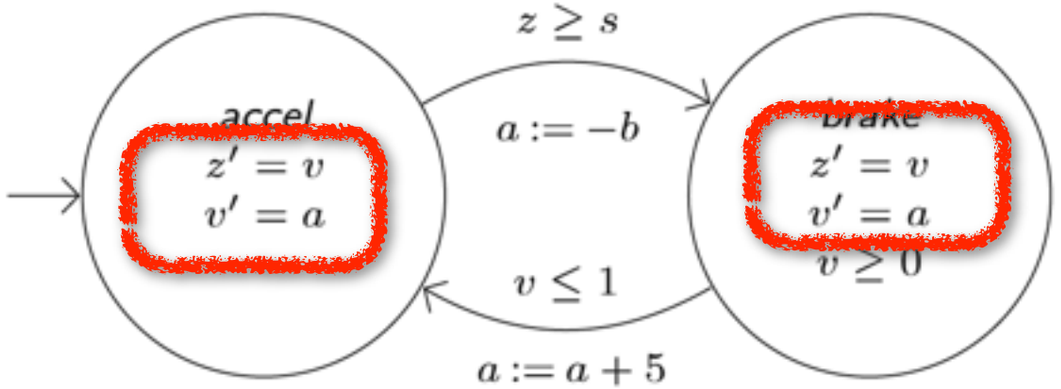

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"Turn Flow into Jump"

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t := 0 ;
while (t ≤ 1) do {
   t := t + dt
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Hasuo (Tokyo)

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* Infinitesimal number dt

* "Infinitely small" :

  0 < dt < r

  for **any** positive real r

Hasuo (Tokyo)

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Hasuo (Tokyo)

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* Non-standard analysis!

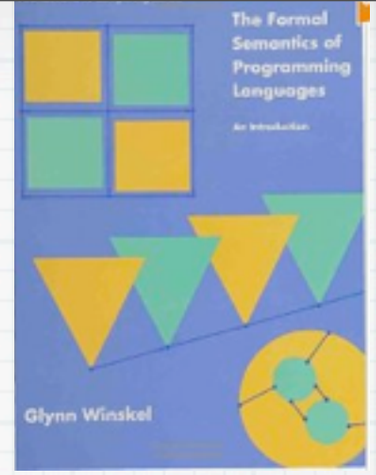
  [Robinson '60s]

Hasuo (Tokyo)

Theoretical Framework [Suenaga&H., ICALP'11]

Theoretical Framework

[Suenaga&H., ICALP'11]



The standard textbook [Winskel]

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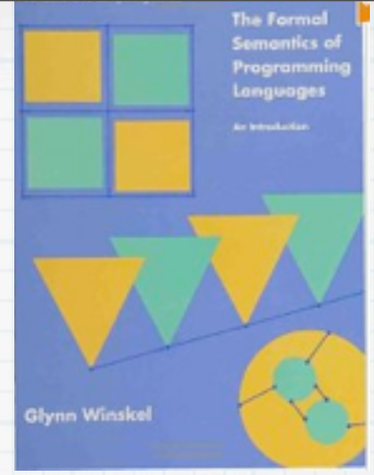
Theoretical Framework

[Suenaga&H., ICALP'11]



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Hasuo (Tokyo)

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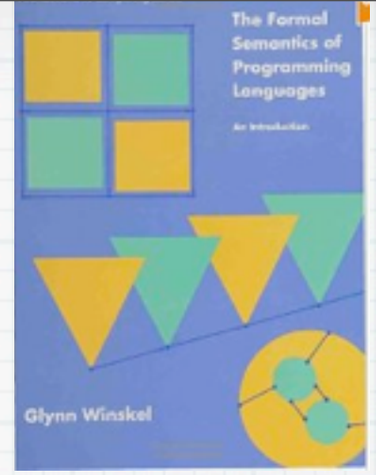

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First-order assertion lang.

$\exists z(x=2*z \wedge y=3*z)$

Hasuo (Tokyo)

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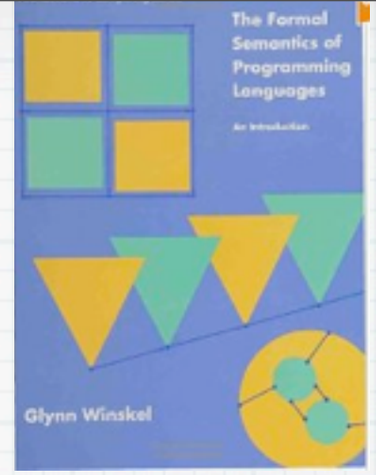

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Hoare

Hoare-style program logic

$$\frac{\{A \wedge b\}\, c\, \{A\}}{\{A\}\ \texttt{while}\ b\ \texttt{do}\ c\ \{A \wedge \neg b\}}$$

Hasuo (Tokyo)

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Rigorous semantics by non-standard analysis

Hasuo (Tokyo)

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- **Hoare**^{dt} : sound and relatively complete

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Rigorous semantics by non-standard analysis

- **Hoare**^{dt} : sound and relatively complete
- Program verification/static analysis of hybrid systems
- Actual verification with NSA

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Static Analysis

Nonstandard Analysis

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Static Analysis

Program Verif. Techniques

* Esp. invariant discovery

Nonstandard Analysis

Infinitesimal dt

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Nonstandard Static Analysis

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Nonstandard Static Analysis

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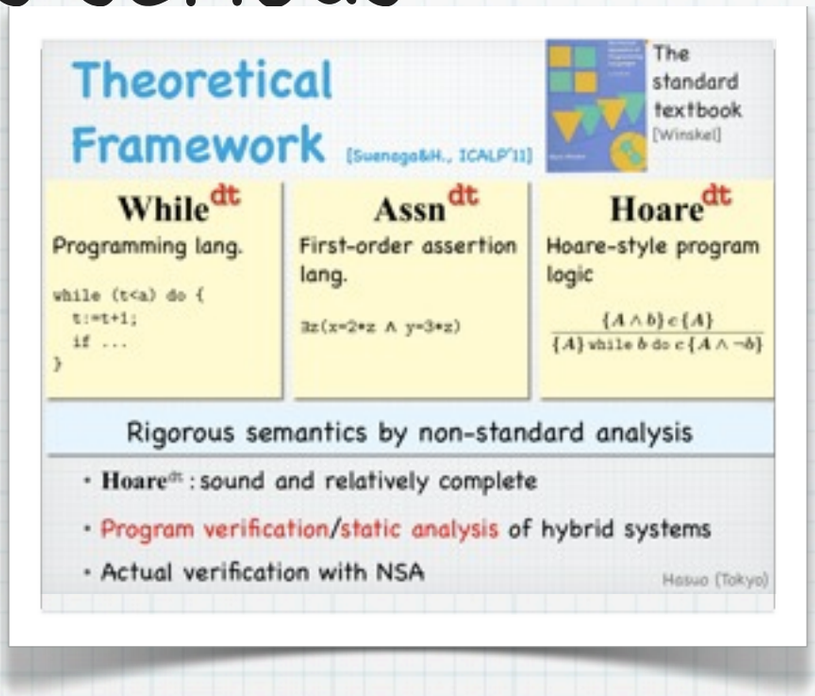
Nonstandard Static Analysis

* Towards serious use of



Nonstandard Static Analysis

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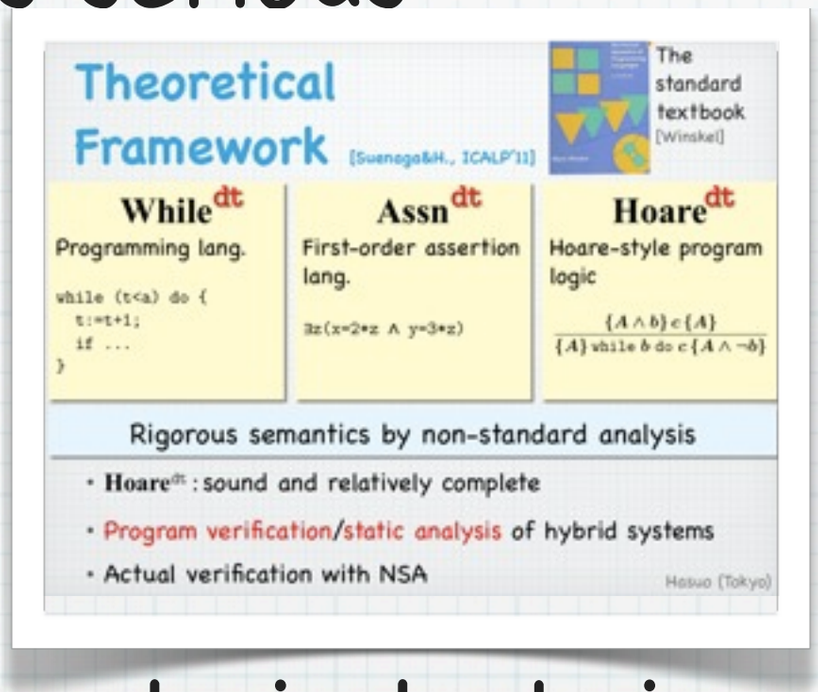

* Static analysis techniques **transferred** to hybrid appl.

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Nonstandard Static Analysis

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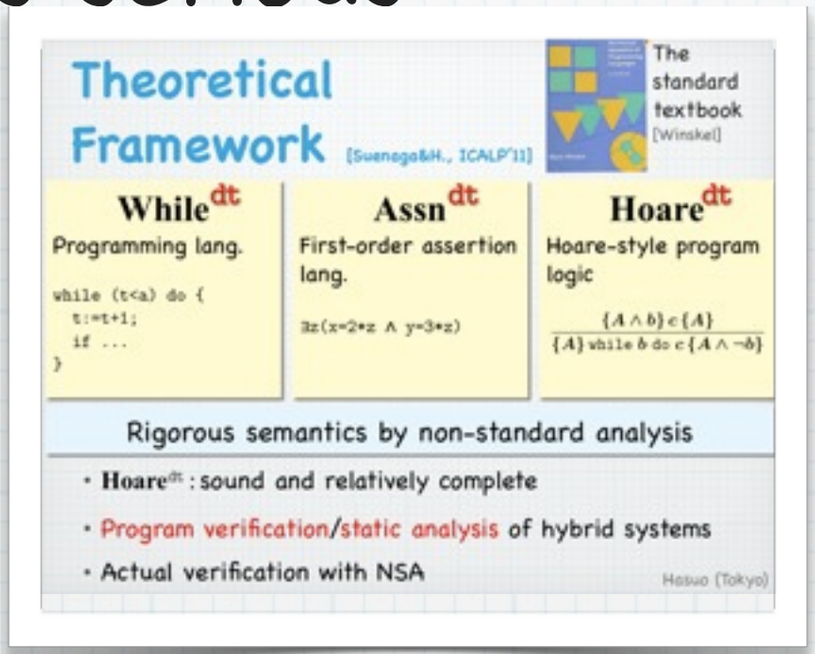

Exactly as they are!

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Hasuo (Tokyo)

Nonstandard Static Analysis

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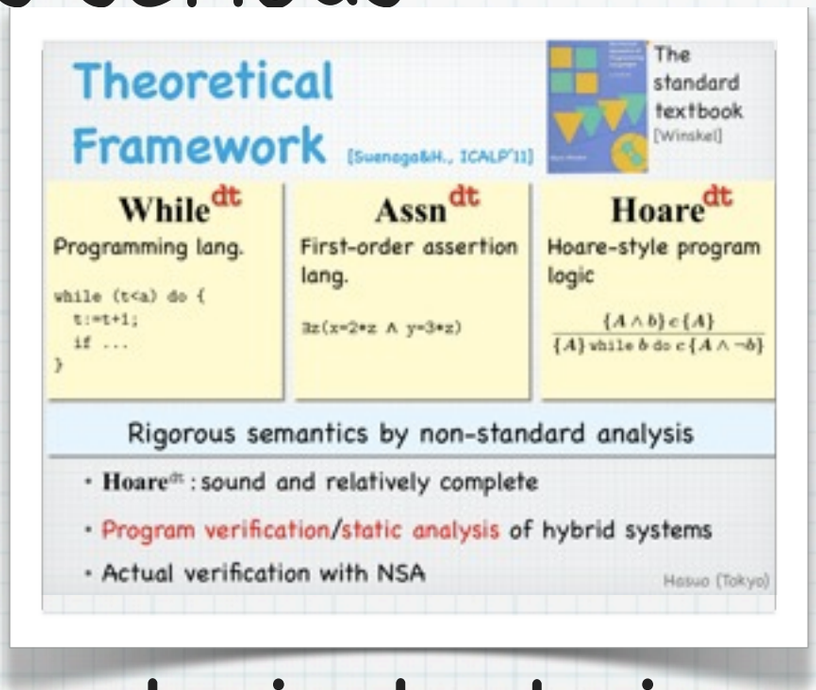
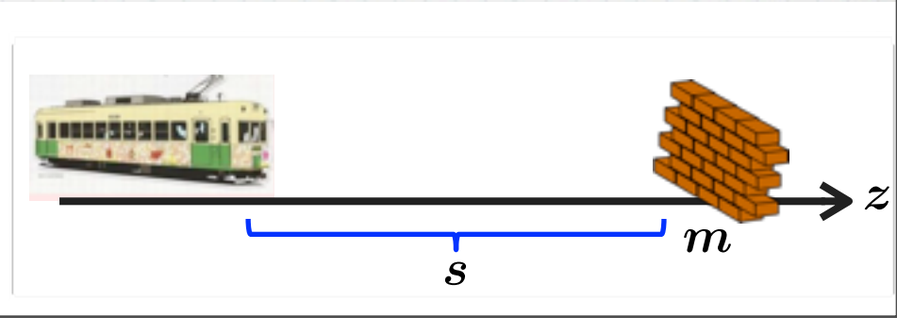


Exactly as they are!

* Static analysis techniques **transferred** to hybrid appl.
* Leading example: ETCS



Prototype Automatic Prover

Hasuo (Tokyo)

Prototype Automatic Prover

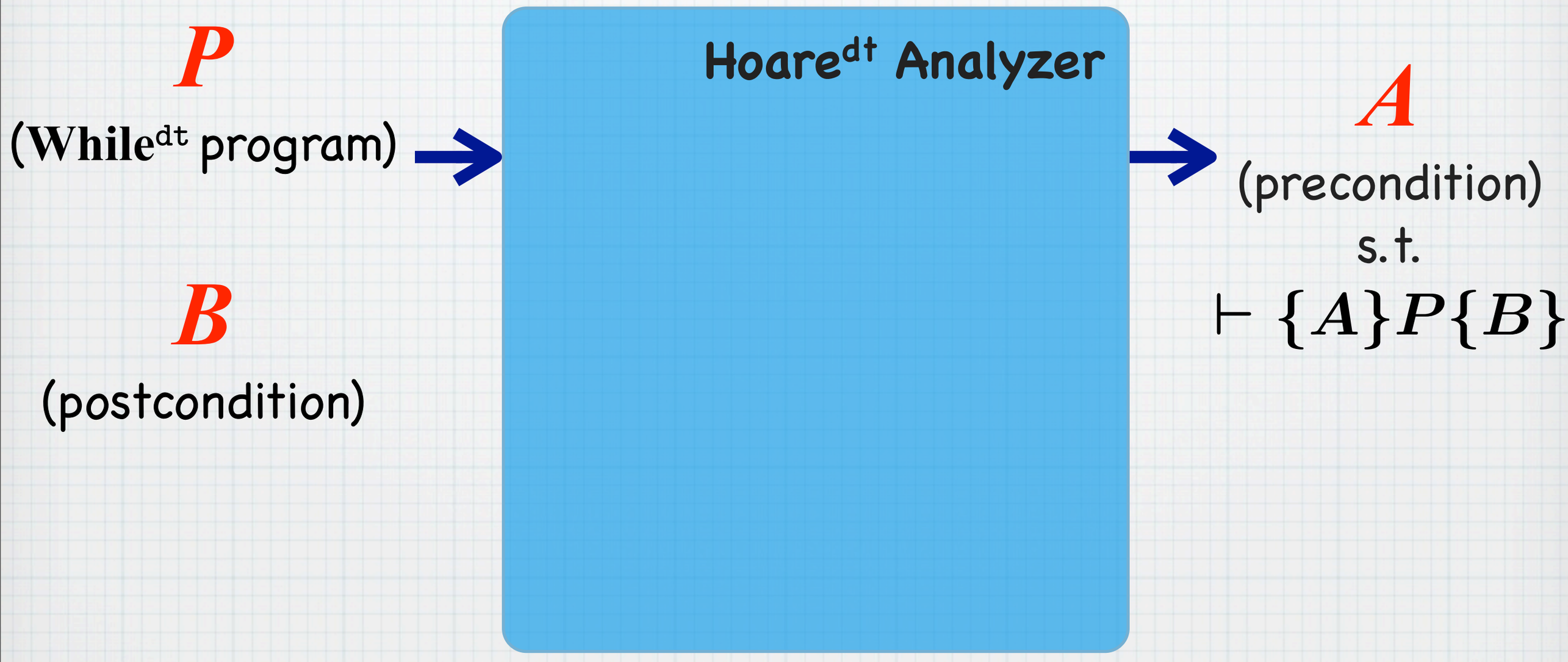
P

(**While**^{dt} program)

B

(postcondition)

Prototype Automatic Prover



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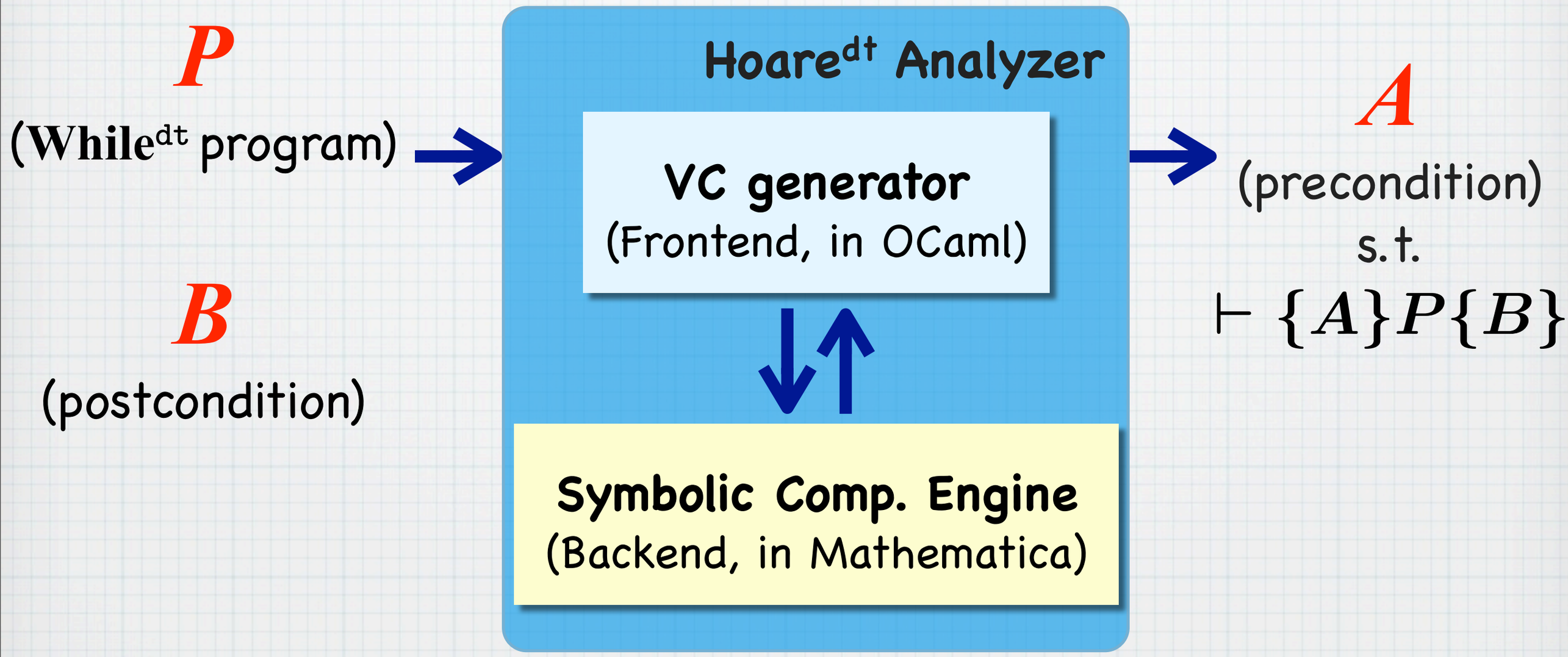

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Prototype Automatic Prover

P

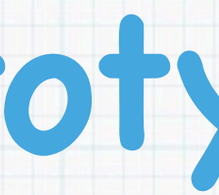
(While^{dt} program)

B

(postcondition)

Hoare^{dt} Analyzer

VC generator
(Frontend, in OCaml)



Symbolic Comp. Engine
(Backend, in Mathematica)

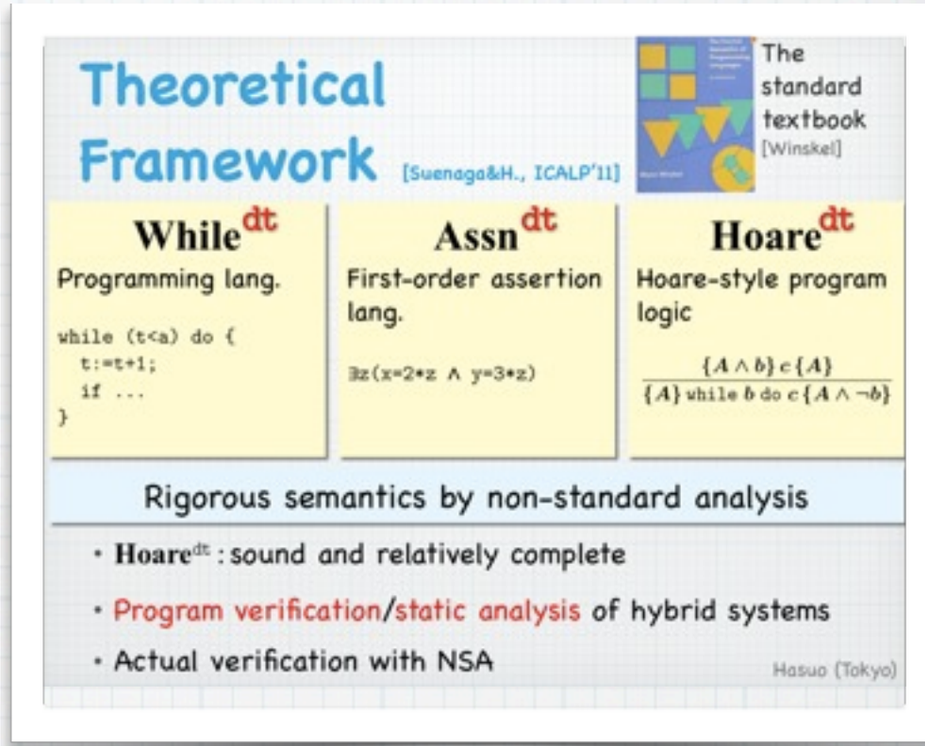
A

(precondition)
s.t.
$\vdash \{A\}P\{B\}$

For reals, not hyperreals
➜ justified by the
transfer principle

Outline

- **Theoretical foundations**
  - **While^{dt}, Assn^{dt}, Hoare^{dt}**
  - Rigorous semantics via NSA
    - Transfer principle, "sectionwise lemmas"
- **Static analysis techniques, transferred as they are**
  - Phase split [Sharma,Dillig,Dillig,Aiken; CAV'11] [Balakrishnan,Sankaranarayanan,Ivancic,Gupta; EMSOFT'09] [Gopan,Reps; SAS'07]
  - Differential invariant [Platzer,Clarke; CAV'08]
  - … and more!



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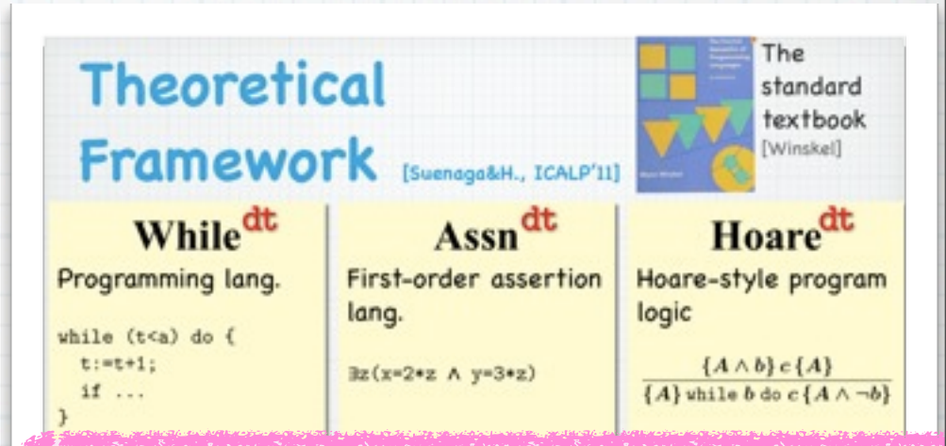

w/ or w/o dt ...

➜ logically "**the same**"

Hasuo (Tokyo)

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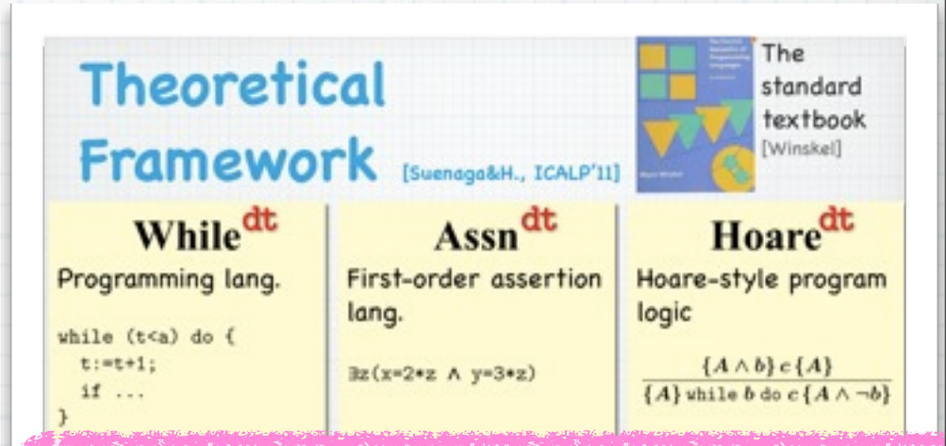

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Hasuo (Tokyo)

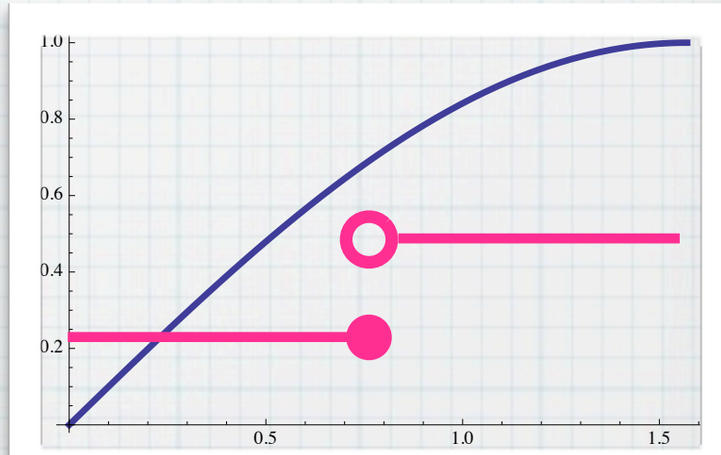
Part I:

Theoretical Foundations

Nonstandard Analysis

* Analysis with an **infinitesimal** δ, e.g.

$$\boldsymbol{f} \text{ is continuous} \iff \left(\begin{array}{l} |\boldsymbol{x} - \boldsymbol{x'}| \text{ is infinitesimal} \\ \Longrightarrow\ |\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{x'})| \text{ is infinitesimal} \end{array} \right)$$



* Cf. Leibniz's **monad**
* Done naively ➜ contradiction!

Hasuo (Tokyo)

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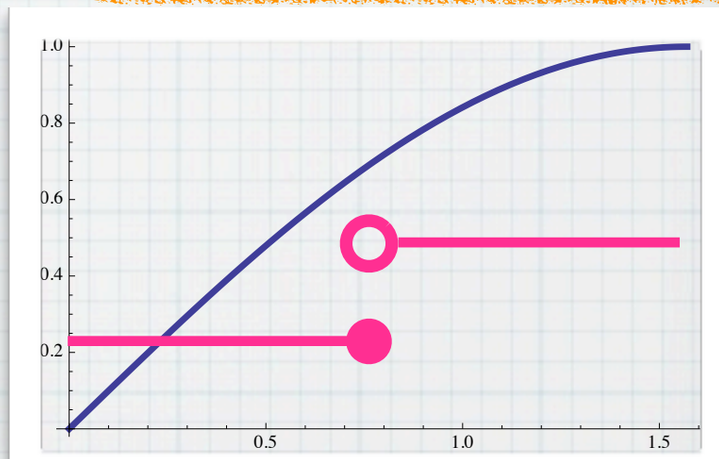
"Infinitely small"

$$0 < \delta < r$$

$$(\forall r \in \mathbb{R}_+)$$



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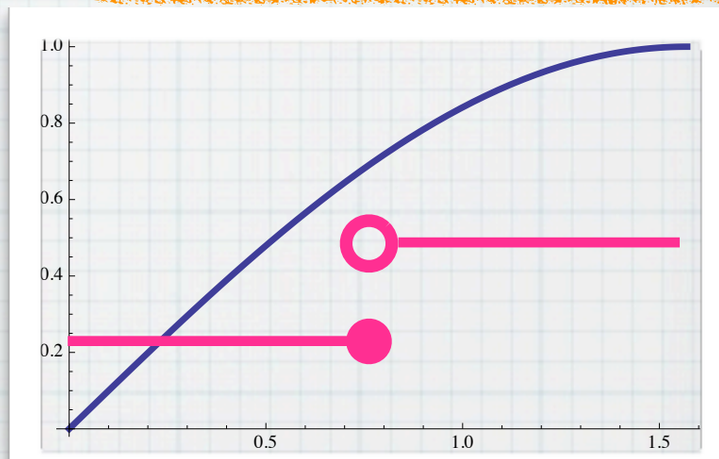

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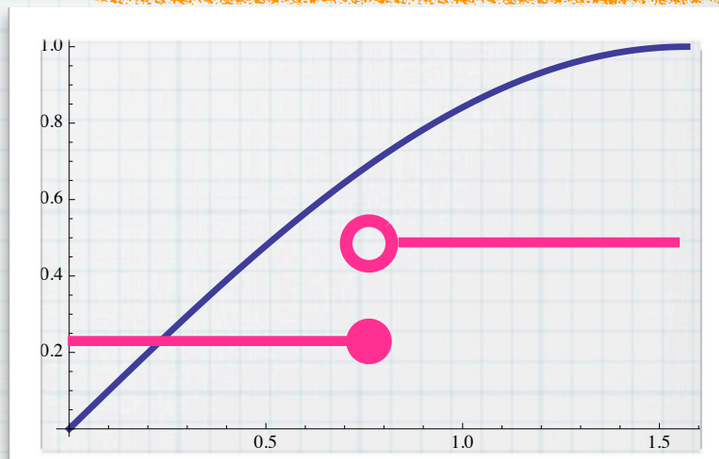

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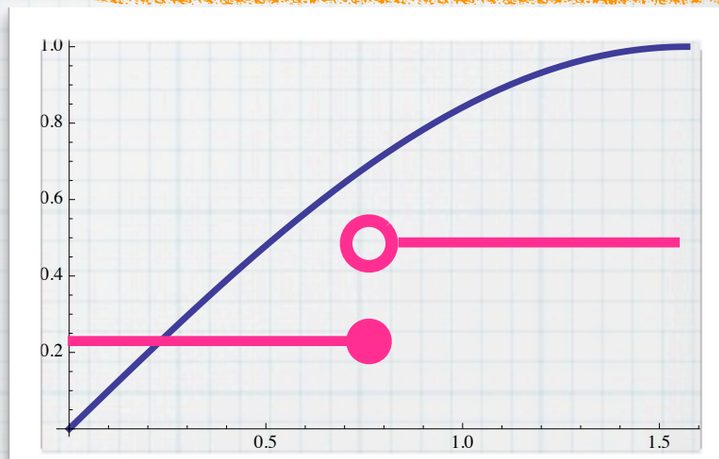

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Logical foundation via an ultrafilter

[Robinson,1960]

Hyperreals

= Reals + Infinitesimals + ...

Defn.

The set of *hyperreal numbers* is

$$^*\mathbb{R} := \mathbb{R}^{\mathbb{N}} / \sim_{\mathcal{F}}$$

Hasuo (Tokyo)

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Ignore

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Ignore

0th section

1st section

2nd section

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0th section

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Ignore

* Operations: sectionwise

$$\begin{array}{rl} & [(a_0, a_1, \dots)] \\ + & [(b_0, b_1, \dots)] \\ = & [(a_0 + b_0, a_1 + b_1, \dots)] \end{array}$$

Hyperreals

= Reals + Infinitesimals + ...

Defn.

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0th section 1st section 2nd section

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- Reals are hyperreals

$$\mathbb{R} \hookrightarrow {}^*\mathbb{R},$$

$$r \mapsto \left[(r, r, \dots)\right]$$

Hasuo (Tokyo)

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OK! ^ ✗ ✗ ✗ ✗ ^ ^ ...

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(existence by AC)

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An *ultrafilter* $\mathcal{F} \subseteq \mathcal{P}(\mathbb{N})$ is such that:

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is in $\mathcal{F}$.
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Thm. (Transfer Principle)
A: a first-order formula.
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$$\mathbb{R} \models A \iff {}^*\mathbb{R} \models {}^*A \ .$$

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$\mathbb{R}$ and $^*\mathbb{R}$ are "logically the same"

Theoretical Framework

[Suenaga&H., ICALP'11]

The standard textbook [Winskel]

While^{dt}

Programming lang.

```
while (t<a) do {
  t:=t+1;
  if ...
}
```

Assn^{dt}

First-order assertion lang.

$\exists z(x=2*z \wedge y=3*z)$

Hoare^{dt}

Hoare-style program logic

$$\frac{\{A \wedge b\}\, c\, \{A\}}{\{A\}\ \texttt{while}\ b\ \texttt{do}\ c\ \{A \wedge \neg b\}}$$

Rigorous semantics by non-standard analysis

Hasuo (Tokyo)

Syntax

While^{dt}

$$
\begin{array}{lll}
\textbf{AExp} \ni & a & ::= \quad x \mid \mathtt{c}_r \mid a_1 \ \mathtt{aop} \ a_2 \mid \mathtt{dt} \\
& \multicolumn{2}{l}{\text{where } \mathtt{c}_r \text{ is a const. for } r \in \mathbb{R},\ \mathtt{aop} \in \{+, -, \cdot, \hat{}, /\}} \\
\textbf{BExp} \ni & b & ::= \quad \mathtt{true} \mid \mathtt{false} \mid b_1 \wedge b_2 \mid \neg b \mid a_1 < a_2 \\
\textbf{Cmd} \ni & c & ::= \quad \mathtt{skip} \mid x := a \mid c_1 ; c_2 \\
& & \quad \mid \mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2 \mid \mathtt{while}\ b\ \mathtt{do}\ c
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$$

Assn^{dt}

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A \quad ::= \quad \mathtt{true} \mid \mathtt{false} \mid A_1 \wedge A_2 \mid \neg A \mid a_1 < a_2 \mid \forall x \in {}^*\mathbb{N}.\, A \mid \forall x \in {}^*\mathbb{R}.\, A
$$

Hoare^{dt}

$$
\frac{}{\{A\}\ \mathtt{skip}\ \{A\}}\ (\textsc{Skip}) \qquad \frac{}{\{\, A[a/x] \,\}\ x := a\ \{A\}}\ (\textsc{Assign})
$$

$$
\frac{\{A\}\ c_1\ \{C\} \quad \{C\}\ c_2\ \{B\}}{\{A\}\ c_1 ; c_2\ \{B\}}\ (\textsc{Seq}) \qquad \frac{\{A \wedge b\}\ c_1\ \{B\} \quad \{A \wedge \neg b\}\ c_2\ \{B\}}{\{A\}\ \mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2\ \{B\}}\ (\textsc{If})
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$$
\frac{\{A \wedge b\}\ c\ \{A\}}{\{A\}\ \mathtt{while}\ b\ \mathtt{do}\ c\ \{A \wedge \neg b\}}\ (\textsc{While}) \qquad \frac{\models A \Rightarrow A' \quad \{A'\}\ c\ \{B'\} \quad \models B' \Rightarrow B}{\{A\}\ c\ \{B\}}\ (\textsc{Conseq})
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$$\textbf{Cmd} \ni c ::= \mathtt{skip} \mid x := a \mid c_1; c_2 \mid \mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2 \mid \mathtt{while}\ b\ \mathtt{do}\ c$$

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$$\frac{\{A\}\,c_1\,\{C\} \quad \{C\}\,c_2\,\{B\}}{\{A\}\,c_1; c_2\,\{B\}}\ (\textsc{Seq}) \qquad \frac{\{A \wedge b\}\,c_1\,\{B\} \quad \{A \wedge \neg b\}\,c_2\,\{B\}}{\{A\}\,\mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2\,\{B\}}\ (\textsc{If})$$

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$$\textbf{Cmd} \ni \quad c \quad ::= \quad \mathtt{skip} \mid x := a \mid c_1 ; c_2 \mid \mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2 \mid \mathtt{while}\ b\ \mathtt{do}\ c$$

$$\frac{}{\{A\}\,\mathtt{skip}\,\{A\}}\ (\textsc{Skip}) \qquad \frac{}{\{A[a/x]\}\,x := a\,\{A\}}\ (\textsc{Assign})$$

$$\frac{\{A\}\,c_1\,\{C\} \quad \{C\}\,c_2\,\{B\}}{\{A\}\,c_1 ; c_2\,\{B\}}\ (\textsc{Seq}) \qquad \frac{\{A \wedge b\}\,c_1\,\{B\} \quad \{A \wedge \neg b\}\,c_2\,\{B\}}{\{A\}\,\mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2\,\{B\}}\ (\textsc{If})$$

$$\frac{\{A \wedge b\}\,c\,\{A\}}{\{A\}\,\mathtt{while}\ b\ \mathtt{do}\ c\,\{A \wedge \neg b\}}\ (\textsc{While}) \qquad \frac{\models A \Rightarrow A' \quad \{A'\}\,c\,\{B'\} \quad \models B' \Rightarrow B}{\{A\}\,c\,\{B\}}\ (\textsc{Conseq})$$

Tokyo)

Syntax

While$^{\mathtt{dt}}$

While + dt

$$\textbf{AExp} \ni \quad a \quad ::= \quad x \mid \mathtt{c}_r \mid a_1 \ \mathtt{aop} \ a_2 \mid \mathtt{dt}$$

where $\mathtt{c}_r$ is a const. for $r \in \mathbb{R}$, $\mathtt{aop} \in \{+, -, \cdot, \hat{}, /\}$

$$\textbf{BExp} \ni \quad b \quad ::= \quad \mathtt{true} \mid \mathtt{false} \mid b_1 \wedge b_2 \mid \neg b \mid a_1 < a_2$$

$$\textbf{Cmd} \ni \quad c \quad ::= \quad \mathtt{skip} \mid x := a \mid c_1 ; c_2 \mid \mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2 \mid \mathtt{while}\ b\ \mathtt{do}\ c$$

$$\frac{}{\{A\}\,\mathtt{skip}\,\{A\}}\ (\textsc{Skip}) \qquad \frac{}{\{A[a/x]\}\,x := a\,\{A\}}\ (\textsc{Assign})$$

$$\frac{\{A\}\,c_1\,\{C\} \quad \{C\}\,c_2\,\{B\}}{\{A\}\,c_1; c_2\,\{B\}}\ (\textsc{Seq}) \qquad \frac{\{A \wedge b\}\,c_1\,\{B\} \quad \{A \wedge \neg b\}\,c_2\,\{B\}}{\{A\}\,\mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2\,\{B\}}\ (\textsc{If})$$

$$\frac{\{A \wedge b\}\,c\,\{A\}}{\{A\}\,\mathtt{while}\ b\ \mathtt{do}\ c\,\{A \wedge \neg b\}}\ (\textsc{While}) \qquad \frac{\models A \Rightarrow A' \quad \{A'\}\,c\,\{B'\} \quad \models B' \Rightarrow B}{\{A\}\,c\,\{B\}}\ (\textsc{Conseq})$$

Tokyo)

Syntax

While^{dt}

While + dt

$$
\begin{array}{lll}
\textbf{AExp} \ni & a & ::= \quad x \mid \mathtt{c}_r \mid a_1 \ \mathtt{aop} \ a_2 \mid \mathtt{dt} \\
& & \text{where } \mathtt{c}_r \text{ is a const. for } r \in \mathbb{R},\ \mathtt{aop} \in \{+, -, \cdot, \hat{}, /\} \\
\textbf{BExp} \ni & b & ::= \quad \mathtt{true} \mid \mathtt{false} \mid b_1 \wedge b_2 \mid \neg b \mid a_1 < a_2 \\
\textbf{Cmd} \ni & c & ::= \quad \mathtt{skip} \mid x := a \mid c_1 ; c_2 \\
& & \quad \mid \mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2 \mid \mathtt{while}\ b\ \mathtt{do}\ c
\end{array}
$$

Assn^{dt}

$$
\begin{array}{lll}
A & ::= & \mathtt{true} \mid \mathtt{false} \mid A_1 \wedge A_2 \mid \neg A \mid a_1 < a_2 \mid \\
& & \forall x \in {}^*\mathbb{N}.\, A \mid \forall x \in {}^*\mathbb{R}.\, A
\end{array}
$$

Hoare^{dt}

$$\frac{}{\{A\}\ \mathtt{skip}\ \{A\}}\ (\textsc{Skip}) \qquad \frac{}{\{\,A[a/x]\,\}\ x := a\ \{A\}}\ (\textsc{Assign})$$

$$\frac{\{A\}\ c_1\ \{C\} \quad \{C\}\ c_2\ \{B\}}{\{A\}\ c_1; c_2\ \{B\}}\ (\textsc{Seq}) \qquad \frac{\{A \wedge b\}\ c_1\ \{B\} \quad \{A \wedge \neg b\}\ c_2\ \{B\}}{\{A\}\ \mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2\ \{B\}}\ (\textsc{If})$$

$$\frac{\{A \wedge b\}\ c\ \{A\}}{\{A\}\ \mathtt{while}\ b\ \mathtt{do}\ c\ \{A \wedge \neg b\}}\ (\textsc{While}) \qquad \frac{\models A \Rightarrow A' \quad \{A'\}\ c\ \{B'\} \quad \models B' \Rightarrow B}{\{A\}\ c\ \{B\}}\ (\textsc{Conseq})$$

Tokyo)

Syntax

While$^{\mathtt{dt}}$

While + dt

$$
\begin{array}{lllll}
\textbf{AExp} \ni & a & ::= & x \mid \mathtt{c}_r \mid a_1 \ \mathtt{aop} \ a_2 \mid \mathtt{dt} & \\
& \text{where } \mathtt{c}_r \text{ is a const. for } r \in \mathbb{R},\ \mathtt{aop} \in \{+, -, \cdot, \hat{}, /\} & & & \\
\textbf{BExp} \ni & b & ::= & \mathtt{true} \mid \mathtt{false} \mid b_1 \wedge b_2 \mid \neg b \mid a_1 < a_2 & \\
\textbf{Cmd} \ni & c & ::= & \mathtt{skip} \mid x := a \mid c_1 ; c_2 & \\
& & & \mid \mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2 \mid \mathtt{while}\ b\ \mathtt{do}\ c &
\end{array}
$$

Assn$^{\mathtt{dt}}$

$$
\begin{array}{lll}
A & ::= & \mathtt{true} \mid \mathtt{false} \mid A_1 \wedge A_2 \mid \neg A \mid a_1 < a_2 \mid \\
& & \forall x \in {}^*\mathbb{N}.\, A \mid \forall x \in {}^*\mathbb{R}.\, A
\end{array}
$$

Hoare$^{\mathtt{dt}}$

$$\frac{}{\{A\}\,\mathtt{skip}\,\{A\}}\ (\textsc{Skip}) \qquad \frac{}{\{\,A[a/x]\,\}\,x := a\,\{A\}}\ (\textsc{Assign})$$

$$\frac{\{A\}\,c_1\,\{C\} \quad \{C\}\,c_2\,\{B\}}{\{A\}\,c_1; c_2\,\{B\}}\ (\textsc{Seq}) \qquad \frac{\{A \wedge b\}\,c_1\,\{B\} \quad \{A \wedge \neg b\}\,c_2\,\{B\}}{\{A\}\,\mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2\,\{B\}}\ (\textsc{If})$$

$$\frac{\{A \wedge b\}\,c\,\{A\}}{\{A\}\,\mathtt{while}\ b\ \mathtt{do}\ c\,\{A \wedge \neg b\}}\ (\textsc{While}) \qquad \frac{\models A \Rightarrow A' \quad \{A'\}\,c\,\{B'\} \quad \models B' \Rightarrow B}{\{A\}\,c\,\{B\}}\ (\textsc{Conseq})$$

Tokyo)

Syntax

While^dt

While + dt

$$
\begin{array}{lll}
\textbf{AExp} \ni & a & ::= \quad x \mid \mathtt{c}_r \mid a_1 \ \mathtt{aop}\ a_2 \mid \mathtt{dt} \\
& & \text{where } \mathtt{c}_r \text{ is a const. for } r \in \mathbb{R},\ \mathtt{aop} \in \{+, -, \cdot, \hat{}, /\} \\
\textbf{BExp} \ni & b & ::= \quad \mathtt{true} \mid \mathtt{false} \mid b_1 \wedge b_2 \mid \neg b \mid a_1 < a_2 \\
\textbf{Cmd} \ni & c & ::= \quad \mathtt{skip} \mid x := a \mid c_1; c_2 \\
& & \quad \mid \mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2 \mid \mathtt{while}\ b\ \mathtt{do}\ c
\end{array}
$$

Assn^dt

$$
\begin{array}{lll}
A & ::= & \mathtt{true} \mid \mathtt{false} \mid A_1 \wedge A_2 \mid \neg A \mid a_1 < a_2 \mid \\
& & \forall x \in {}^*\mathbb{N}.\ A \mid \forall x \in {}^*\mathbb{R}.\ A
\end{array}
$$

Hoare^dt

$$\frac{}{\{A\}\ \mathtt{skip}\ \{A\}}\ (\textsc{Skip}) \qquad \frac{}{\{\,A[a/x]\,\}\ x := a\ \{A\}}\ (\textsc{Assign})$$

$$\frac{\{A\}\ c_1\ \{C\} \quad \{C\}\ c_2\ \{B\}}{\{A\}\ c_1; c_2\ \{B\}}\ (\textsc{Seq}) \qquad \frac{\{A \wedge b\}\ c_1\ \{B\} \quad \{A \wedge \neg b\}\ c_2\ \{B\}}{\{A\}\ \mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2\ \{B\}}\ (\textsc{If})$$

$$\frac{\{A \wedge b\}\ c\ \{A\}}{\{A\}\ \mathtt{while}\ b\ \mathtt{do}\ c\ \{A \wedge \neg b\}}\ (\textsc{While}) \qquad \frac{\models A \Rightarrow A' \quad \{A'\}\ c\ \{B'\} \quad \models B' \Rightarrow B}{\{A\}\ c\ \{B\}}\ (\textsc{Conseq})$$

Tokyo)

Syntax

While + dt

While^{dt}

$$
\begin{array}{lll}
\textbf{AExp} \ni & a & ::= \quad x \mid \mathtt{c}_r \mid a_1 \ \mathtt{aop} \ a_2 \mid \mathtt{dt} \\
& \multicolumn{2}{l}{\text{where } \mathtt{c}_r \text{ is a const. for } r \in \mathbb{R},\ \mathtt{aop} \in \{+, -, \cdot, \hat{}, /\}} \\
\textbf{BExp} \ni & b & ::= \quad \mathtt{true} \mid \mathtt{false} \mid b_1 \wedge b_2 \mid \neg b \mid a_1 < a_2 \\
\textbf{Cmd} \ni & c & ::= \quad \mathtt{skip} \mid x := a \mid c_1 ; c_2 \\
& & \quad \mid \mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2 \mid \mathtt{while}\ b\ \mathtt{do}\ c
\end{array}
$$

Assn^{dt}

$$
A \quad ::= \quad \mathtt{true} \mid \mathtt{false} \mid A_1 \wedge A_2 \mid \neg A \mid a_1 < a_2 \mid \forall x \in {}^*\mathbb{N}.\, A \mid \forall x \in {}^*\mathbb{R}.\, A
$$

$$
\frac{}{\{A\}\ \mathtt{skip}\ \{A\}}\ (\textsc{Skip}) \qquad \frac{}{\{A[a/x]\}\ x := a\ \{A\}}\ (\textsc{Assign})
$$

$$
\frac{\{A\}\ c_1\ \{C\} \quad \{C\}\ c_2\ \{B\}}{\{A\}\ c_1; c_2\ \{B\}}\ (\textsc{Seq}) \qquad \frac{\{A \wedge b\}\ c_1\ \{B\} \quad \{A \wedge \neg b\}\ c_2\ \{B\}}{\{A\}\ \mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2\ \{B\}}\ (\textsc{If})
$$

$$
\frac{\{A \wedge b\}\ c\ \{A\}}{\{A\}\ \mathtt{while}\ b\ \mathtt{do}\ c\ \{A \wedge \neg b\}}\ (\textsc{While}) \qquad \frac{\models A \Rightarrow A' \quad \{A'\}\ c\ \{B'\} \quad \models B' \Rightarrow B}{\{A\}\ c\ \{B\}}\ (\textsc{Conseq})
$$

Tokyo)

Syntax

While + dt

While^{dt}

$$
\begin{array}{lll}
\textbf{AExp} \ni & a & ::= \quad x \mid \mathtt{c}_r \mid a_1 \ \mathtt{aop} \ a_2 \mid \mathtt{dt} \\
& \text{where } \mathtt{c}_r \text{ is a const. for } r \in \mathbb{R},\ \mathtt{aop} \in \{+, -, \cdot, \hat{}, /\} & \\
\textbf{BExp} \ni & b & ::= \quad \mathtt{true} \mid \mathtt{false} \mid b_1 \wedge b_2 \mid \neg b \mid a_1 < a_2 \\
\textbf{Cmd} \ni & c & ::= \quad \mathtt{skip} \mid x := a \mid c_1 ; c_2 \\
& & \quad \mid \mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2 \mid \mathtt{while}\ b\ \mathtt{do}\ c
\end{array}
$$

Assn^{dt}

$$
A \quad ::= \quad \mathtt{true} \mid \mathtt{false} \mid A_1 \wedge A_2 \mid \neg A \mid a_1 < a_2 \mid \forall x \in {}^*\mathbb{N}.\, A \mid \forall x \in {}^*\mathbb{R}.\, A
$$

$$
\frac{}{\{A\}\ \mathtt{skip}\ \{A\}}\ (\textsc{Skip}) \qquad \frac{}{\{A[a/x]\}\ x := a\ \{A\}}\ (\textsc{Assign})
$$

$$
\frac{\{A\}\ c_1\ \{C\} \quad \{C\}\ c_2\ \{B\}}{\{A\}\ c_1; c_2\ \{B\}}\ (\textsc{Seq}) \qquad \frac{\{A \wedge b\}\ c_1\ \{B\} \quad \{A \wedge \neg b\}\ c_2\ \{B\}}{\{A\}\ \mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2\ \{B\}}\ (\textsc{If})
$$

$$
\frac{\{A \wedge b\}\ c\ \{A\}}{\{A\}\ \mathtt{while}\ b\ \mathtt{do}\ c\ \{A \wedge \neg b\}}\ (\textsc{While}) \qquad \frac{\models A \Rightarrow A' \quad \{A'\}\ c\ \{B'\} \quad \models B' \Rightarrow B}{\{A\}\ c\ \{B\}}\ (\textsc{Conseq})
$$

Tokyo)

Syntax

While^dt

While + dt

$$
\begin{array}{lllll}
\textbf{AExp} \ni & a & ::= & x \mid \mathtt{c}_r \mid a_1 \ \mathtt{aop}\ a_2 \mid \mathtt{dt} & \\
& \text{where } \mathtt{c}_r \text{ is a const. for } r \in \mathbb{R},\ \mathtt{aop} \in \{+, -, \cdot, \hat{}, /\} & & & \\
\textbf{BExp} \ni & b & ::= & \mathtt{true} \mid \mathtt{false} \mid b_1 \wedge b_2 \mid \neg b \mid a_1 < a_2 & \\
\textbf{Cmd} \ni & c & ::= & \mathtt{skip} \mid x := a \mid c_1 ; c_2 & \\
& & & \mid \mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2 \mid \mathtt{while}\ b\ \mathtt{do}\ c &
\end{array}
$$

Assn^dt

Assn, *-transformed

$$
\begin{array}{lll}
A & ::= & \mathtt{true} \mid \mathtt{false} \mid A_1 \wedge A_2 \mid \neg A \mid a_1 < a_2 \\
& & \forall x \in {}^*\mathbb{N}.\, A \mid \forall x \in {}^*\mathbb{R}.\, A
\end{array}
$$

$$\frac{}{\{A\}\ \mathtt{skip}\ \{A\}}\ (\textsc{Skip})$$

$$\frac{}{\{A[a/x]\}\ x := a\ \{A\}}\ (\textsc{Assign})$$

$$\frac{\{A\}\ c_1\ \{C\} \quad \{C\}\ c_2\ \{B\}}{\{A\}\ c_1; c_2\ \{B\}}\ (\textsc{Seq})$$

$$\frac{\{A \wedge b\}\ c_1\ \{B\} \quad \{A \wedge \neg b\}\ c_2\ \{B\}}{\{A\}\ \mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2\ \{B\}}\ (\textsc{If})$$

$$\frac{\{A \wedge b\}\ c\ \{A\}}{\{A\}\ \mathtt{while}\ b\ \mathtt{do}\ c\ \{A \wedge \neg b\}}\ (\textsc{While})$$

$$\frac{\models A \Rightarrow A' \quad \{A'\}\ c\ \{B'\} \quad \models B' \Rightarrow B}{\{A\}\ c\ \{B\}}\ (\textsc{Conseq})$$

Tokyo)

Syntax

While^dt

While + dt

$$
\begin{array}{lll}
\textbf{AExp} \ni & a & ::= \quad x \mid \mathtt{c}_r \mid a_1 \ \mathtt{aop}\ a_2 \mid \mathtt{dt} \\
& & \text{where } \mathtt{c}_r \text{ is a const. for } r \in \mathbb{R},\ \mathtt{aop} \in \{+, -, \cdot, \hat{}, /\} \\
\textbf{BExp} \ni & b & ::= \quad \mathtt{true} \mid \mathtt{false} \mid b_1 \wedge b_2 \mid \neg b \mid a_1 < a_2 \\
\textbf{Cmd} \ni & c & ::= \quad \mathtt{skip} \mid x := a \mid c_1; c_2 \\
& & \quad \mid \mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2 \mid \mathtt{while}\ b\ \mathtt{do}\ c
\end{array}
$$

Assn^dt

Assn, *-transformed

$$
\begin{array}{lll}
A & ::= & \mathtt{true} \mid \mathtt{false} \mid A_1 \wedge A_2 \mid \neg A \mid a_1 < a_2 \mid \\
& & \forall x \in {}^*\mathbb{N}.\, A \mid \forall x \in {}^*\mathbb{R}.\, A
\end{array}
$$

Hoare^dt

$$\frac{}{\{A\}\ \mathtt{skip}\ \{A\}}\ (\textsc{Skip}) \qquad \frac{}{\{\,A[a/x]\,\}\ x := a\ \{A\}}\ (\textsc{Assign})$$

$$\frac{\{A\}\ c_1\ \{C\} \quad \{C\}\ c_2\ \{B\}}{\{A\}\ c_1; c_2\ \{B\}}\ (\textsc{Seq}) \qquad \frac{\{A \wedge b\}\ c_1\ \{B\} \quad \{A \wedge \neg b\}\ c_2\ \{B\}}{\{A\}\ \mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2\ \{B\}}\ (\textsc{If})$$

$$\frac{\{A \wedge b\}\ c\ \{A\}}{\{A\}\ \mathtt{while}\ b\ \mathtt{do}\ c\ \{A \wedge \neg b\}}\ (\textsc{While}) \qquad \frac{\models A \Rightarrow A' \quad \{A'\}\ c\ \{B'\} \quad \models B' \Rightarrow B}{\{A\}\ c\ \{B\}}\ (\textsc{Conseq})$$

Tokyo)

Syntax

While^dt

While + dt

$$
\begin{array}{lll}
\textbf{AExp} \ni & a & ::= \quad x \mid \mathtt{c}_r \mid a_1 \ \mathtt{aop}\ a_2 \mid \mathtt{dt} \\
& & \text{where } \mathtt{c}_r \text{ is a const. for } r \in \mathbb{R},\ \mathtt{aop} \in \{+, -, \cdot, \hat{}, /\} \\
\textbf{BExp} \ni & b & ::= \quad \mathtt{true} \mid \mathtt{false} \mid b_1 \wedge b_2 \mid \neg b \mid a_1 < a_2 \\
\textbf{Cmd} \ni & c & ::= \quad \mathtt{skip} \mid x := a \mid c_1; c_2 \\
& & \quad \mid \mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2 \mid \mathtt{while}\ b\ \mathtt{do}\ c
\end{array}
$$

Assn^dt

Assn, *-transformed

$$
\begin{array}{lll}
A & ::= & \mathtt{true} \mid \mathtt{false} \mid A_1 \wedge A_2 \mid \neg A \mid a_1 < a_2 \mid \\
& & \forall x \in {}^*\mathbb{N}.\, A \mid \forall x \in {}^*\mathbb{R}.\, A
\end{array}
$$

Hoare^dt

$$\frac{}{\{A\}\ \mathtt{skip}\ \{A\}}\ (\textsc{Skip}) \qquad \frac{}{\{\,A[a/x]\,\}\ x := a\ \{A\}}\ (\textsc{Assign})$$

$$\frac{\{A\}\ c_1\ \{C\} \quad \{C\}\ c_2\ \{B\}}{\{A\}\ c_1; c_2\ \{B\}}\ (\textsc{Seq}) \qquad \frac{\{A \wedge b\}\ c_1\ \{B\} \quad \{A \wedge \neg b\}\ c_2\ \{B\}}{\{A\}\ \mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2\ \{B\}}\ (\textsc{If})$$

$$\frac{\{A \wedge b\}\ c\ \{A\}}{\{A\}\ \mathtt{while}\ b\ \mathtt{do}\ c\ \{A \wedge \neg b\}}\ (\textsc{While}) \qquad \frac{\models A \Rightarrow A' \quad \{A'\}\ c\ \{B'\} \quad \models B' \Rightarrow B}{\{A\}\ c\ \{B\}}\ (\textsc{Conseq})$$

Tokyo)

Syntax

While^dt

While + dt

$$
\begin{array}{lll}
\textbf{AExp} \ni & a & ::= \quad x \mid \mathtt{c}_r \mid a_1 \ \mathtt{aop}\ a_2 \mid \mathtt{dt} \\
& \multicolumn{2}{l}{\text{where } \mathtt{c}_r \text{ is a const. for } r \in \mathbb{R},\ \mathtt{aop} \in \{+, -, \cdot, \hat{}, /\}} \\
\textbf{BExp} \ni & b & ::= \quad \mathtt{true} \mid \mathtt{false} \mid b_1 \wedge b_2 \mid \neg b \mid a_1 < a_2 \\
\textbf{Cmd} \ni & c & ::= \quad \mathtt{skip} \mid x := a \mid c_1; c_2 \\
& & \quad \mid \mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2 \mid \mathtt{while}\ b\ \mathtt{do}\ c
\end{array}
$$

Assn^dt

Assn, *-transformed

$$
\begin{array}{lll}
A & ::= & \mathtt{true} \mid \mathtt{false} \mid A_1 \wedge A_2 \mid \neg A \mid a_1 < a_2 \mid \\
& & \forall x \in {}^*\mathbb{N}.\ A \mid \forall x \in {}^*\mathbb{R}.\ A
\end{array}
$$

Hoare^dt

$$\frac{}{\{A\}\ \mathtt{skip}\ \{A\}}\ (\textsc{Skip}) \qquad \frac{}{\{\,A[a/x]\,\}\ x := a\ \{A\}}\ (\textsc{Assign})$$

$$\frac{\{A\}\ c_1\ \{C\} \quad \{C\}\ c_2\ \{B\}}{\{A\}\ c_1; c_2\ \{B\}}\ (\textsc{Seq}) \qquad \frac{\{A \wedge b\}\ c_1\ \{B\} \quad \{A \wedge \neg b\}\ c_2\ \{B\}}{\{A\}\ \mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2\ \{B\}}\ (\textsc{If})$$

$$\frac{\{A \wedge b\}\ c\ \{A\}}{\{A\}\ \mathtt{while}\ b\ \mathtt{do}\ c\ \{A \wedge \neg b\}}\ (\textsc{While}) \qquad \frac{\models A \Rightarrow A' \quad \{A'\}\ c\ \{B'\} \quad \models B' \Rightarrow B}{\{A\}\ c\ \{B\}}\ (\textsc{Conseq})$$

Tokyo)

While^{dt}

While + dt

Syntax

$$\textbf{AExp} \ni a ::= x \mid \mathtt{c}_r \mid a_1 \ \mathtt{aop} \ a_2 \mid \mathtt{dt}$$

where $\mathtt{c}_r$ is a const. for $r \in \mathbb{R}$, $\mathtt{aop} \in \{+, -, \cdot, \hat{}, /\}$

$$\textbf{BExp} \ni b ::= \mathtt{true} \mid \mathtt{false} \mid b_1 \wedge b_2 \mid \neg b \mid a_1 < a_2$$

$$\frac{}{\{A\}\,\mathtt{skip}\,\{A\}}\ (\textsc{Skip}) \qquad \frac{}{\{A[a/x]\}\,x := a\,\{A\}}\ (\textsc{Assign})$$

$$\frac{\{A\}\,c_1\,\{C\} \quad \{C\}\,c_2\,\{B\}}{\{A\}\,c_1; c_2\,\{B\}}\ (\textsc{Seq}) \qquad \frac{\{A \wedge b\}\,c_1\,\{B\} \quad \{A \wedge \neg b\}\,c_2\,\{B\}}{\{A\}\,\mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2\,\{B\}}\ (\textsc{If})$$

$$\frac{\{A \wedge b\}\,c\,\{A\}}{\{A\}\,\mathtt{while}\ b\ \mathtt{do}\ c\,\{A \wedge \neg b\}}\ (\textsc{While}) \qquad \frac{\models A \Rightarrow A' \quad \{A'\}\,c\,\{B'\} \quad \models B' \Rightarrow B}{\{A\}\,c\,\{B\}}\ (\textsc{Conseq})$$

Hoare^{dt}

Hasuo (Tokyo)

Syntax

While^{dt}

While + dt

$$\begin{array}{lll} \textbf{AExp} \ni & a & ::= \quad x \mid \mathtt{c}_r \mid a_1 \ \mathtt{aop} \ a_2 \mid \mathtt{dt} \\ & \text{where } \mathtt{c}_r \text{ is a const. for } r \in \mathbb{R},\ \mathtt{aop} \in \{+, -, \cdot, \hat{}, /\} \\ \textbf{BExp} \ni & b & ::= \quad \mathtt{true} \mid \mathtt{false} \mid b_1 \wedge b_2 \mid \neg b \mid a_1 < a_2 \end{array}$$

$$\frac{}{\{A\}\,\mathtt{skip}\,\{A\}}\ (\textsc{Skip}) \qquad \frac{}{\{\,A[a/x]\,\}\,x := a\,\{A\}}\ (\textsc{Assign})$$

$$\frac{\{A\}\,c_1\,\{C\} \quad \{C\}\,c_2\,\{B\}}{\{A\}\,c_1; c_2\,\{B\}}\ (\textsc{Seq}) \qquad \frac{\{A \wedge b\}\,c_1\,\{B\} \quad \{A \wedge \neg b\}\,c_2\,\{B\}}{\{A\}\,\mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2\,\{B\}}\ (\textsc{If})$$

$$\frac{\{A \wedge b\}\,c\,\{A\}}{\{A\}\,\mathtt{while}\ b\ \mathtt{do}\ c\,\{A \wedge \neg b\}}\ (\textsc{While}) \qquad \frac{\models A \Rightarrow A' \quad \{A'\}\,c\,\{B'\} \quad \models B' \Rightarrow B}{\{A\}\,c\,\{B\}}\ (\textsc{Conseq})$$

Hoare^{dt}

Precisely the same rules

Hasuo (Tokyo)

Syntax

While^{dt}

While + dt

$$
\begin{array}{lll}
\textbf{AExp} \ni & a & ::= \quad x \mid \mathtt{c}_r \mid a_1 \ \mathtt{aop} \ a_2 \mid \mathtt{dt} \\
& \multicolumn{2}{l}{\text{where } \mathtt{c}_r \text{ is a const. for } r \in \mathbb{R},\ \mathtt{aop} \in \{+, -, \cdot, \hat{}, /\}} \\
\textbf{BExp} \ni & b & ::= \quad \mathtt{true} \mid \mathtt{false} \mid b_1 \wedge b_2 \mid \neg b \mid a_1 < a_2 \\
\textbf{Cmd} \ni & c & ::= \quad \mathtt{skip} \mid x := a \mid c_1 ; c_2 \\
& & \quad \mid \mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2 \mid \mathtt{while}\ b\ \mathtt{do}\ c
\end{array}
$$

Assn^{dt}

Assn, *-transformed

$$
\begin{array}{lll}
A & ::= & \mathtt{true} \mid \mathtt{false} \mid A_1 \wedge A_2 \mid \neg A \mid a_1 < a_2 \mid \\
& & \forall x \in {}^*\mathbb{N}.\ A \mid \forall x \in {}^*\mathbb{R}.\ A
\end{array}
$$

Hoare^{dt}

Precisely the same rules

$$\frac{}{\{A\}\ \mathtt{skip}\ \{A\}}\ (\textsc{Skip}) \qquad \frac{}{\{\,A[a/x]\,\}\ x := a\ \{A\}}\ (\textsc{Assign})$$

$$\frac{\{A\}\ c_1\ \{C\} \quad \{C\}\ c_2\ \{B\}}{\{A\}\ c_1; c_2\ \{B\}}\ (\textsc{Seq}) \qquad \frac{\{A \wedge b\}\ c_1\ \{B\} \quad \{A \wedge \neg b\}\ c_2\ \{B\}}{\{A\}\ \mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2\ \{B\}}\ (\textsc{If})$$

$$\frac{\{A \wedge b\}\ c\ \{A\}}{\{A\}\ \mathtt{while}\ b\ \mathtt{do}\ c\ \{A \wedge \neg b\}}\ (\textsc{While}) \qquad \frac{\models A \Rightarrow A' \quad \{A'\}\ c\ \{B'\} \quad \models B' \Rightarrow B}{\{A\}\ c\ \{B\}}\ (\textsc{Conseq})$$

Tokyo)

Syntax

While^{dt}

While + dt

$$
\begin{array}{lllll}
\textbf{AExp} \ni & a & ::= & x \mid \mathtt{c}_r \mid a_1 \ \mathtt{aop} \ a_2 \mid \mathtt{dt} & \\
& \text{where } \mathtt{c}_r \text{ is a const. for } r \in \mathbb{R},\ \mathtt{aop} \in \{+, -, \cdot, \hat{}, /\} & & & \\
\textbf{BExp} \ni & b & ::= & \mathtt{true} \mid \mathtt{false} \mid b_1 \wedge b_2 \mid \neg b \mid a_1 < a_2 & \\
& c & ::= & \mathtt{skip} \mid x := a \mid c_1 ; c_2 & \\
& & & \mid \mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2 \mid \mathtt{while}\ b\ \mathtt{do}\ c &
\end{array}
$$

Thm.

Hoare^{dt} rules are *sound* and *relatively complete*.

Hoare^{dt}

Precisely the same rules

$$\frac{}{\{A\}\ \mathtt{skip}\ \{A\}}\ (\textsc{Skip}) \qquad \frac{}{\{\,A[a/x]\,\}\ x := a\ \{A\}}\ (\textsc{Assign})$$

$$\frac{\{A\}\ c_1\ \{C\} \quad \{C\}\ c_2\ \{B\}}{\{A\}\ c_1; c_2\ \{B\}}\ (\textsc{Seq}) \qquad \frac{\{A \wedge b\}\ c_1\ \{B\} \quad \{A \wedge \neg b\}\ c_2\ \{B\}}{\{A\}\ \mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2\ \{B\}}\ (\textsc{If})$$

$$\frac{\{A \wedge b\}\ c\ \{A\}}{\{A\}\ \mathtt{while}\ b\ \mathtt{do}\ c\ \{A \wedge \neg b\}}\ (\textsc{While}) \qquad \frac{\models A \Rightarrow A' \quad \{A'\}\ c\ \{B'\} \quad \models B' \Rightarrow B}{\{A\}\ c\ \{B\}}\ (\textsc{Conseq})$$

Tokyo)

Denotational Semantics: Challenge

```
t := 0 ;
while (t ≤ 1) do {
   t := t + dt
}
```

Denotational Semantics: Challenge

```
t := 0 ;
while (t ≤ 1) do {
   t := t + dt
}
```

```
t := 0 ;
while (true) do {
   t := t + dt
}
```

Hasuo (Tokyo)

Denotational Semantics: Challenge

```
t := 0 ;
while (t ≤ 1) do {
   t := t + dt
}
```

$t = 1 + \mathrm{dt}$

```
t := 0 ;
while (true) do {
   t := t + dt
}
```

$\perp$ (divergence)

Hasuo (Tokyo)

Denotational Semantics: Challenge

```
t := 0 ;
while (t ≤ 1) do {
   t := t + dt
}
```

$t = 1 + \mathrm{dt}$

```
t := 0 ;
while (true) do {
   t := t + dt
}
```

$\perp$ (divergence)

* Semantics by "sectionwise execution"

Hasuo (Tokyo)

Denotational Semantics

* Execute sectionwise and bundle up the outcomes!

```
t := 0;
while (t < 1)
  t := t + dt;
```

Hasuo (Tokyo)

Denotational Semantics

* Execute sectionwise and bundle up the outcomes!

```
t := 0;
while (t < 1)
   t := t + dt;
```

Hasuo (Tokyo)

Denotational Semantics

* Execute sectionwise and bundle up the outcomes!

```
t := (0,0,0,...);
while (t < (1,1,1,...))
    t := t + (1, 1/2, 1/3, ...) ;
```

Hasuo (Tokyo)

Denotational Semantics

* Execute sectionwise and bundle up the outcomes!

0th section

```
t := 0;
while (t < 1)

  t := t + 1 ;
```

1st section

```
t := 0;
while (t < 1)

  t := t + 1/2 ;
```

2nd section

```
t := 0;
while (t < 1)

  t := t + 1/3 ;
```

...

Hasuo (Tokyo)

Denotational Semantics

* Execute sectionwise and bundle up the outcomes!

0th section	1st section	2nd section	

`t := 0;`
`while (t < 1)`
`  t := t + 1 ;`
	`t := 0;`
	`while (t < 1)`
	`  t := t + 1/2 ;`
		`t := 0;`
		`while (t < 1)`
		`  t := t + 1/3 ;`
			…

`t = 1`	`t = 1`	`t = 1`	…

Hasuo (Tokyo)

Denotational Semantics

* Execute sectionwise and bundle up the outcomes!

```
t := (0,0,0,...);
while (t < (1,1,1,...))
    t := t + (1, 1/2, 1/3, ...) ;
```

```
t = (1,1,1,...)
```

Hasuo (Tokyo)

Denotational Semantics

* Execute sectionwise and bundle up the outcomes!

```
t := 0;
while (t < 1)
  t := t + dt;
```

```
t = 1
```

Hasuo (Tokyo)

Denotational Semantics

* Execute sectionwise and bundle up the outcomes!

```
t := 0;
while (t <= 1)
  t := t + dt;
```

Hasuo (Tokyo)

Denotational Semantics

* Execute sectionwise and bundle up the outcomes!

```
t := 0;
while (t <= 1)
   t := t + dt;
```

Hasuo (Tokyo)

Denotational Semantics

* Execute sectionwise and bundle up the outcomes!

```
t := (0,0,0,...);
while (t <= (1,1,1,...))
  t := t + (1, 1/2, 1/3, ...) ;
```

Hasuo (Tokyo)

Denotational Semantics

* Execute sectionwise and bundle up the outcomes!

0th section

```
t := 0;
while (t <= 1)

  t := t + 1 ;
```

1st section

```
t := 0;
while (t <= 1)

  t := t + 1/2 ;
```

2nd section

```
t := 0;
while (t <= 1)

  t := t + 1/3 ;
```

...

Hasuo (Tokyo)

Denotational Semantics

* Execute sectionwise and bundle up the outcomes!

0th section	1st section	2nd section	

`t := 0;`
`while (t <= 1)`
`  t := t + 1 ;`
	`t := 0;`
	`while (t <= 1)`
	`  t := t + 1/2 ;`
		`t := 0;`
		`while (t <= 1)`
		`  t := t + 1/3 ;`
			…

`t = 1 + 1`	`t = 1 + 1/2`	`t = 1 + 1/3`	…

Yasuo (Tokyo)

Denotational Semantics

* Execute sectionwise and bundle up the outcomes!

```
t := (0,0,0,...);
while (t <= (1,1,1,...))
   t := t + (1, 1/2, 1/3, ...) ;
```

$$t = (1,1,1,\ldots) + \left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right)$$

Hasuo (Tokyo)

Denotational Semantics

* Execute sectionwise and bundle up the outcomes!

```
t := 0;
while (t <= 1)
  t := t + dt;
```

```
t = 1 + dt
```

Hasuo (Tokyo)

Denotational Semantics

* Execute sectionwise and bundle up the outcomes!

```
t := 0;
while (true)
  t := t + dt;
```

Hasuo (Tokyo)

Denotational Semantics

* Execute sectionwise and bundle up the outcomes!

```
t := 0;
while (true)
   t := t + dt;
```

Hasuo (Tokyo)

Denotational Semantics

* Execute sectionwise and bundle up the outcomes!

```
t := (0,0,0,...);
while (true)

  t := t + (1, 1/2, 1/3, ...) ;
```

Hasuo (Tokyo)

Denotational Semantics

* Execute sectionwise and bundle up the outcomes!

0th section

```
t := 0;
while (true)

  t := t + 1 ;
```

1st section

```
t := 0;
while (true)

  t := t + 1/2 ;
```

2nd section

```
t := 0;
while (true)

  t := t + 1/3 ;
```

...

Hasuo (Tokyo)

Denotational Semantics

* Execute sectionwise and bundle up the outcomes!

0th section	1st section	2nd section	
`t := 0; while (true) t := t + 1;`	`t := 0; while (true) t := t + 1/2;`	`t := 0; while (true) t := t + 1/3;`	…
$\bot$	$\bot$	$\bot$	…

Hasuo (Tokyo)

Denotational Semantics

* Execute sectionwise and bundle up the outcomes!

```
t := (0,0,0,...);
while (true)
   t := t + (1, 1/2, 1/3, ...) ;
```

$$t = (\bot, \bot, \bot, \ldots)$$

Denotational Semantics

* Execute sectionwise and bundle up the outcomes!

```
t := 0;
while (true)
   t := t + dt;
```

⊥

Denotational Semantics

$$
\begin{array}{lcl}
[\![x]\!]\sigma & := & \sigma(x) \\
[\![a_1 \ \mathtt{aop}\ a_2]\!]\sigma & := & [\![a_1]\!]\sigma \ \mathtt{aop}\ [\![a_2]\!]\sigma \\
[\![\mathtt{dt}]\!]\sigma & := & \omega^{-1} = \left[(1, \tfrac{1}{2}, \tfrac{1}{3}, \dots) \right]
\end{array}
\qquad
\begin{array}{lcl}
[\![\mathtt{c}_r]\!]\sigma & := & r \quad \text{for each } r \in \mathbb{R} \\
[\![\infty]\!]\sigma & := & \omega = \left[(1, 2, 3, \dots) \right]
\end{array}
$$

$$
\begin{array}{lcl}
[\![\mathtt{true}]\!]\sigma & := & \mathrm{tt} \\
[\![b_1 \wedge b_2]\!]\sigma & := & [\![b_1]\!]\sigma \wedge [\![b_2]\!]\sigma \\
[\![a_1 < a_2]\!]\sigma & := & [\![a_1]\!]\sigma < [\![a_2]\!]\sigma
\end{array}
\qquad
\begin{array}{lcl}
[\![\mathtt{false}]\!]\sigma & := & \mathrm{ff} \\
[\![\neg b]\!]\sigma & := & \neg([\![b]\!]\sigma)
\end{array}
$$

$$
[\![\mathtt{skip}]\!]\sigma := \sigma \qquad [\![x := a]\!]\sigma := \sigma\left[x \mapsto [\![a]\!]\sigma \right] \qquad [\![c_1; c_2]\!]\sigma := [\![c_2]\!]\left([\![c_1]\!]\sigma \right)
$$

$$
[\![\mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2]\!]\sigma := \begin{cases} [\![c_1]\!]\sigma & \text{if } [\![b]\!]\sigma = \mathrm{tt} \\ [\![c_2]\!]\sigma & \text{if } [\![b]\!]\sigma = \mathrm{ff} \end{cases}
$$

$$
[\![\mathtt{while}\ b\ \mathtt{do}\ c]\!]\sigma := \left([\![(\mathtt{while}\ b\ \mathtt{do}\ c)|_i]\!](\sigma|_i) \right)_{i \in \mathbb{N}}
$$

Denotational Semantics

Hyperstate (stores hyperreals)

$$
\begin{array}{lcl}
[\![x]\!]\sigma & := & \sigma(x) \\
[\![a_1 \text{ aop } a_2]\!]\sigma & := & [\![a_1]\!]\sigma \text{ aop } [\![a_2]\!]\sigma \\
[\![\mathtt{dt}]\!]\sigma & := & \omega^{-1} = \left[(1, \tfrac{1}{2}, \tfrac{1}{3}, \dots) \right]
\end{array}
\qquad
\begin{array}{lcl}
[\![\mathtt{c}_r]\!]\sigma & := & r \quad \text{for each } r \in \mathbb{R} \\
[\![\infty]\!]\sigma & := & \omega = \left[(1, 2, 3, \dots) \right]
\end{array}
$$

$$
\begin{array}{lcl}
[\![\mathtt{true}]\!]\sigma & := & \mathrm{tt} \\
[\![b_1 \wedge b_2]\!]\sigma & := & [\![b_1]\!]\sigma \wedge [\![b_2]\!]\sigma \\
[\![a_1 < a_2]\!]\sigma & := & [\![a_1]\!]\sigma < [\![a_2]\!]\sigma
\end{array}
\qquad
\begin{array}{lcl}
[\![\mathtt{false}]\!]\sigma & := & \mathrm{ff} \\
[\![\neg b]\!]\sigma & := & \neg([\![b]\!]\sigma)
\end{array}
$$

$$
[\![\mathtt{skip}]\!]\sigma := \sigma \qquad [\![x := a]\!]\sigma := \sigma\left[x \mapsto [\![a]\!]\sigma \right] \qquad [\![c_1; c_2]\!]\sigma := [\![c_2]\!]\left([\![c_1]\!]\sigma \right)
$$

$$
[\![\mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2]\!]\sigma := \begin{cases} [\![c_1]\!]\sigma & \text{if } [\![b]\!]\sigma = \mathrm{tt} \\ [\![c_2]\!]\sigma & \text{if } [\![b]\!]\sigma = \mathrm{ff} \end{cases}
$$

$$
[\![\mathtt{while}\ b\ \mathtt{do}\ c]\!]\sigma := \left([\![(\mathtt{while}\ b\ \mathtt{do}\ c)|_i]\!](\sigma|_i) \right)_{i \in \mathbb{N}}
$$

Denotational Semantics

Hyperstate (stores hyperreals)

$$
\begin{array}{lcl}
[\![x]\!]\sigma & := & \sigma(x) \\
[\![a_1 \text{ aop } a_2]\!]\sigma & := & [\![a_1]\!]\sigma \text{ aop } [\![a_2]\!]\sigma \\
[\![\mathtt{dt}]\!]\sigma & := & \omega^{-1} = \left[(1, \tfrac{1}{2}, \tfrac{1}{3}, \dots) \right]
\end{array}
\qquad
\begin{array}{lcl}
[\![\mathtt{c}_r]\!]\sigma & := & r \quad \text{for each } r \in \mathbb{R} \\
[\![\infty]\!]\sigma & := & \omega = \left[(1, 2, 3, \dots) \right]
\end{array}
$$

$$
\begin{array}{lcl}
[\![\mathtt{true}]\!]\sigma & := & \mathrm{tt} \\
[\![b_1 \wedge b_2]\!]\sigma & := & [\![b_1]\!]\sigma \wedge [\![b_2]\!]\sigma \\
[\![a_1 < a_2]\!]\sigma & := & [\![a_1]\!]\sigma < [\![a_2]\!]\sigma
\end{array}
\qquad
\begin{array}{lcl}
[\![\mathtt{false}]\!]\sigma & := & \mathrm{ff} \\
[\![\neg b]\!]\sigma & := & \neg([\![b]\!]\sigma)
\end{array}
$$

$$
[\![\mathtt{skip}]\!]\sigma := \sigma \qquad [\![x := a]\!]\sigma := \sigma\left[x \mapsto [\![a]\!]\sigma \right] \qquad [\![c_1; c_2]\!]\sigma := [\![c_2]\!]\left([\![c_1]\!]\sigma \right)
$$

$$
[\![\mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2]\!]\sigma := \begin{cases} [\![c_1]\!]\sigma & \text{if } [\![b]\!]\sigma = \mathrm{tt} \\ [\![c_2]\!]\sigma & \text{if } [\![b]\!]\sigma = \mathrm{ff} \end{cases}
$$

$$
[\![\mathtt{while}\ b\ \mathtt{do}\ c]\!]\sigma := \left([\![(\mathtt{while}\ b\ \mathtt{do}\ c)|_i]\!](\sigma|_i) \right)_{i \in \mathbb{N}}
$$

Section of a program

Hasuo (Tokyo)

Denotational Semantics

Hyperstate (stores hyperreals)

$$
\begin{array}{lcl}
[\![x]\!]\sigma & := & \sigma(x) \\
[\![a_1 \text{ aop } a_2]\!]\sigma & := & [\![a_1]\!]\sigma \text{ aop } [\![a_2]\!]\sigma \\
[\![\mathtt{dt}]\!]\sigma & := & \omega^{-1} = \left[(1, \tfrac{1}{2}, \tfrac{1}{3}, \dots) \right]
\end{array}
\qquad
\begin{array}{lcl}
[\![\mathtt{c}_r]\!]\sigma & := & r \quad \text{for each } r \in \mathbb{R} \\
[\![\infty]\!]\sigma & := & \omega = \left[(1, 2, 3, \dots) \right]
\end{array}
$$

$$
\begin{array}{lcl}
[\![\mathtt{true}]\!]\sigma & := & \mathrm{tt} \\
[\![b_1 \wedge b_2]\!]\sigma & := & [\![b_1]\!]\sigma \wedge [\![b_2]\!]\sigma \\
[\![a_1 < a_2]\!]\sigma & := & [\![a_1]\!]\sigma < [\![a_2]\!]\sigma
\end{array}
\qquad
\begin{array}{lcl}
[\![\mathtt{false}]\!]\sigma & := & \mathrm{ff} \\
[\![\neg b]\!]\sigma & := & \neg([\![b]\!]\sigma)
\end{array}
$$

$$
[\![\mathtt{skip}]\!]\sigma := \sigma \qquad [\![x := a]\!]\sigma := \sigma\left[x \mapsto [\![a]\!]\sigma \right] \qquad [\![c_1; c_2]\!]\sigma := [\![c_2]\!]\left([\![c_1]\!]\sigma \right)
$$

$$
[\![\mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2]\!]\sigma := \begin{cases} [\![c_1]\!]\sigma & \text{if } [\![b]\!]\sigma = \mathrm{tt} \\ [\![c_2]\!]\sigma & \text{if } [\![b]\!]\sigma = \mathrm{ff} \end{cases}
$$

$$
[\![\mathtt{while}\ b\ \mathtt{do}\ c]\!]\sigma := \left([\![(\mathtt{while}\ b\ \mathtt{do}\ c)|_i]\!] (\sigma|_i) \right)_{i \in \mathbb{N}}
$$

Section of a program

Applied to a section of a memory state

Hasuo (Tokyo)

Denotational Semantics

Hyperstate (stores hyperreals)

$$
\begin{array}{lcl}
[\![x]\!]\sigma & := & \sigma(x) \\
[\![a_1 \text{ aop } a_2]\!]\sigma & := & [\![a_1]\!]\sigma \text{ aop } [\![a_2]\!]\sigma \\
[\![\mathtt{dt}]\!]\sigma & := & \omega^{-1} = \left[(1, \tfrac{1}{2}, \tfrac{1}{3}, \dots) \right]
\end{array}
\qquad
\begin{array}{lcl}
[\![\mathtt{c}_r]\!]\sigma & := & r \quad \text{for each } r \in \mathbb{R} \\
[\![\infty]\!]\sigma & := & \omega = \left[(1, 2, 3, \dots) \right]
\end{array}
$$

$$
\begin{array}{lcl}
[\![\mathtt{true}]\!]\sigma & := & \mathrm{tt} \\
[\![b_1 \wedge b_2]\!]\sigma & := & [\![b_1]\!]\sigma \wedge [\![b_2]\!]\sigma \\
[\![a_1 < a_2]\!]\sigma & := & [\![a_1]\!]\sigma < [\![a_2]\!]\sigma
\end{array}
\qquad
\begin{array}{lcl}
[\![\mathtt{false}]\!]\sigma & := & \mathrm{ff} \\
[\![\neg b]\!]\sigma & := & \neg([\![b]\!]\sigma)
\end{array}
$$

$$
[\![\mathtt{skip}]\!]\sigma := \sigma \qquad [\![x := a]\!]\sigma := \sigma\left[x \mapsto [\![a]\!]\sigma \right] \qquad [\![c_1; c_2]\!]\sigma := [\![c_2]\!]\left([\![c_1]\!]\sigma \right)
$$

$$
[\![\mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2]\!]\sigma := \begin{cases} [\![c_1]\!]\sigma & \text{if } [\![b]\!]\sigma = \mathrm{tt} \\ [\![c_2]\!]\sigma & \text{if } [\![b]\!]\sigma = \mathrm{ff} \end{cases}
$$

$$
[\![\mathtt{while}\ b\ \mathtt{do}\ c]\!]\sigma := \left([\![(\mathtt{while}\ b\ \mathtt{do}\ c)|_i]\!] (\sigma|_i) \right)_{i \in \mathbb{N}}
$$

Bundled up

Section of a program

Applied to a section of a memory state

Hasuo (Tokyo)

Denotational Semantics

Hyperstate (stores hyperreals)

$$
\begin{array}{lcl}
[\![x]\!]\sigma & := & \sigma(x) \\
[\![a_1 \text{ aop } a_2]\!]\sigma & := & [\![a_1]\!]\sigma \text{ aop } [\![a_2]\!]\sigma \\
[\![\mathtt{dt}]\!]\sigma & := & \omega^{-1} = \left[(1, \tfrac{1}{2}, \tfrac{1}{3}, \ldots) \right]
\end{array}
$$

$$
\begin{array}{lcl}
[\![\mathtt{true}]\!]\sigma & := & \mathrm{tt} \\
[\![b_1 \wedge b_2]\!]\sigma & := & [\![b_1]\!]\sigma \wedge [\![b_2]\!]\sigma \\
[\![a_1 < a_2]\!]\sigma & := & [\![a_1]\!]\sigma < [\![a_2]\!]\sigma
\end{array}
$$

$$
[\![\mathtt{skip}]\!]\sigma := \sigma \qquad [\![x := a]\!]\sigma := \sigma\left[x \mapsto [\![a]\!]\sigma \right] \qquad [\![c_1; c_2]\!]\sigma := [\![c_2]\!]\left([\![c_1]\!]\sigma \right)
$$

$$
[\![\mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2]\!]\sigma := \begin{cases} [\![c_1]\!]\sigma & \text{if } [\![b]\!]\sigma = \mathrm{tt} \\ [\![c_2]\!]\sigma & \text{if } [\![b]\!]\sigma = \mathrm{ff} \end{cases}
$$

$$
[\![\mathtt{while}\ b\ \mathtt{do}\ c]\!]\sigma := \left([\![(\mathtt{while}\ b\ \mathtt{do}\ c)|_i]\!] (\sigma|_i) \right)_{i \in \mathbb{N}}
$$

Def.
The *i-th section* of a While$^{\mathtt{dt}}$ expression e is

$$
e|_i \quad :\equiv \quad e\left[\frac{1}{i+1} \,/\, \mathtt{dt} \right] .
$$

Section of a program

Applied to a section of a memory state

Bundled up

Hasuo (Tokyo)

Denot

$$\left[\begin{array}{l} t := 0\,; \\ \texttt{while}\ (t \le 1)\ \texttt{do} \\ \quad t := t + \mathrm{dt} \end{array}\right] \overset{i\text{-th section}}{\longmapsto} \left[\begin{array}{l} t := 0\,; \\ \texttt{while}\ (t \le 1)\ \texttt{do} \\ \quad t := t + \frac{1}{i+1} \end{array}\right]$$

Hyperstate (stores hyperreals)

$$\begin{array}{lcl}
[\![x]\!]\sigma & := & \sigma(x) \\
[\![a_1 \ \texttt{aop}\ a_2]\!]\sigma & := & [\![a_1]\!]\sigma \ \texttt{aop}\ [\![a_2]\!]\sigma \\
[\![\mathrm{dt}]\!]\sigma & := & \omega^{-1} = [\,(1, \tfrac{1}{2}, \tfrac{1}{3}, \ldots)\,]
\end{array}$$

$$\begin{array}{lcl}
[\![\texttt{true}]\!]\sigma & := & \mathrm{tt} \\
[\![b_1 \wedge b_2]\!]\sigma & := & [\![b_1]\!]\sigma \wedge [\![b_2]\!]\sigma \\
[\![a_1 < a_2]\!]\sigma & := & [\![a_1]\!]\sigma < [\![a_2]\!]\sigma
\end{array}$$

$$[\![\texttt{skip}]\!]\sigma := \sigma \qquad [\![x := a]\!]\sigma := \sigma[\,x \mapsto [\![a]\!]\sigma\,] \qquad [\![c_1; c_2]\!]\sigma := [\![c_2]\!]([\![c_1]\!]\sigma)$$

$$[\![\texttt{if}\ b\ \texttt{then}\ c_1\ \texttt{else}\ c_2]\!]\sigma := \begin{cases} [\![c_1]\!]\sigma & \text{if } [\![b]\!]\sigma = \mathrm{tt} \\ [\![c_2]\!]\sigma & \text{if } [\![b]\!]\sigma = \mathrm{ff} \end{cases}$$

$$[\![\texttt{while}\ b\ \texttt{do}\ c]\!]\sigma := \Big([\![(\texttt{while}\ b\ \texttt{do}\ c)|_i]\!](\sigma|_i) \Big)_{i \in \mathbb{N}}$$

Def.
The *i-th section* of a WHILE$^{\mathrm{dt}}$ expression e is

$$e|_i \quad :\equiv \quad e\left[\,\frac{1}{i+1} \,/\, \mathrm{dt}\,\right] .$$

Section of a program

Applied to a section of a memory state

Bundled up

Hasuo (Tokyo)

"Sectionwise Lemmas"

Sectionwise Execution Lemma.
For any expr. e and $i \in \mathbb{N}$,

$$[\![e]\!]\sigma = \left[\left([\![e|_i]\!](\sigma|_i) \right)_{i \in \mathbb{N}} \right] .$$

Sectionwise Satisfaction Lemma.
For any hyperstate σ and an $\textsc{Assn}^{\mathtt{dt}}$ formula φ:

$$\sigma \models \varphi \iff \sigma|_i \models \varphi|_i \quad \text{for almost every } i.$$

Hasuo (Tokyo)

"Sectionwise Lemmas"

Sectionwise Execution Lemma.
For any expr. e and $i \in \mathbb{N}$,

$$[\![e]\!]\sigma = \left[\left([\![e|_i]\!](\sigma|_i) \right)_{i \in \mathbb{N}} \right] .$$

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Łoś'

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Lem. (Sectionwise validity of Hoare triples)

$$\models \{A\}c\{B\} \iff \models \{A|_i\}\, c|_i\, \{B|_i\} \text{ for almost every } i.$$

Hasuo (Tokyo)

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Interface for **transferring** static analysis techniques

Q. Is a While^{dt} program executable?

* A. Not exactly.
  * A **modeling** language
    * Not numerical approx., but **exact** modeling
    * Advantage:
      close to a common programming style

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* Static analysis ➜ **no need to execute**!
  * Mathematical semantics suffices

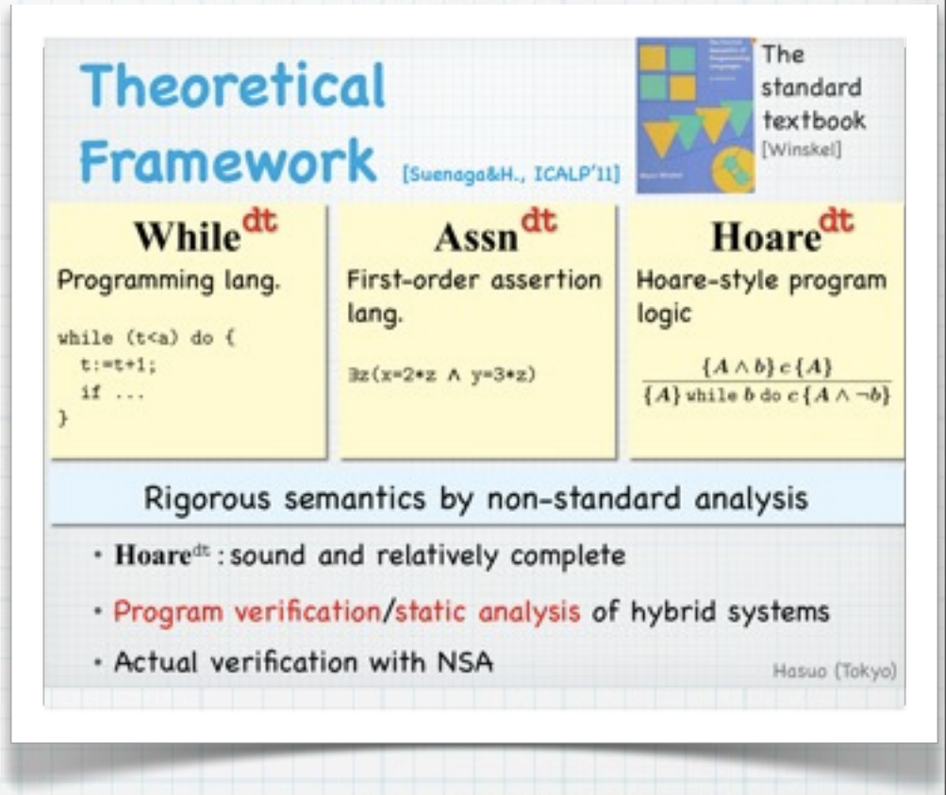
Hasuo (Tokyo)

Outline

Suenaga & H., ICALP'11

* **Theoretical foundations**
  * **While^{dt}, Assn^{dt}, Hoare^{dt}**
  * Rigorous semantics via NSA
  * Transfer principle, "sectionwise lemmas"
* **Static analysis techniques, transferred as they are**
  * Phase split [Sharma,Dillig,Dillig,Aiken; CAV'11] [Balakrishnan,Sankaranarayanan,Ivancic,Gupta; EMSOFT'09] [Gopan,Reps; SAS'07]
  * Differential invariant [Platzer,Clarke; CAV'08]
  * ... and more!

H. & Suenaga, CAV'12



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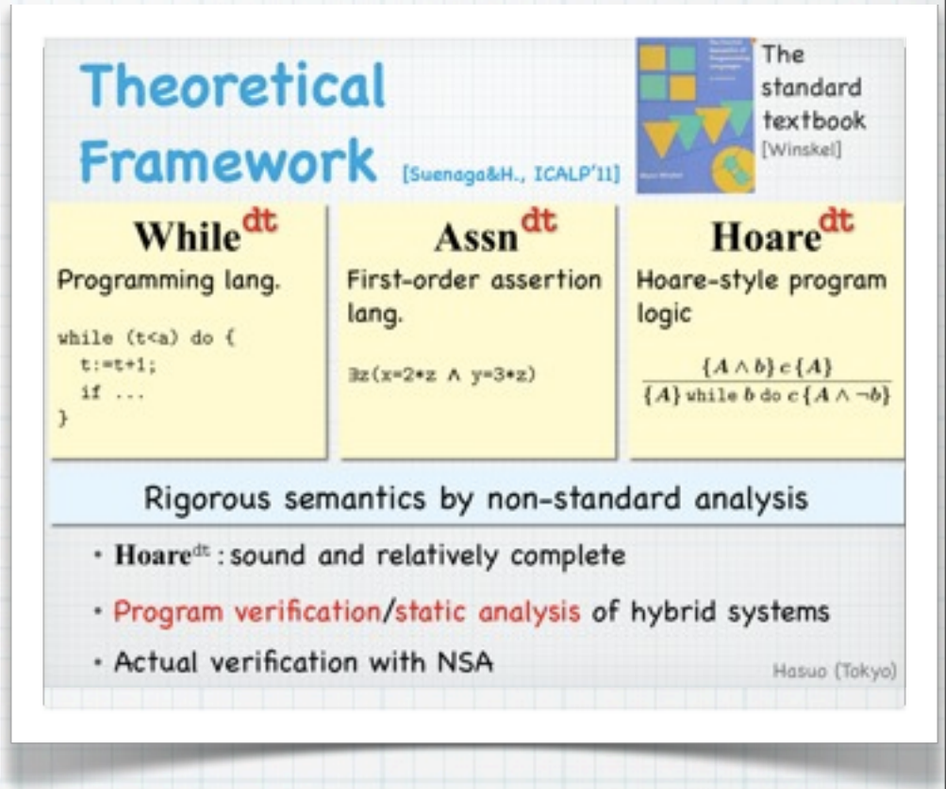

Done

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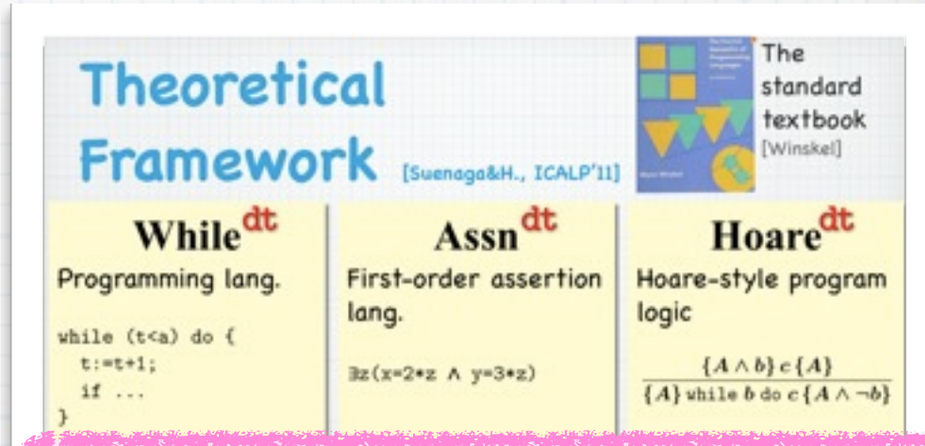

w/ or w/o dt ...

➜ logically "**the same**"

Done

H. & Suenaga, CAV'12

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Part II:

Exercises in Nonstandard Static Analysis

Exercise 1.1


```
while t < ε do {
    t := t + dt;
    v := v + a · dt;
    z := z + v · dt
}
```

Hasuo (Tokyo)

Exercise 1.1


```
while t < ε do {
    t := t + dt;
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```
while v > 0 do {
    t := 0;
    if m − z < s then a := −b else a := a0;
    while t < ε do {
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        z := z + v · dt
    }}
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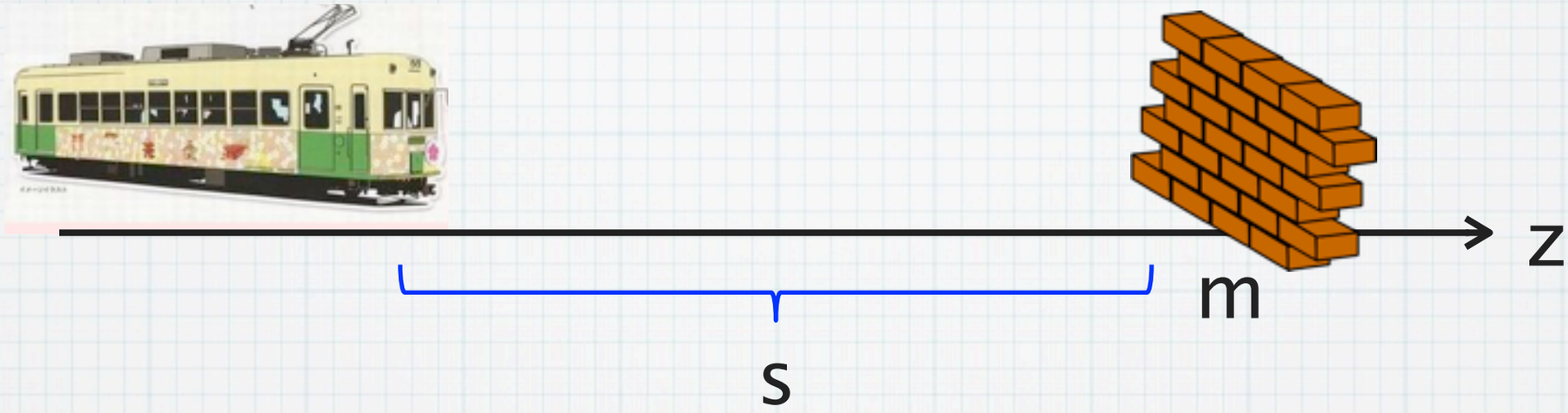


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(Tiny) fragment of
Euro. Train Ctrl. Sys. (ETCS)


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ETCS_0

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Q. Find A s.t. $\models \{A\}\,\mathrm{ETCS}_0\,\{z < m\}$

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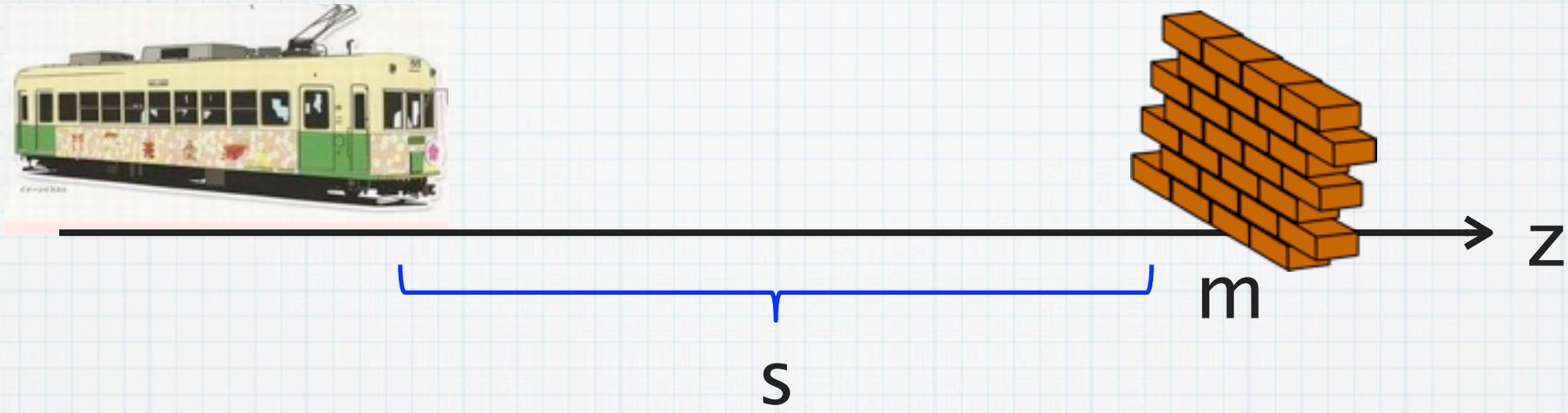

s: big enough
b: big enough
a_0: small enough
…

$$
\begin{array}{l}
\texttt{while } v > 0 \texttt{ do } \{ \\
\quad t := 0\,; \\
\quad \texttt{if } m - z < s \texttt{ then } a := -b \texttt{ else } a := a_0\,; \\
\quad \texttt{while } t < \varepsilon \texttt{ do } \{ \\
\qquad t := t + \mathrm{dt}\,; \\
\qquad v := v + a \cdot \mathrm{dt}\,; \\
\qquad z := z + v \cdot \mathrm{dt} \\
\quad \}\}
\end{array}
$$

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Q. Find A s.t. $\models \{A\}\,\mathrm{ETCS}_0\,\{z < m\}$

Hasuo (Tokyo)

```
while (v > 0) {
  if m - z < s
    then a := -b
    else a := a0;
  t := 0;
  while (t < eps && v > 0) {
    z := z + v * dt;
    v := v + a * dt;
    t := t + dt }}


{z < m}
```

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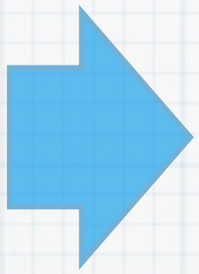

```
while (v > 0 && m - z >= s) {
  a := a0;    t := 0;
  while (t < eps && v > 0) {
    z := z + v * dt;
    v := v + a0 * dt;
    t := t + dt }};
while (v > 0 && m - z < s) {
  a := -b;    t := 0;
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accel.

brake

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Strategy1 "Phase split"

[Sharma,Dillig,Dillig,Aiken; CAV'11]
[Balakrishnan,Sankaranarayanan,Ivancic,Gupta; EMSOFT'09] [Gopan,Reps; SAS'07]

Phase Split
(Standard Ver., for While & Hoare)

[Sharma,Dillig,Dillig,Aiken; CAV'11]

Defn.
The set of *holed commands* $\mathbf{Cmd}_{[\_]}$ is:

$$\mathbf{Cmd}_{[\_]} \ni h \ ::= \ \texttt{if } [\_] \texttt{ then } c_1 \texttt{ else } c_2 \mid h; c \mid c; h \mid \texttt{if } b \texttt{ then } h \texttt{ else } c \mid \texttt{if } b \texttt{ then } c \texttt{ else } h$$

For each holed command h, its *pre-hole fragment* $\overline{h}$ is:

$$\begin{array}{lll} \overline{\texttt{if } [\_] \texttt{ then } c_1 \texttt{ else } c_2} & :\equiv & \texttt{skip} \\ \overline{h; c} \ :\equiv \ \overline{h} \qquad \overline{c; h} & :\equiv & c; \overline{h} \\ \overline{\texttt{if } b \texttt{ then } h \texttt{ else } c} & :\equiv & \texttt{assert } b \,; \overline{h} \\ \overline{\texttt{if } b \texttt{ then } c \texttt{ else } h} & :\equiv & \texttt{assert } \neg b \,; \overline{h} \end{array}$$

Lem.
If a Boolean expression $b_s \in \mathbf{BExp}$ satisfies

$$\models \{b_s\}\, \overline{h}\, \{b_c\} \ , \quad \models \{\neg b_s\}\, \overline{h}\, \{\neg b_c\} \ , \quad \text{and} \quad \models \{b_g \wedge b_s\}\, h[b_c]\, \{\neg b_g \vee b_s\} \ ,$$

then we have

$$[\![\texttt{while } b_g \texttt{ do } h[b_c]]\!] = \left[\!\!\left[\begin{array}{l} \texttt{while } (b_g \wedge \neg b_s) \texttt{ do } h[\texttt{false}] \,; \\ \texttt{while } (b_g \wedge b_s) \texttt{ do } h[\texttt{true}] \end{array} \right]\!\!\right] .$$

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h[_] is a command containing

```
if [ _ ] then ... else ...
```

Hasuo (Tokyo)

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$\texttt{while } b_g \texttt{ do } \boxed{\ldots(\texttt{if}\ldots)\ldots}$

into $\left[\begin{array}{l} \texttt{while } b_g \wedge \neg b_s \texttt{ do } \boxed{\ldots} \ ; \\ \texttt{while } b_g \wedge b_s \texttt{ do } \boxed{\ldots} \end{array}\right]$

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Proof.

$\models \{b_{\mathbf{s}}\}\,\overline{h}\,\{b_{\mathbf{c}}\}$

$\models \{\neg b_{\mathbf{s}}\}\,\overline{h}\,\{\neg b_{\mathbf{c}}\}$

$\models \{b_{\mathbf{g}} \wedge b_{\mathbf{s}}\}\,h[b_{\mathbf{c}}]\,\{\neg b_{\mathbf{g}} \vee b_{\mathbf{s}}\}$

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The set of *holed commands* $\mathbf{Cmd}_{[\_]}$ is:

$$\mathbf{Cmd}_{[\_]} \ni h \ ::= \ \texttt{if } [\_] \texttt{ then } c_1 \texttt{ else } c_2 \mid h;c \mid c;h \mid \texttt{if } b \texttt{ then } h \texttt{ else } c \mid \texttt{if } b \texttt{ then } c \texttt{ else } h$$

For each holed command h, its *pre-hole fragment* $\overline{h}$ is:

$$\begin{array}{lll} \overline{\texttt{if } [\_] \texttt{ then } c_1 \texttt{ else } c_2} & :\equiv & \texttt{skip} \\ \overline{h;c} \ :\equiv \ \overline{h} \qquad \overline{c;h} & :\equiv & c;\overline{h} \\ \overline{\texttt{if } b \texttt{ then } h \texttt{ else } c} & :\equiv & \texttt{assert } b \,;\overline{h} \\ \overline{\texttt{if } b \texttt{ then } c \texttt{ else } h} & :\equiv & \texttt{assert } \neg b \,;\overline{h} \end{array}$$

Lem.
If a Boolean expression $b_{\mathbf{s}} \in \mathbf{BExp}$ satisfies

$$\models \{b_{\mathbf{s}}\}\,\overline{h}\,\{b_{\mathbf{c}}\}\ , \quad \models \{\neg b_{\mathbf{s}}\}\,\overline{h}\,\{\neg b_{\mathbf{c}}\}\ , \quad \text{and} \quad \models \{b_{\mathbf{g}} \wedge b_{\mathbf{s}}\}\,h[b_{\mathbf{c}}]\,\{\neg b_{\mathbf{g}} \vee b_{\mathbf{s}}\}\ ,$$

then we have

$$[\![\ \texttt{while } b_{\mathbf{g}} \texttt{ do } h[b_{\mathbf{c}}]\ ]\!] = \left[\!\!\left[\begin{array}{l} \texttt{while } (b_{\mathbf{g}} \wedge \neg b_{\mathbf{s}}) \texttt{ do } h[\texttt{false}]\,; \\ \texttt{while } (b_{\mathbf{g}} \wedge b_{\mathbf{s}}) \texttt{ do } h[\texttt{true}] \end{array}\right]\!\!\right] .$$

Proof.

$$\begin{array}{l} \models \{b_{\mathbf{s}}\}\,\overline{h}\,\{b_{\mathbf{c}}\} \\ \models \{\neg b_{\mathbf{s}}\}\,\overline{h}\,\{\neg b_{\mathbf{c}}\} \\ \models \{b_{\mathbf{g}} \wedge b_{\mathbf{s}}\}\,h[b_{\mathbf{c}}]\,\{\neg b_{\mathbf{g}} \vee b_{\mathbf{s}}\} \end{array}$$

$\Leftrightarrow$ sectionwise

$$\begin{array}{l} \models \{b_{\mathbf{s}}|_i\}\,\overline{h|_i}\,\{b_{\mathbf{c}}|_i\} \\ \models \{\neg b_{\mathbf{s}}|_i\}\,\overline{h|_i}\,\{\neg b_{\mathbf{c}}|_i\} \\ \models \{b_{\mathbf{g}}|_i \wedge b_{\mathbf{s}}|_i\}\,h|_i[b_{\mathbf{c}}|_i]\,\{\neg b_{\mathbf{g}}|_i \vee b_{\mathbf{s}}|_i\} \end{array}$$

$\vdots$

(for almost all i)

Hasuo (Tokyo)

Phase Split
(Nonstandard Ver., for While^{dt} & Hoare^{dt})

Defn.
The set of *holed commands* $\mathbf{Cmd}_{[\_]}$ is:

$$\mathbf{Cmd}_{[\_]} \ni h \ ::= \ \texttt{if } [\_] \texttt{ then } c_1 \texttt{ else } c_2 \mid h;c \mid c;h \mid \texttt{if } b \texttt{ then } h \texttt{ else } c \mid \texttt{if } b \texttt{ then } c \texttt{ else } h$$

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$$\begin{array}{lll} \overline{\texttt{if } [\_] \texttt{ then } c_1 \texttt{ else } c_2} & :\equiv & \texttt{skip} \\ \overline{h;c} \ :\equiv \ \overline{h} \qquad \overline{c;h} & :\equiv & c;\overline{h} \\ \overline{\texttt{if } b \texttt{ then } h \texttt{ else } c} & :\equiv & \texttt{assert } b \,;\overline{h} \\ \overline{\texttt{if } b \texttt{ then } c \texttt{ else } h} & :\equiv & \texttt{assert } \neg b \,;\overline{h} \end{array}$$

Lem.
If a Boolean expression $b_{\mathbf{s}} \in \mathbf{BExp}$ satisfies

$$\models \{b_{\mathbf{s}}\}\,\overline{h}\,\{b_{\mathbf{c}}\}\ , \quad \models \{\neg b_{\mathbf{s}}\}\,\overline{h}\,\{\neg b_{\mathbf{c}}\}\ , \quad \text{and} \quad \models \{b_{\mathbf{g}} \wedge b_{\mathbf{s}}\}\,h[b_{\mathbf{c}}]\,\{\neg b_{\mathbf{g}} \vee b_{\mathbf{s}}\}\ ,$$

then we have

$$[\![\ \texttt{while } b_{\mathbf{g}} \texttt{ do } h[b_{\mathbf{c}}]\ ]\!] = \left[\!\!\left[\begin{array}{l} \texttt{while } (b_{\mathbf{g}} \wedge \neg b_{\mathbf{s}}) \texttt{ do } h[\texttt{false}]\,; \\ \texttt{while } (b_{\mathbf{g}} \wedge b_{\mathbf{s}}) \texttt{ do } h[\texttt{true}] \end{array}\right.\right] .$$

Proof.

$$\models \{b_{\mathbf{s}}\}\,\overline{h}\,\{b_{\mathbf{c}}\}$$
$$\models \{\neg b_{\mathbf{s}}\}\,\overline{h}\,\{\neg b_{\mathbf{c}}\}$$
$$\models \{b_{\mathbf{g}} \wedge b_{\mathbf{s}}\}\,h[b_{\mathbf{c}}]\,\{\neg b_{\mathbf{g}} \vee b_{\mathbf{s}}\}$$

$\Leftrightarrow$ sectionwise

$$\models \{b_{\mathbf{s}}|_i\}\,\overline{h|_i}\,\{b_{\mathbf{c}}|_i\}$$
$$\models \{\neg b_{\mathbf{s}}|_i\}\,\overline{h|_i}\,\{\neg b_{\mathbf{c}}|_i\}$$
$$\models \{b_{\mathbf{g}}|_i \wedge b_{\mathbf{s}}|_i\}\,h|_i[b_{\mathbf{c}}|_i]\,\{\neg b_{\mathbf{g}}|_i \vee b_{\mathbf{s}}|_i\}$$

(for almost all i)

$\Rightarrow\Rightarrow\Rightarrow$ std. ver.

$$[\![\ \texttt{while } b_{\mathbf{g}}|_i \texttt{ do } h|_i[b_{\mathbf{c}}|_i]\ ]\!] = \left[\!\!\left[\begin{array}{l} \texttt{while } (b_{\mathbf{g}}|_i \wedge \neg b_{\mathbf{s}}|_i) \texttt{ do } h|_i[\texttt{false}]\,; \\ \texttt{while } (b_{\mathbf{g}}|_i \wedge b_{\mathbf{s}}|_i) \texttt{ do } h|_i[\texttt{true}] \end{array}\right.\right]$$

Hasuo (Tokyo)

Phase Split
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Defn.
The set of *holed commands* $\mathbf{Cmd}_{[\_]}$ is:

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For each holed command h, its *pre-hole fragment* $\overline{h}$ is:

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Lem.
If a Boolean expression $b_s \in \mathbf{BExp}$ satisfies

$$\models \{b_s\}\,\overline{h}\,\{b_c\}\ , \quad \models \{\neg b_s\}\,\overline{h}\,\{\neg b_c\}\ , \quad \text{and} \quad \models \{b_g \wedge b_s\}\,h[b_c]\,\{\neg b_g \vee b_s\}\ ,$$

then we have

$$[\![\ \texttt{while } b_g \texttt{ do } h[b_c]\ ]\!] = \left[\!\!\left[\begin{array}{l} \texttt{while } (b_g \wedge \neg b_s) \texttt{ do } h[\texttt{false}]\,; \\ \texttt{while } (b_g \wedge b_s) \texttt{ do } h[\texttt{true}] \end{array}\right]\!\!\right] .$$

Proof.

$$\models \{b_s\}\,\overline{h}\,\{b_c\}$$
$$\models \{\neg b_s\}\,\overline{h}\,\{\neg b_c\}$$
$$\models \{b_g \wedge b_s\}\,h[b_c]\,\{\neg b_g \vee b_s\}$$

$\Leftrightarrow$ sectionwise

$$\models \{b_s|_i\}\,\overline{h|_i}\,\{b_c|_i\}$$
$$\models \{\neg b_s|_i\}\,\overline{h|_i}\,\{\neg b_c|_i\}$$
$$\models \{b_g|_i \wedge b_s|_i\}\,h|_i[b_c|_i]\,\{\neg b_g|_i \vee b_s|_i\}$$

$\vdots$ (for almost all i)

$\Rightarrow\Rightarrow\Rightarrow$ std. ver.

$$[\![\ \texttt{while } b_g|_i \texttt{ do } h|_i[b_c|_i]\ ]\!] = \left[\!\!\left[\begin{array}{l} \texttt{while } (b_g|_i \wedge \neg b_s|_i) \texttt{ do } h|_i[\texttt{false}]\,; \\ \texttt{while } (b_g|_i \wedge b_s|_i) \texttt{ do } h|_i[\texttt{true}] \end{array}\right]\!\!\right]$$

$\vdots$

sectionwise $\Leftrightarrow$

$$[\![\ \texttt{while } b_g \texttt{ do } h[b_c]\ ]\!] = \left[\!\!\left[\begin{array}{l} \texttt{while } (b_g \wedge \neg b_s) \texttt{ do } h[\texttt{false}]\,; \\ \texttt{while } (b_g \wedge b_s) \texttt{ do } h[\texttt{true}] \end{array}\right]\!\!\right]$$

Transferring Static Analysis Strategies

$$\models \{b_s\}\, \overline{h}\, \{b_c\}$$
$$\models \{\neg b_s\}\, \overline{h}\, \{\neg b_c\}$$
$$\models \{b_g \wedge b_s\}\, h[b_c]\, \{\neg b_g \vee b_s\}$$

⇕ sectionwise

$$\models \{b_s|_i\}\, \overline{h|_i}\, \{b_c|_i\}$$
$$\models \{\neg b_s|_i\}\, \overline{h|_i}\, \{\neg b_c|_i\}$$
$$\models \{b_g|_i \wedge b_s|_i\}\, h|_i[b_c|_i]\, \{\neg b_g|_i \vee b_s|_i\}$$

⋮

(for almost all i)

⇒⇒⇒ ⋮ std. ver.

$$[\![\texttt{while}\ b_g|_i\ \texttt{do}\ h|_i[b_c|_i]]\!]$$
$$= [\![\texttt{while}\ (b_g|_i \wedge \neg b_s|_i)\ \texttt{do}\ h|_i[\texttt{false}]\ ;\ \texttt{while}\ (b_g|_i \wedge b_s|_i)\ \texttt{do}\ h|_i[\texttt{true}]]\!]$$

⋮

sectionwise ⇕

$$[\![\texttt{while}\ b_g\ \texttt{do}\ h[b_c]]\!]$$
$$= [\![\texttt{while}\ (b_g \wedge \neg b_s)\ \texttt{do}\ h[\texttt{false}]\ ;\ \texttt{while}\ (b_g \wedge b_s)\ \texttt{do}\ h[\texttt{true}]]\!]$$

* Doesn't matter what "std. ver." is
* ➜ **modular method** for transfer

Hasuo (Tokyo)

```
while (v > 0) {
  if m - z < s
    then a := -b
    else a := a0;
  t := 0;
  while (t < eps && v > 0) {
    z := z + v * dt;
    v := v + a * dt;
    t := t + dt }}

{z < m}
```

```
while (v > 0 && m - z >= s) {
  a := a0;    t := 0;
  while (t < eps && v > 0) {
    z := z + v * dt;
    v := v + a0 * dt;
    t := t + dt }};
while (v > 0 && m - z < s) {
  a := -b;    t := 0;
  while (t < eps && v > 0) {
    z := z + v * dt;
    v := v - b * dt;
    t := t + dt }}

{z < m}
```

accel.

brake

Strategy 1 "Phase split"

[Sharma,Dillig,Dillig,Aiken; CAV'11]
[Balakrishnan,Sankaranarayanan,Ivancic,Gupta; EMSOFT'09]
[Gopan,Reps; SAS'07]

```
while (v > 0 && m - z >= s) {
  a := a0;   t := 0;
  while (t < eps && v > 0) {
    z := z + v * dt;
    v := v + a0 * dt;
    t := t + dt }};
while (v > 0 && m - z < s) {
  a := -b;   t := 0;
  while (t < eps && v > 0) {
    z := z + v * dt;
    v := v - b * dt;
    t := t + dt }}
```

{z < m}

```
while (v > 0 && m - z >= s) {
  a := a0;    t := 0;
  while (t < eps && v > 0) {
    z := z + v * dt;
    v := v + a0 * dt;
    t := t + dt }};
while (v > 0 && m - z < s) {
  a := -b;    t := 0;
  while (t < eps && v > 0) {
    z := z + v * dt;
    v := v - b * dt;
    t := t + dt }}

{z < m}
```

```
if (v > 0)
  then
    while (m - z >= s) {
      a := a0;    t := 0;
      while (t < eps) {
        z := z + v * dt;
        v := v + a0 * dt;
        t := t + dt }}
  else skip;
while (v > 0) {
  a := -b;
  z := z + v * dt;
  v := v - b * dt }

{z < m}
```

```
while (v > 0 && m - z >= s) {
  a := a0;    t := 0;
  while (t < eps && v > 0) {
    z := z + v * dt;
    v := v + a0 * dt;
    t := t + dt }};
while (v > 0 && m - z < s) {
  a := -b;    t := 0;
  while (t < eps && v > 0) {
    z := z + v * dt;
    v := v - b * dt;
    t := t + dt }}

{z < m}
```

```
if (v > 0)
  then
    while (m - z >= s) {
      a := a0;    t := 0;
      while (t < eps) {
        z := z + v * dt;
        v := v + a0 * dt;
        t := t + dt }}
  else skip;
while (v > 0) {
  a := -b;
  z := z + v * dt;
  v := v - b * dt }

{z < m}
```

Startegies 2,3

"Superfluous guard elim." "Time elapse"

```
while (v > 0 && m - z >= s) {
  a := a0;    t := 0;
  while (t < eps && v > 0) {
    z := z + v * dt;
    v := v + a0 * dt;
    t := t + dt }};
while (v > 0 && m - z < s) {
  a := -b;    t := 0;
  while (t < eps && v > 0) {
    z := z + v * dt;
    v := v - b * dt;
    t := t + dt }}

{z < m}
```

```
if (v > 0)
  then
    while (m - z >= s) {
      a := a0;    t := 0;
      while (t < eps) {
        z := z + v * dt;
        v := v + a0 * dt;
        t := t + dt }}
  else skip;
while (v > 0) {
  a := -b;
```

Strategy 4

"Differential invariant"

[Platzer,Clarke; CAV'08]

Startegies 2,3

"Superfluous guard elim." "Time elapse"

```
if (v > 0)
  then
    while (m - z >= s) {
      a := a0;   t := 0;
      while (t < eps) {
        z := z + v * dt;
        v := v + a0 * dt;
        t := t + dt }}
  else skip;
while (v > 0) {
  a := -b;
  z := z + v * dt;
  v := v - b * dt }

{z < m}
```

```
if (v > 0)
  then
    while (m - z >= s) {
      a := a0;   t := 0;
      while (t < eps) {
        z := z + v * dt;
        v := v + a0 * dt;
        t := t + dt }}
  else skip;
while (v > 0) {
  a := -b;
  z := z + v * dt;
  v := v - b * dt }

{z < m}
```

```
if (v > 0)
  then
    while (m - z >= s) {
      a := a0;   t := 0;
      while (t < eps) {
        z := z + v * dt;
        v := v + a0 * dt;
        t := t + dt }}
  else skip;
```

$(v > 0 \vee m > z) \wedge$

$\{ \; (b^2 \mathrm{dt}^2 + 4b\mathrm{dt}v + 8bz + 4v^2 < 8bm \quad \}$

$\quad \vee b\mathrm{dt}v + 2bz + v^2 \leq 2bm)$

```
while (v > 0) {
  a := -b;
  z := z + v * dt;
  v := v - b * dt }
```

```
if (v > 0)
  then
    while (m - z >= s) {
      a := a0;   t := 0;
      while (t < eps) {
        z := z + v * dt;
        v := v + a0 * dt;
        t := t + dt }}
  else skip;
while (v > 0) {
  a := -b;
  z := z + v * dt;
  v := v - b * dt }

{z < m}
```

```
if (v > 0)
  then
    while (m - z >= s) {
      a := a0;   t := 0;
      while (t < eps) {
        z := z + v * dt;
        v := v + a0 * dt;
        t := t + dt }}
  else skip;
  (v > 0 ∨ m > z)∧
{ (b²dt² + 4bdtv + 8bz + 4v² < 8bm        }
    ∨ bdtv + 2bz + v² ≤ 2bm)
while (v > 0) {
  a := -b;
  z := z + v * dt;
  v := v - b * dt }
```

Strategy 5

"QE Invariant"

QE Invariant

Lem. In $\textsc{Hoare}^{\mathtt{dt}}$,

$$\vdash \left\{ \begin{array}{l} (\neg \boldsymbol{b} \Rightarrow \boldsymbol{A}) \wedge \\ \forall \boldsymbol{y} \in {}^*\mathbb{N}.\big(\big(\boldsymbol{b}[\boldsymbol{a}/\boldsymbol{x}]^{\boldsymbol{y}} \wedge \neg \boldsymbol{b}[\boldsymbol{a}/\boldsymbol{x}]^{\boldsymbol{y}+1} \big) \Rightarrow \boldsymbol{A}[\boldsymbol{a}/\boldsymbol{x}]^{\boldsymbol{y}+1} \big) \end{array} \right\}$$

$$\mathtt{while}\ \boldsymbol{b}\ \mathtt{do}\ \boldsymbol{x} := \boldsymbol{a}\ \{\boldsymbol{A}\}\ .$$

Hasuo (Tokyo)

QE Invariant

Lem. In $\textsc{Hoare}^{\mathrm{dt}}$,

$$\vdash \left\{ \begin{array}{l} (\neg b \Rightarrow A) \wedge \\ \forall y \in {}^*\mathbb{N}.\big(\big(b[a/x]^y \wedge \neg b[a/x]^{y+1} \big) \Rightarrow A[a/x]^{y+1} \big) \end{array} \right\}$$

$$\texttt{while}\ b\ \texttt{do}\ x := a\ \{A\}\ .$$

quantifier must go!
(to manage complexity)

Hasuo (Tokyo)

QE Invariant

Lem. In $\textsc{Hoare}^{\mathrm{dt}}$,

$$\vdash \left\{ \begin{array}{l} (\neg b \Rightarrow A) \wedge \\ \forall y \in {}^*\mathbb{N}.\big(\big(b[a/x]^y \wedge \neg b[a/x]^{y+1} \big) \Rightarrow A[a/x]^{y+1} \big) \end{array} \right\} \texttt{while } b \texttt{ do } x := a \ \{A\} \ .$$

quantifier must go!
(to manage complexity)

* **Quantifier elimination**
  * Tarski, CAD algorithm, `Resolve` in Mathematica

QE Invariant

Lem. In $\textsc{Hoare}^{\mathrm{dt}}$,

$$\vdash \left\{ \begin{array}{l} (\neg b \Rightarrow A) \wedge \\ \forall y \in {}^*\mathbb{N}.\big(\big(b[a/x]^y \wedge \neg b[a/x]^{y+1} \big) \Rightarrow A[a/x]^{y+1} \big) \end{array} \right\} \texttt{while}\ b\ \texttt{do}\ x := a\ \{A\}\ .$$

quantifier must go!
(to manage complexity)

* **Quantifier elimination**
  * Tarski, CAD algorithm, `Resolve` in Mathematica
  * e.g. $\models\ \forall x \in \mathbb{R}.\,(x^2 + ax + b > 0) \iff a^2 - 4b < 0$

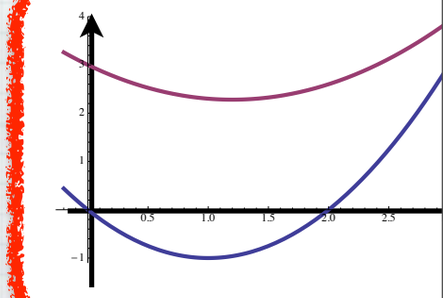
Hasuo (Tokyo)

QE Invariant

Lem. In $\textsc{Hoare}^{\mathrm{dt}}$,

$$\vdash \left\{ \begin{array}{l} (\neg b \Rightarrow A) \wedge \\ \forall y \in {}^*\mathbb{N}.\big(\big(b[a/x]^y \wedge \neg b[a/x]^{y+1} \big) \Rightarrow A[a/x]^{y+1} \big) \end{array} \right\} \texttt{while } b \texttt{ do } x := a \ \{A\} \ .$$

quantifier must go!
(to manage complexity)



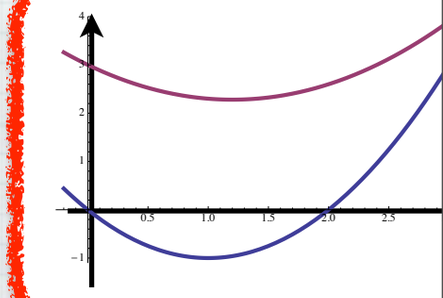
* **Quantifier elimination**
  * Tarski, CAD algorithm, `Resolve` in Mathematica
  * e.g. $\models \forall x \in \mathbb{R}.\, (x^2 + ax + b > 0) \iff a^2 - 4b < 0$

QE Invariant

Lem. In $\textsc{Hoare}^{\mathrm{dt}}$,

$$\vdash \left\{ \begin{array}{l} (\neg b \Rightarrow A) \wedge \\ \forall y \in {}^*\mathbb{N}.\big(\big(b[a/x]^y \wedge \neg b[a/x]^{y+1} \big) \Rightarrow A[a/x]^{y+1} \big) \end{array} \right\} \texttt{while}\ b\ \texttt{do}\ x := a\ \{A\}\ .$$

quantifier must go!
(to manage complexity)



* **Quantifier elimination**
  * Tarski, CAD algorithm, `Resolve` in Mathematica
  * e.g. $\models\ \forall x \in \mathbb{R}.\,(x^2 + ax + b > 0) \iff a^2 - 4b < 0$
  * then $\models\ \forall x \in {}^*\mathbb{R}.\,(x^2 + ax + b > 0) \iff a^2 - 4b < 0$

by **transfer**!

Hasuo (Tokyo)

```
if (v > 0)
  then
    while (m - z >= s) {
      a := a0;   t := 0;
      while (t < eps) {
        z := z + v * dt;
        v := v + a0 * dt;
        t := t + dt }}
  else skip;
while (v > 0) {
  a := -b;
  z := z + v * dt;
  v := v - b * dt }

{z < m}
```

```
if (v > 0)
  then
    while (m - z >= s) {
      a := a0;   t := 0;
      while (t < eps) {
        z := z + v * dt;
        v := v + a0 * dt;
        t := t + dt }}
  else skip;
  (v > 0 ∨ m > z)∧
{ (b²dt² + 4bdtv + 8bz + 4v² < 8bm        }
    ∨ bdtv + 2bz + v² ≤ 2bm)
while (v > 0) {
  a := -b;
  z := z + v * dt;
  v := v - b * dt }
```

Strategy 5

"QE Invariant"

```
if (v > 0)
  then
    while (m - z >= s) {
      a := a0;    t := 0;
      while (t < eps) {
        z := z + v * dt;
        v := v + a0 * dt;
        t := t + dt }}
  else skip;
```

$(v > 0 \vee m > z) \wedge$

$\{\ (b^2 \mathrm{dt}^2 + 4b\mathrm{dt}v + 8bz + 4v^2 < 8bm \quad \}$

$\vee\ b\mathrm{dt}v + 2bz + v^2 \leq 2bm)$

```
while (v > 0) {
  a := -b;
  z := z + v * dt;
  v := v - b * dt }
```

```
if (v > 0)
  then
    while (m - z >= s) {
      a := a0;    t := 0;
      while (t < eps) {
        z := z + v * dt;
        v := v + a0 * dt;
        t := t + dt }}
  else skip;
```

$(v > 0 \lor m > z) \land$

$\{\ (b^2 \mathrm{dt}^2 + 4b\mathrm{dt}v + 8bz + 4v^2 < 8bm \quad \}$

$\quad \lor\ b\mathrm{dt}v + 2bz + v^2 \leq 2bm)$

```
while (v > 0) {
  a := -b;
  z := z + v * dt;
  v := v - b * dt }
```

+ some fwd. propagation

```
{ ... (long fml. with dt) }
while (m - z >= s) {
  a := a0;    t := 0;
  while (t < eps) {
    z := z + v * dt;
    v := v + a0 * dt;
    t := t + dt }}
{ ... }
while (v > 0) {
  a := -b;
  z := z + v * dt;
  v := v - b * dt }
```

```
if (v > 0)
  then
    while (m - z >= s) {
      a := a0;    t := 0;
      while (t < eps) {
        z := z + v * dt;
        v := v + a0 * dt;
        t := t + dt }}
  else skip;
```

$(v > 0 \vee m > z) \wedge$

$\{ \; (b^2 \mathrm{dt}^2 + 4b\mathrm{dt}v + 8bz + 4v^2 < 8bm \quad \}$

$\quad \vee b\mathrm{dt}v + 2bz + v^2 \leq 2bm)$

```
while (v > 0) {
  a := -b;
  z := z + v * dt;
  v := v - b * dt }
```

+ some fwd. propagation

```
{ ... (long fml. with dt) }
while (m - z >= s) {
  a := a0;    t := 0;
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    z := z + v * dt;
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{ ... }
while (v > 0) {
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```

Strategy 6

"Iteration count"

```
if (v > 0)
  then
    while (m - z >= s) {
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  else skip;
```

$(v > 0 \vee m > z) \wedge$

$\{\ (b^2 dt^2 + 4b dt v + 8bz + 4v^2 < 8bm \quad \}$

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Strategy 6

"Iteration count"

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x := 0;
while (x < x0) {
  x := x + a
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Strategy 6

"Iteration count"

```
x := 0;
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  x := x + a
}
```

iteration: x0/a times?

* approximated by ⌊x0/a⌋ or ⌈x0/a⌉
* ➜ monotonicity reqm. must be discharged

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long fml. w/o `dt`, whose core is

$$a_0(2\varepsilon\sqrt{2a_0(m-s-z_0)+v_0^2}+b\epsilon^2+2m-2s-2z_0)$$
$$+2b\varepsilon\sqrt{2a_0(m-s-z_0)+v_0^2}+a_0^2\epsilon^2+v_0^2<2bs$$

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Strategy 7

"Cast to shadow"

(Eliminates `dt`, strengthens the precond.)

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Lem. If:

1. $\boldsymbol{a}$ is *closed*
2. $\boldsymbol{r} \mapsto [\![\boldsymbol{a}[\boldsymbol{r}/\mathtt{dt}]]\!]$ is continuous at $\boldsymbol{r} = \mathbf{0}$,

then $\models \boldsymbol{a}[\mathbf{0}/\mathtt{dt}] < \mathbf{0} \Longrightarrow \boldsymbol{a} < \mathbf{0}$.

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the final outcome

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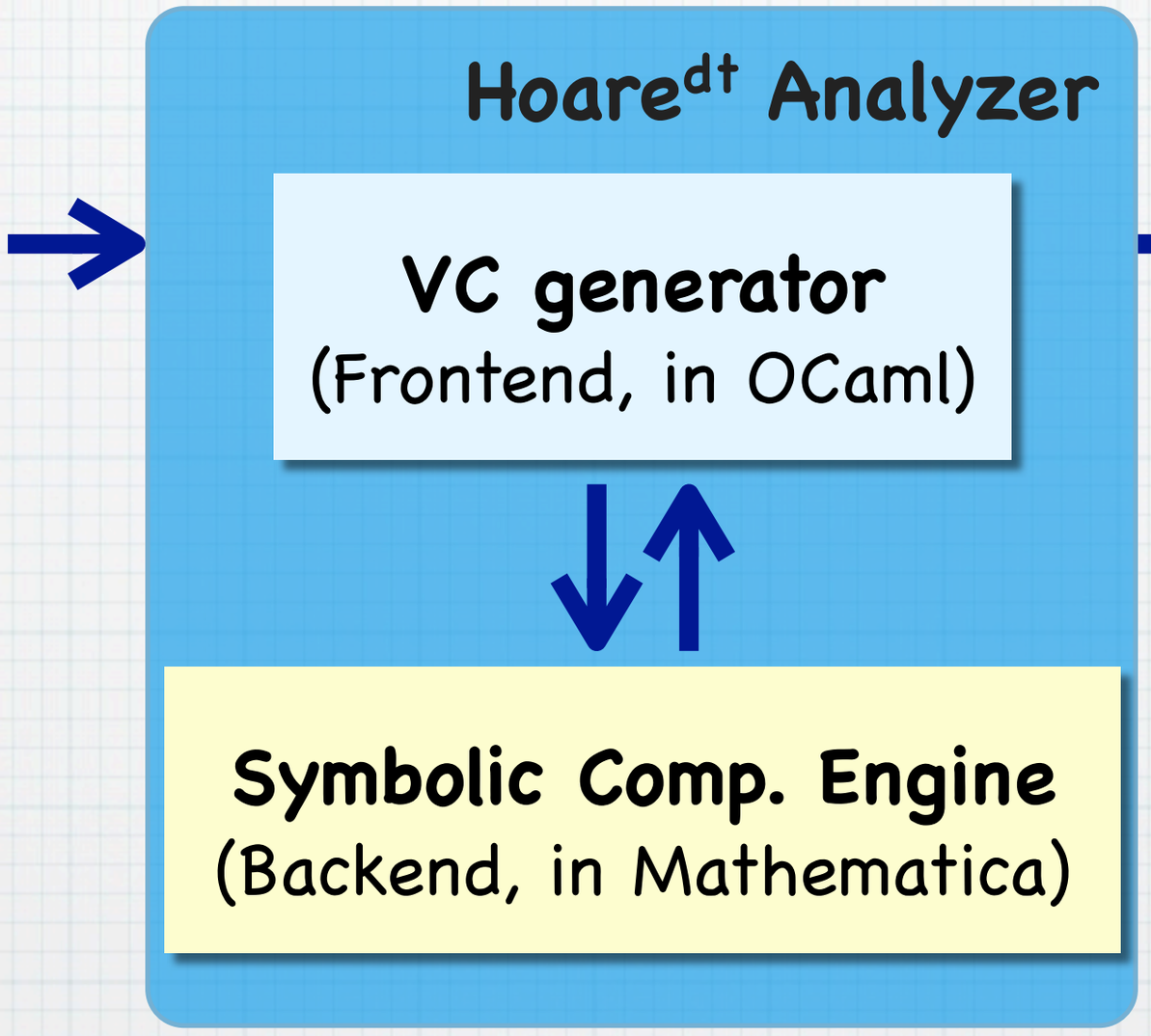
Prototype Automatic Prover

P

(While^{dt} program)

B

(postcondition)



A

(precondition)

s.t.

$\vdash \{A\}P\{B\}$

* Fujitsu HX600 with Quad Core AMD Opteron 2.3GHz CPU, 32GB memory. Mathematica 7.0 for Linux x86 (64-bit)
  * ETCS: 40.96 sec.
  * Bouncing ball: runs with one manual insertion of invariants

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Prototype Automatic Prover

P

(While^{dt} program)

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Hoare^{dt} Analyzer

VC generator
(Frontend, in OCaml)

Symbolic Comp. Engine
(Backend, in Mathematica)

A

(precondition)
s.t.
$\vdash \{A\}P\{B\}$

Totally symbolic
(crucial for transfer)

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Related Work

- **Deductive verification** of hybrid sys. [Platzer, '10] [Platzer, LICS'12]
  - Automatic prover KeYmaera
- **Static analysis techniques**
  - A **LOT** in CAV, SAS, VMCAI, ...
  - Applied to hybrid systems (w/ diff. eq.)
    [Rodriguez-Carbonell, Tiwari; HSCC'05] [Sankaranarayanan; HSCC'10]
    [Sankaranarayanan, Sipma, Manna; Formal Methods Sys. Design '08]
- Use of **NSA for hybrid systems**
  [Benveniste, Bourke, Caillaud, Pouzet; J. Comput. Syst. Sci. '12]
  [Bliudze, Krob; Fundam. Inform. '09] [Gamboa, Kaufmann; J. Autom. Reason. '01]
- **Continuous techniques applied to discrete appl.**
  [Chaudhuri, Gulwani, Lublinerman, NavidPour; FSE '11]
  - Not contending! Combination?

Conclusions

While^{dt}

Programming lang.

```
while (t<a) do {
  t:=t+1;
  if ...
}
```

Assn^{dt}

First-order assertion lang.

$\exists z(x=2*z \wedge y=3*z)$

Hoare^{dt}

Hoare-style program logic

$$\frac{\{A \wedge b\}\, c\, \{A\}}{\{A\}\ \texttt{while}\ b\ \texttt{do}\ c\ \{A \wedge \neg b\}}$$

Rigorous semantics by non-standard analysis

Hasuo (Tokyo)

Conclusions

Nonstandard Static Analysis

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Rigorous semantics by non-standard analysis

- Tool's effectivity. More heuristics?
- (Any discrete frmwk.)^{dt} ?
- Simulink as stream processing?
- With (explicit) differential equations?

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Conclusions

Nonstandard Static Analysis

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Thank you for your attention!

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