[1] KS and IH. **Programming with Infinitesimals: A While-Language for Hybrid System Modeling**. Proc. ICALP 2011, Track B.
[2] IH and KS. **Exercises in Nonstandard Static Analysis of Hybrid Systems**. Proc. CAV 2012.

Nonstandard Static Analysis

Transfer Verification to Hybrid Systems

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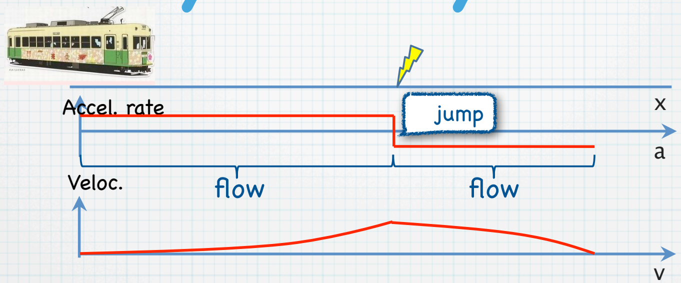
京都大学 KYOTO UNIVERSITY

Hybrid System



* Flow & jump
  * Digital control in a physical environment
  * Component of cyber-physical systems

Hybrid System

Formal verification
(computer science)

Hybrid!

- Flow?
- With minimal cost?

Discrete
"jump"

and

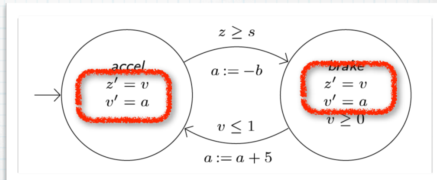
Continuous
"flow"

Hybrid!

Control theory
(applied analysis)

Formal Verification Approaches

* Hybrid automata
  [Alur, Henzinger, ...; '90s–]



* Differential dynamic logic
  [Platzer & others, '07–]

$$[\dot{x} = 1 \text{ while } x \le 3]\varphi$$

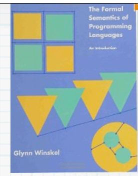
* Differential equations, explicitly
  ➜ distinction jump vs. flow

"Turn Flow into Jump"

$t := 0$;
while $(t \le 1)$ do {
  $t := t + \mathrm{dt}$
}

- Infinitesimal number dt
  - "Infinitely small" :
    0 < dt < r
    for **any** positive real r
- t = 1 after the execution?
- Non-standard analysis!
  [Robinson '60s]

Theoretical Framework [Suenaga&H., ICALP'11]



The standard textbook [Winskel]

While$^{\mathrm{dt}}$	**Assn**$^{\mathrm{dt}}$	**Hoare**$^{\mathrm{dt}}$

Programming lang.
`while (t<a) do {`
`  t:=t+1;`
`  if ...`
`}`
	First-order assertion lang.
	`∃z(x=2*z ∧ y=3*z)`
		Hoare-style program logic
		$\dfrac{\{A\wedge b\}\,c\,\{A\}}{\{A\}\ \mathtt{while}\ b\ \mathtt{do}\ c\ \{A\wedge\neg b\}}$

Rigorous semantics by non-standard analysis

- **Hoare**$^{\mathrm{dt}}$: sound and relatively complete
- Program verification/static analysis of hybrid systems
- Actual verification with NSA

Program Verif. Techniques
- Esp. invariant discovery

Static Analysis

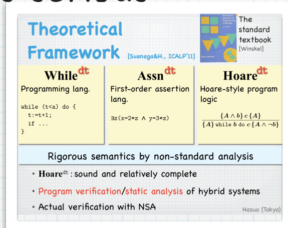
Nonstandard Static Analysis

Nonstandard Analysis

Infinitesimal dt

Nonstandard Static Analysis

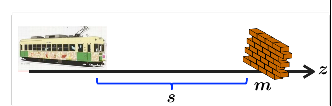
- Towards serious use of



- Static analysis techniques transferred to hybrid appl.

Exactly as they are!

- Leading example: ETCS



Prototype Automatic Prover

P (While^{dt} program) →

B (postcondition)

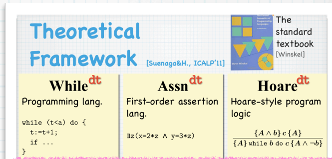
Hoare^{dt} Analyzer

- **VC generator** (Frontend, in OCaml)
- ↓↑
- **Symbolic Comp. Engine** (Backend, in Mathematica)

→ ***A*** (precondition) s.t. $\vdash \{A\}P\{B\}$

For reals, not hyperreals
➜ justified by the **transfer principle**

Outline



w/ or w/o dt ...
➜ logically "the same"

* **Theoretical foundations**
  * While^{dt}, Assn^{dt}, Hoare^{dt}
  * Rigorous semantics via NSA
  * Transfer principle, "sectionwise lemmas"
* **Static analysis techniques, transferred as they are**
  * Phase split [Sharma,Dillig,Dillig,Aiken; CAV'11] [Balakrishnan,Sankaranarayanan,Ivancic,Gupta; EMSOFT'09] [Gopan,Reps; SAS'07]
  * Differential invariant [Platzer,Clarke; CAV'08]
  * ... and more!

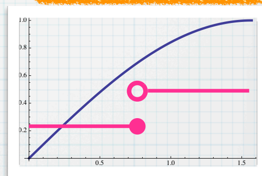
Part I: Theoretical Foundations

Nonstandard Analysis

* Analysis with an **infinitesimal** δ, e.g.

"Infinitely small" $0 < \delta < r$ $(\forall r \in \mathbb{R}_+)$

$$f \text{ is continuous} \iff \left(\begin{array}{l} |x - x'| \text{ is infinitesimal} \\ \implies |f(x) - f(x')| \text{ is infinitesimal} \end{array} \right)$$



* Cf. Leibniz's **monad**
* Done naively ➜ contradiction!

Logical foundation via an ultrafilter
[Robinson,1960]

Hyperreals

= Reals + Infinitesimals + ...

Defn.
The set of *hyperreal numbers* is

$$^*\mathbb{R} \;:=\; \mathbb{R}^{\mathbb{N}} / \sim_{\mathcal{F}} \;\ni\; \big[\,(a_0, a_1, a_2, \dots)\,\big]$$

Ignore

0th section / 1st section / 2nd section

* Operations: sectionwise

$$\begin{array}{ll} & [(a_0, a_1, \dots)] \\ + & [(b_0, b_1, \dots)] \\ = & [(a_0 + b_0, a_1 + b_1, \dots)] \end{array}$$

* Reals are hyperreals

$\mathbb{R} \hookrightarrow {}^*\mathbb{R}$,
$r \mapsto \big[\,(r, r, \dots)\,\big]$

Hyperreals

= Reals + Infinitesimals + ...

Defn.
The set of *hyperreal numbers* is

$$^*\mathbb{R} \;:=\; \mathbb{R}^{\mathbb{N}} / \sim_{\mathcal{F}} \;\ni\; \big[\,(a_0, a_1, a_2, \dots)\,\big]$$

* Predicates: sectionwise, "for almost all i"

$[(a_i)_{i\in\mathbb{N}}] < [(b_i)_{i\in\mathbb{N}}]$
$\iff$ $a_i < b_i$ "for almost every i"
$\Longleftarrow$ $\{i \in \mathbb{N} \mid a_i \not< b_i\}$ is finite

"For sufficiently large i"
"Except for finitely many i"

Precise defn. is via an ultrafilter $\mathcal{F}$:

$[(a_i)_{i\in\mathbb{N}}] < [(b_i)_{i\in\mathbb{N}}]$
$\iff$ $\{i \in \mathbb{N} \mid a_i < b_i\} \in \mathcal{F}$

Hyperreals

= Reals + Infinitesimals + ...

Defn.
The set of *hyperreal numbers* is
$^*\mathbb{R} \;:=\; \mathbb{R}^{\mathbb{N}} / \sim_{\mathcal{F}}$

$[(a_i)_{i\in\mathbb{N}}] < [(b_i)_{i\in\mathbb{N}}]$
$\iff$ $a_i < b_i$ "for almost every i"
$\Longleftarrow$ $\{i \in \mathbb{N} \mid a_i \not< b_i\}$ is finite

Prop. $\omega^{-1} = \big[\,(1, \frac{1}{2}, \frac{1}{3}, \dots)\,\big]$ is infinitesimal.

$$\omega^{-1} = (1, \tfrac{1}{2}, \tfrac{1}{3}, \dots, \tfrac{1}{N}, \tfrac{1}{N+1}, \dots)$$

OK! ^ ✖ ✖ ✖ ✖ ^ ^ ...

$$\frac{1}{N} = (\tfrac{1}{N}, \tfrac{1}{N}, \tfrac{1}{N}, \dots, \tfrac{1}{N}, \tfrac{1}{N}, \dots)$$

Hyperreals

= Reals + Infinitesimals + ...

Defn.
The set of *hyperreal numbers* is
$^*\mathbb{R} \;:=\; \mathbb{R}^{\mathbb{N}} / \sim_{\mathcal{F}}$

$[(a_i)_{i\in\mathbb{N}}] < [(b_i)_{i\in\mathbb{N}}]$
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$\Longleftarrow$ $\{i \in \mathbb{N} \mid a_i \not< b_i\}$ is finite

Prop. $\omega^{-1} = \big[\,(1, \frac{1}{2}, \frac{1}{3}, \dots)\,\big]$ is infinitesimal.

Prop. $\omega = \big[\,(1, 2, 3, \dots)\,\big]$ is infinite.

Hyperreals = Reals + Inf

Ultrafilter (existence by AC)

Defn. An *ultrafilter* $\mathcal{F} \subseteq \mathcal{P}(\mathbb{N})$ is such that:

1. For each $X \subseteq \mathbb{N}$, exactly one of X and $\mathbb{N} \setminus X$ is in $\mathcal{F}$.
2. $X, Y \in \mathcal{F} \Longrightarrow X \cap Y \in \mathcal{F}$
3. $X \in \mathcal{F}, X \subseteq Y \Longrightarrow Y \in \mathcal{F}$
4. $\emptyset \notin \mathcal{F}$

Defn. The set of *hyperreal numbers* is

$$^*\mathbb{R} \;:=\; \mathbb{R}^{\mathbb{N}} / \sim_{\mathcal{F}}$$

$$[(a_i)_{i\in\mathbb{N}}] < [(b_i)_{i\in\mathbb{N}}] \iff \{i \in \mathbb{N} \mid a_i < b_i\} \in \mathcal{F}$$

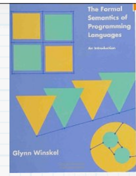
Thm. (Transfer Principle)
A: a first-order formula.
$^*A$: its $*$-*transform*. Then

$$\mathbb{R} \models A \iff {}^*\mathbb{R} \models {}^*A \,.$$

Same as A, except:
$\forall x \in \mathbb{R}$ in A is
$\forall x \in {}^*\mathbb{R}$ in $^*A$

$\mathbb{R}$ and $^*\mathbb{R}$ are "logically the same"

Theoretical Framework [Suenaga&H., ICALP'11]



The standard textbook [Winskel]

While^{dt}
Programming lang.

```
while (t<a) do {
  t:=t+1;
  if ...
}
```

Assn^{dt}
First-order assertion lang.

∃z(x=2*z ∧ y=3*z)

Hoare^{dt}
Hoare-style program logic

$$\frac{\{A \wedge b\}\, c\, \{A\}}{\{A\}\ \mathtt{while}\ b\ \mathtt{do}\ c\ \{A \wedge \neg b\}}$$

Rigorous semantics by non-standard analysis

Hasuo (Tokyo)

Syntax

While^{dt} — While + dt

AExp $\ni$ a ::= $x \mid c_r \mid a_1\ \mathsf{aop}\ a_2 \mid \mathsf{dt}$
where c_r is a const. for $r \in \mathbb{R}$, aop $\in \{+, -, \cdot, \hat{}\,, /\}$
BExp $\ni$ b ::= $\mathsf{true} \mid \mathsf{false} \mid b_1 \wedge b_2 \mid \neg b \mid a_1 < a_2$
Cmd $\ni$ c ::= $\mathsf{skip} \mid x := a \mid c_1; c_2$
$\mid \mathsf{if}\ b\ \mathsf{then}\ c_1\ \mathsf{else}\ c_2 \mid \mathsf{while}\ b\ \mathsf{do}\ c$

Assn^{dt}

A ::= $\mathsf{true} \mid \mathsf{false} \mid A_1 \wedge A_2 \mid \neg A \mid a_1 < a_2 \mid$
$\forall x \in {}^*\mathbb{N}.\, A \mid \forall x \in {}^*\mathbb{R}.\, A$

Hoare^{dt}

$$\frac{}{\{A\}\ \mathsf{skip}\ \{A\}}\ (\textsc{Skip}) \qquad \frac{}{\{A[a/x]\}\ x := a\ \{A\}}\ (\textsc{Assign})$$

$$\frac{\{A\}\, c_1\, \{C\} \quad \{C\}\, c_2\, \{B\}}{\{A\}\, c_1; c_2\, \{B\}}\ (\textsc{Seq}) \qquad \frac{\{A \wedge b\}\, c_1\, \{B\} \quad \{A \wedge \neg b\}\, c_2\, \{B\}}{\{A\}\ \mathsf{if}\ b\ \mathsf{then}\ c_1\ \mathsf{else}\ c_2\ \{B\}}\ (\textsc{If})$$

$$\frac{\{A \wedge b\}\, c\, \{A\}}{\{A\}\ \mathsf{while}\ b\ \mathsf{do}\ c\ \{A \wedge \neg b\}}\ (\textsc{While}) \qquad \frac{\models A \Rightarrow A' \quad \{A'\}\, c\, \{B'\} \quad \models B' \Rightarrow B}{\{A\}\, c\, \{B\}}\ (\textsc{Conseq})$$

Tokyo)

Syntax

While^{dt} — While + dt

AExp $\ni$ a ::= $x \mid c_r \mid a_1\ \mathsf{aop}\ a_2 \mid \mathsf{dt}$
where c_r is a const. for $r \in \mathbb{R}$, aop $\in \{+, -, \cdot, \hat{}\,, /\}$
BExp $\ni$ b ::= $\mathsf{true} \mid \mathsf{false} \mid b_1 \wedge b_2 \mid \neg b \mid a_1 < a_2$
Cmd $\ni$ c ::= $\mathsf{skip} \mid x := a \mid c_1; c_2$
$\mid \mathsf{if}\ b\ \mathsf{then}\ c_1\ \mathsf{else}\ c_2 \mid \mathsf{while}\ b\ \mathsf{do}\ c$

Assn^{dt} — Assn, $*$-transformed

A ::= $\mathsf{true} \mid \mathsf{false} \mid A_1 \wedge A_2 \mid \neg A \mid a_1 < a_2 \mid$
$\forall x \in {}^*\mathbb{N}.\, A \mid \forall x \in {}^*\mathbb{R}.\, A$

Hoare^{dt}

$$\frac{}{\{A\}\ \mathsf{skip}\ \{A\}}\ (\textsc{Skip}) \qquad \frac{}{\{A[a/x]\}\ x := a\ \{A\}}\ (\textsc{Assign})$$

$$\frac{\{A\}\, c_1\, \{C\} \quad \{C\}\, c_2\, \{B\}}{\{A\}\, c_1; c_2\, \{B\}}\ (\textsc{Seq}) \qquad \frac{\{A \wedge b\}\, c_1\, \{B\} \quad \{A \wedge \neg b\}\, c_2\, \{B\}}{\{A\}\ \mathsf{if}\ b\ \mathsf{then}\ c_1\ \mathsf{else}\ c_2\ \{B\}}\ (\textsc{If})$$

$$\frac{\{A \wedge b\}\, c\, \{A\}}{\{A\}\ \mathsf{while}\ b\ \mathsf{do}\ c\ \{A \wedge \neg b\}}\ (\textsc{While}) \qquad \frac{\models A \Rightarrow A' \quad \{A'\}\, c\, \{B'\} \quad \models B' \Rightarrow B}{\{A\}\, c\, \{B\}}\ (\textsc{Conseq})$$

Tokyo)

Syntax

While$^{\mathrm{dt}}$

While + dt

AExp $\ni$ a ::= $x \mid c_r \mid a_1 \; \mathtt{aop} \; a_2 \mid \mathtt{dt}$

where c_r is a const. for $r \in \mathbb{R}$, $\mathtt{aop} \in \{+, -, \cdot, \hat{}, /\}$

BExp $\ni$ b ::= $\mathtt{true} \mid \mathtt{false} \mid b_1 \wedge b_2 \mid \neg b \mid a_1 < a_2$

c ::= $\mathtt{skip} \mid x := a \mid c_1 ; c_2$ $\mid \mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2 \mid \mathtt{while}\ b\ \mathtt{do}\ c$

Thm. HOARE$^{\mathrm{dt}}$ rules are *sound* and *relatively complete*.

Hoare$^{\mathrm{dt}}$

Precisely the same ...

$$\frac{}{\{A\}\,\mathtt{skip}\,\{A\}}\ (\textsc{Skip}) \qquad \frac{}{\{\,A[a/x]\,\}\ x := a\ \{A\}}\ (\textsc{Assign})$$

$$\frac{\{A\}\,c_1\,\{C\} \quad \{C\}\,c_2\,\{B\}}{\{A\}\,c_1 ; c_2\,\{B\}}\ (\textsc{Seq}) \qquad \frac{\{A \wedge b\}\,c_1\,\{B\} \quad \{A \wedge \neg b\}\,c_2\,\{B\}}{\{A\}\ \mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2\ \{B\}}\ (\textsc{If})$$

$$\frac{\{A \wedge b\}\,c\,\{A\}}{\{A\}\ \mathtt{while}\ b\ \mathtt{do}\ c\ \{A \wedge \neg b\}}\ (\textsc{While}) \qquad \frac{\models A \Rightarrow A' \quad \{A'\}\,c\,\{B'\} \quad \models B' \Rightarrow B}{\{A\}\,c\,\{B\}}\ (\textsc{Conseq})$$

Denotational Semantics: Challenge

$t := 0\,;$
$\mathtt{while}\ (t \leq 1)\ \mathtt{do}\ \{$
$\quad t := t + \mathtt{dt}$
$\}$

$\Downarrow$

$t = 1 + \mathtt{dt}$

$t := 0\,;$
$\mathtt{while}\ (\mathtt{true})\ \mathtt{do}\ \{$
$\quad t := t + \mathtt{dt}$
$\}$

$\Downarrow$

$\perp$ (divergence)

- Semantics by "sectionwise execution"

Denotational Semantics

- Execute sectionwise and bundle up the outcomes!

```
t := 0;
while (t < 1)
  t := t + dt;
```


Denotational Semantics

- Execute sectionwise and bundle up the outcomes!

```
t := 0;
while (t < 1)
  t := t + dt;
```

Denotational Semantics

* Execute sectionwise and bundle up the outcomes!

```
t := (0,0,0,...);
while (t < (1,1,1,...))
  t := t + (1, 1/2, 1/3, ...) ;
```


Denotational Semantics

* Execute sectionwise and bundle up the outcomes!

0th section	1st section	2nd section	
`t := 0; while (t < 1) t := t + 1 ;`	`t := 0; while (t < 1) t := t + 1/2 ;`	`t := 0; while (t < 1) t := t + 1/3 ;`	...
`t = 1`	`t = 1`	`t = 1`	...

Denotational Semantics

* Execute sectionwise and bundle up the outcomes!

```
t := (0,0,0,...);
while (t < (1,1,1,...))
  t := t + (1, 1/2, 1/3, ...) ;
```

```
t = (1,1,1,...)
```


Denotational Semantics

* Execute sectionwise and bundle up the outcomes!

```
t := 0;
while (t < 1)
  t := t + dt;
```

```
t = 1
```

Denotational Semantics

* Execute sectionwise and bundle up the outcomes!

```
t := 0;
while (t <= 1)
  t := t + dt;
```


Denotational Semantics

* Execute sectionwise and bundle up the outcomes!

```
t := 0;
while (t <= 1)
  t := t + dt;
```


Denotational Semantics

* Execute sectionwise and bundle up the outcomes!

t := (0,0,0,...);
while (t <= (1,1,1,...))
t := t + $(1, \frac{1}{2}, \frac{1}{3}, \ldots)$;

Denotational Semantics

* Execute sectionwise and bundle up the outcomes!

0th section	1st section	2nd section	

t := 0;
while (t <= 1)
t := t + 1 ;
	t := 0;
	while (t <= 1)
	t := t + $\frac{1}{2}$;
		t := 0;
		while (t <= 1)
		t := t + $\frac{1}{3}$;
			...

t = 1 + 1	t = 1 + $\frac{1}{2}$	t = 1 + $\frac{1}{3}$	...

Denotational Semantics

* Execute sectionwise and bundle up the outcomes!

```
t := (0,0,0,...);
while (t <= (1,1,1,...))
  t := t + (1, 1/2, 1/3, ...) ;
```

$$t = (1,1,1,\ldots) + \left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right)$$

Denotational Semantics

* Execute sectionwise and bundle up the outcomes!

```
t := 0;
while (t <= 1)
  t := t + dt;
```

```
t = 1 + dt
```


Denotational Semantics

* Execute sectionwise and bundle up the outcomes!

```
t := 0;
while (true)
  t := t + dt;
```


Denotational Semantics

* Execute sectionwise and bundle up the outcomes!

```
t := 0;
while (true)
  t := t + dt;
```

Denotational Semantics

* Execute sectionwise and bundle up the outcomes!

```
t := (0,0,0,...);
while (true)
  t := t + (1, 1/2, 1/3, ...) ;
```


Denotational Semantics

* Execute sectionwise and bundle up the outcomes!

0th section

```
t := 0;
while (true)
  t := t + 1 ;
```

1st section

```
t := 0;
while (true)
  t := t + 1/2 ;
```

2nd section

```
t := 0;
while (true)
  t := t + 1/3 ;
```

...

⊥ ⊥ ⊥ ...

Denotational Semantics

* Execute sectionwise and bundle up the outcomes!

```
t := (0,0,0,...);
while (true)
  t := t + (1, 1/2, 1/3, ...) ;
```

```
t = (⊥,⊥,⊥,...)
```


Denotational Semantics

* Execute sectionwise and bundle up the outcomes!

```
t := 0;
while (true)
  t := t + dt;
```

⊥

Denot

$$\left[\begin{array}{l} t := 0\,; \\ \texttt{while}\ (t \le 1)\ \texttt{do} \\ \quad t := t + \mathtt{dt} \end{array}\right] \overset{i\text{-th section}}{\longmapsto} \left[\begin{array}{l} t := 0\,; \\ \texttt{while}\ (t \le 1)\ \texttt{do} \\ \quad t := t + \frac{1}{i+1} \end{array}\right]$$

Hyperstate (stores hyperreals)

$$\begin{aligned} [\![x]\!]\sigma &:= \sigma(x) \\ [\![a_1 \;\texttt{aop}\; a_2]\!]\sigma &:= [\![a_1]\!]\sigma \;\texttt{aop}\; [\![a_2]\!]\sigma \\ [\![\mathtt{dt}]\!]\sigma &:= \omega^{-1} = [\,(1, \tfrac{1}{2}, \tfrac{1}{3}, \dots)\,] \end{aligned}$$

$$\begin{aligned} [\![\mathtt{true}]\!]\sigma &:= \mathtt{tt} \\ [\![b_1 \wedge b_2]\!]\sigma &:= [\![b_1]\!]\sigma \wedge [\![b_2]\!]\sigma \\ [\![a_1 < a_2]\!]\sigma &:= [\![a_1]\!]\sigma < [\![a_2]\!]\sigma \end{aligned}$$

$[\![\mathtt{fa}$ … $[\![\neg$ …

Def. The *i-th section* of a WHILE$^{\mathrm{dt}}$ expression e is

$$e|_i \;:\equiv\; e\left[\tfrac{1}{i+1} \,/\, \mathtt{dt}\right] .$$

$$[\![\mathtt{skip}]\!]\sigma := \sigma \qquad [\![x := a]\!]\sigma := \sigma[\,x \mapsto [\![a]\!]\sigma\,] \qquad [\![c_1; c_2]\!]\sigma := [\![c_2]\!]([\![c_1]\!]\sigma)$$

$$[\![\texttt{if}\ b\ \texttt{then}\ c_1\ \texttt{else}\ c_2]\!]\sigma := \begin{cases} [\![c_1]\!]\sigma & \text{if } [\![b]\!]\sigma = \mathtt{tt} \\ [\![c_2]\!]\sigma & \text{if } [\![b]\!]\sigma = \mathtt{ff} \end{cases}$$

$$[\![\texttt{while}\ b\ \texttt{do}\ c]\!]\sigma := \left(\,[\![\,(\texttt{while}\ b\ \texttt{do}\ c)|_i\,]\!](\sigma|_i)\,\right)_{i \in \mathbb{N}}$$

Bundled up

Section of a program

Applied to a section of a memory state

"Sectionwise Lemmas"

Sectionwise Execution Lemma. For any expr. e and $i \in \mathbb{N}$,

$$[\![e]\!]\sigma = [\,(\,[\![e|_i]\!](\sigma|_i)\,)_{i\in\mathbb{N}}\,] .$$

Sectionwise Satisfaction Lemma. For any hyperstate σ and an ASSN$^{\mathrm{dt}}$ formula φ:

$$\sigma \models \varphi \iff \sigma|_i \models \varphi|_i \quad \text{for almost every } i.$$

Łos'

"Sectionwise Lemmas"

Lem. (Sectionwise validity of Hoare triples)

$$\models \{A\}c\{B\} \iff \models \{A|_i\}\; c|_i\; \{B|_i\} \quad \text{for almost every } i.$$

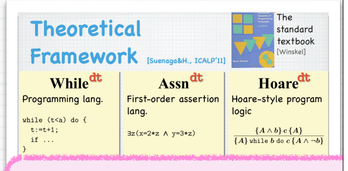
Interface for transferring static analysis techniques

Q. Is a While$^{\mathrm{dt}}$ program executable?

* A. Not exactly.
  * A modeling language
    * Not numerical approx., but exact modeling
    * Advantage: close to a common programming style
  * Static analysis ➜ no need to execute!
    * Mathematical semantics suffices

Outline

Suenaga & H., ICALP'11



- **Theoretical foundations**
  - **While^dt, Assn^dt, Hoare^dt**
  - Rigorous semantics via NSA
  - Transfer principle, "sectionwise lemmas"

w/ or w/o dt ...
→ logically "the same"

Done

H. & Suenaga, CAV'12

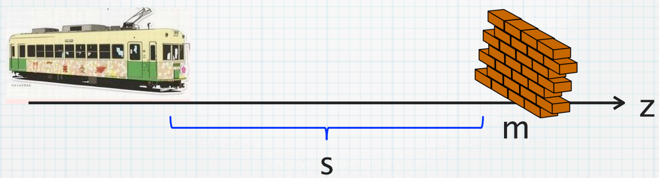
- **Static analysis techniques, transferred as they are**
  - Phase split [Sharma,Dillig,Dillig,Aiken; CAV'11] [Balakrishnan,Sankaranarayanan,Ivancic,Gupta; EMSOFT'09] [Gopan,Reps; SAS'07]
  - Differential invariant [Platzer,Clarke; CAV'08]
  - ... and more!

Part II:

Exercises in Nonstandard Static Analysis

Exercise 1.1

(Tiny) fragment of Euro. Train Ctrl. Sys. (ETCS)



s: big enough
b: big enough
a_0: small enough
...

while $v > 0$ do {
  $t := 0$;
  if $m - z < s$ then $a := -b$ else $a := a_0$;
  while $t < \varepsilon$ do {
    $t := t + \mathrm{dt}$;
    $v := v + a \cdot \mathrm{dt}$;
    $z := z + v \cdot \mathrm{dt}$
  }}

ETCS_0

Q. Find A s.t. $\models \{A\}\,\mathrm{ETCS}_0\,\{z < m\}$


```
while (v > 0) {
  if m - z < s
    then a := -b
    else a := a0;
  t := 0;
  while (t < eps && v > 0) {
    z := z + v * dt;
    v := v + a * dt;
    t := t + dt }}

{z < m}
```

```
while (v > 0 && m - z >= s) {
  a := a0;    t := 0;
  while (t < eps && v > 0) {
    z := z + v * dt;
    v := v + a0 * dt;
    t := t + dt }};
while (v > 0 && m - z < s) {
  a := -b;    t := 0;
  while (t < eps && v > 0) {
    z := z + v * dt;
    v := v - b * dt;
    t := t + dt }}

{z < m}
```

accel.

brake

Strategy1 **"Phase split"**
[Sharma,Dillig,Dillig,Aiken; CAV'11]
[Balakrishnan,Sankaranarayanan,Ivancic,Gupta; EMSOFT'09] [Gopan,Reps; SAS'07]

Phase Split

(Standard Ver., for While & Hoare)

[Sharma,Dillig,Dillig,Aiken; CAV'11]

Defn.
The set of *holed commands* $\mathbf{Cmd}_{[\_]}$ is:

$$\mathbf{Cmd}_{[\_]} \ni h \;::=\; \texttt{if}\ [\_]\ \texttt{then}\ c_1\ \texttt{else}\ c_2 \mid h;c \mid c;h \mid \texttt{if}\ b\ \texttt{then}\ h\ \texttt{else}\ c \mid \texttt{if}\ b\ \texttt{then}\ c\ \texttt{else}\ h$$

For each holed command h, its *pre-hole fragment* $\overline{h}$ is:

$$\overline{\texttt{if}\ [\_]\ \texttt{then}\ c_1\ \texttt{else}\ c_2} :\equiv \texttt{skip}$$
$$\overline{h;c} :\equiv \overline{h} \qquad \overline{c;h} :\equiv c;\overline{h}$$
$$\overline{\texttt{if}\ b\ \texttt{then}\ h\ \texttt{else}\ c} :\equiv \texttt{assert}\ b\,;\overline{h}$$
$$\overline{\texttt{if}\ b\ \texttt{then}\ c\ \texttt{else}\ h} :\equiv \texttt{assert}\ \neg b\,;\overline{h}$$

$\texttt{while}\ b_g\ \texttt{do}\ \boxed{\ldots(\texttt{if}\ldots)\ldots}$

into $\left[\begin{array}{l}\texttt{while}\ b_g \wedge \neg b_s\ \texttt{do}\ \boxed{\ldots}\ ;\\ \texttt{while}\ b_g \wedge b_s\ \texttt{do}\ \boxed{\ldots}\end{array}\right]$

Lem.
If a Boolean expression $b_s \in \mathbf{BExp}$ satisfies

$$\models \{b_s\}\,\overline{h}\,\{b_c\}\ ,\quad \models \{\neg b_s\}\,\overline{h}\,\{\neg b_c\}\ ,\quad \text{and} \quad \models \{b_g \wedge b_s\}\,h[b_c]\,\{\neg b_g \vee b_s\}\ ,$$

then we have

$$[\![\,\texttt{while}\ b_g\ \texttt{do}\ h[b_c]\,]\!] = \left[\!\!\left[\begin{array}{l}\texttt{while}\ (b_g \wedge \neg b_s)\ \texttt{do}\ h[\texttt{false}]\ ;\\ \texttt{while}\ (b_g \wedge b_s)\ \texttt{do}\ h[\texttt{true}]\end{array}\right]\!\!\right] .$$

$h[\,\_\,]$ is a command containing

`if [ _ ] then ... else ...`

Phase Split

(Nonstandard Ver., for While^{dt} & Hoare^{dt})

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The set of *holed commands* $\mathbf{Cmd}_{[\_]}$ is:

$$\mathbf{Cmd}_{[\_]} \ni h \;::=\; \texttt{if}\ [\_]\ \texttt{then}\ c_1\ \texttt{else}\ c_2 \mid h;c \mid c;h \mid \texttt{if}\ b\ \texttt{then}\ h\ \texttt{else}\ c \mid \texttt{if}\ b\ \texttt{then}\ c\ \texttt{else}\ h$$

For each holed command h, its *pre-hole fragment* $\overline{h}$ is:

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$$\overline{h;c} :\equiv \overline{h} \qquad \overline{c;h} :\equiv c;\overline{h}$$
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Proof.

$\models \{b_s\}\,\overline{h}\,\{b_c\}$
$\models \{\neg b_s\}\,\overline{h}\,\{\neg b_c\}$
$\models \{b_g \wedge b_s\}\,h[b_c]\,\{\neg b_g \vee b_s\}$

⇔ sectionwise

$\models \{b_s|_i\}\,\overline{h}|_i\,\{b_c|_i\}$
$\models \{\neg b_s|_i\}\,\overline{h}|_i\,\{\neg b_c|_i\}$
$\models \{b_g|_i \wedge b_s|_i\}\,h|_i[b_c|_i]\,\{\neg b_g|_i \vee b_s|_i\}$

(for almost all i)

⇒ std. ver.

$[\![\,\texttt{while}\ b_g|_i\ \texttt{do}\ h|_i[b_c|_i]\,]\!] = \left[\!\!\left[\begin{array}{l}\texttt{while}\ (b_g|_i \wedge \neg b_s|_i)\ \texttt{do}\ h|_i[\texttt{false}]\ ;\\ \texttt{while}\ (b_g|_i \wedge b_s|_i)\ \texttt{do}\ h|_i[\texttt{true}]\end{array}\right]\!\!\right]$

sectionwise ⇕

$[\![\,\texttt{while}\ b_g\ \texttt{do}\ h[b_c]\,]\!] = \left[\!\!\left[\begin{array}{l}\texttt{while}\ (b_g \wedge \neg b_s)\ \texttt{do}\ h[\texttt{false}]\ ;\\ \texttt{while}\ (b_g \wedge b_s)\ \texttt{do}\ h[\texttt{true}]\end{array}\right]\!\!\right]$

Transferring

Static Analysis Strategies

$\models \{b_s\}\,\overline{h}\,\{b_c\}$
$\models \{\neg b_s\}\,\overline{h}\,\{\neg b_c\}$
$\models \{b_g \wedge b_s\}\,h[b_c]\,\{\neg b_g \vee b_s\}$

⇔ sectionwise

$\models \{b_s|_i\}\,\overline{h}|_i\,\{b_c|_i\}$
$\models \{\neg b_s|_i\}\,\overline{h}|_i\,\{\neg b_c|_i\}$
$\models \{b_g|_i \wedge b_s|_i\}\,h|_i[b_c|_i]\,\{\neg b_g|_i \vee b_s|_i\}$

(for almost all i)

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sectionwise ⇕

$[\![\,\texttt{while}\ b_g\ \texttt{do}\ h[b_c]\,]\!] = \left[\!\!\left[\begin{array}{l}\texttt{while}\ (b_g \wedge \neg b_s)\ \texttt{do}\ h[\texttt{false}]\ ;\\ \texttt{while}\ (b_g \wedge b_s)\ \texttt{do}\ h[\texttt{true}]\end{array}\right]\!\!\right]$

* Doesn't matter what "std. ver." is
* → **modular method** for transfer

```
while (v > 0) {
  if m - z < s
    then a := -b
    else a := a0;
  t := 0;
  while (t < eps && v > 0) {
    z := z + v * dt;
    v := v + a * dt;
    t := t + dt }}

{z < m}
```

⇒

```
while (v > 0 && m - z >= s) {
  a := a0;   t := 0;
  while (t < eps && v > 0) {
    z := z + v * dt;
    v := v + a0 * dt;
    t := t + dt }};
while (v > 0 && m - z < s) {
  a := -b;   t := 0;
  while (t < eps && v > 0) {
    z := z + v * dt;
    v := v - b * dt;
    t := t + dt }}

{z < m}
```

accel.

brake

Strategy 1 **"Phase split"**

[Sharma,Dillig,Dillig,Aiken; CAV'11]
[Balakrishnan,Sankaranarayanan,Ivancic,Gupta; EMSOFT'09]
[Gopan,Reps; SAS'07]

```
while (v > 0 && m - z >= s) {
  a := a0;   t := 0;
  while (t < eps && v > 0) {
    z := z + v * dt;
    v := v + a0 * dt;
    t := t + dt }};
while (v > 0 && m - z < s) {
  a := -b;   t := 0;
  while (t < eps && v > 0) {
    z := z + v * dt;
    v := v - b * dt;
    t := t + dt }}

{z < m}
```

```
if (v > 0)
  then
    while (m - z >= s) {
      a := a0;   t := 0;
      while (t < eps) {
        z := z + v * dt;
        v := v + a0 * dt;
        t := t + dt }}
  else skip;
while (v > 0) {
  a := -b;
```

Strategy 4
"Differential invariant"
[Platzer,Clarke; CAV'08]

Startegies 2,3
"Superfluous guard elim."   "Time elapse"

```
if (v > 0)
  then
    while (m - z >= s) {
      a := a0;   t := 0;
      while (t < eps) {
        z := z + v * dt;
        v := v + a0 * dt;
        t := t + dt }}
  else skip;
while (v > 0) {
  a := -b;
  z := z + v * dt;
  v := v - b * dt }

{z < m}
```

```
if (v > 0)
  then
    while (m - z >= s) {
      a := a0;   t := 0;
      while (t < eps) {
        z := z + v * dt;
        v := v + a0 * dt;
        t := t + dt }}
  else skip;
```
$$\left\{ \begin{array}{l} (v > 0 \lor m > z) \land \\ (b^2\mathrm{dt}^2 + 4b\mathrm{dt}v + 8bz + 4v^2 < 8bm \\ \quad \lor\, b\mathrm{dt}v + 2bz + v^2 \leq 2bm) \end{array} \right\}$$
```
while (v > 0) {
  a := -b;
  z := z + v * dt;
  v := v - b * dt }
```

Strategy 5
"QE Invariant"

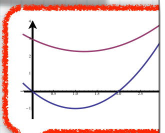
QE Invariant

Lem. In HOARE$^{\mathrm{dt}}$,

$$\vdash \left\{ \begin{array}{l} (\neg b \Rightarrow A) \land \\ \forall y \in {}^*\mathbb{N}.\big(\big(b[a/x]^y \land \neg b[a/x]^{y+1}\big) \Rightarrow A[a/x]^{y+1}\big) \end{array} \right\} \ \texttt{while}\ b\ \texttt{do}\ x := a\ \ \{A\}\ .$$

quantifier must go!
(to manage complexity)

- **Quantifier elimination**
  - Tarski, CAD algorithm, `Resolve` in Mathematica
  - e.g. $\models \forall x \in \mathbb{R}.\, (x^2 + ax + b > 0) \iff a^2 - 4b < 0$
  - then $\models \forall x \in {}^*\mathbb{R}.\, (x^2 + ax + b > 0) \iff a^2 - 4b < 0$



by **transfer**!


```
if (v > 0)
  then
    while (m - z >= s) {
      a := a0;   t := 0;
      while (t < eps) {
        z := z + v * dt;
        v := v + a0 * dt;
        t := t + dt }}
  else skip;
while (v > 0) {
  a := -b;
  z := z + v * dt;
  v := v - b * dt }

{z < m}
```

```
if (v > 0)
  then
    while (m - z >= s) {
      a := a0;   t := 0;
      while (t < eps) {
        z := z + v * dt;
        v := v + a0 * dt;
        t := t + dt }}
  else skip;
```
$$\left\{ \begin{array}{l} (v > 0 \lor m > z) \land \\ (b^2\mathrm{dt}^2 + 4b\mathrm{dt}v + 8bz + 4v^2 < 8bm \\ \quad \lor\, b\mathrm{dt}v + 2bz + v^2 \leq 2bm) \end{array} \right\}$$
```
while (v > 0) {
  a := -b;
  z := z + v * dt;
  v := v - b * dt }
```

Strategy 5
"QE Invariant"

```
if (v > 0)
  then
    while (m - z >= s) {
      a := a0;    t := 0;
      while (t < eps) {
        z := z + v * dt;
        v := v + a0 * dt;
        t := t + dt }}
  else skip;
```

$\{ (v > 0 \lor m > z) \land (b^2\mathrm{dt}^2 + 4b\mathrm{dt}v + 8bz + 4v^2 < 8bm \lor b\mathrm{dt}v + 2bz + v^2 \leq 2bm) \}$

```
while (v > 0) {
  a := -b;
  z := z + v * dt;
  v := v - b * dt }
```

+ some fwd. propagation

```
{ ... (long fml. with dt) }
while (m - z >= s) {
  a := a0;    t := 0;
  while (t < eps) {
    z := z + v * dt;
    v := v + a0 * dt;
    t := t + dt }}
{ ... }
while (v > 0) {
  a := -b;
  z := z + v * dt;
  v := v - b * dt }
```

Strategy 6

"Iteration count"

```
x := 0;
while (x < x0) {
  x := x + a
}
```

iteration: x0/a times?

- approximated by ⌊x0/a⌋ or ⌈x0/a⌉
- ➜ monotonicity reqm. must be discharged

```
{ ... (long fml. with dt) }
while (m - z >= s) {
  a := a0;    t := 0;
  while (t < eps) {
    z := z + v * dt;
    v := v + a0 * dt;
    t := t + dt }}
{ ... }
while (v > 0) {
  a := -b;
  z := z + v * dt;
  v := v - b * dt }
```

long fml. w/o dt, whose core is

$$a_0(2\varepsilon\sqrt{2a_0(m-s-z_0)+v_0^2} + b\epsilon^2 + 2m - 2s - 2z_0) + 2b\varepsilon\sqrt{2a_0(m-s-z_0)+v_0^2} + a_0^2\epsilon^2 + v_0^2 < 2bs$$

the final outcome

Lem. If:

1. a is *closed*
2. $r \mapsto [\![a[r/\mathrm{dt}]]\!]$ is continuous at $r = 0$,

then $\models a[0/\mathrm{dt}] < 0 \implies a < 0$.

Strategy 7

"Cast to shadow"

(Eliminates dt, strengthens the precond.)

Prototype Automatic Prover

P (While[dt] program) →

Hoare[dt] Analyzer

VC generator (Frontend, in OCaml)

↓↑

Symbolic Comp. Engine (Backend, in Mathematica)

→ A (precondition) s.t. $\vdash \{A\}P\{B\}$

B (postcondition)

Totally symbolic (crucial for transfer)

- Fujitsu HX600 with Quad Core AMD Opteron 2.3GHz CPU, 32GB memory. Mathematica 7.0 for Linux x86 (64-bit)
  - ETCS: 40.96 sec.
  - Bouncing ball: runs with one manual insertion of invariants

Related Work

- **Deductive verification** of hybrid sys. [Platzer, '10] [Platzer, LICS'12]
  - Automatic prover KeYmaera
- **Static analysis techniques**
  - A **LOT** in CAV, SAS, VMCAI, ...
  - Applied to hybrid systems (w/ diff. eq.) [Rodriguez-Carbonell, Tiwari; HSCC'05] [Sankaranarayanan; HSCC'10] [Sankaranarayanan, Sipma, Manna; Formal Methods Sys. Design '08]
- Use of **NSA for hybrid systems** [Benveniste, Bourke, Caillaud, Pouzet; J. Comput. Syst. Sci. '12] [Bliudze, Krob; Fundam. Inform. '09] [Gamboa, Kaufmann; J. Autom. Reason. '01]
- **Continuous techniques applied to discrete appl.** [Chaudhuri, Gulwani, Lublinerman, NavidPour; FSE '11]
  - Not contending! Combination?

Conclusions

Nonstandard Static Analysis

While^{dt}
Programming lang.

```
while (t<a) do {
  t:=t+1;
  if ...
}
```

Assn^{dt}
First-order assertion lang.

```
∃z(x=2*z ∧ y=3*z)
```

Hoare^{dt}
Hoare-style program logic

$$\frac{\{A \wedge b\}\, c\, \{A\}}{\{A\}\, \mathtt{while}\ b\ \mathtt{do}\ c\, \{A \wedge \neg b\}}$$

Rigorous semantics by non-standard analysis

- Tool's effectivity. More heuristics?
- (Any discrete frmwk.)^{dt} ?
- Simulink as stream processing?
- With (explicit) differential equations?

Thank you for your attention!

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