Completeness results for the ZX-calculus

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Outline

Motivation

Approximate completeness for single qubits

Completeness for stabilizer diagrams

Conclusions & Outlook

Motivation

- ZX-calculus is incomplete, even for single qubits; not obvious how to complete it
- instead: look for fragments of the general calculus which are complete
- approximate universality: small sets of operators suffice to approximate arbitrary unitaries to any accuracy
- *approximate completeness*: completeness for such a set

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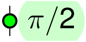
The single-qubit Clifford+T group

An approximately universal group, generated by:

- single-qubit Clifford group $\mathcal{C}_1 = \langle S, H \rangle$, where

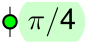
$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$



$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

- T gate

$$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$$



The ZX-calculus for the single-qubit Clifford+T group

Generated by $\pi/2$, H, and $\pi/4$—or $\pi/4$ and $\pi/2$

- *single-qubit*: restrict diagrams to line graphs
- *Clifford+T group*: restrict phases to multiples of $\pi/4$

(Ignore global phases.)

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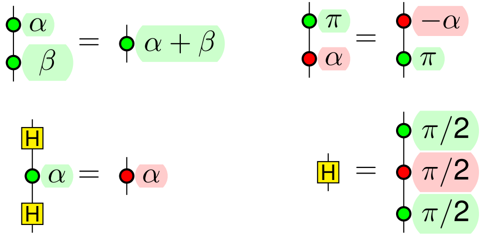
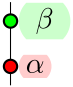
Rules:



and $2n\pi$ $=$ | for integer n

Single-qubit Clifford diagrams

Any single-qubit Clifford diagram can be written uniquely as one of the following, with $\alpha, \beta, \gamma \in \{0, \pi/2, \pi, -\pi/2\}$:



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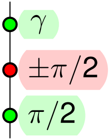

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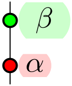

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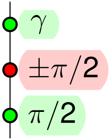
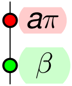


or, equivalently, one of



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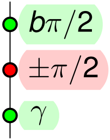
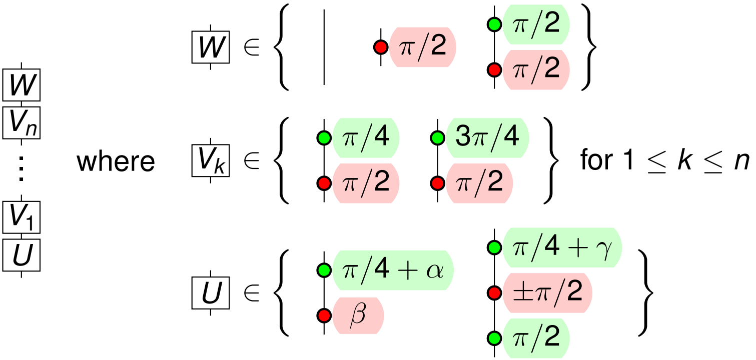


where $a, b \in \{0, 1\}$ and β, γ as above.

The normal form for single-qubit Clifford+T diagrams

Following [Matsumoto & Amano 2008], any single-qubit Clifford+T diagram is either pure Clifford or it can be written as



with n a non-negative integer and $\alpha, \beta, \gamma \in \{0, \pi/2, \pi, -\pi/2\}$.

Pushing phases down, part 1

Diagrams are rewritten into normal form by pushing phases towards the bottom of the diagram as much as possible.

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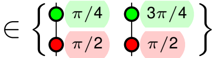
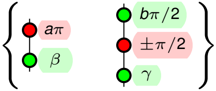
- can push π and π past $\boxed{V}$ 

Write Clifford operators as 

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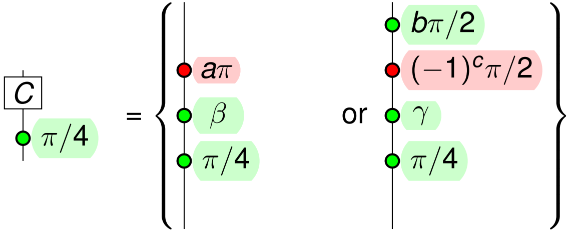

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$$\begin{matrix} \boxed{C} \\ \pi/4 \end{matrix} = \left\{ \begin{matrix} a\pi \\ \beta + \pi/4 \end{matrix} \quad \text{or} \quad \begin{matrix} b\pi/2 \\ (-1)^c\pi/2 \\ \gamma \\ \pi/4 \end{matrix} \right\}$$

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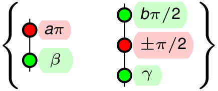
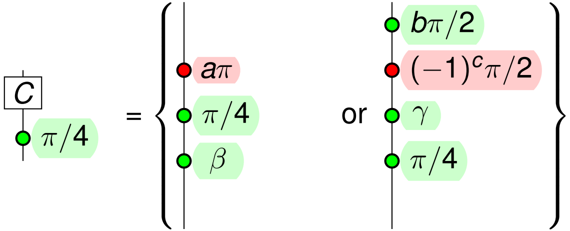
Write Clifford operators as , then



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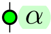
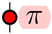
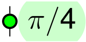
- can push 

 and 

 past 

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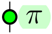
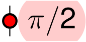
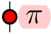
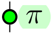
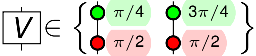
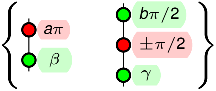
 and 

 past 

- can push 

 and 

 past 

Write Clifford operators as 

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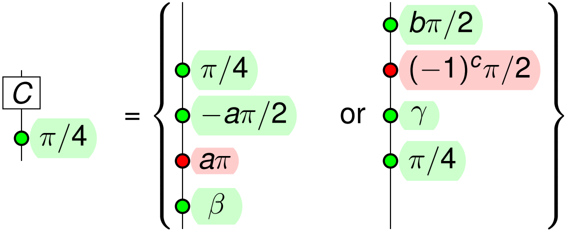

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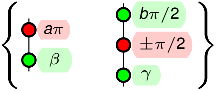
Write Clifford operators as 

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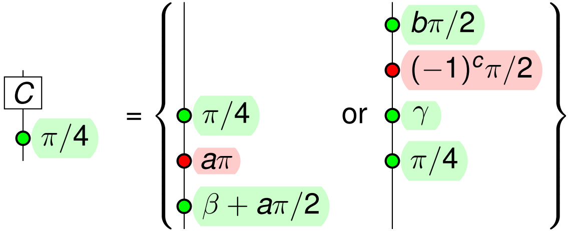

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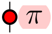
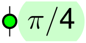
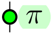
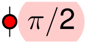
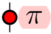
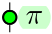
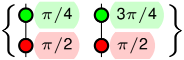
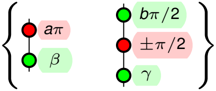
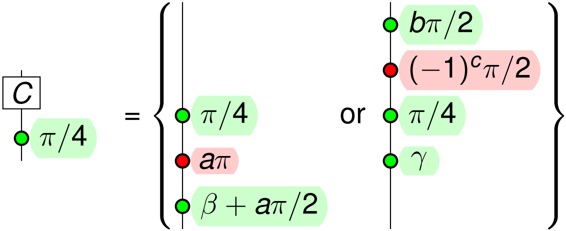
- can push  and  past 
- can push  and  past 
- can push  and  past  $\in$ 

Write Clifford operators as , then



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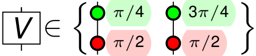
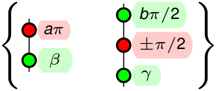
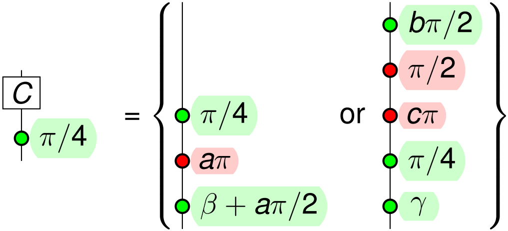
- can push π and π past $V \in$ 

Write Clifford operators as , then



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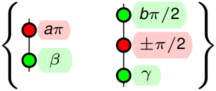
Write Clifford operators as 

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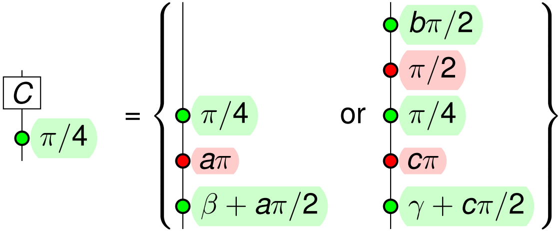

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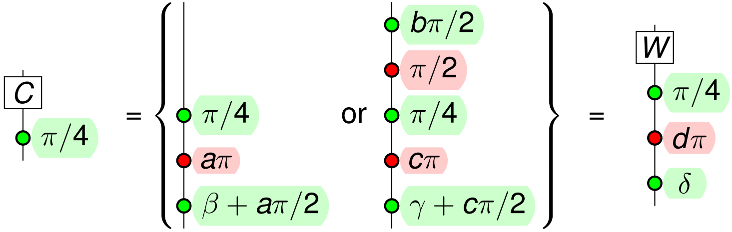
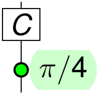
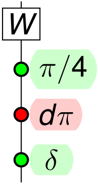


Pushing phases down, part 2

Have 

= 

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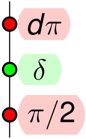
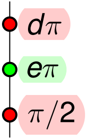
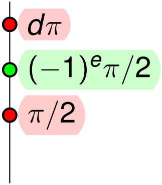


= 

or 

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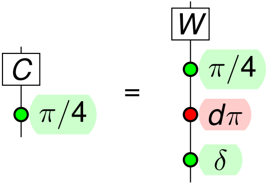
Have 

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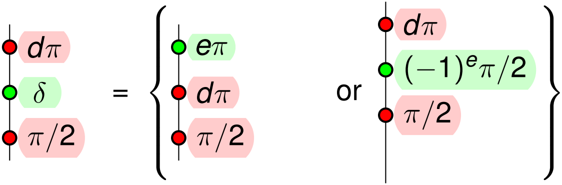

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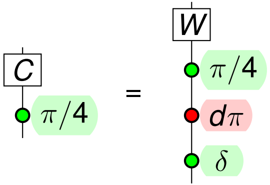
Have 

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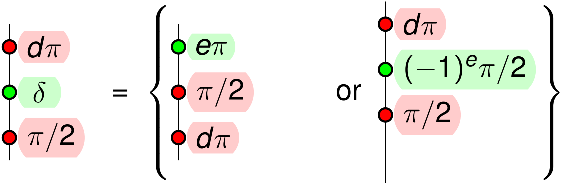

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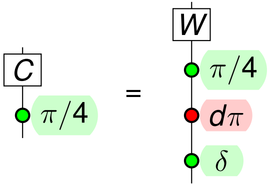

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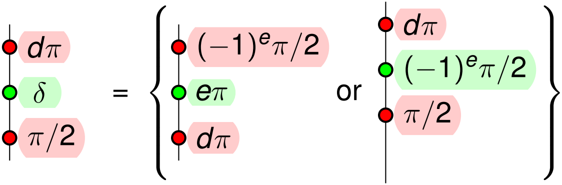

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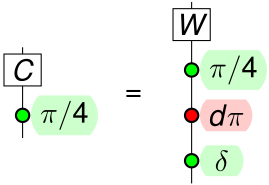

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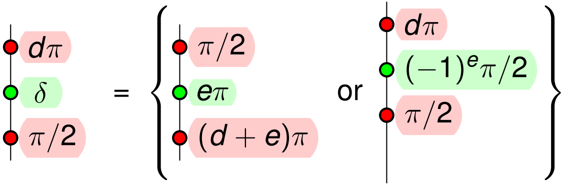

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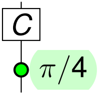

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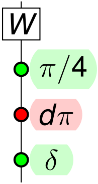

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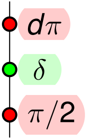

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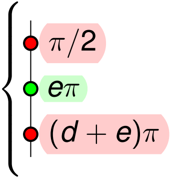

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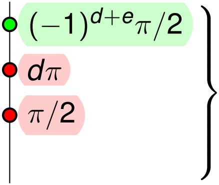
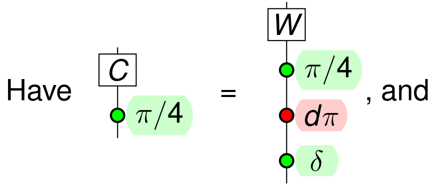
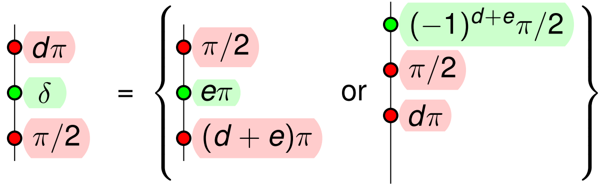


Pushing phases down, part 2





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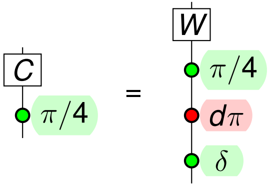

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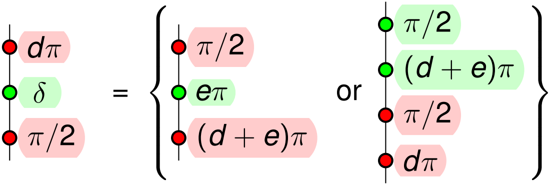

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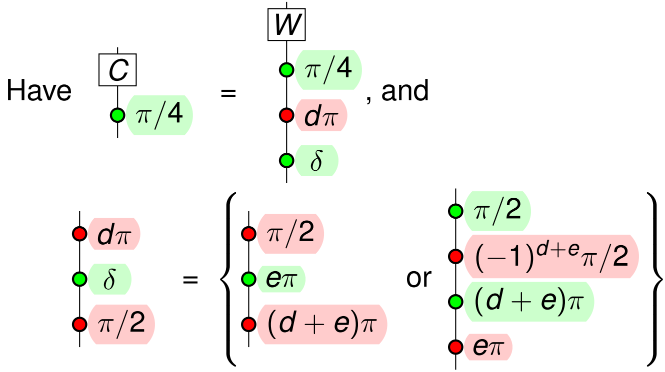

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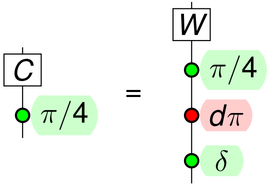

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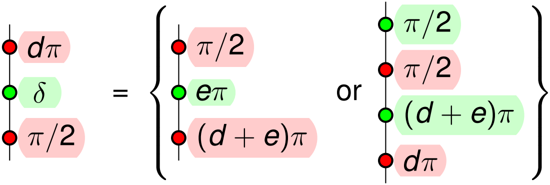

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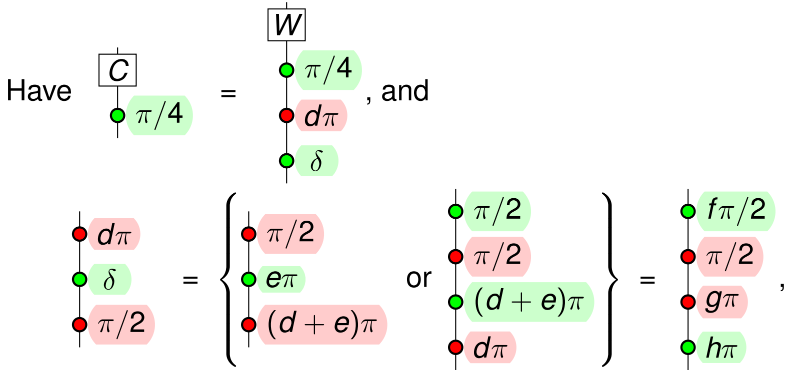

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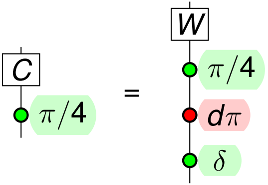

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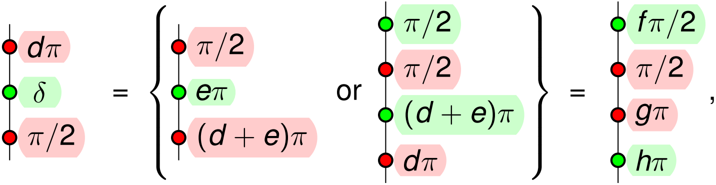

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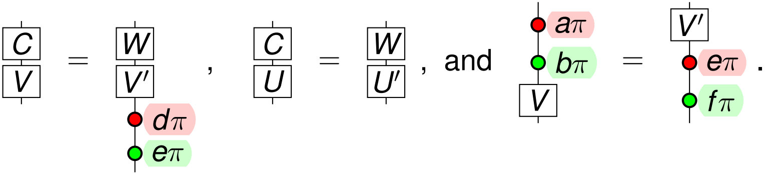
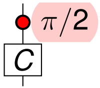


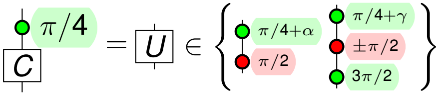
Rewriting diagrams into normal form, starting with a Clifford diagram

The single-qubit Clifford+T group is generated by $\pi/4$ and $\pi/2$ so it suffices to check what happens when a pure Clifford operator or a normal form diagram is composed with one of these.

Now for any Clifford unitary $\boxed{C}$,



is pure Clifford, and



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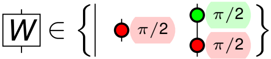
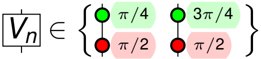
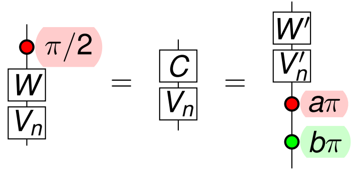
Rewriting diagrams into normal form, starting with a normal form diagram

For any  and ,



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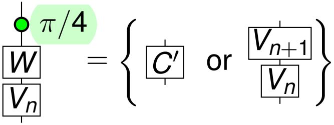

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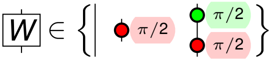
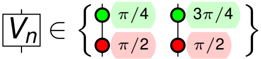
For any 

and 

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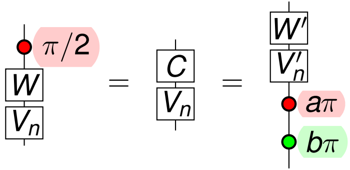

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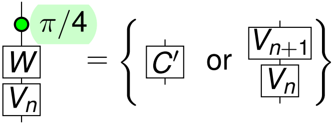

Case $n = 0$ is similar.

No normal form diagram represents the identity operator

Can write any single-qubit density operator as $xX + yY + zZ$, where $x, y, z \in \mathbb{R}$ and X, Y, Z are the Pauli matrices.
Clifford unitaries act on the vectors (x, y, z) by permuting the elements and adding minus signs; T sends

$$(x, y, z) \mapsto \frac{1}{\sqrt{2}} \left(x - y, x + y, z\sqrt{2} \right) .$$

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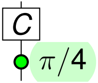
States resulting from application of a normal form operator to $|0\rangle$ will have vectors of the form

$$\frac{1}{\sqrt{2^m}}\left(x_1 + x_2\sqrt{2}, y_1 + y_2\sqrt{2}, z_1 + z_2\sqrt{2}\right)$$

where $x_1, x_2, y_1, y_2, z_1, z_2 \in \mathbb{Z}$. By parity arguments, can show none of them represent $|0\rangle$, thus no normal form operator is equal to the identity.

The inverse of a normal form diagram has the same number of $\pi/4$ phase shifts as the original

We have 

$=$ 

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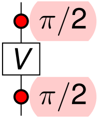
$=$ 

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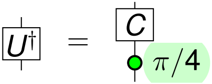
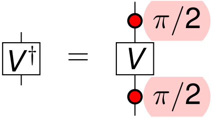
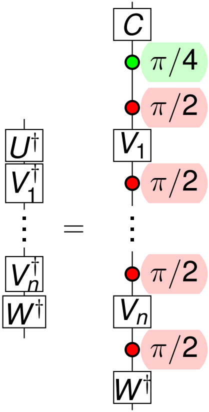
We have  and  , so



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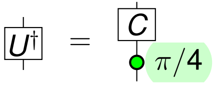
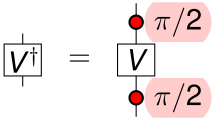
We have 

and 

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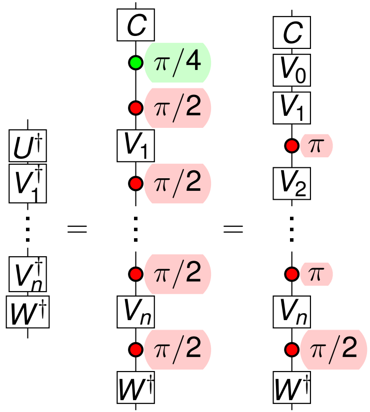

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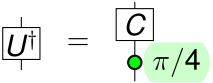
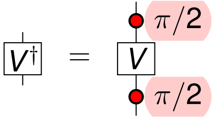
We have 

and 

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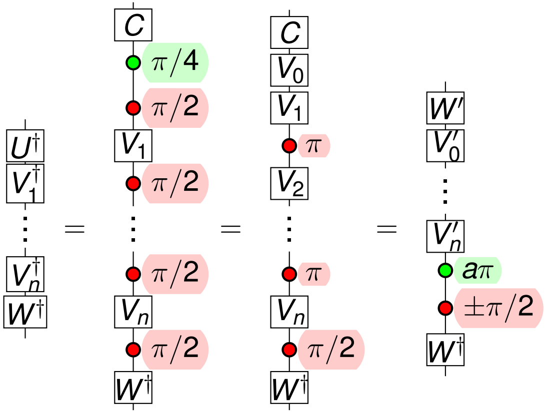

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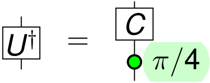

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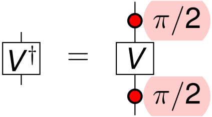

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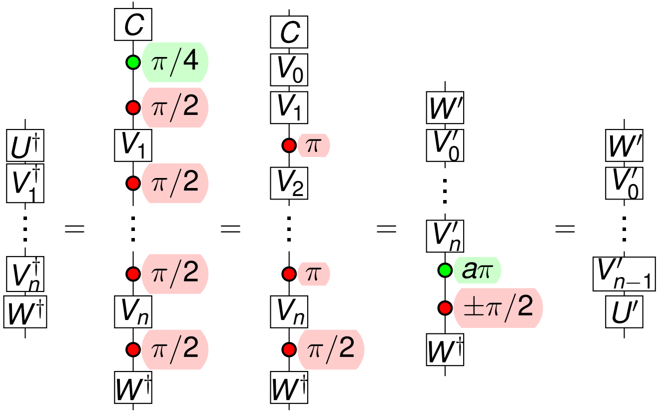

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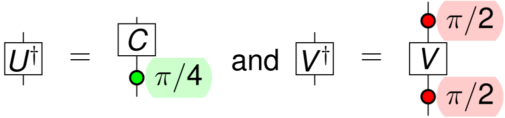

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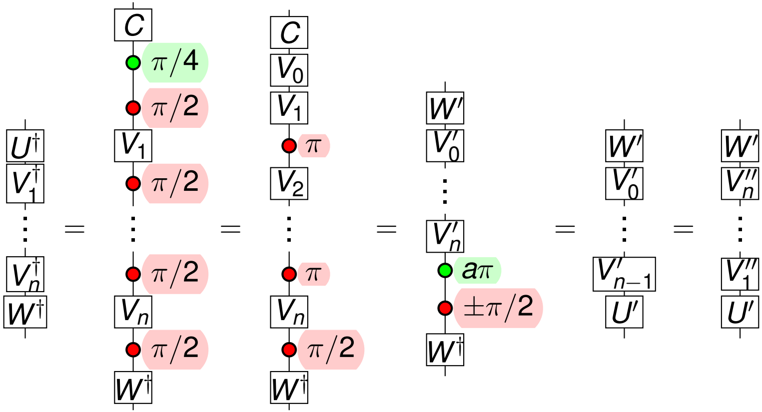

Normal forms are unique

Given two normal form diagrams that are not identical, composing one diagram with the inverse of the other will yield a non-trivial diagram.

- Assume diagrams are such that the topmost nodes differ (in colour or phase, or both).
- Inverting a diagram does not change the number of $\pi/4$ phases.
- Distinguish cases to see that the resulting diagram never collapses to the trivial one.

Outline

Stabilizer quantum mechanics

Stabilizer operations:

- preparation of qubits in state $|0\rangle$
- Clifford unitaries, generated by

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \quad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \Lambda X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

- measurements in computational basis

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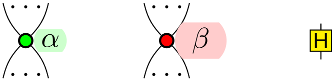
ZX-calculus:



where α, β are multiples of $\pi/2$

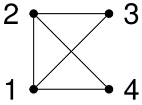
Graph states in the ZX-calculus

Definition

Let G be a finite simple undirected graph. The ZX-calculus diagram for the corresponding graph state consists of:

- for each node in G, a green node with one output, and
- for each edge in G, an edge with a Hadamard node on it.

E.g.



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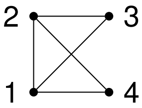
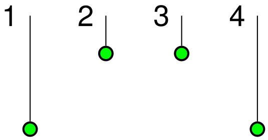


$\mapsto$



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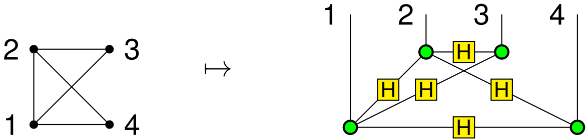

Local complementations

The local Clifford group consists of all tensor products of the single-qubit Clifford operators $\langle S, H \rangle$.

Theorem (Van den Nest et. al, 2004)

Any stabilizer state is local Clifford-equivalent to some graph state.

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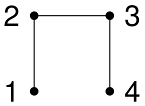
Two graph states are local Clifford-equivalent if and only if they are related by a sequence of local complementations.

A local complementation about a vertex v inverts the subgraph generated by the neighbourhood of v: e.g.



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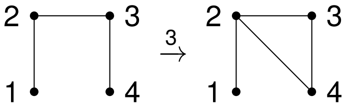

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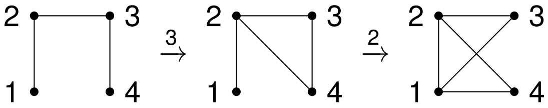
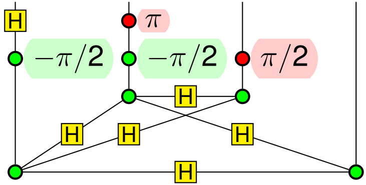


Stabilizer completeness proof (overview)

[Duncan & Perdrix, 2009] show that local complementations can be derived from the rules of the ZX-calculus.
Use this to show that the results from [Van den Nest et al., 2004] hold in the ZX-calculus:

- Any stabilizer ZX-calculus diagram can be rewritten into a graph state with local Clifford operators, e.g.



- There exists a terminating algorithm that, given two stabilizer diagrams, will rewrite them to be identical if possible.

See arXiv:1307.7025 for details.

Conclusions & Outlook

- ZX-calculus is not complete in general but fragments of it are complete, e.g.
  - line graphs where all phases are multiples of $\pi/4$ (single-qubit Clifford+T group)
  - diagrams where all phases are multiples of $\pi/2$ (stabilizer quantum mechanics)
- can these results be combined to multi-qubit Clifford+T operators?
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Thank you!