Contextuality and Noncommutative Geometry

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Overview

Gel’fand duality and quantum theory

- Gel’fand duality establishes an equivalence between (geometry) compact, Hausdorff topological spaces and (algebra) *commutative*, unital C^*-algebras
- Physically, it is the duality between pure classical state spaces and algebras of observables
- Goal: Find the geometric dual for noncommutative C^*-algebras, i.e. those used in quantum theory

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NC geometry and the NC dictionary

- The 'geometry' of noncommutative C^*-algebras have been indirectly studied for decades by mathematicians via algebra

Geometry	**Algebra**
continuous real function	self-adjoint operator
closed set	closed ideal
compact	unital
metric space	separable
Borel measure	positive functional
cartesian product	tensor product
vector bundle	finite, projective module
Riemannian spin manifold	spectral triple

Conceptual commutative diagram

$$\begin{tikzcd}
 & \begin{array}{c}\textbf{Geometry}\\ \text{(States)}\end{array} & \begin{array}{c}\textbf{Algebra}\\ \text{(Observables)}\end{array} \\
\begin{array}{c}\textbf{Commutative}\\ \text{(Classical)}\end{array} & \begin{array}{c}\text{Topological}\\ \text{spaces}\end{array} \arrow[r, "\Sigma"] \arrow[d, "F \circ -"'] & \begin{array}{c}\text{Commutative}\\ C^*\text{-algebras}\end{array} \arrow[d, "\iota"] \\
\begin{array}{c}\textbf{Noncommutative}\\ \text{(Quantum)}\end{array} & \mathbf{G} \arrow[r, "\lim"] & \begin{array}{c}\text{Noncommutative}\\ C^*\text{-algebras}\end{array}
\end{tikzcd}$$

Spatial diagrams

- Replace "topological space" with "diagram of topological spaces" as a generalized notion of spectrum (I-B)
- Functorially associate to a unital C^*-algebra $\mathcal{A}$ a contravariant functor whose codomain is compact, Hausdorff spaces
- Consider the subcategory $S(\mathcal{A})$:

  Objects contexts of $\mathcal{A}$ (commutative, unital sub-C^*-algebras $V \subset \mathcal{A}$)

  Arrows inner automorphisms of $\mathcal{A}$ restricted to a context ($\phi_u|_V : V \to W$ where $\phi|_u$ is conjugation by a unitary $u \in \mathcal{A}$ and $\phi(V) \subset W$)
- The diagram $G(\mathcal{A})$ is the spectrum functor composed with the inclusion functor of $S(\mathcal{A})$: contextual state spaces

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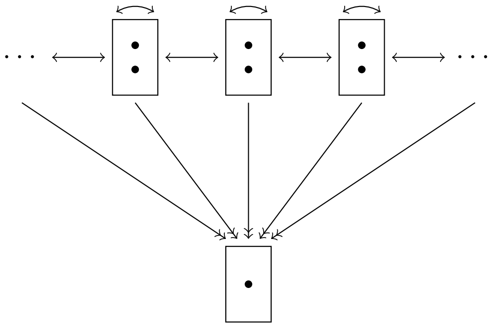
Spatial diagrams: $M_2(\mathbb{C})$



Extending a topological functor

- Given a functor $F : KHaus \to \mathcal{C}$ with (co)complete $\mathcal{C}$ we get an extension $\tilde{F} : uC^* \to \mathcal{C}$
  1. Apply F to the diagram $G(\mathcal{A})$
  2. Take the (co)limit: $\tilde{F}(\mathcal{A}) = \lim F \circ G(\mathcal{A})$
- Intuitively: like decomposing a noncommutative space into its quotient spaces, applying the functor F to the ones which are genuine topological spaces, and pasting together the results

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K-theory, topological to noncommutative

- We tried this with K-theory, a significant topological cohomology theory based on vector bundles which has a well-studied noncommutative geometric generalization
- Operator K-theory is defined in terms of finite, projective modules over $\mathcal{A}$ and is a classifying invariant of C^*-algebras
- It is open whether $\tilde{K} \simeq K_0$ on the nose

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Conjecture: open sets to ideals

Suppose $\tau : KHaus \longrightarrow Lat$ is the functor assigning to a topological space its topological lattice, i.e. closed sets under containment, and to a continuous function the lattice homomorphism of direct image

Conjecture (QPL 2013)

$\tilde{\tau} : uC^* \longrightarrow Lat$ is the functor assigning to a C^*-algebra its lattice of closed, two-sided ideals

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Motivation

- The lattice of closed, 2-sided ideals of $\mathcal{A}$ is the same as the hull-kernel/Jacobson/Zariski topological lattice of $Prim(\mathcal{A})$ the primary ideal space of $\mathcal{A}$
- The points of $Prim(\mathcal{A})$ are the kernels of irreducible $*$-representations of $\mathcal{A}$
- The primary ideal space and Gel'fand spectrum coincide in the commutative case

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- The primary ideal space is a C^*-algebraic version of the spectrum functor $Spec(R)$
  - For a commutative ring R, the spectrum of R is the prime ideals of R together with the Zariski topology
  - In algebraic geometry, one gives $Spec(R)$ a structure sheaf and studies R by studying this locally ringed space
- Speculation: Can G be considered an enriched C^*-algebraic version of $Spec$? The basis for introducing sheaf-theoretic techniques into NCG?

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Motivation

Theorem (Dauns–Hoffman '68)

Let $\mathcal{A}$ be a unital C^*-algebra. Then $\mathcal{A}$ is a $C(Prim(\mathcal{A}))$-module in the following sense: for each $a \in \mathcal{A}$ and $f \in C(Prim(A)$, there is an element $fa \in C(Prim(\mathcal{A}))$ such that $fa \equiv f(P)a \bmod P$ for all $P \in Prim(\mathcal{A})$

Partial ideals

Definition

A partial ideal of $\mathcal{A}$ is a choice of ideal I_V from each context $V \subset \mathcal{A}$ such that whenever $V \subset V'$, the ideal I_V can be recovered from $I_{V'}$ as $I_V = I_{V'} \cap V$

- Every (total) ideal $I \subset \mathcal{A}$ gives rise to a partial ideal: $I_V = I \cap V$
- The elements of the lattice $\tilde{\tau}(\mathcal{A})$ are simply partial ideals which are fixed by unitary rotation

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Conjecture: total ideals from partial ideals

Conjecture (QPL 2013)

A partial ideal of a C^*-algebra $\mathcal{A}$ arises from a total ideal if and only if it is fixed by any unitary rotation

Proof strategy: Consider first the enveloping von Neumann algebra $\mathcal{A}^{**}$ of $\mathcal{A}$ where ideals are generated by projections. There is a close link between contexts/ideals of $\mathcal{A}^{**}$ and contexts/ideals of $\mathcal{A}$.

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Theorem (–, Soares Barbosa)

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Summary

- Introduced a context-indexed diagram of spaces $G(\mathcal{A})$ associated to a C*-algebra $\mathcal{A}$ as a proposed geometric dual object
- Showed that this leads to automatic generalizations of topological concepts to algebraic ones which seems to agree with the canonical noncommutative geometric generalizations
- This justifies regarding $G(\mathcal{A})$ as a generalization of Gel'fand spectrum
- Accounting for contextuality appears to be effective in providing a structural explanation for the unreasonable effectiveness of geometric tools in the study of noncommutative algebras