# Completeness of Hardy Non-locality:

## Consequences & Applications

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QPL 2014

# Overview

## Theorem*

For all $(2,k,2)$ and $(2,2,l)$ scenarios,

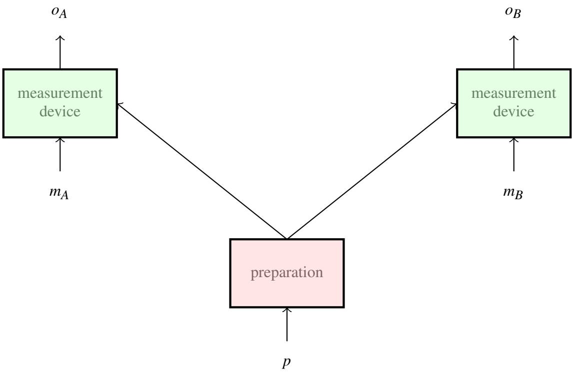
$$\boxed{\text{Hardy non-locality} \Longleftrightarrow \text{Logical non-locality}}$$

## Consequences & Applications

1. Hardy subsumes other paradoxes
2. Complexity results for logical non-locality
3. Bell states are anomalous
4. Hardy non-locality can be realised with certainty

*S Mansfield, T Fritz - Foundations of Physics, 2012

# Non-locality




Bell-CHSH Inequality:

$$\left| E(m_A, m_B) + E(m_A, m'_B) + E(m'_A, m_B) - E(m'_A, m'_B) \right| \leq 2$$

# Logical Non-locality

A more intuitive approach to non-locality

- Probabilities $\longrightarrow$ Truth values (possibilities)
- Inequalities $\longrightarrow$ Logical deductions

$$\text{Logical } NL > NL$$

Examples:

- Hardy, GHZ, KS, etc.
- Hardy's argument is considered to be the simplest

# Hardy's Non-locality Paradox

Bob

| Alice | | $\uparrow$ | $\downarrow$ | $G$ | $W$ |
|---|---|---|---|---|---|
| | $\uparrow$ | 1 | | | 0 |
| | $\downarrow$ | | | | |
| | $G$ | | | 0 | |
| | $W$ | 0 | | | |

- Outcome $(\uparrow,\uparrow)$ is possible
- If $A$ measures *spin* and $B$ measures *colour*, or vice versa, the outcomes $(\uparrow,W)$ or $(W,\uparrow)$ are never obtained
- When *spin* $\uparrow$ is recorded, the other subsystem must have *colour* $G$
- Since $(\uparrow,\uparrow)$ is possible, then $(G,G)$ must be possible
- Contradiction!

# Generalisations of Hardy Non-locality

Measurements have up to $l$ outcomes

| | $o'_1$ | $\cdots$ | $o'_l$ | $o_1 \cdots o_{m_2}$ | $o_{m_2+1} \cdots o_l$ |
|---|---|---|---|---|---|
| $o'_1$ | $1$ | | | | $0 \quad \cdots \quad 0$ |
| $\vdots$ | | | | | |
| $o'_l$ | | | | | |
| $o_1$ | | | | $0 \quad \cdots \quad 0$ | |
| $\vdots$ | | | | $\vdots \quad \ddots \quad \vdots$ | |
| $o_{m_1}$ | | | | $0 \quad \cdots \quad 0$ | |
| $o_{m_1+1}$ | $0$ | | | | |
| $\vdots$ | $\vdots$ | | | | |
| $o_l$ | $0$ | | | | |

# Generalisations of Hardy Non-locality

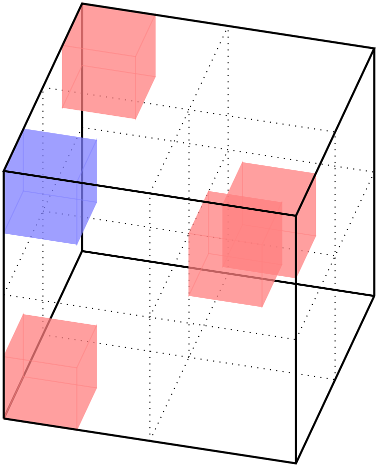
$k$ measurement settings per party

| | | | | |
|---|---|---|---|---|
| | 1 | 0 | | |
| | 0 | 0 | $\ddots$ | |
| | | $\ddots$ | | |
| | | | | |

# Generalisations of Hardy Non-locality

$n > 2$ parties

Figure : The $n = 3$ Hardy paradox. Blue $\leftrightarrow$ truth value ‘1’, red $\leftrightarrow$ ‘0’



# Completeness of Hardy Non-locality

Hardy non-locality can be defined for all $(n, k, l)$ scenarios.

- $n$ parties
- $k$ measurement settings per party
- $l$ outcomes to each measurement

Theorem*

For all $(2, k, 2)$ and $(2, 2, l)$ scenarios,

$$\boxed{\text{Hardy non-locality} \iff \text{Logical non-locality}}$$

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# Hardy Subsumes Other Paradoxes

The *Chen et al. paradox** occurs if at least one starred entry is non-zero. Relevant entries are either above or below the diagonal.

$$
\begin{array}{c|ccc|ccc|}
 & & & & & & \\ \hline
 & * & \cdots & * & 0 & \cdots & 0 \\
 & & \ddots & \vdots & & \ddots & \vdots \\
 & & & * & & & 0 \\ \hline
 & 0 & \cdots & 0 & & & \\
 & & \ddots & \vdots & 0 & & \\
 & & & 0 & \vdots & \ddots & \\
 & & & & 0 & \cdots & 0 \\ \hline
\end{array}
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\begin{array}{c|c|c|}
 & & \\
\hline
 & \begin{matrix} 1 & \cdots & * \\ & \ddots & \vdots \\ & & * \end{matrix} & \begin{matrix} 0 & \cdots & 0 \\ & \ddots & \vdots \\ & & 0 \end{matrix} \\
\hline
 & \begin{matrix} 0 & \cdots & 0 \\ & \ddots & \vdots \\ & & 0 \end{matrix} & \begin{matrix} 0 & & \\ \vdots & \ddots & \\ 0 & \cdots & 0 \end{matrix} \\
\hline
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# Complexity of Logical Non-locality

$$\boxed{\text{Hardy non-locality} \Longleftrightarrow \text{Logical non-locality}}$$

So, in relevant scenarios, one has only to search for Hardy paradoxes

### Proposition

Polynomial algorithms can be given for deciding logical non-locality in $(2,2,l)$ and $(2,k,2)$ scenarios.

# Bell States are Anomalous

Are all entangled states logically non-local?

**Logically Non-local**

- Hardy: all *non-maximally* entangled 2-qubit states
- Abramsky, Constantin & Ying: all entangled $n$-qubit states
- GHZ, Cabello: Many maximally entangled $n > 2$ qubit states

**Exception!**

- Bell States (maximally entangled 2-qubit states)

$$\frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right), \text{ etc.}$$

# Bell States Are Anomalous: Proof (Sketch)

- Need only consider

$$\frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right), \text{ etc.}$$

- Projective measurements necessarily lead to $(2,k,2)$ scenarios

## Claim

For any observables $\{A_1, A_2, B_3, B_4\}$ there is no Hardy paradox

# Bell States Are Anomalous: Proof (Sketch)

## Claim

For any observables $\{A_1, A_2, B_3, B_4\}$ there is no Hardy paradox

State:

$$\frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right), \text{ etc.}$$

Observables:

$$\{A_1, A_2, B_3, B_4\}$$

Eigenvectors:

$$|0_i\rangle = \cos\frac{\theta_i}{2}\,|0\rangle + e^{i\phi_i}\sin\frac{\theta_i}{2}$$

$$|1_i\rangle = \sin\frac{\theta_i}{2}\,|0\rangle + e^{-i\phi_i}\cos\frac{\theta_i}{2}$$

Outcome probabilities:

$$\langle 0_j 0_k|\psi\rangle = \frac{1}{\sqrt{2}}\left(\cos\frac{\theta_j}{2}\cos\frac{\theta_k}{2} + e^{-i\left(\phi_j+\phi_k\right)}\sin\frac{\theta_j}{2}\sin\frac{\theta_k}{2}\right)$$

$$\langle 0_j 1_k|\psi\rangle = \frac{1}{\sqrt{2}}\left(\cos\frac{\theta_j}{2}\sin\frac{\theta_k}{2} + e^{-i\left(\phi_j-\phi_k\right)}\sin\frac{\theta_j}{2}\cos\frac{\theta_k}{2}\right)$$

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Outcome probabilities:

$$p(01 \mid AB) = p(10 \mid AB)$$
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Symmetries + No-signalling + Hardy Paradox:

| | | | | |
|---|---|---|---|---|
| | ½ | 0 | ½ | 0 |
| | 0 | ½ | 0 | ½ |
| | ½ | 0 | $\frac{1-q}{2}$ | $\frac{q}{2}$ |
| | 0 | ½ | $\frac{q}{2}$ | $\frac{1-q}{2}$ |

$0 < q \leq 1$

$q = 0$: Local
$q = 1$: PR box

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Observables: $A_1 = A_2 = B_3 = B_4 = \pm X$

$$\Rightarrow q = 0$$

Contradiction!

# The Paradoxical Probability

| Alice \ Bob | $\uparrow$ | $\downarrow$ | $G$ | $W$ |
|---|---|---|---|---|
| $\uparrow$ | 0.09 | | | 0 |
| $\downarrow$ | | | | |
| $G$ | | | 0 | |
| $W$ | 0 | | | |

- An *almost* probability free non-locality proof
- Experimental motivations for maximising this probability
- Considered a measure of the quality of Hardy non-locality

| **Model** | **Probability** |
|---|---|
| Hardy | $\frac{5\sqrt{5}-11}{2} \approx 0.09$ |
| Hardy Ladder ($k \to \infty$) | 0.5 |
| Ghosh et al. (tripartite) | 0.125 |
| Choudhary (non-quantum, $NS$) | 0.5 |
| Chen et al. ($l \to \infty$) | $\approx 0.4$ |

# Probability Free Hardy Non-locality?

- Recall: Chen et al. sum paradoxical probabilities

| | | |
|---|---|---|
| | | |
| | $\begin{matrix} * & \cdots & * \\ & \ddots & \vdots \\ & & * \end{matrix}$ | $\begin{matrix} 0 & \cdots & 0 \\ & \ddots & \vdots \\ & & 0 \end{matrix}$ |
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# Probability Free Hardy Non-locality?

- Recall: Chen et al. sum paradoxical probabilities
- If we allow this, we can achieve Hardy non-locality *with certainty*!

Example: the PR box

| | | | | |
|---|---|---|---|---|
| | 1 | 0 | 1 | 0 |
| | 0 | 1 | 0 | 1 |
| | 1 | 0 | 0 | 1 |
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# Hardy Non-locality With Certainty

The GHZ model: Local $X$ & $Y$ measurements on

$$|GHZ\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right)$$

| | 000 | 001 | 010 | 011 | 100 | 101 | 110 | 111 |
|---|---|---|---|---|---|---|---|---|
| $X$ $X$ $X$ | 1 | 0 | 0 | 1 | 0 | 1 | 1 | 0 |
| $X$ $Y$ $Y$ | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 1 |
| $Y$ $X$ $X$ | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 1 |
| $Y$ $Y$ $X$ | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 1 |

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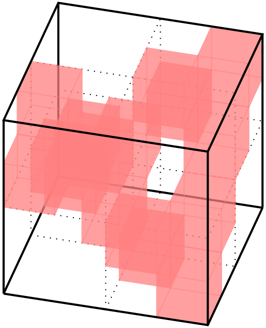

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| | |
| PR box (non-quantum, $NS$) | 1 |
| GHZ | 1 |

# Conclusion

## Theorem*

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