Mixed quantum states in higher categories

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(with Chris Heunen and Jamie Vicary)

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Table of contents

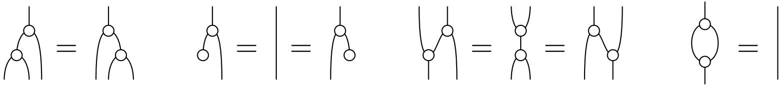
Existing models for classical and quantum data
- Special dagger Frobenius algebras
- 2-categorical quantum mechanics

The construction $2(-)$
- The theory of bimodules
- The $2(-)$ construction
- $2(\mathrm{CP}^*(-))$

Applications
- A unified description of teleportation and classical encryption
- A unified security proof

$\mathrm{CP}^*(-)$

1. Special dagger Frobenius algebras in a monoidal category **C**:



CP*(−)

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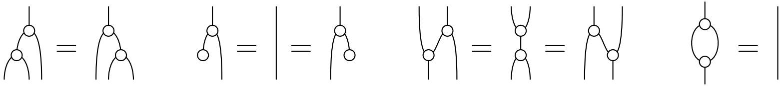
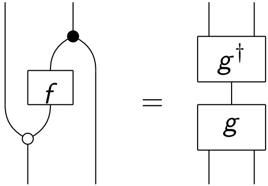


2. Completely positive maps between Frobenius algebras:
   morphisms f in **C**, for which $\exists g$ such that



2-categories and their graphical language

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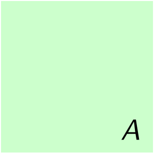
0-cells Regions Classical information



2-categories and their graphical language

0-cells	Regions	Classical information
1-cells	Lines	Quantum systems

$$B \xrightarrow[s]{} A$$

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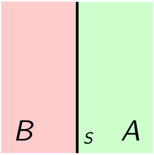

2-categories and their graphical language

0-cells	Regions	Classical information
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2-cells	Vertices	Quantum dynamics

$$\begin{tikzcd} B \arrow[r, "s"'] \arrow[r, bend left=60, "s'"] \arrow[r, phantom, bend left=30, "\alpha\Uparrow"] & A \end{tikzcd}$$

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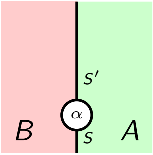

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Horizontal composition

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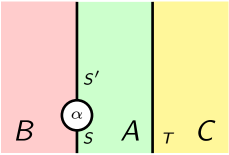

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Vertical composition

$$\begin{tikzcd}
B \arrow[r, "s"'] \arrow[r, bend left=60, "s'"{name=M}] \arrow[r, bend left=90, "s''"{name=U}] \arrow[r, phantom, bend left=30, "\Uparrow \alpha"] \arrow[r, phantom, bend left=75, "\Uparrow \gamma"] & A \arrow[r, "T"'] & C
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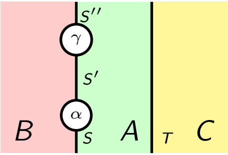

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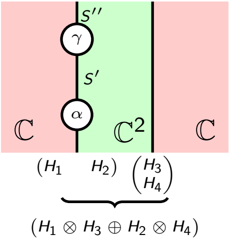


The standard example is **2Hilb**:

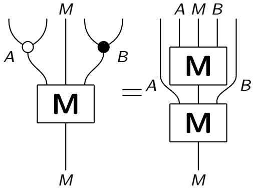
- *0-cells* given by natural numbers
- *1-cells* given by matrices of finite-dimensional Hilbert spaces
- *2-cells* given by matrices of linear maps

Quantum systems interacting with their environment

Quantum systems interacting with their environment

Let $(A, \text{mult}, \text{unit})$ and $(B, \text{mult}, \text{unit})$ be classical structures in $\mathbf{C}$.

A *dagger C-D-bimodule* is a morphism $\mathbf{M}$ satisfying:



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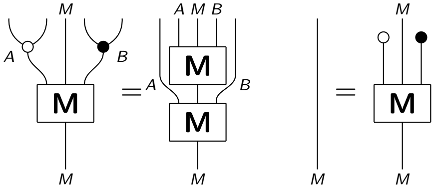

Quantum systems interacting with their environment

Let $(A, \text{(white comultiplication)}, \text{(white counit)})$ and $(B, \text{(black comultiplication)}, \text{(black counit)})$ be classical structures in $\mathbf{C}$.

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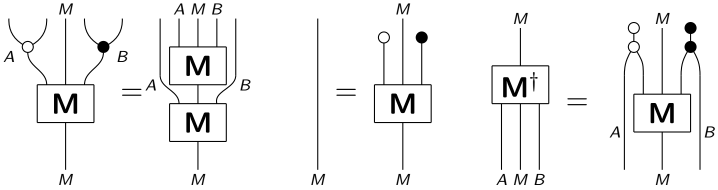

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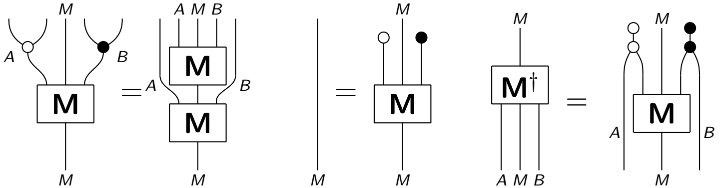
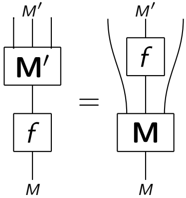


A bimodule homomorphism is a morphism $f \in \mathbf{C}$, such that



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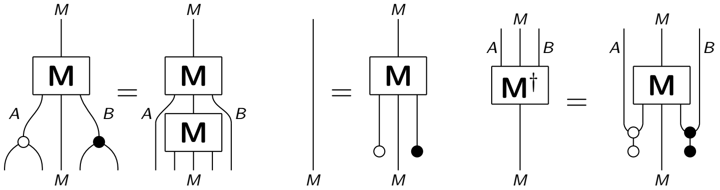

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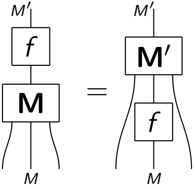

The $2(-)$ construction

How can we construct the 2-category $2(\mathbf{C})$ from $\mathbf{C}$?

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For proofs see LW (2013), Masters's thesis, 'Categorical Models for Quantum Computing'.

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Horizontal composition is defined by the following coequaliser in **C**:

$$
\begin{tikzcd}
M \otimes B \otimes N \arrow[r, shift left, "\mathbf{M}_B \otimes id_N"] \arrow[r, shift right, "id_M \otimes {}_B\mathbf{N}"'] & M \otimes N \arrow[r, "\pi"] \arrow[dr, "f"'] & M \otimes_B N \arrow[d, dashed, "\tilde{f}"] \\
& & K
\end{tikzcd}
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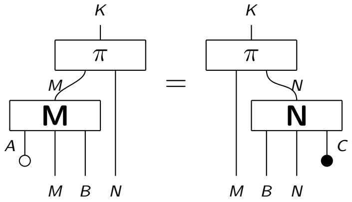

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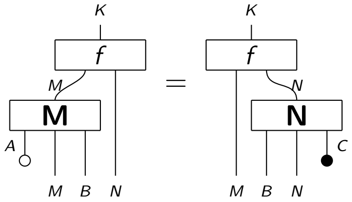

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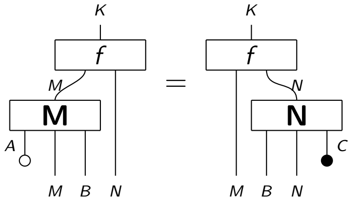

Can we find this module explicitly?

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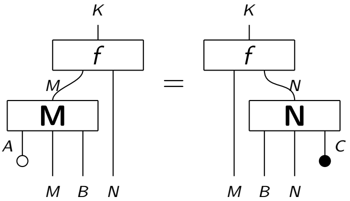
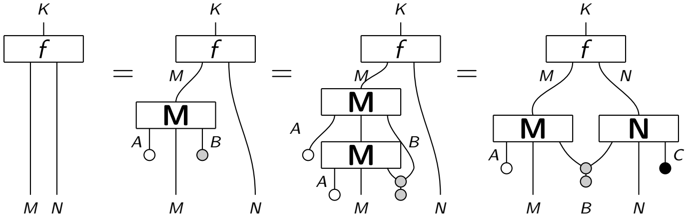


Can we find this module explicitly? Yes!

Horizontal composition in terms of dagger splittings

Any such f factorizes through **M**∘**N**:



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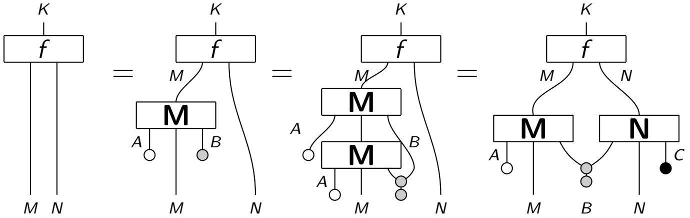
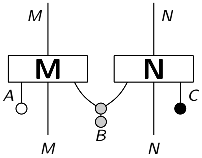


Theorem

Finding the dagger coequaliser is equivalent to finding a dagger splitting of the following morphism:



$2(\mathrm{CP}^*(-))$

We would like to understand the 2-category '?'

$$\begin{tikzcd}
\textbf{FHilb} \arrow[r, "\mathrm{CP}^*(-)"] \arrow[d, "2(-)"'] & \mathrm{CP}^*(\textbf{FHilb}) \arrow[d, "2(-)"] \\
2(\textbf{FHilb}) \arrow[r, dashed, "\mathrm{CP}^*(-)"'] & ?
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- There is a correspondence between special dagger Frobenius algebras on classical structures in **FHilb** and finite groupoids.

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- This required a classification of classical structures in $\mathrm{CP}^*(\mathbf{FHilb})$.
- There is a correspondence between special dagger Frobenius algebras on classical structures in **FHilb** and finite groupoids.
- $\mathrm{CP}^*(\mathbf{FHilb})$ does not have all coequalisers.

Modelling POVM's

The following subcategory of $2(\mathrm{CP}^*(\textbf{FHilb}))$ is a sufficient model for modelling communication protocols:

- 0-cells: natural numbers
- 1-cells: matrices of dagger Frobenius algebras
- 2-cells: matrices of completely positive maps

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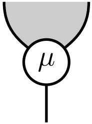
Measurements are defined as counit-preserving 2-cells of type:



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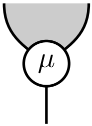


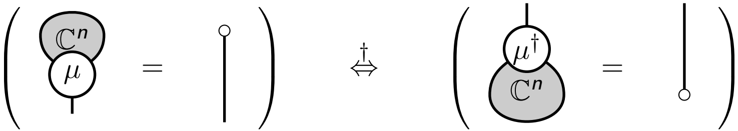
Theorem

Measurements on algebras $\mathbb{C}^n$ are exactly stochastic maps.
Measurements on algebras $B(H)$ are exactly POVMs.

Modelling POVM's

Proof.

The counit preserving condition gives us



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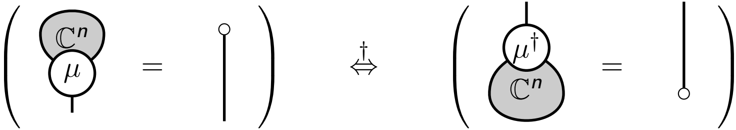
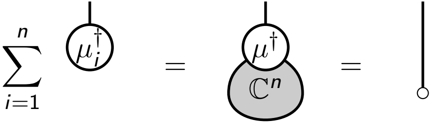


So we have the following equalities of positive elements:



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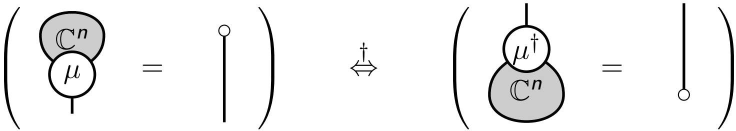

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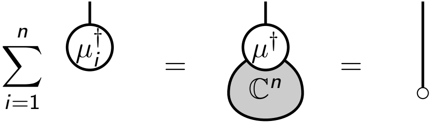

- On $\mathbb{C}^n$ this corresponds to a stochastic map

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Proof.

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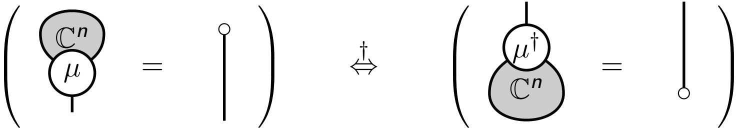
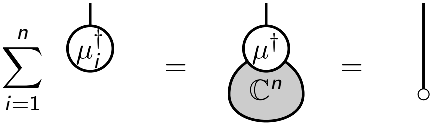


So we have the following equalities of positive elements:

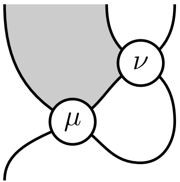


- On $\mathbb{C}^n$ this corresponds to a stochastic map
- On $B(\mathbb{C}^n)$ this corresponds to a POVM

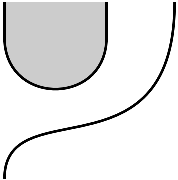
□

Classical encryption and quantum teleportation

Quantum teleportation and classical encryption are solutions to the following equation with μ a measurement and ν unitary 2-cell:



$$=$$



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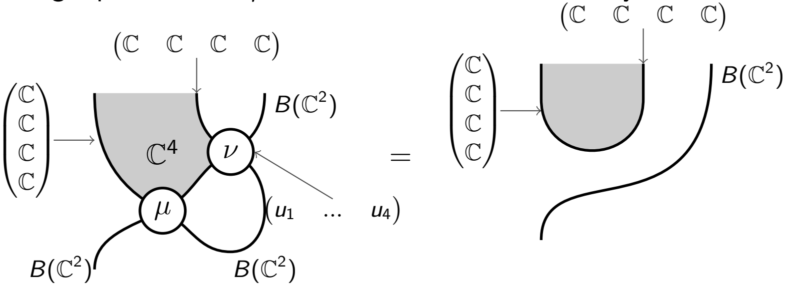

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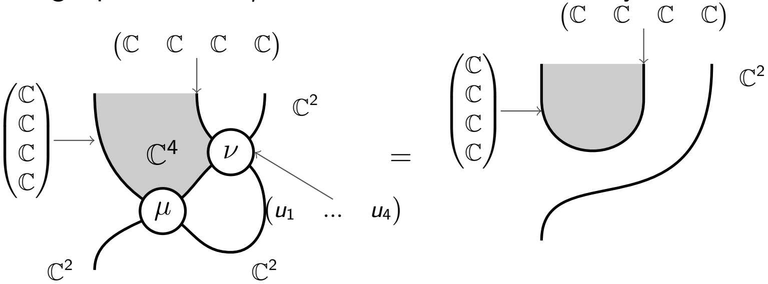

This equation corresponds to:

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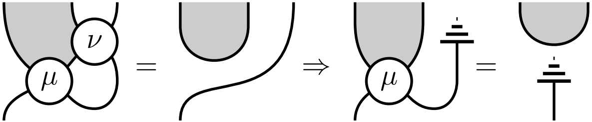
A unified security proof

When the output is destroyed, all information is lost:



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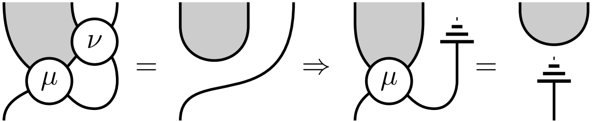

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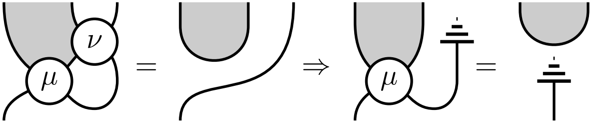

- We apply the trace map on both sides of the equation
- On the left-hand-side: ν is a family invertible completely positive maps, which are trace preserving.

So this give a unified security proof

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Thank you!