Towards a Unified Method for Termination*

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Abstract

The question of how to ensure programs terminate has been for decades attracting remarkable attention of computer scientists, resulting in a great number of techniques for proving termination of term rewriting and other models of computation. Nowadays it has become hard for new-comers to come up with new termination techniques/tools, since there are so many to learn/implement before inventing a new one. In this talk, I present my past and on-going work towards unified method for termination, that allow one to learn/implement a single idea and obtain many well-known techniques as instances.

1 Preliminaries

An *abstract reduction system (ARS)*, following Klop [10], consists of a set T and a family $\{\xrightarrow[\rho]{}\}_{\rho\in\mathcal{R}}$ of binary relations over T. Our interest is proving that $\xrightarrow[\mathcal{R}]{} := \bigcup_{\rho\in\mathcal{R}} \xrightarrow[\rho]{}$ is *terminating*, i.e., there is no infinite sequence of form $s_1 \xrightarrow[\mathcal{R}]{} s_2 \xrightarrow[\mathcal{R}]{} \cdots$.

Termination can be incrementally proved by a function $[\![\cdot]\!] : T \to A$ to a well-founded ordered set $\langle A, \succsim, \succ\rangle$. Let us define $[{}_{(}\succsim_{)}] := \{\rho \mid s \xrightarrow[\rho]{} t \implies [\![s]\!] \,{}_{(}\succsim_{)}\, [\![t]\!]\}$.

▶ Proposition 1. If $\mathcal{R} \subseteq [\succsim]$, then $\xrightarrow[\mathcal{R}]{}$ is terminating if $\xrightarrow[\mathcal{R}\setminus[\succ]]{}$ is. ◀

In term rewriting, *reduction orders* are a famous approach for termination, which use identity $[\![\cdot]\!]$ and impose conditions on orderings so that $[\succ] = \succ$. To minimize definitions, let us formulate only *interpretation*-based approach.

▶ **Definition 2** (sorted terms and term rewriting)**.** A *sorted signature* $\mathcal{F}$ consists of a set $\mathcal{S}_\mathcal{F}$ of sorts and a family $\{\mathcal{F}_\tau\}_{\tau\in\mathcal{S}_\mathcal{F}^*\times\mathcal{S}_\mathcal{F}}$ of function symbols. $\mathcal{F}$ is *single sorted* if $\mathcal{S}_\mathcal{F}$ is singleton. The *arity* of $f \in \mathcal{F}_{\vec{\sigma},\sigma}$ is the length of $\vec{\sigma}$. Given a family $\{\mathcal{V}_\sigma\}_{\sigma\in\mathcal{S}_\mathcal{F}}$ of variables, the set $\mathcal{T}_\sigma(\mathcal{F},\mathcal{V})$ of *terms* of sort σ are defined as usual. A *term rewrite system (TRS)* is a set $\mathcal{R}$, where each $\rho \in \mathcal{R}$ is a pair $\langle l, r\rangle$ of (single-sorted) terms with $l \notin \mathcal{V}$ and $\mathsf{Var}(l) \supseteq \mathsf{Var}(r)$. The ARSs $\xrightarrow[\rho]{}$, $\xrightarrow[\rho]{\epsilon}$ and $\xrightarrow[\rho]{>\epsilon}$, are defined as usual. ◀

▶ **Definition 3** (algebras)**.** For a sorted signature $\mathcal{F}$, an $\mathcal{F}$*-algebra* $[\![\cdot]\!]$ assigns each sort $\sigma \in \mathcal{S}_\mathcal{F}$ a set $[\![\sigma]\!]$ and each symbol $f \in \mathcal{F}_{[\sigma_1,\dots,\sigma_n],\sigma}$ a mapping $[\![f]\!] : [\![\sigma_1]\!] \times \cdots \times [\![\sigma_n]\!] \to [\![\sigma]\!]$. The interpretation $[\![s]\!]\alpha \in [\![\sigma]\!]$ of term $s \in \mathcal{T}_\sigma(\mathcal{F},\mathcal{V})$ under assignment α is defined as usual. An $\mathcal{F}$*-logic* is an $\mathcal{F}$-algebra with a special sort $\mathtt{bool} \in \mathcal{S}_\mathcal{F}$ and standard logic symbols $\wedge, \vee, \Rightarrow \in \mathcal{F}_{[\mathtt{bool},\mathtt{bool}],\mathtt{bool}}$ etc. with expected interpretations. We say $\phi \in \mathcal{T}_{\mathtt{bool}}(\mathcal{F},\mathcal{V})$ is *valid*, written $[\![\phi]\!]$, if $[\![\phi]\!]\alpha = \textsc{True}$ for any assignment α. ◀

▶ **Definition 4** (ordered algebras)**.** An *ordered* $\mathcal{F}$*-algebra* is a logic $[\![\cdot]\!]$, where the domain is quasi-ordered and the signature is $\mathcal{F}$ extended with logic symbols and $\geq, > \in \mathcal{F}_{[\sigma,\sigma'],\mathtt{bool}}$, all interpreted as expected. We say $[\![\cdot]\!]$ is

- *well-founded* if $>$ is well-founded,
- *(weakly) monotone* if $[\![f]\!]$ is monotone w.r.t. ${}_{(}\geq_{)}$ in every argument for every $f \in \mathcal{F}$, and

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- *(weakly) simple* if $[\![f]\!](a_1,\dots,a_n) \mathrel{{}_{(}{\geq}_{)}} a_i$ for every $f \in \mathcal{F}$ and i. ◀

Let us write $s \mathrel{[_{(}{\geq}_{)}]} t :\Longleftrightarrow [\![s \mathrel{{}_{(}{\geq}_{)}} t]\!]$. A reduction order can be characterized by $[>]$ of a well-founded monotone algebra, a *simplifiaction order* [2] by a simple monotone algebra, and *reduction pair* [1] by a well-founded weakly monotone algebra in the same manner.

2 Monotone WPO

Here we present the basic version of the *weighted path order (WPO)* [14].

▶ **Definition 5** (monotone WPO). Let $[\![\cdot]\!]$ be a well-founded $\mathcal{F}$-algebra, and $\succsim$ a well-founded quasi-order on $\mathcal{F}$. We define relations ${}_{(}{\succsim}_{)}{}_{\mathsf{WPO}}$ as follows: $s = f(s_1,\dots,s_n) \mathrel{{}_{(}{\succsim}_{)}{}_{\mathsf{WPO}}} t$ iff

1. $[\![s > t]\!]$, or
2. $[\![s \geq t]\!]$ and
   - a. $\exists i \in \{1,\dots,n\}.\ s_i \succsim_{\mathsf{WPO}} t$, or
   - b. $t = g(t_1,\dots,t_m)$, $\forall j \in \{1,\dots,m\}.\ s \succ_{\mathsf{WPO}} t_j$ and either
     - i. $f \succ g$ or
     - ii. $f \sim g$ and $\langle s_1,\dots,s_n\rangle \mathrel{{}_{(}{\succsim}_{)}{}^{\mathsf{lex}}_{\mathsf{WPO}}} \langle t_1,\dots,t_m\rangle$. ◀

▶ **Theorem 6.** *WPO is a simplification order if $[\![\cdot]\!]$ is weakly monotone and weakly simple.* ◀

- Dropping lines **1** and **2** results in LPO. The same effect can be achieved by choosing a trivial (singleton-carrier) algebra as $[\![\cdot]\!]$ or by interpretation $[\![f]\!](a_1,\dots,a_n) = \max\{a_1,\dots,a_n\}$ on a usual carrier like $\mathbb{N}$. Hence, WPO subsumes LPO.
- Dropping lines **a** and **b** results in GKBO, but for the resulting order to be well-founded, it is required to strengthen the weak simplicity condition to strict simplicity. When this condition is satisfied, WPO coincides with GKBO.
- The definition of KBO has a similar structure as Definition 5, with a particular condition in **a** (which can in fact be simplified). Similar to GKBO, KBO requires the "admissibility" condition on $[\![\cdot]\!]$, and under this condition, WPO coincides with KBO. For a general and detailed account, see [13, Chapter 3].
- As a reduction (simplification) order, WPO also subsumes monotone interpretations over totally ordered carrier, in the sense that $[>] \subseteq \succ_{\mathsf{WPO}}$. The side condition of Theorem 6 is known to be satisfied [15]. Further, WPO can be seen as a stretch of [16, Proposition 12], which indicates that a weakly simple and weakly monotone well-founded algebra can be extended to a simplification order.

3 Weakly Monotone WPO

The *dependency pair method* [1] reduces the termination of $\xrightarrow[\mathcal{R}]{}$ to the *finiteness* of *DP problem* $\langle \mathsf{DP}(\mathcal{R}), \mathcal{R}\rangle$, i.e., the termination of ARS[1] $\{\xrightarrow[\rho]{\epsilon} \circ \xrightarrow[\mathcal{R}]{>\epsilon}{}^*\}_{\rho\in\mathsf{DP}(\mathcal{R})}$, where $\mathsf{DP}(\mathcal{R}) := \{\langle l, r\rangle \mid \langle l, C[r]\rangle \in \mathcal{R}, \mathsf{root}(r) \in \mathcal{D}\}$ for $\mathcal{D}$ consisting of the root symbols of the left-hand sides of $\mathcal{R}$. A big merit of this reduction is that the monotonicity of reduction orders can be relaxed:

▶ **Theorem 7** ([7, 5]). *Let $\langle \succsim, \succ\rangle$ be a reduction pair such that $\mathcal{R} \cup \mathcal{P} \subseteq \succsim$. Then $\xrightarrow[\mathcal{P}]{\epsilon} \circ \xrightarrow[\mathcal{R}]{>\epsilon}{}^*$ is terminating if $\xrightarrow[\mathcal{P}\setminus\succ]{\epsilon} \circ \xrightarrow[\mathcal{R}]{>\epsilon}{}^*$ is.* ◀

[1] More precisely, $\xrightarrow[\rho]{\epsilon}$ here is restricted to $\xrightarrow[\mathcal{R}]{>\epsilon}$-terminating terms.

Although WPO is a simplification order, its well-foundedness is directly proved by an inductive argument inspired by Jouannaud and Rubio [9]. There, the key is to ensure $[\![s \geq s_i]\!]$ for those s_i's that are used in recursive comparison in lines **a** and **ii**. In fact, WPO is still well-founded if we restrict these recursively compared arguments to those which $[\![s \geq s_i]\!]$ is ensured. This has a similar effect as *argument filtering*, but we can additionally take the weights of dropped arguments into account.

▶ **Definition 8** (weakly monotone WPO)**.** Let π be a mapping that assigns each n-ary symbol $f \in \mathcal{F}$ a subset $\pi(f) \subseteq \{1,\dots,n\}$ of its argument positions. Abusing notation, we see $\pi(f)$ also as an index-filtering operation over lists. We refine WPO as follows: $s = f(s_1,\dots,s_n) \mathrel{{}_{(\succsim)}}_{\mathsf{WPO}}\, t$ iff

1. $[\![s > t]\!]$, or
2. $[\![s \geq t]\!]$ and
   - **a.** $\exists i \in \pi(f).\ s_i \succsim_{\mathsf{WPO}} t$, or
   - **b.** $t = g(t_1,\dots,t_m)$, $\forall j \in \pi(g).\ s \succ_{\mathsf{WPO}} t_j$ and either
     - **i.** $f \succ g$ or
     - **ii.** $f \sim g$ and $\pi(f)[s_1,\dots,s_n] \mathrel{{}_{(\succsim)}}^{\mathsf{lex}}_{\mathsf{WPO}} \pi(g)[t_1,\dots,t_m]$. ◀

Since now some arguments may be dropped in the comparison of line **ii**, $\succ_{\mathsf{WPO}}$ is not closed under context anymore, but it is no problem in the DP framework.

▶ **Theorem 9** ([14])**.** *WPO forms a reduction pair if* $[\![\cdot]\!]$ *is weakly monotone and* π-simple*:* $[\![f]\!](a_1,\dots,a_n) \geq a_i$ *whenever* $i \in \pi(f)$*.* ◀

With some small refinements [14, Section 4.2], one can get $[\geq] \subseteq \succsim_{\mathsf{WPO}}$ and $[>] \subseteq \succ_{\mathsf{WPO}}$ by setting $\pi(f) := \emptyset$ and $\succsim := \mathcal{F} \times \mathcal{F}$. In this sense, weakly monotone WPO subsumes weakly monotone interpretations.

4 Non-Monotone WPO

Now we further generalize WPO so that non-monotone algebras can be used. Such algebras are still useful for proving innermost termination [6], and probably full termination [3].

▶ **Definition 10.** Let μ be a mapping that assigns each n-ary symbol f and $i \in \{1,\dots,n\}$ a subset of $\{\geq,\leq\}$. We say an ordered algebra $[\![\cdot]\!]$ is *μ-monotone* if $a_i \geq a_i'$ implies $[\![f]\!](\dots,a_i,\dots) \sqsupseteq [\![f]\!](\dots,a_i',\dots)$ whenever $\sqsupseteq \in \mu(f,i)$. For a TRS $\mathcal{R}$, we define set $U_{\mathcal{R},\mu}(s)$ of pairs of terms so that $U_{\mathcal{R},\mu}(f(s_1,\dots,s_n))$ is a superset of

1. $\{\langle l,r\rangle\} \cup U_{\mathcal{R},\mu}(r)$ for every $\langle l,r\rangle \in \mathcal{R}$ with $\mathsf{root}(l) = f$,
2. $U_{\mathcal{R},\mu}(s_i)$ if $\sqsubseteq \notin \mu(f,i)$, and
3. $U_{\mathcal{R},\mu}(s_i)^{-1}$ if $\sqsupseteq \notin \mu(f,i)$.

▶ **Theorem 11** ([6])**.** *Let* $[\![\cdot]\!]$ *be a well-founded μ-monotone algebra such that* $\mathcal{P} \cup \left(\bigcup_{\langle l,r\rangle \in \mathcal{P}} U_{\mathcal{R},\mu}(r)\right) \subseteq [\geq]$*. Then*[2] $\xrightarrow[\mathcal{P}]{\epsilon} \circ \xrightarrow[\mathcal{R}]{>\epsilon}{}^{!}$ *is terminating if* $\xrightarrow[\mathcal{P}\setminus[>]]{\epsilon} \circ \xrightarrow[\mathcal{R}]{>\epsilon}{}^{!}$ *is.* ◀

WPO is already applicable to μ-monotone algebras, in the following sense:

▶ **Theorem 12.** *WPO forms a well-founded μ-monotone term algebra, if*

1. $[\![\cdot]\!]$ *is μ-monotone and π-simple, and*

[2] More precisely, $\xrightarrow[\rho]{\epsilon}$ here is restricted to $\xrightarrow[\mathcal{R}]{>\epsilon}$-normal forms.

2. *$\mu(f,i) = \{\geq\}$ for every $f \in \mathcal{F}$ and $i \in \pi(f)$.*

Proof Sketch. Only μ-monotonicity has to be proved. The interesting case is $\leq \in \mu(f,i)$. Then $s_i \gtrsim_{\mathsf{WPO}} s'_i$ implies $[\![f(\dots,s_i,\dots) \leq f(\dots,s'_i,\dots)]\!]$. If this is not "strict", then it is easy to see that case **ii** is applied, and compared argument lists are identical. ◀

It is tempting to exploit anti-monotonicity by comparing some arguments in line **ii** in reverse direction. Then μ-monotonicity will still be preserved, but unfortunately, it turns out that well-foundedness and even *non-inifinitesimality* [6] will be broken.

5 Constrained WPO

WPO combines the syntactic termination argument of path orders and the semantic termination argument of algebraic interpretations. Hence, we expect WPO to be useful for term rewriting combined with algebraic semantics [4, 12].

▶ **Definition 13.** Let $\mathcal{F}$ be a signature, partitioned into $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$. We fix the semantics of $\mathcal{B}$ by a $\mathcal{B}$-logic $[\![\cdot]\!]_{\mathcal{B}}$, and assume a terminating $\mathcal{B}$-TRS $\mathcal{S}$ such that[3] $\langle l,r\rangle \in \mathcal{S}$ implies $[\![l]\!]_{\mathcal{B}} = [\![r]\!]_{\mathcal{B}}$. A *constrained TRS* $\mathcal{R}$ is a set where each $\rho \in \mathcal{R}$ is a triple $\langle l,\phi,r\rangle$, such that[4] $l,r \in \mathcal{T}(\mathcal{F},\mathcal{V})$, $\phi \in \mathcal{T}_{\mathsf{bool}}(\mathcal{B},\mathcal{V})$, $\mathsf{root}(l) \in \mathcal{D}$ and $\mathsf{Var}(l) \cup \mathsf{Var}(\phi) \supseteq \mathsf{Var}(r)$. The relation $\xrightarrow[\rho]{\epsilon}$ is characterized by $l\theta \xrightarrow[\rho]{\epsilon} r\theta$ where $\phi\theta \in \mathcal{T}_{\mathsf{bool}}(\mathcal{B},\emptyset)$ is valid, and extended to $\xrightarrow[\rho]{>\epsilon}$ and $\xrightarrow[\rho]{}$ as usual. The termination problem is a relative termination problem: $\xrightarrow[\mathcal{R}]{}/\xrightarrow[\mathcal{S}]{}$. ◀

Kop [11] imported the dependency pair method to (logically) constrained TRSs.

▶ **Theorem 14.** *Constrained TRS $\mathcal{R}$ is terminating if $\mathcal{S}$ is non-duplicating and $\xrightarrow[\mathsf{DP}(\mathcal{R})]{\epsilon} \circ \xrightarrow[\mathcal{R}\cup\mathcal{S}]{>\epsilon}{}^*$ is terminating, where* $\mathsf{DP}(\mathcal{R}) := \{\, \langle l,\phi,r\rangle \mid \langle l,\phi,C[r]\rangle \in \mathcal{R}, \mathsf{root}(r) \in \mathcal{D} \,\}$.

Proof. Let $\overline{\mathcal{R}} := \{\langle l\theta, r\theta\rangle \mid \langle l,\phi,r\rangle \in \mathcal{R}, [\![\phi\theta]\!]_{\mathcal{B}}\}$. We have $\xrightarrow[\overline{\mathcal{R}}]{>\epsilon} = \xrightarrow[\mathcal{R}]{>\epsilon}$ and $\xrightarrow[\mathsf{DP}(\overline{\mathcal{R}})]{\epsilon} = \xrightarrow[\mathsf{DP}(\mathcal{R})]{\epsilon}$. Iborra et al. [8] shows that $\xrightarrow[\overline{\mathcal{R}}]{}/\xrightarrow[\mathcal{S}]{}$ is terminating if $\xrightarrow[\mathsf{DP}(\overline{\mathcal{R}})]{\epsilon} \circ \xrightarrow[\mathcal{S}\cup\overline{\mathcal{R}}]{>\epsilon}{}^*$ is. ◀

The key ingredient of constrained TRSs is, obviously, constraints. Hence it is of essential importance to exploit information from constraints.

▶ **Definition 15** (constrained WPO)**.** For $\phi \in \mathcal{T}_{\mathsf{bool}}(\mathcal{B},\mathcal{V})$, we define relations $_{(}\gtrsim_{)\mathsf{WPO}\,[\phi]}$ as follows: $s = f(s_1,\dots,s_n) \;_{(}\gtrsim_{)\mathsf{WPO}\,[\phi]}\; t$ iff

1. $[\![\phi \Rightarrow s > t]\!]$, or
2. $[\![\phi \Rightarrow s \geq t]\!]$ and
   - **a.** $\exists i \in \pi(f).\ s_i \gtrsim_{\mathsf{WPO}\,[\phi]} t$, or
   - **b.** $t = g(t_1,\dots,t_m)$, $\forall j \in \pi(g).\ s \succ_{\mathsf{WPO}\,[\phi]} t_j$ and either
     - **i.** $f \succ g$ or
     - **ii.** $f \sim g$ and $\pi(f)[s_1,\dots,s_n] \;_{(}\gtrsim_{)\mathsf{WPO}\,[\phi]}^{\mathsf{lex}}\; \pi(g)[t_1,\dots,t_m]$. ◀

As one naturally expects, $[\![\cdot]\!]$ used above should *respect* $[\![\cdot]\!]_{\mathcal{B}}$ used for inducing rewrite relation, i.e., $[\![s]\!] = [\![s]\!]_{\mathcal{B}}$ for any $s \in \mathcal{T}(\mathcal{B},\mathcal{V})$. Under this assumption, we may write $[_{(}\gtrsim_{)\mathsf{WPO}}] := \{\langle l,\phi,r\rangle \mid l \;_{(}\gtrsim_{)\mathsf{WPO}\,[\phi]}\; r\}$; this notation, corresponding to the notation in Proposition 1, is justified by the following fact:

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[3] Kop and Nishida [12] assumes $\mathcal{B}$ to contain all values of $[\![\cdot]\!]_{\mathcal{B}}$, and fixes $\mathcal{S}$ to be the *calculation* step. For the purpose of this talk, we do not need these assumptions.

[4] Here we ignore the sensible assumption that l and r should be of the same sort.

▶ **Lemma 16.** *If* $[\![\cdot]\!]$ *respects* $[\![\cdot]\!]_{\mathcal{B}}$*, then* $\langle l, \phi, r\rangle \in [(\succsim)_{\mathsf{WPO}}]$ *implies* $\xrightarrow[\langle l,\phi,r\rangle]{\epsilon} \subseteq (\succsim)_{\mathsf{WPO}}$.

Proof Sketch. Consider $l\theta \xrightarrow[\langle l,\phi,r\rangle]{\epsilon} r\theta$, so $[\![\phi\theta]\!]_{\mathcal{B}}$. Let us assume that $l \succ_{\mathsf{WPO}\,[\phi]} r$ is derived from case **1**. Then we have $[\![\phi \Rightarrow l > r]\!]$, hence $[\![\phi\theta \Rightarrow l\theta > r\theta]\!]$. Since $[\![\phi\theta]\!] = [\![\phi\theta]\!]_{\mathcal{B}} = \textsc{True}$, we get $[\![l\theta > r\theta]\!]$, deriving $l\theta \succ_{\mathsf{WPO}} r\theta$. Other cases go as well. ◀

Since $[\![\cdot]\!]_{\mathcal{B}}$ is often the integer arithmetic, which is not monotone, we cannot assume (weak) monotonicity on $[\![\cdot]\!]$. So we now restrict our interest to innermost termination. We extend $U_{\mathcal{R},\mu}(\mathcal{P})$ for constrained TRSs by replacing item **1** of Definition 10 with

1. $\{\langle l, \phi, r\rangle\} \cup U_{\mathcal{R},\mu}(r)$ for every $\langle l, \phi, r\rangle \in \mathcal{R}$ with $\mathsf{root}(l) = f$.

▶ **Theorem 17.** *Let* $[\![\cdot]\!]$ *respect* $[\![\cdot]\!]_{\mathcal{B}}$*, and* $\mathcal{P} \cup \left(\bigcup_{\langle l,\phi,r\rangle\in\mathcal{P}} U_{\mathcal{R}\cup\mathcal{S},\mu}(r)\right) \subseteq [\succsim_{\mathsf{WPO}}]$*. Then* $\xrightarrow[\mathcal{P}]{\epsilon} \circ \xrightarrow[\mathcal{R}\cup\mathcal{S}]{>\epsilon}{}^{!}$ *is terminating if* $\xrightarrow[\mathcal{P}\setminus[\succ_{\mathsf{WPO}}]]{\epsilon} \circ \xrightarrow[\mathcal{R}\cup\mathcal{S}]{>\epsilon}{}^{!}$ *is.*

Proof Sketch. We take the same approach as Theorem 14 to reduce to Theorem 11. ◀

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