Bisimulations and unfolding in $\mathcal{P}$-accessible categorical models

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Abstract

We propose a categorical framework for bisimulations and unfoldings that unifies the classical approach from Joyal and al. via open maps and unfoldings. This is based on a notion of categories accessible with respect to a subcategory of path shapes, i.e., for which one can define a nice notion of trees as glueings of paths. We show that transition systems and presheaf models are instances of our framework. We also prove that in our framework, several notions of bisimulation coincide, in particular an "operational one" akin to the standard definition in transition systems. Also, our notion of accessibility is preserved by coreflections. This also leads us to a notion of unfolding that behaves well in the accessible case: it is a right adjoint and is a universal covering, i.e., it is initial among the morphisms that have the unique lifting property with respect to path shapes. As an application, we prove that the universal covering of a groupoid, a standard construction in algebraic topology, is an unfolding, when the category of path shapes is well chosen.

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1 Introduction

Bisimulations were introduced in [12] as a way to express that two concurrent systems are "equivalent", in a way that would reflect not only trace equivalence, but also the branching structure of executions. Later, Joyal, Nielsen and Winskel [6] developed a theory of bisimulations using open maps, which are particular morphisms in some category of models, which satisfy lifting properties with respect to a specified subcategory of execution paths. They made explicit links between this abstract view of bisimulations and the classical relational definition, for some models of concurrency, including transition systems and event structures.

In some other line of work, Nielsen, Plotkin and Winskel [10] introduced a notion of unfolding for 1-safe Petri nets. The unfolding produces an "equivalent" Petri net, which is infinite in general (in the absence of cut-rules) and is non-looping. This is at the basis of numerous verification methods on Petri nets [2]. Later, Winskel [13] developed the categorical framework of Petri nets and unfoldings by relating them to coreflections (special cases of adjoint functors) between some categories of concurrent models, more particularly occurence nets and event structures. Winskel also developed a classification of concurrent models by coreflections [14].

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As open maps, unfoldings are closely related to prominent concepts in algebraic topology. Unfoldings are closely linked to coverings, which are nice fibered spaces which (also) satisfy unique lifting properties (see e.g. [4]). Coverings are closely related to partial unfoldings of the state space (see e.g. [13]). The universal covering can actually be defined, in ordinary algebraic topology, as a complete delooping of paths in topological spaces, which makes it very similar to unfoldings of transition systems. Although the analogy is enlightening, formalizing it stumbles on various difficulties. In this paper, we propose to unify those theories, by putting lifting properties of "paths" at the center of the picture.

In Section 2 we recall the case of transition systems, and the classical notion of bisimilarity [12]. The categorical approaches to bisimulation via open maps and via "strong path-bisimulation" of Joyal, Nielsen and Winskel [6] are equivalent to this classical notion of bisimulation in the particular case of transition systems, but are not equivalent in general, if we do not set up the proper context.

The framework of $\mathcal{P}$-accessible models we are developing is going to define this context, where those two notions of bisimilarities will be equivalent. We introduce accessible models in Section 3 and prove this result in the same section. Somehow, these categories are the right ones in the sense that their objects are closely tied with the glueing of paths within them. It will be trivially the case for transition systems: they will be accessible models in our sense.

In Section 4, we show that the framework of presheaf models of [6] is also a particular case of our framework, yielding another proof of one of the main results of [6].

Many models are related through coreflections (see, e.g. [11]). In Section 5, we show that, when two models are related this way, then accessibility is transferred from one the other. This makes our notion possibly applicable to a great variety of models.

We then turn to unfoldings in accessible models, in Section 6. Indeed, there is a very nice notion of paths and path extensions in accessible models, making the notion of unfolding very natural. In particular, in Section 6.2, we show that the unfolding of a model is bisimilar to the original model. As a bonus, unfoldings are defined in a canonical manner in accessible categories: they are right adjoints (Section 6.3). Finally, we show that unfoldings in accessible models enjoy unique path lifting properties (Section 7.2) making them similar to universal coverings. In the case of groupoids for instance, we show that unfoldings are universal coverings (recapped in Section 7.1).

2 Categorical models and bisimilarities

We first recall, from [6], two notions of bisimilarity in a category with a specified subcategory of path shapes.

2.1 Category of models, subcategory of paths

We consider a category $\mathcal{M}$ (of **models**) together with a small subcategory (of **path-shapes**) $\mathcal{P}$. We assume that $\mathcal{M}$ and $\mathcal{P}$ have a common initial object I, i.e., an object $I \in \mathcal{P}$ such that for every object A of $\mathcal{P}$ (resp. of $\mathcal{M}$), there is a unique morphism in $\mathcal{P}$ (resp. in $\mathcal{M}$) from I to A. We note ι_A this unique morphism. One typical example is the category of transition systems, that we briefly recap below.

Fix an alphabet Σ. A **transition system** $T = (Q, i, \Delta)$ on Σ is the following data: a set Q (of states); a initial state $i \in Q$; a set of transitions $\Delta \subseteq Q \times \Sigma \times Q$.

A **morphism of transition systems on** Σ $f : T_1 = (Q_1, i_1, \Delta_1) \longrightarrow T_2 = (Q_2, i_2, \Delta_2)$ is a function $f : Q_1 \longrightarrow Q_2$ such that $f(i_1) = i_2$ and for every $(p, a, q) \in \Delta_1$, $(f(p), a, f(q)) \in$

Δ_2.

We note $\mathbf{TS}(\Sigma)$, the category of transition systems on Σ and morphisms of transition systems.

The subcategory of path-shapes will be in this case the category of branches: for $n \in \mathbb{N}$, a **n-branch shape on** Σ is a transition system $([n], 0, \Delta)$ where:

- $[n]$ is the set $\{0, \ldots, n\}$;
- Δ is of the form $\{(i, a_i, i+1) \mid i \in [n-1]\}$ for some a_0, ..., a_{n-1} in Σ.

$$0 \xrightarrow{a_0} 1 \xrightarrow{a_1} 2 \quad \cdots \quad n-1 \xrightarrow{a_{n-1}} n$$

We then take $\mathbf{Br}(\Sigma)$ as the full subcategory of $\mathbf{TS}(\Sigma)$ of n-branch shapes for all $n \in \mathbb{N}$. A common initial object of $\mathbf{TS}(\Sigma)$ and $\mathbf{Br}(\Sigma)$ is then the 0-branch shape $I = ([0], 0, \varnothing)$. We call **$n$-branch of a transition system** T any morphism of transition system from a n-branch form to T.

2.2 A relational bisimilarity of models: path-bisimilarity

Equivalence of transition systems is defined through the notion of **bisimulation**. Classically [12], a bisimulation between $T_1 = (Q_1, i_1, \Delta_1)$ and $T_2 = (Q_2, i_2, \Delta_2)$ is a relation $R \subseteq Q_1 \times Q_2$ such that:

(i) $(i_1, i_2) \in R$;

(ii) if $(q_1, q_2) \in R$ and $(q_1, a, q_1') \in \Delta_1$ then there is $q_2' \in Q_2$ such that $(q_2, a, q_2') \in \Delta_2$ and $(q_1', q_2') \in R$;

(iii) if $(q_1, q_2) \in R$ and $(q_2, a, q_2') \in \Delta_2$ then there is $q_1' \in Q_1$ such that $(q_1, a, q_1') \in \Delta_1$ and $(q_1', q_2') \in R$.

We then say that two transition systems are **bisimilar** if there is a bisimulation between them.

A bisimulation between T_1 and T_2 induces a relation R_n' between n-branches of T_1 and n-branches of T_2 by:

$$R_n' = \{(f_1 : B_1 \longrightarrow T_1, f_2 : B_2 \longrightarrow T_2) \mid \forall i \in [n],\ (f_1(i), f_2(i)) \in R\}$$

These relations satisfy that:

- $(\iota_{T_1}, \iota_{T_2}) \in R_0'$ by (i);
- by (ii), if $(f_1, f_2) \in R_n'$ and if $(f_1(n), a, q_1) \in \Delta_1$ then there is $q_2 \in Q_2$ such that $(f_2(n), a, q_2) \in \Delta_2$ and $(f_1', f_2') \in R_{n+1}'$ where $f_i'(j) = f_i(j)$ if $j \leq n$, q_i otherwise;
- symmetrically with (iii);
- if $(f_1, f_2) \in R_{n+1}'$ then $(f_1', f_2') \in R_n'$ where f_i' is the restriction of f_i to $[n]$.

In fact, bisimilarity of transition systems is equivalent to the existence of such relations on n-branches. This leads us to the general notion of strong path-bisimulation [6].

A **strong path-bisimulation** R between X and Y, objects of $\mathcal{M}$ is a set of elements of the form $X \xleftarrow{f} P \xrightarrow{g} Y$ with P object of $\mathcal{P}$ such that:

(a) $X \xleftarrow{\iota_X} I \xrightarrow{\iota_Y} Y$ belongs to R;

(b) if $X \xleftarrow{f} P \xrightarrow{g} Y$ belongs to R then for every **path extension** of X, i.e, every morphism p in $\mathcal{P}$ such that:

$$\begin{array}{ccc} P & \xrightarrow{f} & X \\ {\scriptstyle p}\downarrow & \nearrow_{f'} & \\ Q & & \end{array}$$

commutes then there exists a path extension of Y

$$\begin{array}{ccc} P & \xrightarrow{g} & Y \\ {\scriptstyle p}\downarrow & \nearrow_{g'} & \\ Q & & \end{array}$$

such that $X \xleftarrow{f'} Q \xrightarrow{g'} Y$ belongs to R;

(c) symmetrically;

(d) if $X \xleftarrow{f} P \xrightarrow{g} Y$ belongs to R and if we have a morphism $p : Q \longrightarrow P \in \mathcal{P}$ then $X \xleftarrow{f \circ p} Q \xrightarrow{g \circ p} Y$ belongs to R;

We say that X and Y are **strong path bisimilar** iff there exists a strong path bisimulation between them.

2.3 A fibrational bisimilarity of models: $\mathcal{P}$-bisimilarity

Lifting properties are a useful ingredient in category theory and algebraic topology. In [6], they permit to design an abstract notion of bisimilarity via morphisms which satisfy lifting properties with respect to paths, recovering a large variety of models and motivating the use of presheaf models by the work on pretopoi in [5].

We say that a morphism $f : X \longrightarrow Y$ of $\mathcal{M}$ is **($\mathcal{P}$-)open** iff for all commutative diagrams:

$$\begin{array}{ccc} P & \xrightarrow{x} & X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle f} \\ Q & \xrightarrow[y]{} & Y \end{array}$$

with $p : P \longrightarrow Q \in \mathcal{P}$, there exists a morphism $\theta : Q \longrightarrow X$ such that the following diagram commutes:

$$\begin{array}{ccc} P & \xrightarrow{x} & X \\ {\scriptstyle p}\downarrow & \nearrow^{\theta} & \downarrow{\scriptstyle f} \\ Q & \xrightarrow[y]{} & Y \end{array}$$

We then say that two objects X and Y of $\mathcal{M}$ are **$\mathcal{P}$-bisimilar** iff there exists a span $f : Z \longrightarrow X$ and $g : Z \longrightarrow Y$ where f and g are $\mathcal{P}$-open.

It is known that if X and Y are $\mathcal{P}$-bisimilar then they are strong path bisimilar [6]. The converse also holds in the case of transition systems (both $\mathcal{P}$ and path bisimilarities coincide with the classical bisimilarity), but there is no general result for the converse. The purpose of the following section is to investigate a general framework in which those two notions of bisimilarity will coincide.

3 Accessible models and equivalence of bisimilarities

For the converse, we must build a span of open maps from a strong path-bisimulation. It requires in particular that we construct an object of $\mathcal{M}$, which will be the tip of the span. One way of doing so is to glue the elements of the bisimulation in order to obtain an "object of bisimilar paths". Categorically, a glueing is a colimit, so a natural hypothesis should be the existence of some colimits in $\mathcal{M}$.

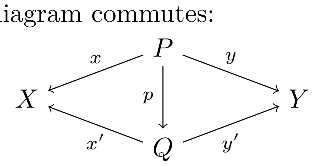
Concretely, a $\mathcal{P}$**-tree** in $\mathcal{M}$ is a colimit in $\mathcal{M}$ of a small diagram with values in $\mathcal{P}$, i.e., of a functor $D : \mathcal{D} \longrightarrow \mathcal{P}$ where $\mathcal{D}$ is a small category. We say that **all** $\mathcal{P}$**-trees exist in** $\mathcal{M}$ if every small diagram with values in $\mathcal{P}$ has a colimit in $\mathcal{M}$. In the category of transition systems, $\mathbf{Br}(\Sigma)$-trees are exactly synchronization trees, i.e., a transition system $T = (Q, i, \Delta)$ such that:

- every state in Q is accessible, i.e., for every $q \in Q$, there is a n-branch $f : B \longrightarrow T$ for some $n \in \mathbb{N}$ such that $f(n) = q$;
- T is acyclic, i.e., for every branch $f : B \longrightarrow T$, there is no $i \neq j$ such that $f(i) = f(j)$;
- T is non-joining, i.e., if (q_1, a, p) and $(q_2, b, p) \in \Delta$ then $a = b$ and $q_1 = q_2$.

In particular, all $\mathbf{Br}(\Sigma)$-trees exists in $\mathbf{TS}(\Sigma)$. We note $\mathbf{Tree}(\mathcal{M}, \mathcal{P})$ for the full subcategory of $\mathcal{M}$ of $\mathcal{P}$-trees.

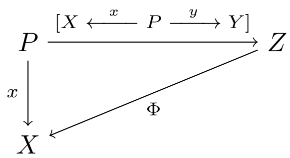
Let R be a strong path bisimulation between X and Y and assume that all $\mathcal{P}$-trees exist. Let us construct a span of maps between X and Y. First, we construct the tip of the span as the colimit of a particular diagram with values in $\mathcal{P}$, defined from R. Let $\mathcal{C}$ be the following category:

- objects of $\mathcal{C}$ are elements of R;
- morphisms from $X \xleftarrow{x} P \xrightarrow{y} Y$ to $X \xleftarrow{x'} Q \xrightarrow{y'} Y$ are morphisms $p : P \longrightarrow Q$ of $\mathcal{P}$ such that the following diagram commutes:



Then define the diagram $F : \mathcal{C} \longrightarrow \mathcal{P}$ which maps every $X \xleftarrow{x} P \xrightarrow{y} Y \in R$ to P and every p to itself. Since $\mathcal{P}$-trees exist (F is small because R is a set), let $(Z, ([\alpha])_{\alpha \in R})$ be the colimit of F, where the $[X \xleftarrow{x} P \xrightarrow{y} Y] : P = F(X \xleftarrow{x} P \xrightarrow{y} Y) \longrightarrow Z$ are the maps from the colimit.

Z will be the tip of our span. Now we need to construct maps $\Phi : Z \longrightarrow X$ and $\Psi : Z \longrightarrow Y$. Let us do it for Φ: since $(X, (F(X \xleftarrow{x} P \xrightarrow{y} Y) \xrightarrow{x} X))$ is a cocone of F, there exists a unique morphism $\Phi : Z \longrightarrow X$ such that for all $X \xleftarrow{x} P \xrightarrow{y} Y \in R$ the following diagram commutes:



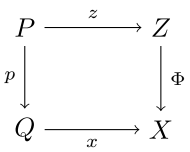
To prove that strong path-bisimilarity implies $\mathcal{P}$-bisimilarity, we just need to prove that Φ is open. But it does not hold in general. We will need that we do not create more paths in a tree than the ones we used in the glueing. In the case of transition systems, this says that every path in a tree seen as the colimit of a certain diagram D with values in $\mathcal{P}$ is a subbranch of some $D(i)$. More generally, we will say that $\mathcal{M}$ is $\mathcal{P}$**-accessible** if :

- all $\mathcal{P}$-trees exist;
- every morphism $f : P \longrightarrow Z$ where $P \in \mathcal{P}$ and $(Z, (\eta_d)_{d \in \mathcal{D}})$ is the colimit of a non-empty small diagram $D : \mathcal{D} \longrightarrow \mathcal{P}$ factorizes as $f = \eta_d \circ p$ for some $d \in \mathcal{D}$ with $p : P \longrightarrow D(d) \in \mathcal{P}$.

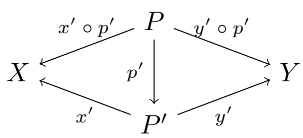
In particular, $\mathbf{TS}(\Sigma)$ is $\mathbf{Br}(\Sigma)$-accessible.

► Remark. The name "accessible" is a reference to κ-accessible categories [8] where κ is a cardinal, which is a very similar property of a category, requiring the existence of some colimits (in this case, filtered colimits) and the same kind of factorizations for morphisms whose codomain is such a colimit.

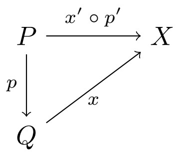
Assuming that $\mathcal{M}$ is $\mathcal{P}$-accessible, we can now prove that Φ is open. Consider a commutative diagram of the form:



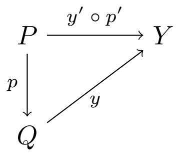
with p in $\mathcal{P}$. As Z is a colimit of a non-empty (because R is non-empty) small diagram, then by $\mathcal{P}$-accessibility, $z : P \longrightarrow Z$ factorizes as $[X \xleftarrow{x'} P' \xrightarrow{y'} Y] \circ p'$ for some $X \xleftarrow{x'} P' \xrightarrow{y'} Y \in R$ and $p' : P \longrightarrow P' \in \mathcal{P}$. Then, by condition (d) of a strong path bisimulation, $X \xleftarrow{x' \circ p'} P \xrightarrow{y' \circ p'} Y$ belongs to R. Moreover, the following diagram commutes:



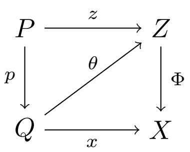
Then, $z = [X \xleftarrow{x'} P' \xrightarrow{y'} Y] \circ p' = [X \xleftarrow{x' \circ p'} P \xrightarrow{y' \circ p'} Y]$.
So, $x \circ p = \Phi \circ z = \Phi \circ [X \xleftarrow{x' \circ p'} P \xrightarrow{y' \circ p'} Y] = x' \circ p'$ by definition of Φ. This means that we have the following commutative diagram:



Then, by condition (b) of a strong path bisimulation, there is a path extension of Y:



such that $X \xleftarrow{x} Q \xrightarrow{y} Y$ belongs to R.
Then the morphism $\theta = [X \xleftarrow{x} Q \xrightarrow{y} Y] : Q \longrightarrow Z$ is the lifting we were looking for:



So we deduce:

▶ **Theorem 1.** *If $\mathcal{M}$ is $\mathcal{P}$-accessible and if X and Y are strong path bisimilar then they are $\mathcal{P}$-bisimilar.*

4 Presheaf models

Presheaf models were introduced in [6], motivated by the work on pretopoi in [5]. We prove in this section that presheaf models are a particular case of accessible models.

Assume given a small category Δ with an initial object J. A **rooted presheaf on** Δ is a functor F from Δ^{op} to **Set** such that $F(J)$ is a singleton. Let $[\Delta^{op}, Set]_*$ be the category of rooted presheaves on Δ and natural transformations. We have a functor (called the **Yoneda embedding**) $\mathfrak{Y} : \Delta \longrightarrow [\Delta^{op}, Set]_*$:

- we associate an object P of Δ with the rooted presheaf $\mathfrak{Y}(P)$ which maps:
  - every object Q of Δ to $\Delta(Q, P)$
  - every morphism $p : Q \longrightarrow Q'$ of Δ to the function $\mathfrak{Y}(P)(p) : \Delta(Q', P) \longrightarrow \Delta(Q, P)$ $\quad f \mapsto f \circ p$
- we associate a morphism $p : P \longrightarrow P'$ with the natural transformation $\mathfrak{Y}(p) : \mathfrak{Y}(P) \longrightarrow \mathfrak{Y}(P')$ defined by

$$\mathfrak{Y}(p)_Q : \Delta(Q, P) \longrightarrow \Delta(Q, P') \qquad f \mapsto p \circ f$$

▶ **Theorem 2.** *Let $\mathcal{P}$ be the image of $\mathfrak{Y}$ and $\mathcal{M} = [\Delta^{op}, Set]_*$. Then $\mathcal{M}$ is $\mathcal{P}$-accessible.*

Proof.

⋆ **$\mathcal{P}$ is a full embedding of $\mathcal{M}$:** by the Yoneda lemma.

⋆ **computation of colimits in $\mathcal{M}$:** consider a small diagram $D : U \longrightarrow \mathcal{M}$. The colimit in $[\Delta^{op}, Set]_*$ of D is the colimit in $[\Delta^{op}, Set]$ (which is cocomplete [1]) of the small (non-empty) diagram $D_\perp : U_\perp \longrightarrow M$ where:
  - $U_\perp$ is the category obtained by adding an object $\perp$ to U with a unique morphism from $\perp$ to any object of U or $\perp$ and no morphism from an object of U to $\perp$
  - $D_\perp$ maps $\perp$ to $\mathfrak{Y}(J)$ (which is the initial object of $\mathcal{M}$ and $\mathcal{P}$ by the Yoneda lemma), any object u of U to $D(u)$, the morphism from $\perp$ to u object of $U_\perp$ to the unique natural transformation η_u from $\mathfrak{Y}(J)$ to $D_\perp(u)$ and any morphism ν of U to $D(\nu)$

⋆ **all trees exist:** consequence of the previous point

⋆ **$\mathcal{P}$-accessibility:** let $D : U \longrightarrow \mathcal{P}$ be a non-empty small diagram and $f : \mathfrak{Y}(P) \longrightarrow$ *colim* D a morphism of $\mathcal{M}$ with P in Δ and *colim* D the colimit of D in $\mathcal{M}$. $(colim\ D)(P)$ is computed as the quotient:

$$\Big(\bigsqcup_{u \in U} D(u)(P) \sqcup \Delta(P, J)\Big)/\sim$$

where $\sim$ is the equivalence relation on $\bigsqcup_{u\in U} D(u)(P) \sqcup \Delta(P, J)$ generated by:
  - for every $\nu : u \longrightarrow u'$ of U, for every $x \in D(u)(P)$, $x \sim D(\nu)_P(x)$
  - for every $x \in \Delta(P, J)$ and every u in U, $x \sim \eta_u(x)$

Since U is non-empty, every x in $\Delta(P, J)$ is equivalent to some element of $\bigsqcup_{u\in U} D(u)(P)$. So, every element of $(colim\ D)(P)$ is the image of one of the projections of an element of some $D(u)(P)$. Let v be an object of U and $x \in D(v)(P)$ such that $f_P(id_P) \in (colim\ D)(P)$ is the image of x by the projection from $D(v)(P)$ to $(colim\ D)(P)$. By the Yoneda lemma, there exists a unique natural transformation $\theta : \mathfrak{Y}(P) \longrightarrow D(v)$ such that $\theta_P(id_P) = x$. θ belongs to $\mathcal{P}$ because $\mathcal{P}$ is a full embedding of $\mathcal{M}$. If $\pi_v : D(v) \longrightarrow colimD$ is the morphism from the universal cocone, then by the Yoneda lemma, $f = \pi_v \circ \theta$.

◀

5 Relationships with coreflections

Coreflections are a nice categorical way to express the fact that a computational model can be simulated by another one. This view was initiated in [13], where it was shown in particular that there is a coreflection from event structures to occurrence nets and so to 1-safe Petri nets. Note that the right adjoints of those coreflections give interesting constructions : in the case of occurrence nets in Petri nets, the right adjoint gives what is called the unfolding of a 1-safe Petri net. In this section, we prove that accessibility is preserved by coreflections.

In fact we can prove the even more general following theorem:

▶ **Theorem 3.** *Let $\mathcal{P}$ (resp. $\mathcal{P}'$) be a subcategory of $\mathcal{M}$ (resp. $\mathcal{M}'$). Assume that:*

- *$\mathcal{M}$ is $\mathcal{P}$-accessible*
- *there is a functor $F : \mathcal{M} \longrightarrow \mathcal{M}'$ such that:*
  - *F preserves trees i.e. for every small diagram $D : U \longrightarrow \mathcal{P}$, the colimit of $F \circ D$ in $\mathcal{M}'$ exists and is equal to $F(colim\ D)$*
  - *F induces an functor from $\mathcal{P}$ to $\mathcal{P}'$*
  - *there is a functor $G : \mathcal{P}' \longrightarrow \mathcal{P}$ and a natural isomorphism $\nu : F \circ G \longrightarrow id_{\mathcal{P}'}$*

Then $\mathcal{M}'$ is $\mathcal{P}'$-accessible.

The preservation of trees holds for example when F is a left adjoint. The other two conditions hold for example when F induces a equivalence between $\mathcal{P}$ and $\mathcal{P}'$. So, we deduce:

▶ **Corollary 4.** *If $F : \mathcal{M} \longrightarrow \mathcal{M}'$ is a coreflection, if $\mathcal{P}'$ is the image of $\mathcal{P}$ by F and if $\mathcal{M}$ is $\mathcal{P}$-accessible then $\mathcal{M}'$ is $\mathcal{P}'$-accessible.*

Proof of Theorem 3. Let $G : \mathcal{P}' \longrightarrow \mathcal{P}$ and $\nu : F \circ G \longrightarrow id_{\mathcal{P}'}$ a natural isomorphism.

⋆ **existence of trees:** let $D : U \longrightarrow \mathcal{P}'$ be a small diagram. By preservation of trees and existence of trees in $\mathcal{M}$, the colimit of $F \circ G \circ D$ in $\mathcal{M}'$ exists and is equal to $F(colim\ G \circ D)$. But ν induces a natural isomorphism between D and $F \circ G \circ D$. Then the colimit of D in $\mathcal{M}'$ exists.

⋆ **$\mathcal{P}'$-accessibility:** Let $z : P' \longrightarrow Z$ morphism of $\mathcal{M}'$ with $P' \in \mathcal{P}'$ and $(Z, (\eta_u)_{u \in U})$ is the colimit of a non-empty small diagram $D : U \longrightarrow \mathcal{P}'$.

By naturality of ν, the following diagram commutes:

$$\begin{CD}
F \circ G(P') @>{F \circ G(z)}>> F \circ G(Z)\\
@A{\nu_{P'}^{-1}}AA @VV{\nu_Z}V\\
P' @>>{z}> Z
\end{CD}$$

By $\mathcal{P}$-accessibility, $G(z) : G(P') \longrightarrow G(colim\ D) = colim\ (G \circ D)$ factorizes as $G(z) = \eta_u \circ p$ with $p : G(P') \longrightarrow G \circ D(u)$ morphism of $\mathcal{P}$ and $\eta_u : G \circ D(u) \longrightarrow colim(G \circ D)$ is from the universal cocone. Then the following diagram commutes:

$$\begin{array}{ccccc}
 & & F \circ G \circ D(u) & & \\
 & \overset{F(p)}{\nearrow} & & \overset{F(\eta_u)}{\searrow} & \\
F \circ G(P') & & \xrightarrow{F \circ G(z)} & & F \circ G(colim\ D)\\
\uparrow{\scriptstyle \nu_{P'}^{-1}} & & & & \downarrow{\scriptstyle \nu_{colim\ D}}\\
P' & & \xrightarrow[z]{} & & colim\ D
\end{array}$$

Then z factorizes as $\eta'_u \circ (\nu_{D(u)} \circ F(p) \circ \nu_{P'}^{-1})$ with $\eta'_u : D(u) \longrightarrow colim\ D$ coming from the universal cocone and $\nu_{D(u)} \circ F(p) \circ \nu_{P'}^{-1} : P' \longrightarrow D(u)$ morphism of $\mathcal{P}'$.

◀

6 Unfoldings in accessible models

6.1 The case of TS(Σ)

The unfolding of a transition system is an equivalent system without loops, obtained by "unfolding" the loops. More precisely, it is a tree which will be bisimilar to the transition system. Given a transition system $T = (Q, i, \Delta)$, the unfolding $\mathrm{Unfold}(T)$ of T is the synchronization tree (P, j, Γ) where:

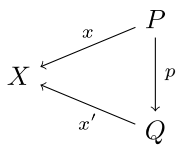
- $P = \{(q_0, a_1, q_1, \ldots, a_n, q_n) \mid q_i \in Q,\ a_i \in \Sigma,\ (q_i, a_{i+1}, q_{i+1}) \in \Delta \wedge q_0 = i\}$
- $j = (i)$
- $\Gamma = \{((q_0, a_1, q_1, \ldots, a_n, q_n), b, (q_0, a_1, q_1, \ldots, a_n, q_n, b, q)) \mid (q_n, b, q) \in \Delta\}$

It is easy to check that $\{(q_n, (q_0, a_1, q_1, \ldots, a_n, q_n)) \mid (q_0, a_1, q_1, \ldots, a_n, q_n) \in P\}$ is a bisimulation between T and $\mathrm{Unfold}(T)$.
Equivalently, the unfolding of T can be defined as a glueing of all branches of T, this is the way we will define more generally the unfolding in a categorical model.

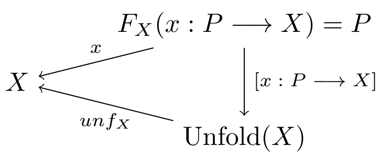
6.2 $\mathcal{P}$-unfolding and bisimilarity

Let $\mathcal{M}$ be a category where all $\mathcal{P}$-trees exist and X an object of $\mathcal{M}$. Let $\mathcal{P} \downarrow X$ be the small comma category whose:

- objects are morphisms $x : P \longrightarrow X$ of $\mathcal{M}$ with P in $\mathcal{P}$
- morphisms from $x : P \longrightarrow X$ to $x' : Q \longrightarrow X$ are morphisms $p : P \longrightarrow Q$ of $\mathcal{P}$ such that the following diagram commutes:



We then define the small diagram $F_X : \mathcal{P} \downarrow X \longrightarrow \mathcal{P}$ which maps every $x : P \longrightarrow X$ to P and every p to itself. Let $\mathrm{Unfold}(X)$ be the colimit of F_X in $\mathcal{M}$. We call it the **($\mathcal{P}$-) unfolding of** X. Since $(X, (x : P \longrightarrow X)_x)$ is a cocone of F_X, there is a unique morphism $unf_X : \mathrm{Unfold}(X) \longrightarrow X$ such that for every $x : P \longrightarrow X$ with $P \in \mathcal{P}$, the following diagram commutes:



where $[x : P \longrightarrow X]$ is the morphism coming from the colimit.

Using a similar argument as in Theorem 1, we have the following:

▶ **Theorem 5.** *When $\mathcal{M}$ is $\mathcal{P}$-accessible, unf_X is $\mathcal{P}$-open and so X and* $\mathrm{Unfold}(X)$ *are $\mathcal{P}$-bisimilar (strong path bisimilar).*

6.3 Unfolding is a right adjoint

The following lemma implies that the unfolding of a tree (and so of an unfolding) is isomorphic to the tree itself:

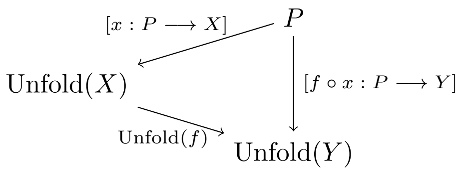
▶ **Lemma 6.**

(i) *When all trees exist in $\mathcal{M}$,* Unfold *extends to a functor* Unfold $: \mathcal{M} \longrightarrow Tree(\mathcal{M}, \mathcal{P})$*.*

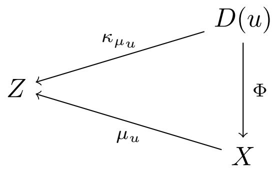
(ii) *When $\mathcal{M}$ is $\mathcal{P}$-accessible, $\mathcal{P}$ is dense in $Tree(\mathcal{M}, \mathcal{P})$ i.e. for all $X \in Tree(\mathcal{M}, \mathcal{P})$, $(X, (x)_{x:P \longrightarrow X})$ is a colimit of F_X.*

Proof.

(i) Let $f : X \longrightarrow Y$ be a morphism of $\mathcal{M}$. Then $(\text{Unfold}(Y), ([f \circ x : P \longrightarrow Y])_{x:P \longrightarrow X})$ is a cocone of F_X. So there is a unique morphism $\text{Unfold}(f) : \text{Unfold}(X) \longrightarrow \text{Unfold}(Y)$ such that for every path $x : P \longrightarrow X$ of X, the following diagram commutes:



(ii) Assume given another cocone $(Z, (\kappa_x : P \longrightarrow Z)_{x:P \longrightarrow X})$ of F_X. We construct a morphism $\Phi : X \longrightarrow Z$ like this: as X is in $Tree(\mathcal{M}, \mathcal{P})$, there is a small non-empty diagram $G : U \longrightarrow \mathcal{P}$ such that $(X, (\mu_u)_{u \in U})$ is a colimit of G for some μ_u. So, for all u, $\mu_u : D(u) \longrightarrow X$ is an object of $\mathcal{P} \downarrow X$. Since $(Z, (\kappa_{\mu_u} : D(u) \longrightarrow Z)_{u \in U})$ is a cocone of D, there is a unique morphism $\Phi : X \longrightarrow Z$ such that for all $u \in U$, the following diagram commutes:



Then, we can check that Φ is a morphism of cocones from $(X, (x)_{x:P \longrightarrow X})$ to $(Z, (\kappa_x : P \longrightarrow Z)_{x:P \longrightarrow X})$ and that it is the unique such morphism.

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From this sort of density property, we deduce that the unfolding is a right adjoint of the inclusion of trees in $\mathcal{M}$. This result is similar to the one from [13] stating that the unfolding is the right adjoint of the inclusion of occurrence nets in 1-safe Petri nets.

▶ **Theorem 7.** *When $\mathcal{M}$ is $\mathcal{P}$-accessible,* Unfold *is a right adjoint of inj : **Tree**$(\mathcal{M}, \mathcal{P}) \longrightarrow \mathcal{M}$, the embedding of **Tree**$(\mathcal{M}, \mathcal{P})$ in $\mathcal{M}$. In particular, the injection of **Tree**$(\mathcal{M}, \mathcal{P})$ in $\mathcal{M}$ is a coreflection.*

Proof.

⋆ **definition of the counit** $\epsilon : inj \circ \text{Unfold} \longrightarrow id_{\mathcal{M}}$**:** $\epsilon_X = unf_X$.

⋆ **definition of the unit** $\eta : id_{Tree(\mathcal{M},\mathcal{P})} \longrightarrow \text{Unfold} \circ inj$**:** by density of $\mathcal{P}$ in $Tree(\mathcal{M}, \mathcal{P})$, for all $X \in Tree(\mathcal{M}, \mathcal{P})$ there is a unique (iso)morphism $\eta_X : X \longrightarrow \text{Unfold}(X)$ such that for all $x : P \longrightarrow X$, $\eta_X \circ x = [x : P \longrightarrow X]$.

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7 Unfoldings and universal coverings

Unfoldings and coverings of spaces [9] are very similar in the sense that they both “unfold” loops (or “kill” the first homotopy group). But it seems that there were no general formal links in the literature between those two structures. We present here a view toward this.

7.1 Coverings of groupoids

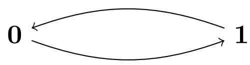
Coverings of groupoids are more natural than coverings of spaces as they are defined by lifting properties and their existence does not assume any hypothesis on the groupoid. They are very close to coverings of spaces since a covering of a space induces a covering of its fundamental groupoid and lots of properties of coverings of spaces can be expressed on the induced coverings of groupoids [9].

A **small pointed connected groupoid** (spc groupoids for short) is a pair $(\mathcal{C}, c)$ of a small connected groupoid $\mathcal{C}$ and an object c of $\mathcal{C}$. A **pointed functor** is a functor $F : (\mathcal{C}, c) \longrightarrow (\mathcal{D}, d)$ between spc groupoids such that $F(c) = d$. We note $\mathbf{Grpd}_\star$ the category of spc groupoids and pointed functors.

A **covering of a spc groupoids** $(\mathcal{C}, c)$ is a pointed functor $F : (\tilde{\mathcal{C}}, \tilde{c}) \longrightarrow (\mathcal{C}, c)$ such that for every morphism $f : c \longrightarrow c'$ of $\mathcal{C}$ there exists a unique object $\tilde{c'}$ of $\tilde{\mathcal{C}}$ and an unique morphism $\tilde{f} : \tilde{c} \longrightarrow \tilde{c'}$ such that $F(\tilde{f}) = f$. We say that a covering is **universal** if $\tilde{\mathcal{C}}(\tilde{c}, \tilde{c}) = \{id_{\tilde{c}}\}$.

Covering are similar to open maps since they satisfy a lifting property. In fact, they are open maps when we consider the following subcategory of paths. Let $\mathcal{I}$ be the full subcategory of $\mathbf{Grpd}_\star$ whose objects are the following to spc groupoids:

- $\mathbf{0}$, the spc groupoid with one object and only the identity as morphism
- $\mathbf{1}$, the spc groupoid with two objects:



pointed on 0.

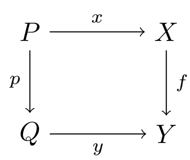
It is easy to check that $\mathbf{Grpd}_\star$ is $\mathcal{I}$-accessible.

Coverings are exactly the open maps whose lifts are unique. Universal coverings are universal in the category of coverings in the following sense [9]: given a universal covering $F : (\tilde{\mathcal{C}}, \tilde{c}) \longrightarrow (\mathcal{C}, c)$ and a covering $G : (\mathcal{D}, d) \longrightarrow (\mathcal{C}, c)$, then there is a unique pointed functor $H : (\tilde{\mathcal{C}}, \tilde{c}) \longrightarrow (\mathcal{D}, d)$ such that $G \circ H = F$. Moreover, H is a covering. This means that universal covering is initial in the category of coverings. In particular, universal coverings are unique up to isomorphism. Contrary to universal coverings of spaces, universal coverings of groupoids always exist [9].

7.2 Unfoldings and unique path lifting property

We have just seen that (universal) coverings are defined by unique lifting property. Now let us see the link between unfoldings and unique liftings.

We say that a morphism $f : X \longrightarrow Y$ is a **($\mathcal{P}$-) covering** if it has the **unique path lifting property**, i.e., if for all commutative diagram:



with $p : P \longrightarrow Q \in \mathcal{P}$, there exists a unique morphism $\theta : Q \longrightarrow X$ such that the following diagram commutes:

$$\begin{array}{ccc} P & \xrightarrow{\ x\ } & X \\ {\scriptstyle p}\big\downarrow & \overset{\theta}{\nearrow} & \big\downarrow{\scriptstyle f} \\ Q & \xrightarrow[\ y\ ]{} & Y \end{array}$$

▶ Remark. This is the same as $\mathcal{P}$-open but with the unicity of the lift.

The following result states that unfolding is a covering and that moreover it is initial among coverings.

▶ **Theorem 8.** *When $\mathcal{M}$ is $\mathcal{P}$-accessible:*

i) *unf_X has the unique path lifting property*

ii) *for every morphism $f : Y \longrightarrow X$ which has the unique lifting property, there is a unique morphism $\tilde{f} : \mathrm{Unfold}(X) \longrightarrow Y$ such that $f \circ \tilde{f} = unf_X$. Moreover, $\tilde{f}$ has the unique path lifting property.*

Proof.

i) This is a consequence of ii) because id_X has the unique path lifting property and $id_X \circ unf_X = unf_X$ and so $unf_X = \widetilde{id_X}$.

ii) ⋆ **construction of $\tilde{f}$:** For every $x : P \longrightarrow X$, by the unique path lifting property, there is a unique $\tilde{x} : P \longrightarrow Y$ such that

$$\begin{array}{ccc} I & \xrightarrow{\ \iota_Y\ } & Y \\ {\scriptstyle \iota_P}\big\downarrow & \overset{\tilde{x}}{\nearrow} & \big\downarrow{\scriptstyle f} \\ P & \xrightarrow[\ x\ ]{} & X \end{array}$$

i.e. a unique $\tilde{x}$ such that $f \circ \tilde{x} = x$. Since $(Y, (\tilde{x})_{x:P\longrightarrow X})$ is a cocone of F_X and since $(\mathrm{Unfold}(X), ([x])_x)$ is a colimit of F_X, there is a unique $\tilde{f} : \mathrm{Unfold}(X) \longrightarrow Y$ such that for every $x : P \longrightarrow X$, $\tilde{f} \circ \iota_x = \tilde{x}$ and so, $f \circ \tilde{f} \circ \iota_x = f \circ \tilde{x} = x = unf_X \circ \iota_x$ and by unicity of the definition of unf_X, $f \circ \tilde{f} = unf_X$.

⋆ **unicity of $\tilde{f}$:** consequence of the unique path lifting property of f.

⋆ **existence of the lift:** The lift of a diagram of the form:

$$\begin{array}{ccc} P & \xrightarrow{\ z\ } & \mathrm{Unfold}(X) \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle \tilde{f}} \\ Q & \xrightarrow[\ y\ ]{} & Y \end{array}$$

with $p \in \mathcal{P}$, is obtained as a lift of the following diagram:

$$\begin{array}{ccc} P & \xrightarrow{\ z\ } & \mathrm{Unfold}(X) \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle unf_X} \\ Q & \xrightarrow[\ f\circ y\ ]{} & X \end{array}$$

coming from the fact that unf_X is $\mathcal{P}$-open.

⋆ **unicity of the lift:** consequence of $\mathcal{P}$-accessibility.

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In the case of $\mathbf{Grpd}_\star$ and $\mathcal{I}$, this implies that the unfolding is a covering and is initial in the category of coverings. So we deduce:

▶ **Corollary 9.** *The universal covering of a spc groupoid coincides with its $\mathcal{I}$-unfolding.*

8 Conclusion

We have generalized Joyal, Nielsen and Winskel's approach of [6] to what we called accessible models. We have shown in particular that presheaf models and transitions systems are particular cases of accessible models. In these models, not only do we have a faithful formulation of bisimulation in the form of open maps, but also, we have a nice characterization of unfoldings, as form of generalized universal covering.

In the future, we would like to exploit this framework on a variety of models. As coreflections produce accessible categories from accessible categories, this is already the case for some interesting models. On top of this, we would like to study the case of 1-safe Petri nets in more detail and also, hybrid and stochastic hybrid models for which notions of bisimulations have been defined in the literature, see e.g. [7, 3].

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