Complete non-orders and fixed points

Akihisa Yamada[1], Jérémy Dubut[1,2]

[1] National Institute of Informatics, Tokyo, Japan
[2] Japanese-French Laboratory for Informatics, Tokyo, Japan

Supported by ERATO HASUO Metamathematics for Systems Design Project (No. JPMJER1603), JST

Introduction

- Interactive Theorem Proving is appreciated for reliability
- But it's also engineering tool for mathematics (esp. Isabelle/jEdit)
  - refactoring proofs and claims
  - sledgehammer
  - quickcheck/nitpick(/nunchaku)
- We develop an Isabelle library of **order theory** (as a case study)

  ⇒ we could generalize many known results, like:
  - completeness conditions: duality and relationships
  - Knaster-Tarski fixed-point theorem
  - Kleene's fixed-point theorem

Order

A binary relation ($\sqsubseteq$)

- **reflexive** $\Leftrightarrow x \sqsubseteq x$
- **transitive** $\Leftrightarrow x \sqsubseteq y$ and $y \sqsubseteq z$ implies $x \sqsubseteq z$
- **antisymmetric** $\Leftrightarrow x \sqsubseteq y$ and $y \sqsubseteq x$ implies $x = y$
- **partial order** $\Leftrightarrow$ reflexive + transitive + antisymmetric

Order

A binary relation (⊑)

```
locale less_eq_syntax = fixes less_eq :: 'a ⇒ 'a ⇒ bool (infix "⊑" 50)
```

- **reflexive** ⟺ $x \sqsubseteq x$

```
locale reflexive = ... assumes "x ⊑ x"
```

- **transitive** ⟺ $x \sqsubseteq y$ and $y \sqsubseteq z$ implies $x \sqsubseteq z$

```
locale transitive = ... assumes "x ⊑ y ⟹ y ⊑ z ⟹ x ⊑ z"
```

- **antisymmetric** ⟺ $x \sqsubseteq y$ and $y \sqsubseteq x$ implies $x = y$

```
locale antisymmetric = ... assumes "x ⊑ y ⟹ y ⊑ x ⟹ x = y"
```

- **partial order** ⟺ reflexive + transitive + antisymmetric

```
locale partial_order = reflexive + transitive + antisymmetric
```

Quasi-order

A binary relation (⊑)

locale less_eq_syntax = **fixes** less_eq :: 'a ⇒ 'a ⇒ bool (infix "⊑" 50)

- **reflexive** ⟺ $x \sqsubseteq x$

  locale reflexive = ... **assumes** "x ⊑ x"

- **transitive** ⟺ $x \sqsubseteq y$ and $y \sqsubseteq z$ implies $x \sqsubseteq z$

  locale transitive = ... **assumes** "x ⊑ y ⟹ y ⊑ z ⟹ x ⊑ z"

- ~~**antisymmetric** ⟺ $x \sqsubseteq y$ and $y \sqsubseteq x$ implies $x = y$~~

  ~~**locale** antisymmetric = ... **assumes** "x ⊑ y ⟹ y ⊑ x ⟹ x = y"~~

- **quasi-order** ⟺ reflexive + transitive

  locale quasi_order = reflexive + transitive

Pseudo-order [Skala 1971]

A binary relation (⊑)

locale less_eq_syntax = **fixes** less_eq :: 'a ⇒ 'a ⇒ bool (infix "⊑" 50)

- **reflexive** ⟺ $x \sqsubseteq x$

  locale reflexive = ... **assumes** "x ⊑ x"

- ~~**transitive** ⟺ $x \sqsubseteq y$ and $y \sqsubseteq z$ implies $x \sqsubseteq z$~~

  ~~**locale** transitive = ... **assumes** "x ⊑ y ⟹ y ⊑ z ⟹ x ⊑ z"~~

- **antisymmetric** ⟺ $x \sqsubseteq y$ and $y \sqsubseteq x$ implies $x = y$

  locale antisymmetric = ... **assumes** "x ⊑ y ⟹ y ⊑ x ⟹ x = y"

- **pseudo order** ⟺ reflexive + antisymmetric

  locale pseudo_order = reflexive + antisymmetric

Non-order

A binary relation (⊑)

    locale less_eq_syntax = fixes less_eq :: 'a ⇒ 'a ⇒ bool (infix "⊑" 50)

- ~~**reflexive** ⟺ $x \sqsubseteq x$~~

    ~~**locale** reflexive = … **assumes** "x ⊑ x"~~

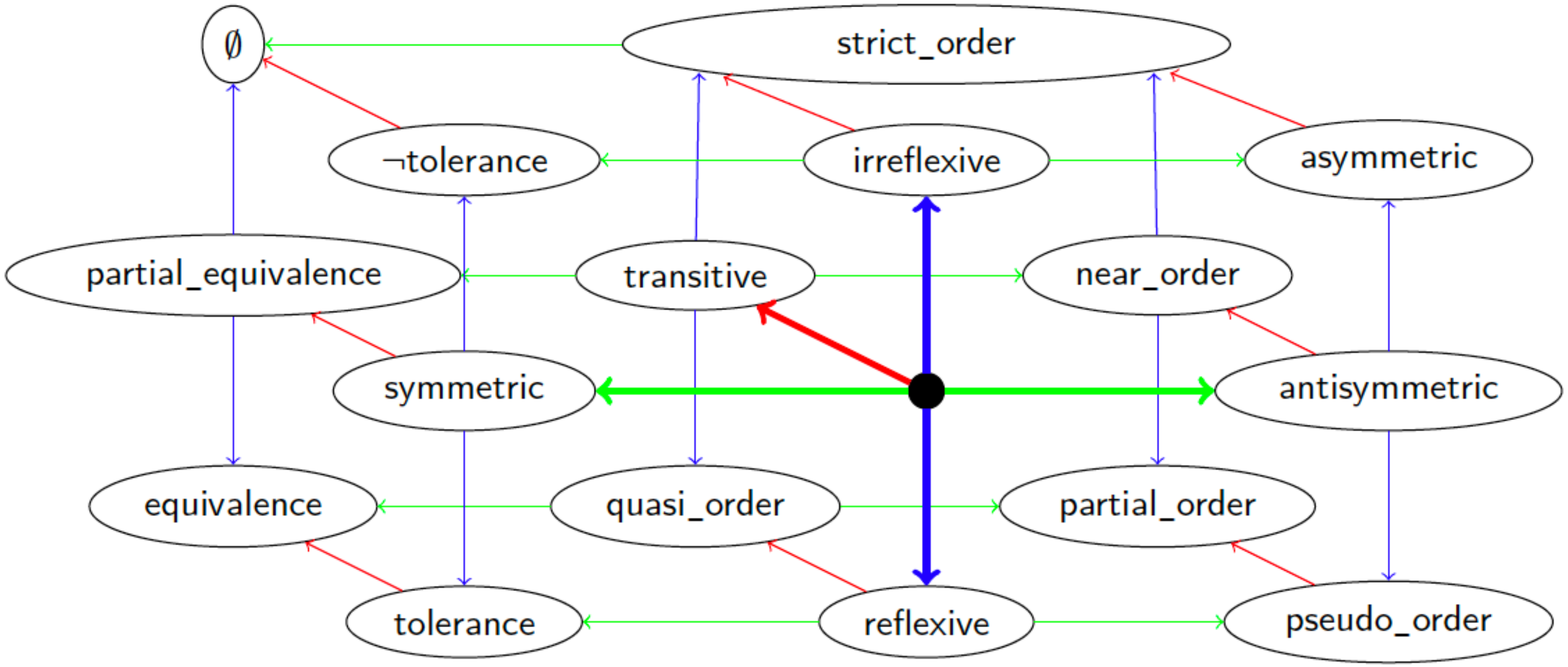
- ~~**transitive** ⟺ $x \sqsubseteq y$ and $y \sqsubseteq z$ implies $x \sqsubseteq z$~~

    ~~**locale** transitive = … **assumes** "x ⊑ y ⟹ y ⊑ z ⟹ x ⊑ z"~~

- ~~**antisymmetric** ⟺ $x \sqsubseteq y$ and $y \sqsubseteq x$ implies $x = y$~~

    ~~**locale** antisymmetric = … **assumes** "x ⊑ y ⟹ y ⊑ x ⟹ x = y"~~

Locale combinations



Complete non-orders

- upper/lower **bounds**:

  definition "bound (⊑) X b ≡ ∀x ∈ X. x ⊑ b"

- **greatest/least** elements:

  definition "extreme (⊑) X e ≡ e ∈ X ∧ (∀x ∈ X. x ⊑ e)"

- **suprema/infima** (l.u.b./g.l.b.):

  abbreviation "extreme_bound (⊑) X s ≡ extreme (⊒) {b. bound (⊑) X b} s"

- **complete** ⟺ any set X of elements has a supremum

  locale complete = **assumes** "∃s. extreme_bound (⊑) X s"

Proposition: The dual of complete **non-**order is complete

sublocale complete ⊆ dual: complete "(⊒)"

Knaster–Tarski fixed points

Knaster–Tarski: Part 1

- **Theorem** (Knaster–Tarski, part 1)

  Any monotone map f on a complete order $\sqsubseteq$ has a fixed point
  (monotone: $x \sqsubseteq y \Longrightarrow f(x) \sqsubseteq f(y)$)
  (fixed point: $f(x) = x$)

- **Theorem** [Stauti & Maaden 2013]

  Any monotone map f on a complete **pseudo**-order $\sqsubseteq$ has a fixed point
  (relaxed transitivity)

Theorem [this work]

Any monotone map f on a complete **non-**order $\sqsubseteq$ has a **quasi-**fixed point
(relaxed transitivity, reflexivity, antisymmetry)

(quasi-fixed point: $f(x) \sim x$ i.e., $f(x) \sqsubseteq x$ and $x \sqsubseteq f(x)$)

Proof sketch (by Stauti & Maaden)

```
definition AA where "AA ≡
  {A. f ` A ⊆ A ∧ (∀B ⊆ A. ⊔ B ∈ A)}"
lemma "∃c ∈ ⋂ AA. f c = c"
proof
  define c where "c ≡ ⊔ (⋂ AA)"
  show "c ∈ ⋂ AA"...
  show "f c = c"
  proof (rule antisym)
    show "f c ⊑ c"...
    show "c ⊑ f c"...
  qed
qed
```

Proof sketch (minus reflexivity)

```
definition AA where "AA ≡
  {A. f ` A ⊆ A ∧ (∀B ⊆ A. ⨆ B ∈ A)}"
lemma "∃c ∈ ⋂ AA. f c = c"
proof
  define c where "c ≡ ⨆ (⋂ AA)"
  show "c ∈ ⋂ AA"...
  show "f c = c"
  proof (rule antisym)
    show "f c ⊑ c"...
    show "c ⊑ f c"...
  qed
qed
```

works!

Proof sketch (**minus antisymmetry**)

supremum is not unique

```
definition AA where "AA ≡
  {A. f ` A ⊆ A ∧ (∀B ⊆ A. ∀s. extreme_bound (⊑) B s ⟶ s ∈ A)}"
lemma "∃c ∈ ⋂ AA. f c ∼ c"
proof-
  obtain c where "extreme_bound (⊑) (⋂ AA) c"...
  show "c ∈ ⋂ AA"...
  show "f c ∼ c"
  proof ~~(rule antisym)~~
    show "f c ⊑ c"...
    show "c ⊑ f c"...
  qed
qed
```

f c ⊑ c and c ⊑ f c doesn't mean f c = c

Knaster–Tarski, Part 1: Existence

- **Main result 1**

```
theorem (in complete)
  assumes "monotone (⊑) (⊑) f" shows "∃x. f x ∼ x"
```

Knaster–Tarski, Part 2: Completeness

- **Theorem** (Knaster–Tarski, Part 2)

  For any monotone map on a complete partial order,
  the set of fixed points is complete

- **Theorem** [Stauti & Maaden 2013]

  Any monotone map on a complete **pseudo order** has a **least** fixed point

- **Conjecture?**

  Any monotone map on a complete **non-order** has a **least** **quasi-**fixed point?

Least quasi-fixed points?

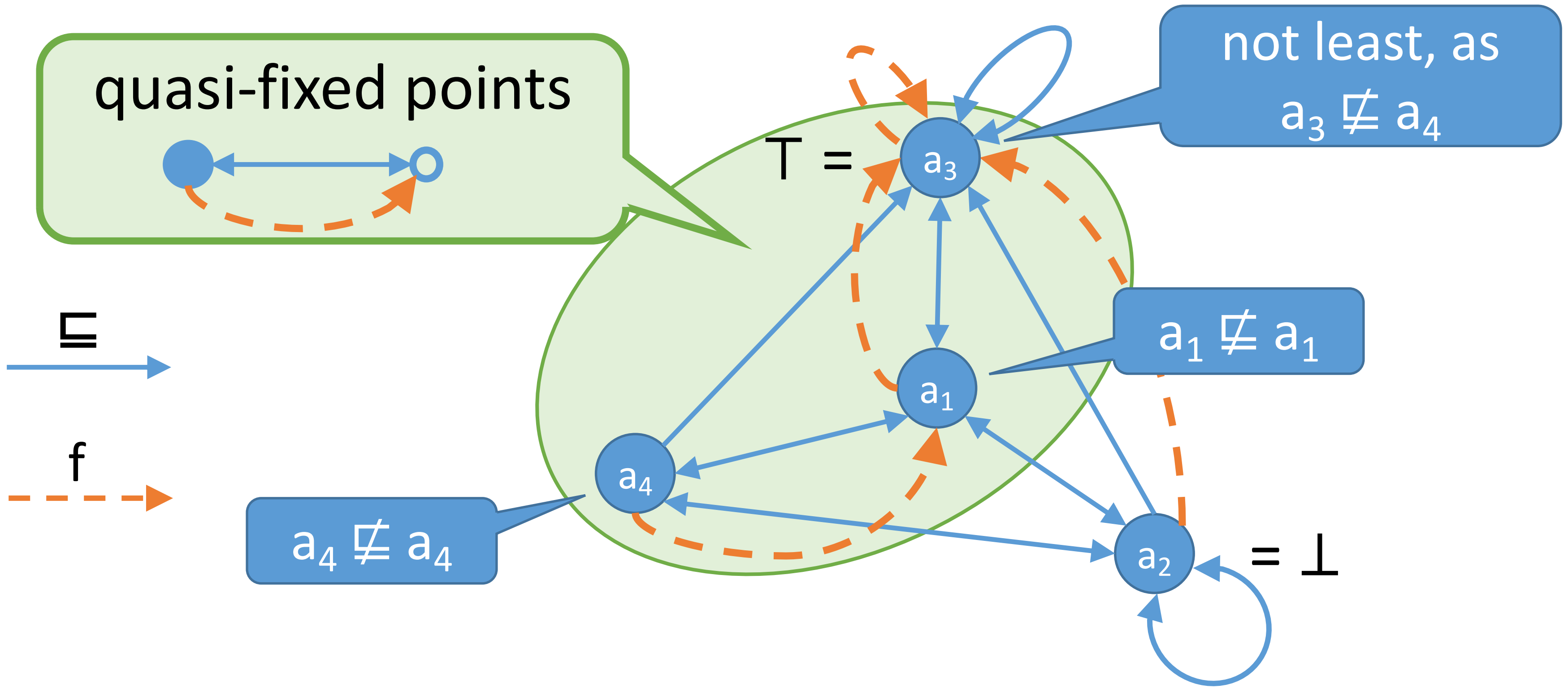
- **Counterexample** [Nitpick]

nontheorem (in complete)
  assumes "monotone (⊑) (⊑) f" shows "∃p. extreme (⊒) {s. f s ∼ s} p"
nitpick

$f = (\lambda x.\ \_)\ (a_1 := a_3\ ,\ a_2 := a_3\ ,\ a_3 := a_3\ ,\ a_4 := a_1\ )$
$(\sqsubseteq) = (\lambda x.\ \_)$
$(\ a_1 := (\lambda x.\ \_)\ (a_1 := \text{False},\ a_2 := \text{True},\ a_3 := \text{True},\ a_4 := \text{True}),$
$a_2 := (\lambda x.\ \_)\ (a_1 := \text{True},\ a_2 := \text{True},\ a_3 := \text{True},\ a_4 := \text{True}),$
$a_3 := (\lambda x.\ \_)\ (a_1 := \text{True},\ a_2 := \text{False},\ a_3 := \text{True},\ a_4 := \text{False}),$
$a_4 := (\lambda x.\ \_)\ (a_1 := \text{True},\ a_2 := \text{True},\ a_3 := \text{True},\ a_4 := \text{False}))$

least quasi-fixed points?

- **Counterexample** [Nitpick]



Argument by Stauti & Maaden

definition AA **where** "AA ≡ {A. f ` A ⊆ A ∧ (∀B ⊆ A. ⨆ B ∈ A)}"

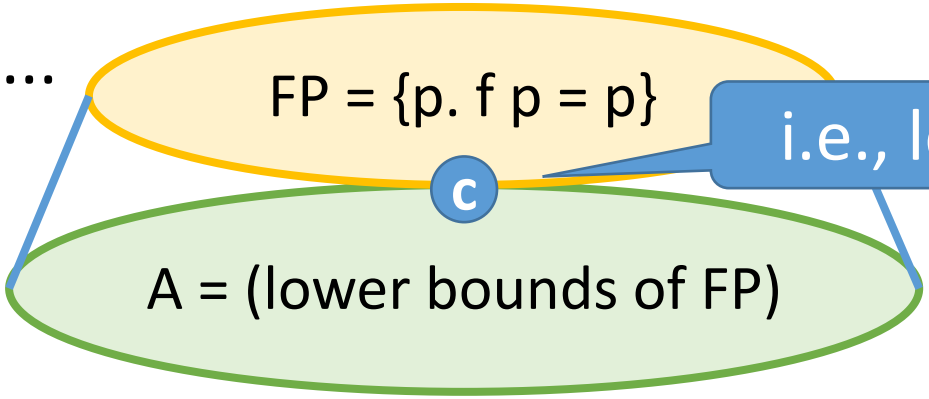
lemma "∃c ∈ ⋂ AA. f c = c" ...

from previous proof

definition A where "A ≡ {a. bound (⊒) {p. f p = p} a}"

lemma "A ∈ AA"
proof
  show "f ` A ⊆ A"...
  show "∀B ⊆ A. ⨆ B ∈ A"...
qed

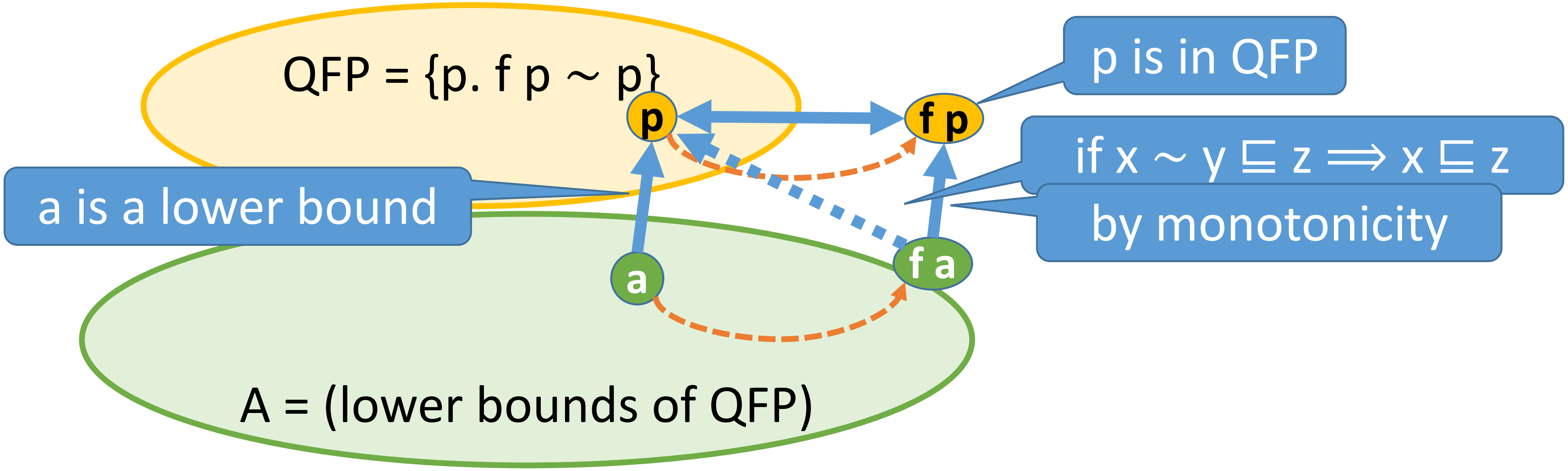
by dropping antisymmetry, proof breaks here!



i.e., least fixed point

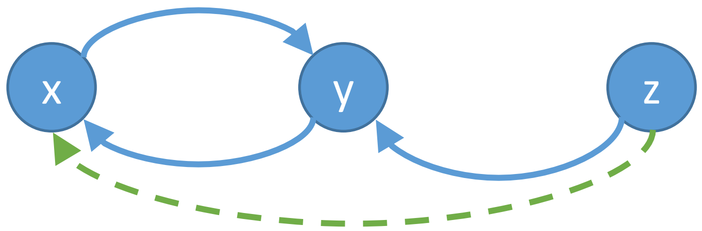
∈ AA !
So c ∈ A (∩ FP)

Proof of "f ` A ⊆ A"



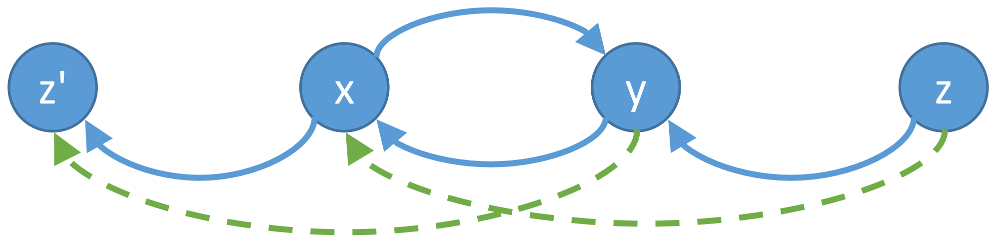
QFP = {p. f p ~ p}
p
f p
p is in QFP
if x ~ y ⊑ z ⟹ x ⊑ z
by monotonicity
a is a lower bound
a
f a
A = (lower bounds of QFP)

Attractivity



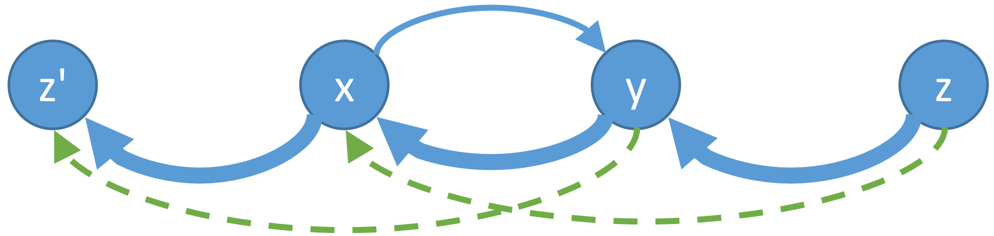
locale semiattractive =
  assumes "x ⊑ y ⟹ y ⊑ x ⟹ y ⊑ z ⟹ x ⊑ z"

Attractivity

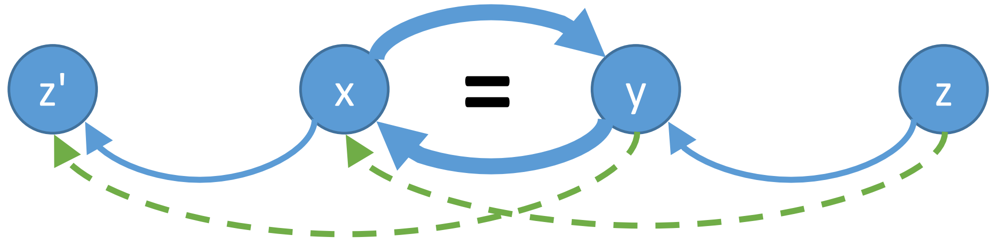


locale attractive =
semiattractive + dual: semiattractive "(⊒)"

sublocale transitive ⊆ attractive



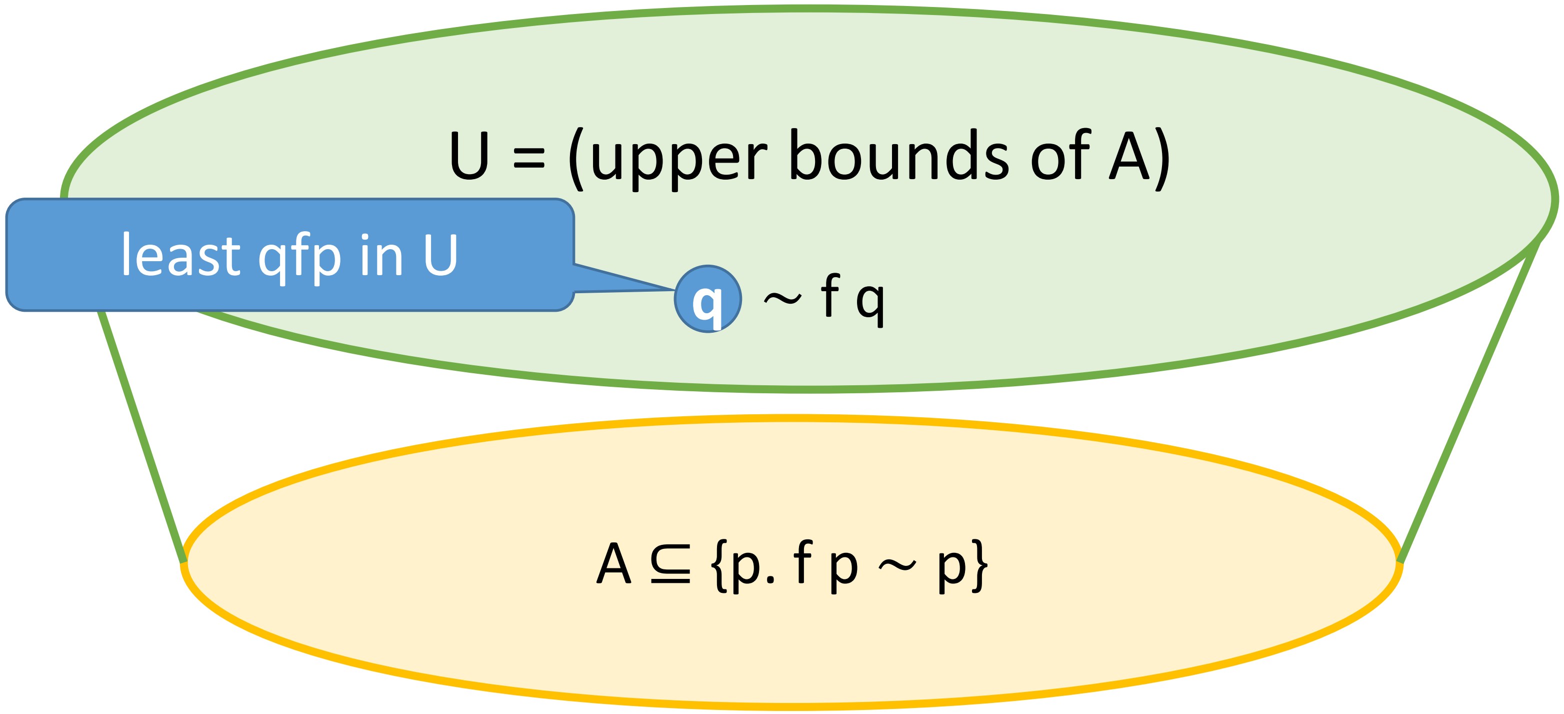
sublocale antisymmetric ⊆ attractive



Knaster-Tarski, part 2: Completeness

- **Main result 2**:

  theorem (**in** complete_attractive)
  assumes "monotone (⊑) (⊑) f" **shows** "complete_in (⊑) {p. f p ∼ p}"



Knaster-Tarski, part 2: Completeness

- **Main result 2**:

```
theorem (in complete_attractive)
  assumes "monotone (⊑) (⊑) f" shows "complete_in (⊑) {p. f p ∼ p}"
```

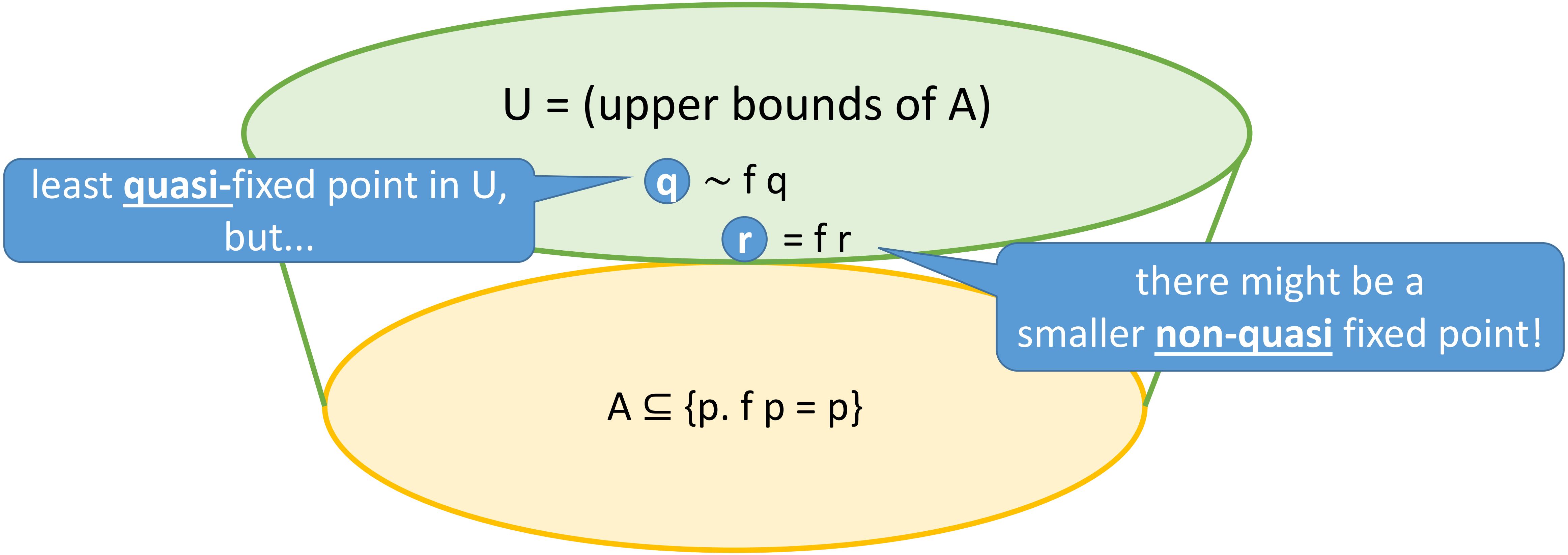
- In pseudo order, $x \sim y \Leftrightarrow x = y$. So

```
corollary (in complete_pseudo_order)
  assumes "monotone (⊑) (⊑) f" shows "complete_in (⊑) {p. f p = p}"
```

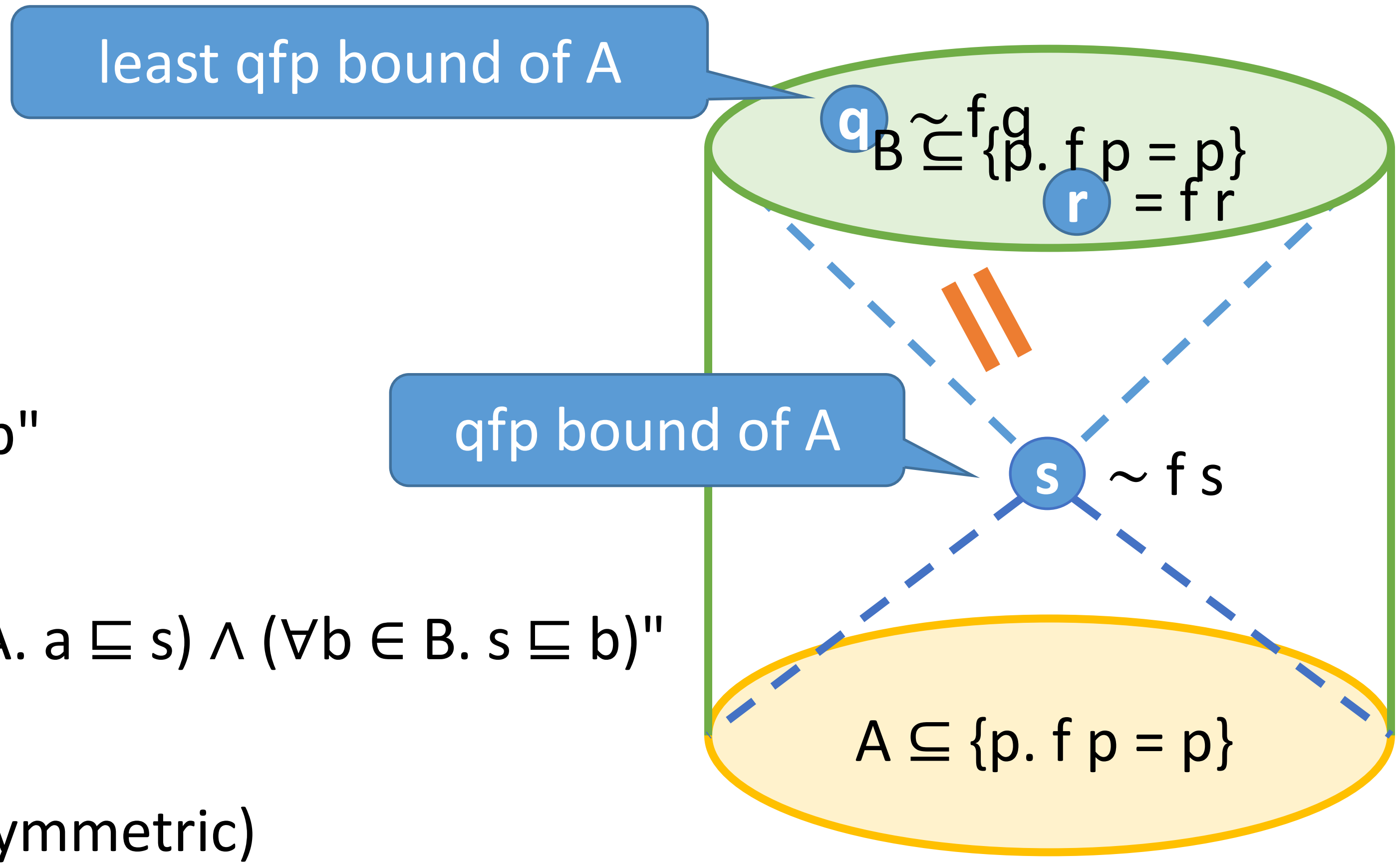
Completes Stauti & Maaden's work!
... but is reflexivity necessary?

Completeness only with antisymmetry

- **conjecture** (**in** complete_antisymmetric)
  assumes "monotone (⊑) (⊑) f" **shows** "complete_in (⊑) {p. f p = p}"



Completeness only with antisymmetry



- **a key lemma**

  lemma qfp_interpolant:
  assumes "complete (⊑)"
  and "monotone (⊑) (⊑) f"
  and "∀a ∈ A. ∀b ∈ B. a ⊑ b"
  and "∀a ∈ A. f a = a"
  and "∀b ∈ B. f b = b"
  shows "∃s. f s ~ s ∧ (∀a ∈ A. a ⊑ s) ∧ (∀b ∈ B. s ⊑ b)"

- **Main result 3**

  theorem (**in** complete_antisymmetric)
  assumes "monotone (⊑) (⊑) f" **shows** "complete_in (⊑) {p. f p = p}"

Kleene fixed points

Kleene fixed points, part 1: Construction

- **Theorem** (Kleene, part 1)

  Let f be a Scott-continuous map on a directed-complete order.
  Then $\bigsqcup_n f^n(\bot)$ exists and is a fixed point.

- **Theorem** [Mashburn 1983]

  Let f be an **ω**-continuous map on a **ω**-complete order.
  Then $\bigsqcup_n f^n(\bot)$ exists and is a fixed point.

Theorem [this work]

Let f be an **ω**-continuous map on a **ω**-complete **non-order**.

Let $\bot$ be **a** least element.

Then $\{f^n(\bot) \mid n \in \mathbb{N}\}$ has suprema, and they are all **quasi**-fixed point.

ω-completeness

- **ω-chain**: a sequence $c_0, c_1, \dots$ s.t. $i \leq j$ implies $c_i \sqsubseteq c_j$

  definition "omega_chain C ≡ ∃c :: nat ⇒ 'a. monotone (≤) (⊑) c ∧ C = range c"

- **ω-complete**: any ω-chain has a supremum

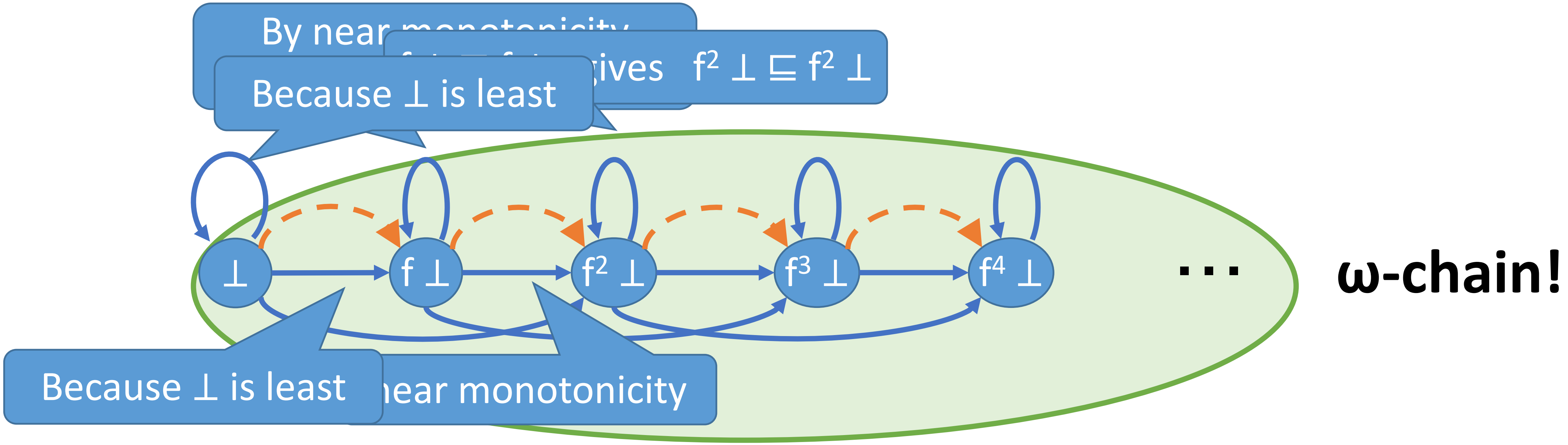
  locale omega_complete =
    assumes "omega_chain C ⟹ ∃s. extreme_bound (⊑) C s"

- **ω-continuity**: f preserves all suprema of ω-chains
  - **definition** "omega_continuous f ≡
    ∀C. omega_chain C ⟶
        ∀s. extreme_bound (⊑) C s ⟶ extreme_bound (⊑) (f ` C) (f s)"

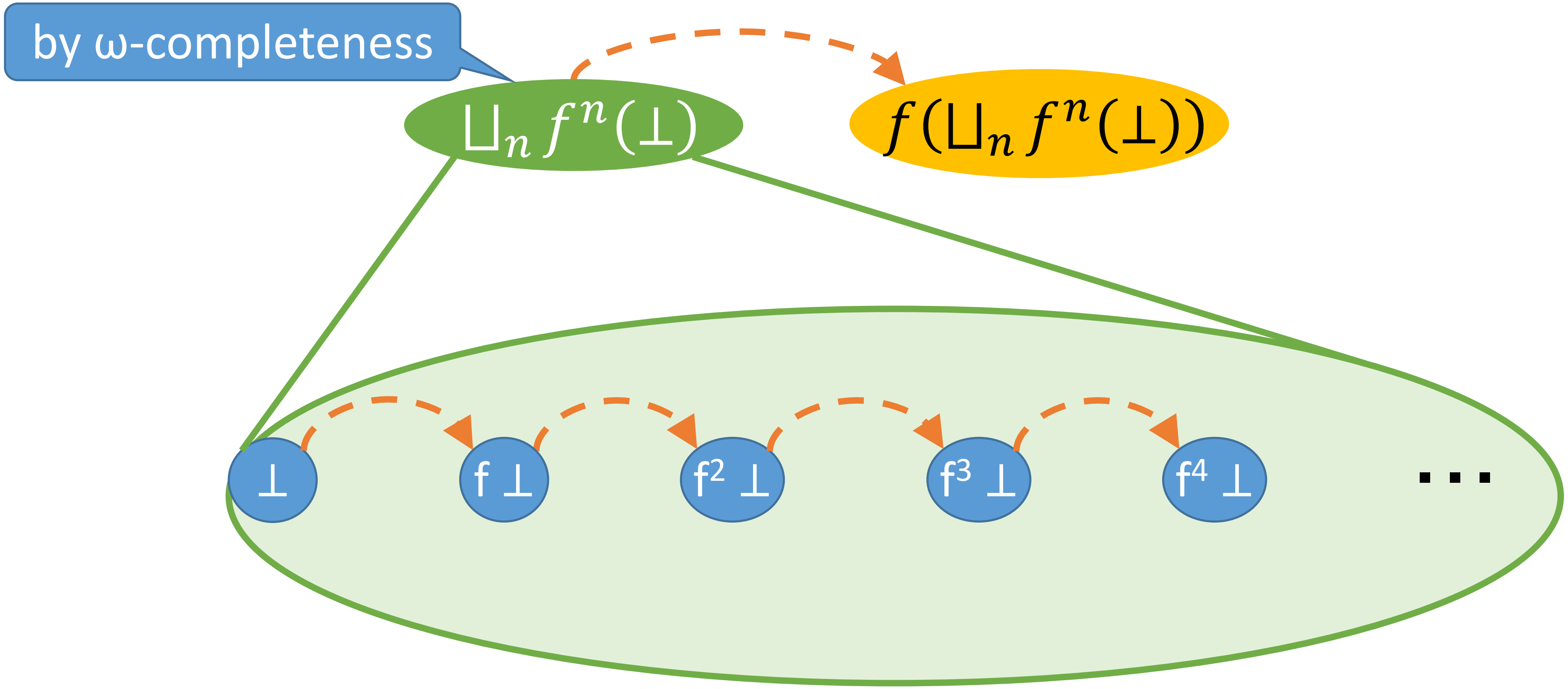
ω-continuity implies "near" monotonicity

- **lemma**
  assumes "omega_continuous f" **and** "x ⊑ y" **and** "x ⊑ x" **and** "y ⊑ y"
  shows "f x ⊑ f y"
  proof-
  have "omega_chain {x, y}"...
  have "extreme_bound {x, y} y"...
  have "extreme_bound (f ` {x, y}) (f y) **using** omega_continuity...
  then show "f x ⊑ f y" **by** auto
  qed

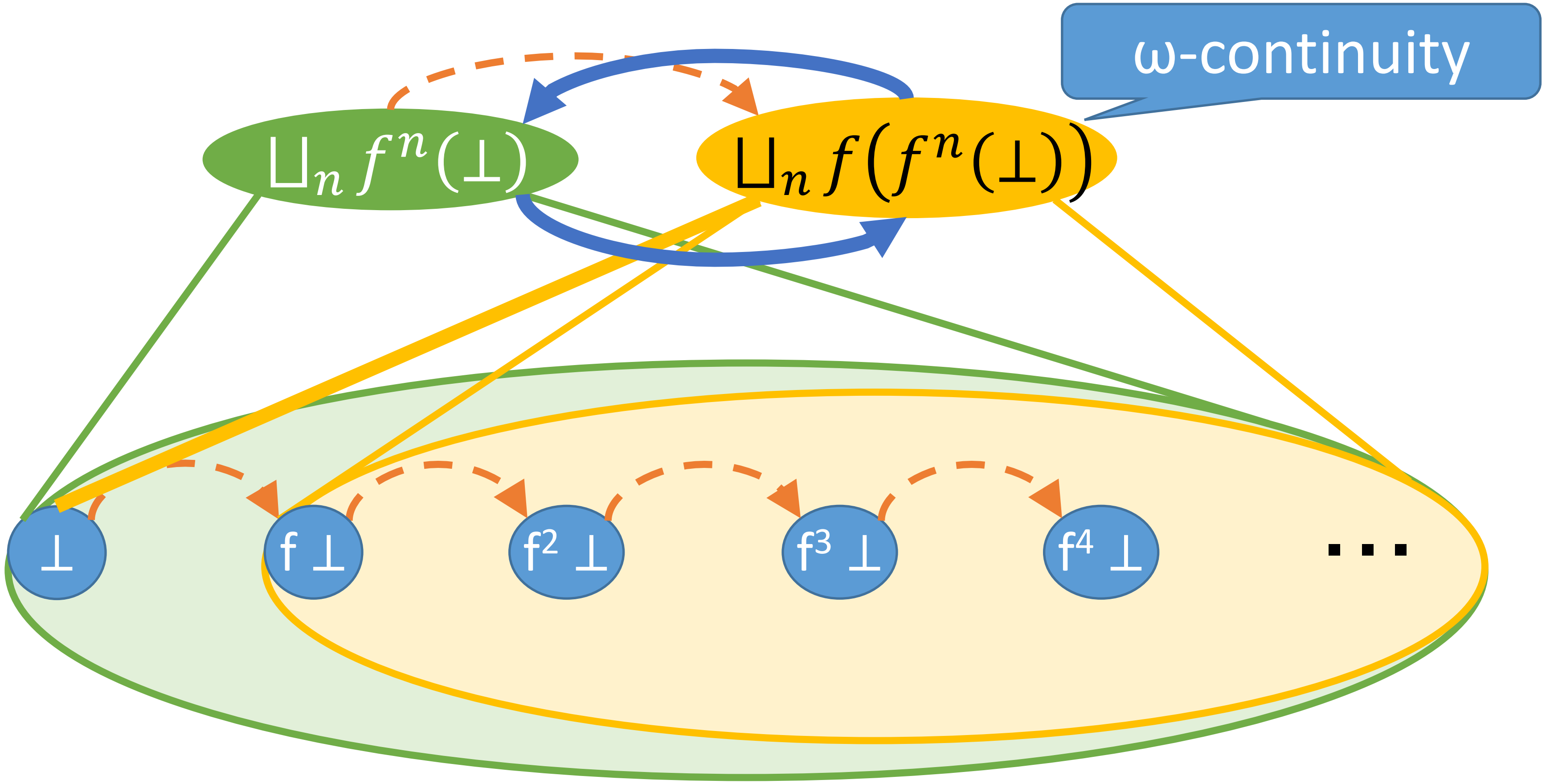
$\{f^n(\bot) \mid n \in \mathbb{N}\}$ is an ω-chain



$\bigsqcup_n f^n(\bot)$ is quasi-fixed; as usual



$\sqcup_n f^n(\bot)$ is quasi-fixed; as usual



Kleene fixed point, part 1: Construction

- Main result 4:

theorem
shows "∃p. extreme_bound (⊑) $\{f^n(\bot) \mid n \in \mathbb{N}\}$ p"
and "extreme_bound (⊑) $\{f^n(\bot) \mid n \in \mathbb{N}\}$ p ⟹ f p ~ p"

there is a supremum for $\{f^n(\bot) \mid n \in \mathbb{N}\}$

and any such is a quasi-fixed point

Kleene fixed point, part 2: Leastness

- **Theorem** (Kleene, part 2)

  Let f be a Scott-continuous map on a directed-complete order.
  Then $\bigsqcup_n f^n(\bot)$ is the **least** fixed point

- **Theorem** [Mashburn 1983]

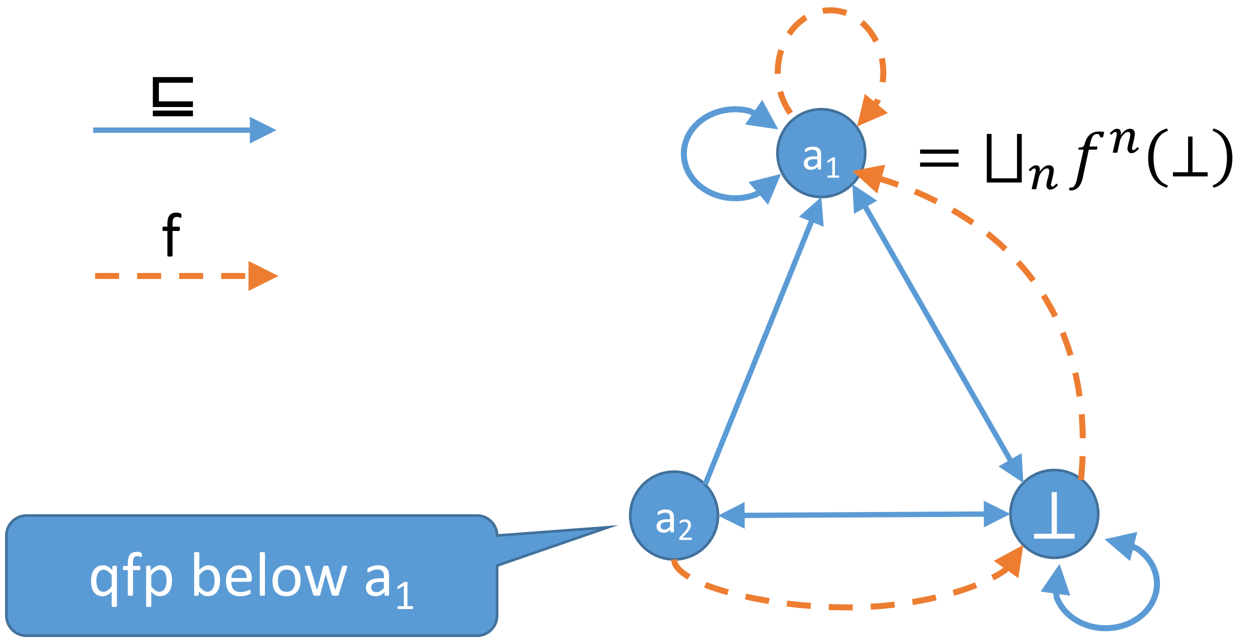
  Let f be an **ω**-continuous map on a **ω**-complete order.
  Then $\bigsqcup_n f^n(\bot)$ is the **least** fixed point.

- **Conjecture**

  Let f be an **ω**-continuous map on a **ω**-complete **non-order**.
  Are suprema of $\{f^n(\bot) \mid n \in \mathbb{N}\}$ **least** **quasi**-fixed points?

Is $\bigsqcup_n f^n(\bot)$ least?

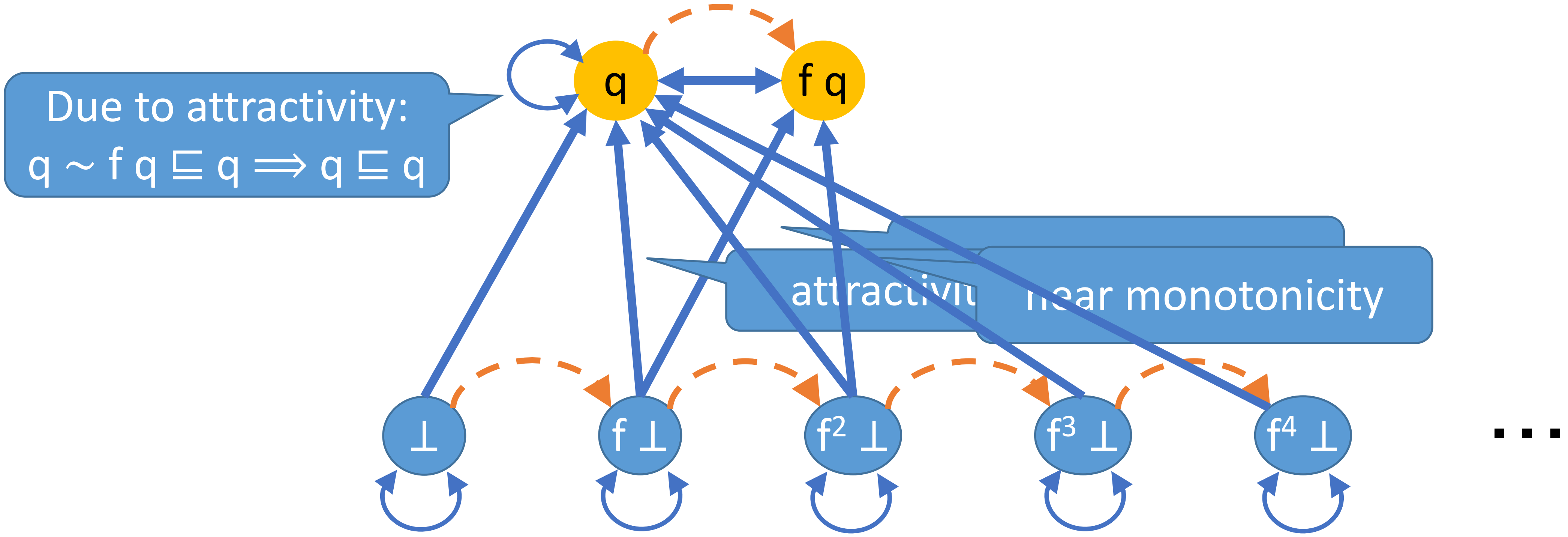
- **Counterexample** [Nitpick]



Attractivity is the key, again!

$\bigsqcup_n f^n(\bot)$ is least under attractivity

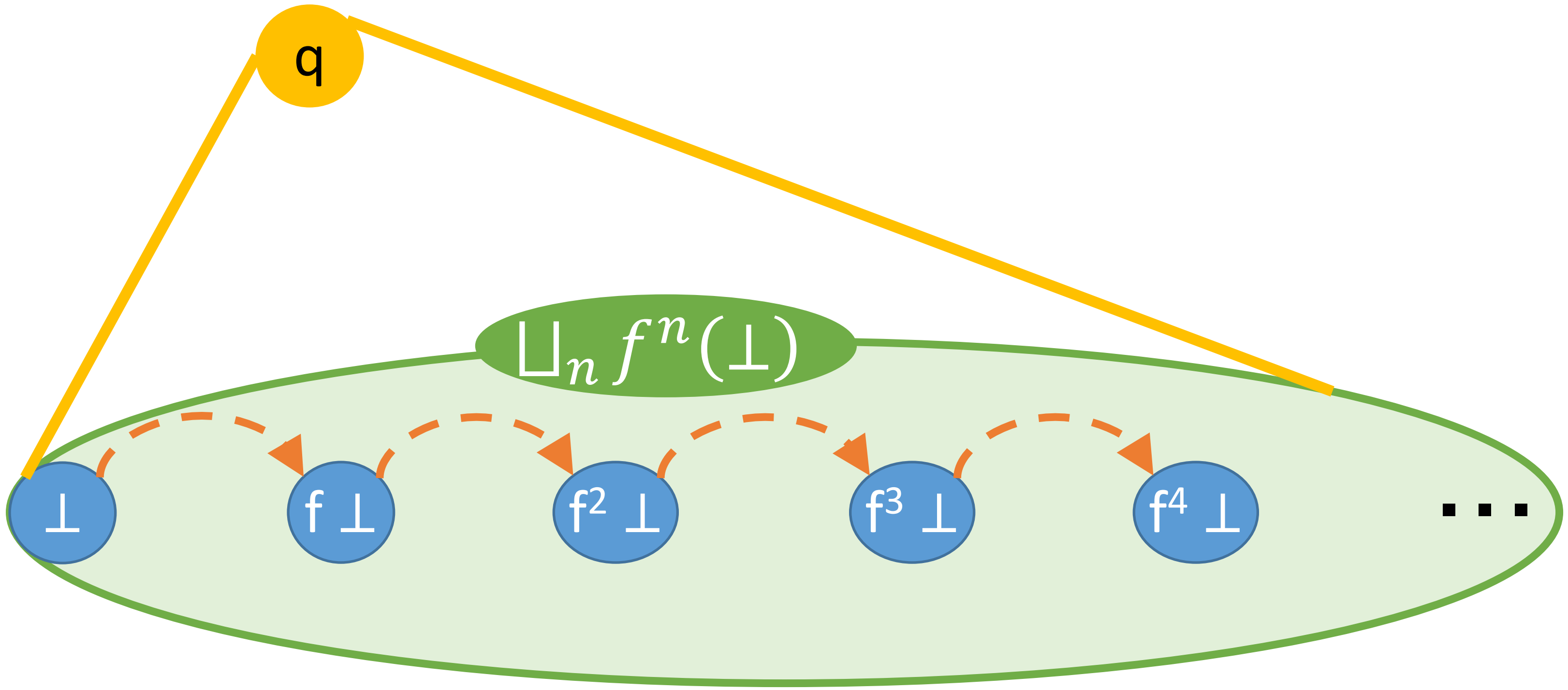
fix q **assume** "q ∼ f q" **have** "f ^{n} ⊥ ⊑ q"



$\bigsqcup_n f^n(\bot)$ is least under attractivity

fix q **assume** "q ~ f q" **have** "f ⁿ ⊥ ⊑ q" **by** ...

then show "$\bigsqcup_n f^n(\bot) \sqsubseteq$ q"...



Kleene fixed point, part 2

Main result 5 (last):

corollary (**in** attractive)
"extreme_bound (⊑) $\{f^n(\bot) \mid n \in \mathbb{N}\}$ s ⟷ extreme (⊒) {q. f q ∼ q} s"

suprema of $\{f^n(\bot) \mid n \in \mathbb{N}\}$ are **the** least quasi-fixed points

Conclusion

- An Isabelle/HOL library for non-orders
- Generalized some (folklore) results on completeness
- Generalized Knaster—Tarski fixed-point theorem
  - monotone map on complete non-order has a quasi-fixed point
  - if attractive, the set of quasi-fixed points is complete
- Generalized Kleene fixed-point theorem
  - for an ω-continuous map on ω-complete non-order, suprema of $\{f^n \perp \mid n \in \mathbb{N}\}$ is a quasi-fixed point
  - if attractive, they are the least quasi-fixed points