Trees in partial Higher Dimensional Automata

Jérémy Dubut

National Institute of Informatics
Japanese-French Laboratory for Informatics

FoSSaCS'19, Prague, April 8th

Fixing partial Higher Dimensional Automata

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Fixing partial Higher Dimensional Automata

Category Theory wins

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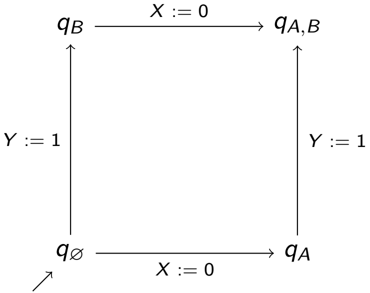
Concurrency vs. true concurrency

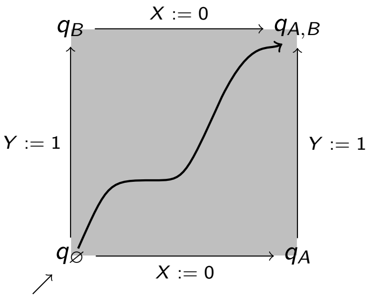
Independent actions



Concurrency

Interleaving behaviors: A then B or B then A

Independent actions



True concurrency

Continuous behaviors: any scheduling of A and B
Refinement **[van Glabbeek, Goltz]**: in reality $X := 0$ and $Y := 1$ are not atomic

Homotopy Theory and Concurrency

Directed Algebraic Topology:

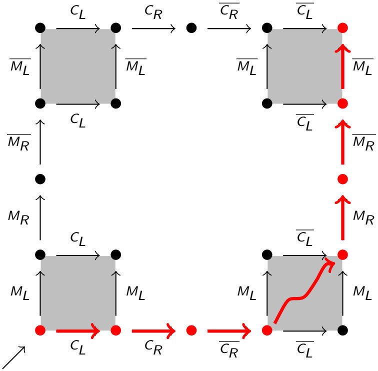
True concurrency has the flavor of
a directed homotopy theory

Original goal of this paper:

Concurrency has the flavor of a homotopy theory

Higher Dimensional Automata

Truly concurrent systems



HDA **[Pratt]** = transition system with higher dimensional data that accounts for true concurrency

Transition systems

Graph

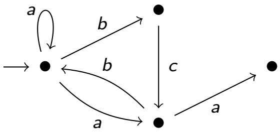
A **graph** is:

- a set V of vertices,
- a set E of edges,
- two functions $s, t : E \longrightarrow V$, the source and the target.

Transition systems

A **transition system** on an alphabet Σ is:

- a graph (V, E, s, t),
- an initial state $i_0 \in V$,
- a labelling function $\lambda : E \longrightarrow \Sigma$.



Extending graphs

Precubical sets

A **precubical set** is:

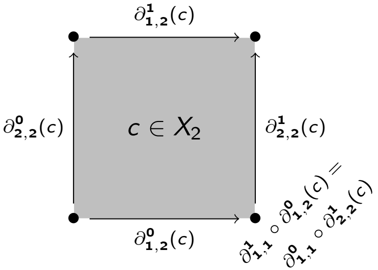
- a collection of sets $(X_n)_{n\geq 0}$,
- a collection of function $(\partial^\alpha_{i,n} : X_n \longrightarrow X_{n-1})_{n>0,\, 1\leq i\leq n,\, \alpha\in\{0,1\}}$.

satisfying for $i > j$:

$$\partial^\beta_{j,n} \circ \partial^\alpha_{i,n+1} = \partial^\alpha_{i-1,n} \circ \partial^\beta_{j,n+1}$$

Graph:

- $X_0 = V$, $X_1 = E$ and $X_{n>1} = \varnothing$,
- $s = \partial^0_{1,1}$ and $t = \partial^1_{1,1}$,
- equations are trivial.



Extending systems

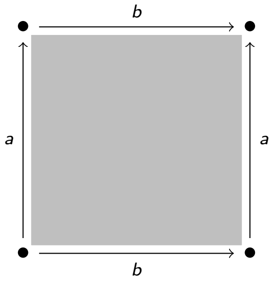
Higher Dimensional Automata **[Pratt]**

An **HDA** on the alphabet on Σ is:

- a precubical set (X, ∂),
- an initial state $i_0 \in X_0$,
- a labelling function $\lambda : X_1 \longrightarrow \Sigma$.

satisfying for every $c \in X_2$:

$$\lambda(\partial_i^1(c)) = \lambda(\partial_i^0(c))$$



Precubical sets, categorically

The cube category

Define $\square$ as the subcategory of **Set** whose:

- objects are $\{0,1\}^n$, for $n \in \mathbb{N}$,
- morphisms are generated by the so-called **co-face maps**:

$$d^{\alpha}_{i,n} : (\beta_1, \ldots, \beta_n) \longmapsto (\beta_1, \ldots, \beta_{i-1}, \alpha, \beta_i, \ldots, \beta_n)$$

[van Glabbeek]

Precubical sets are precisely presheaves on the cube category.

Morphisms of precubical sets

Morphisms

A morphism of precubical sets from (X, ∂) to (Y, δ) is a collection

$$f_n : X_n \longrightarrow Y_n$$

of functions such that:

$$f_{n-1} \circ \partial_{i,n}^{\alpha} = \delta_{i,n}^{\alpha} \circ f_n$$

Morphism are precisely natural transformations between presheaves on the cube category.

Initial state and labelling are morphisms

The one point precubical set

Define $*$ to be the precubical set such that $*_0 = \{\bullet\}$ and $*_{n>0} = \varnothing$.

Specifying an initial state in a precubical set X is the same as specifying a morphism from $*$ to X.

The alphabet precubical set

Given an alphabet Σ, define the precubical set, also noted Σ such that:

- $\Sigma_n = \Sigma^n$,
- $\partial_i^\alpha : \Sigma^n \longrightarrow \Sigma^{n-1}$ which maps $a_1 \dots a_n$ to $a_1 \dots a_{i-1}.a_{i+1} \dots a_n$.

Specifying a labelling function on a precubical set X is the same as specifying a morphism from X to Σ.

The category of HDA

Category of HDA

The category $\mathbf{HDA}_\Sigma$ of HDA has as morphisms from $(X, \partial, i_0, \lambda)$ to (Y, δ, j_0, η) the morphisms of precubical sets f from (X, ∂) to (Y, δ) such that:

- $f_0(i_0) = j_0$
- $\lambda = \eta \circ f_1$

The category of HDA is isomorphic to the double slice category:

$$*/[\square^{op}, \mathbf{Set}]/\Sigma$$

Paths and homotopies in Higher Dimensional Automata

Runs in transition systems

Run

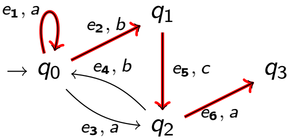
A **run** in a transition system is sequence written as:

$$q_0 \xrightarrow{e_1} q_1 \xrightarrow{e_2} \dots \xrightarrow{e_n} q_n$$

with:

- $q_i \in V$ and $e_i \in E$
- $q_0 = i_0$
- for every i, $s(e_i) = q_{i-1}$ and $t(e_i) = q_i$

$$q_0 \xrightarrow{e_1} q_0 \xrightarrow{e_2} q_1 \xrightarrow{e_5} q_2 \xrightarrow{e_6} q_3$$



Idea: see it as $q_0 \xrightarrow{s} e_1 \xrightarrow{t} q_0 \xrightarrow{s} e_2 \xrightarrow{t} q_1 \xrightarrow{s} e_5 \xrightarrow{t} q_2 \xrightarrow{s} e_6 \xrightarrow{t} q_3$

Runs in HDA

Path [van Glabbeek]

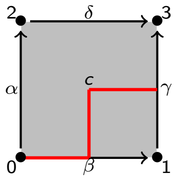
A **path** in a HDA is sequence written as:

$$q_0 \xrightarrow{j_1,\alpha_1} q_1 \xrightarrow{j_2,\alpha_2} \dots \xrightarrow{j_n,\alpha_n} q_n$$

with:

- $q_i \in X$, $j_i \in \mathbb{N}$, $\alpha_i \in \{0,1\}$
- $q_0 = i_0$
- for every i,
  - if $\alpha_i = 0$, $q_{i-1} = \partial_{j_i}^{\alpha_i}(q_i)$
  - if $\alpha_i = 1$, $q_i = \partial_{j_i}^{\alpha_i}(q_{i-1})$

$$0 \xrightarrow{1,0} \beta \xrightarrow{1,0} c \xrightarrow{2,1} \gamma$$

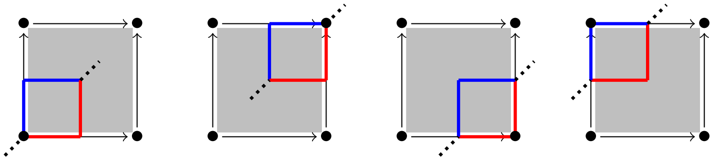


Homotopies

Elementary homotopies [van Glabbeek]

A path $i_0 = q_0 \xrightarrow{j_1, \alpha_1} \ldots \xrightarrow{j_n, \alpha_n} q_n$ is elementary homotopic to $i_0 = q'_0 \xrightarrow{k_1, \beta_1} \ldots \xrightarrow{k_n, \beta_n} q'_n$ if there is $1 \leq l \leq n-1$ such that:

- for every $p \neq l$ $q_p = q'_p$
- for every $r \notin \{l, l+1\}$ $j_r = k_r$, $\alpha_r = \beta_r$
- $\alpha_l = \beta_{l+1}$ and $\alpha_{l+1} = \beta_l$
- $k_l > j_l$, $j_l = k_{l+1}$ and $k_l = j_{l+1} - 1$

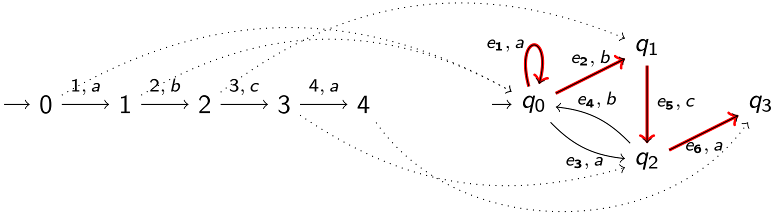


$\alpha_l = 0,\ \beta_l = 0$ $\alpha_l = 1,\ \beta_l = 1$ $\alpha_l = 1,\ \beta_l = 0$ $\alpha_l = 0,\ \beta_l = 1$

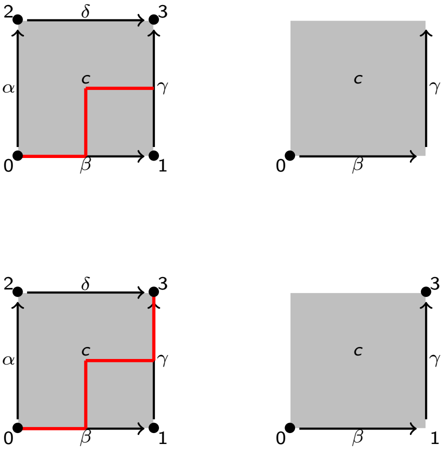
Internalizing runs in transition systems

[Joyal, Nielsen, Winskel]

A run in a transition system T is the same as a morphism from a finite linear system to T.



Internalizing paths and homotopies in HDA?



Fixing the notion of partial HDA

Fahrenberg's definition of partial HDA

Partial precubical sets

A **partial precubical set** is:

- a collection of sets $(X_n)_{n \geq 0}$,
- a collection of partial functions $(\partial_{i,n}^{\alpha} : X_n \longrightarrow X_{n-1})_{n>0,\, 1 \leq i \leq n,\, \alpha \in \{0,1\}}$.

satisfying for $i > j$: $$\partial_{j,n}^{\beta} \circ \partial_{i,n+1}^{\alpha} = \partial_{i-1,n}^{\alpha} \circ \partial_{j,n+1}^{\beta}$$

when both sides are defined.

Partial HDA

A **partial HDA** on the alphabet on Σ is:

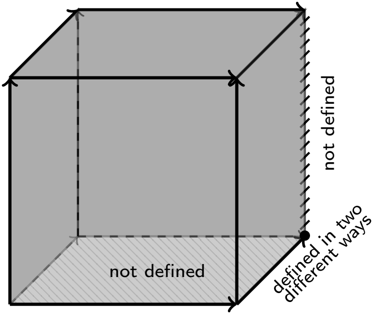
- a partial precubical set (X, ∂),
- an initial state $i_0 \in X_0$,
- a labelling function $\lambda : X_1 \longrightarrow \Sigma$.

satisfying for every $c \in X_2$:

$$\lambda(\partial_i^1(c)) = \lambda(\partial_i^0(c))$$

when both sides are defined.

Problem I: cubes are not cubes



If c is the cube, ∂_1^1 is not defined on $\partial_1^1(c)$ and $\partial_2^1(c)$, ∂_3^0 is not defined on c, then we cannot prove that

$$\partial_1^1 \circ \partial_2^0 \circ \partial_1^1(c) = \partial_1^1 \circ \partial_2^0 \circ \partial_2^1(c)$$

Problem II: morphisms are too general

Morphisms

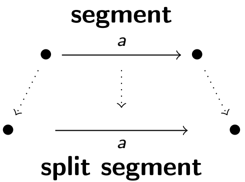
A morphism of partial precubical sets from (X, ∂) to (Y, δ) is a collection

$$f_n : X_n \longrightarrow Y_n$$

of functions such that:

$$f_{n-1} \circ \partial_{i,n}^{\alpha} = \delta_{i,n}^{\alpha} \circ f_n$$

when both sides are defined.



Partial precubical sets, as lax functors

Lax functor [Niefield]

By a lax functor $F : \mathcal{C} \rightharpoonup \mathbf{pSet}$, we mean the following data:

- for every object $c \in \mathcal{C}$, a set Fc
- for every morphism $i : c \longrightarrow c'$ of $\mathcal{C}$, a partial function $Fi : Fc \longrightarrow Fc'$

satisfying that:

- $F\mathrm{id}_c = \mathrm{id}_{Fc}$
- $Fj \circ Fi \subseteq F(j \circ i)$

Partial precubical set [Dubut]

A partial precubical set is a lax functor on the cube category.

Partial precubical sets, concretely

Partial precubical sets

A **partial precubical set** is:

- a collection of sets $(X_n)_{n \geq 0}$,
- a collection of partial functions $\partial_{i_1 < \ldots < i_k}^{\alpha_1, \ldots, \alpha_k} : X_n \longrightarrow X_{n-k}$.

satisfying:

$$\partial_{j_1 < \ldots < j_l}^{\beta_1, \ldots, \beta_l} \circ \partial_{i_1 < \ldots < i_k}^{\alpha_1, \ldots, \alpha_k} \subseteq \partial_{h_1 < \ldots < h_{k+l}}^{\gamma_1, \ldots, \gamma_{n+p}}$$

Ex: for $i > j$, $\partial_j^\beta \circ \partial_i^\alpha \subseteq \partial_{j<i}^{\beta,\alpha}$ and $\partial_{i-1}^\alpha \circ \partial_j^\beta \subseteq \partial_{j<i}^{\beta,\alpha}$

Morphisms of partial precubical sets

[Niefield]

A **function-valued op-lax transformation** between lax functors $F : \mathcal{C} \rightharpoonup \mathbf{pSet}$ to $G : \mathcal{C} \rightharpoonup \mathbf{pSet}$ is a collection of *total* functions $f_c : Fc \longrightarrow Gc$, such that for every $i : c \longrightarrow c'$,

$$f_{c'} \circ F(i) \subseteq G(i) \circ f_c$$

By $\mathbf{Lax(\mathcal{C})}$, we denote the category of lax functors and function-valued op-lax transformations.

The category of partial HDA is the double slice category

$$*/\mathbf{Lax}(\square^{op})/\Sigma = \mathbf{pHDA}_\Sigma$$

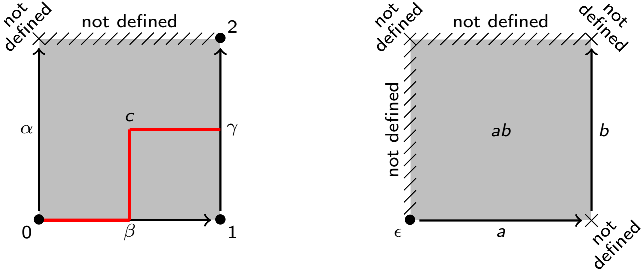
Completing a pHDA

[Dubut]

This process forms a functor $\chi : \mathbf{pHDA}_\Sigma \longrightarrow \mathbf{HDA}_\Sigma$ which is the left adjoint of the embedding of $\mathbf{HDA}_\Sigma$ in $\mathbf{pHDA}_\Sigma$.

For free: a geometric realization for pHDA!

Internalizing paths



Path (right) – path shape representing it (left)

A run in a pHDA X is the same as a morphism from a path shape to X.

We can do something similar for homotopies.

What's next?

Concurrency vs. Homotopy theory

Homotopy	Concurrency
cofibration generators (basic constructions of the theory)	path shapes and extensions
trivial fibration (rlp w.r.t. cofibration generators)	open maps w.r.t. path shapes
cofibrant objects (obtained from basic constructions)	trees
cofibrant replacement (process to obtain a cofibrant object)	unfolding