Natural homology

ICALP track B, Kyoto

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I.

Homotopy, dihomotopy and directed homology

Algebraic topology

SEE WIKIPEDIA

Objective

Objective :

Compare spaces **with a notion of order** up to continuous deformation **that preserves this order**

Problem coming from :

- geometric semantics of truly concurrent systems
  - PV-programs **[Dijkstra 68]**
  - scan/update **[Afek et al. 90]**
  - higher dimensional automata **[Pratt 91]**
- theory of relativity **[Dodson, Poston 97]**

Non directed case : algebraic topology

Non directed case : algebraic topology

Compare spaces ~~with a notion of order~~ up to continuous deformation ~~that preserves this order~~

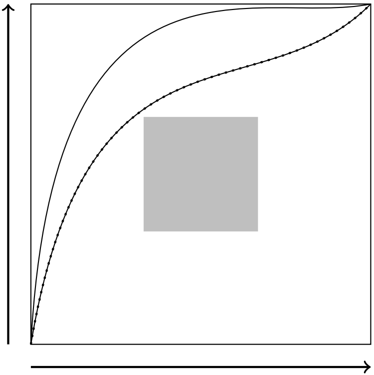
Homology **[Poincaré 1895]** which is :

- sound (invariant of homotopy)
- partially complete **[Hurewicz 52, Whitehead 49]**
- computable **[Poincaré 1900]**
- modular (homology can be expressed from homology of simpler spaces **[Mayer, Vietoris 30]**)

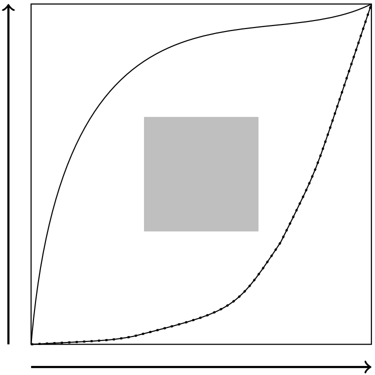
Dihomotopies

Dipaths = **increasing** continuous functions from $[0,1]$ to X

2 dipaths are **di**homotopic = you can deform continuously one into the other **while staying a dipath**

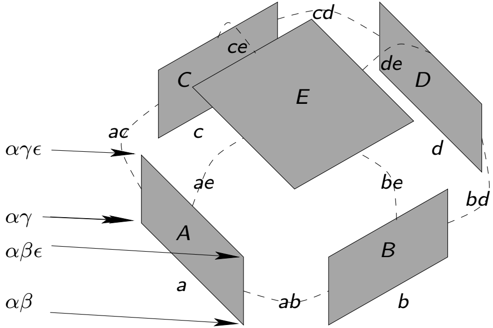


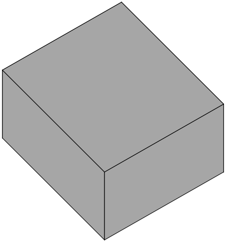
(di)homotopic



non (di)homotopic

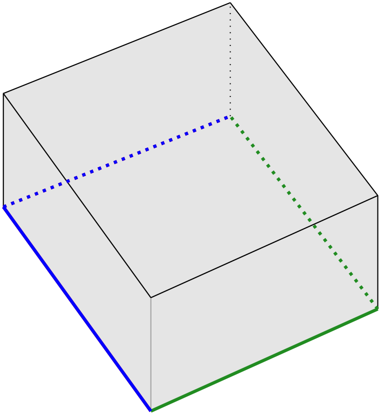
Homotopy vs dihomotopy





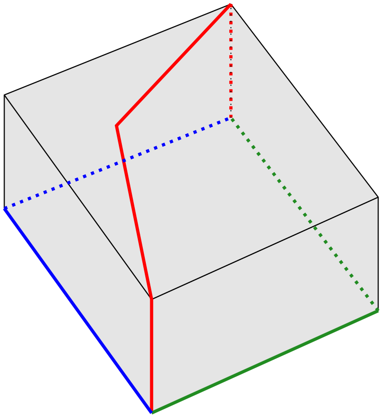
Fahrenberg's matchbox **[Fahrenberg 04]**

Homotopy vs dihomotopy



homotopic...

Homotopy vs dihomotopy



... but not dihomotopic

Related work contribution

Candidates of directed homology :

- past and future homologies **[Goubault 95]**
- ordered homology groups **[Grandis 04]**
- directed homology via ω-categories **[Fahrenberg 04]**
- homology graph **[Kahl 13]**

Not fine enough : do not distinguish Fahrenberg's matchbox from a point

Our contribution (1/4) :

A directed homology fine enough to detect failure of dihomotopy, **natural homology**

II.

Natural Homology

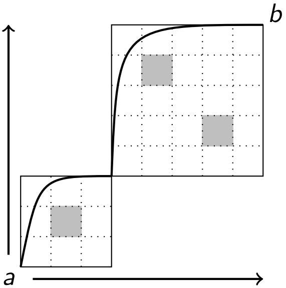
Trace spaces

dipath = continuous increasing map from $[0, 1]$ to X

trace = dipath modulo reparametrization

Trace space from a to b [Raussen 09]

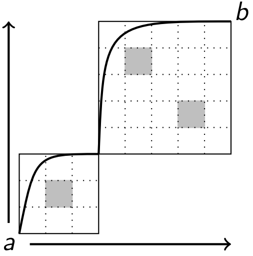
$T(X)(a, b) = \{\text{traces from } a \text{ to } b\}$ with compact open topology

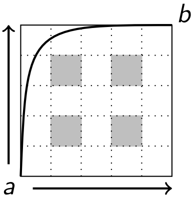


A first idea

Not_so_good directed homology :

$Not\_so\_good(X)$ = classical homology of $T(X)(a,b)$





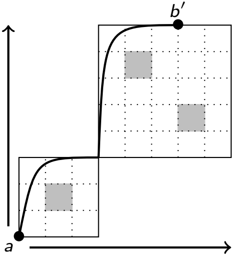
$A = (U \mid\mid S) \bullet (U.S \mid\mid U.S)$

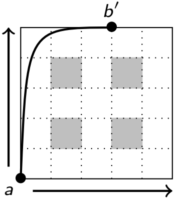
$B = U.U \mid\mid S.S$

$$T(A)(a,b) \simeq 6 \ point \ space \simeq T(B)(a,b)$$

$$Not\_so\_good(A) \simeq \mathbb{Z}^6 \simeq Not\_so\_good(B)$$

A first (not so) bad idea





make a, b vary

$T(A)(a, b') \simeq 4 \; point \; space$

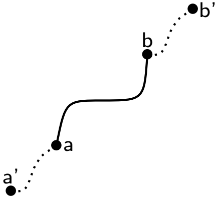
$Not\_so\_good(A) \simeq \mathbb{Z}^4$

no a', b' such that

$T(B)(a', b') \simeq 4 \; point \; space$

$Not\_so\_good(B) \simeq \mathbb{Z}^4$

Natural homology



$\mathcal{F}_X$ = category whose :

- objects are traces
- morphisms are extensions

Natural homology :

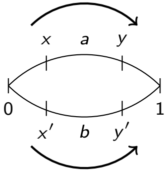
functor $\overrightarrow{H}_n(X) : \mathcal{F}_X \longrightarrow \mathbf{Ab}$

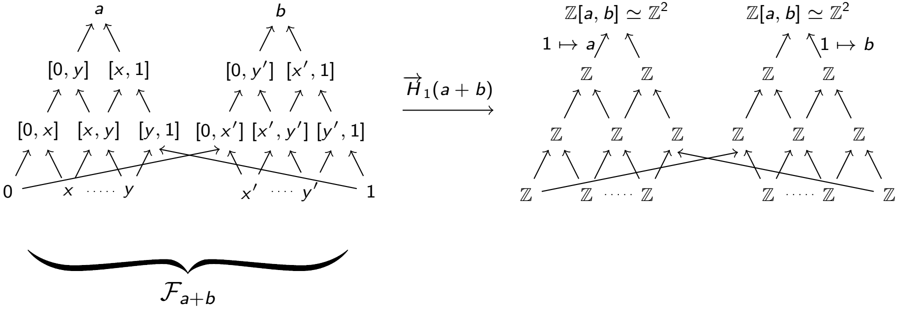
$$(a \xrightarrow{\gamma} b) \longmapsto H_{n-1}(T(X)(a,b))$$

(H_{n-1} = classical homology)

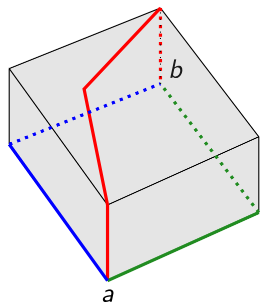
- $\mathcal{F}_X$ = category of factorizations **[Mac Lane 71]**
- $\overrightarrow{H}_n(X)$ = natural system **[Leech 73, Baues, Wirsching 85]**

Example : first natural homology of $a + b$





Natural homology on Fahrenberg's matchbox



2 dipaths non dihomotopic

$\Rightarrow$ $T(X)(a,b) \simeq$ 2 point space

$\Rightarrow$ $H_0(T(X)(a,b)) \simeq \mathbb{Z}^2$

$\Rightarrow$ $\overrightarrow{H}_1(X)$ not $\mathbb{Z}$ everywhere

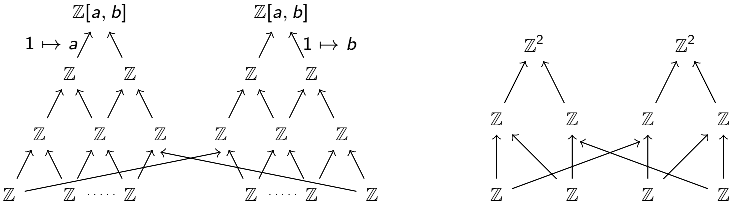
$\Rightarrow$ **natural homology detects failure of dihomotopy in Fahrenberg's matchbox**

III.

Comparison of natural systems

How to compare natural systems ?

$\overrightarrow{H}_n(A) = \overrightarrow{H}_n(B) \Rightarrow A = B$, modulo isomorphism



Our contribution (2/4) :

We define a notion of bisimilarity for natural systems

Similar idea in **[Fiore 00]** for functorial models of concurrent computation

Bisimulation of functors

Bisimulation between $F : C \longrightarrow \textbf{Ab}$ and $G : D \longrightarrow \textbf{Ab}$
$=$ set R of pairs (c, d) such that :

- c is an object of C
- d is an object of D

satisfying :

- for every object c of C, there exists d such that $(c, d) \in R$
-

$$(c, d) \in R$$

$$\begin{array}{ccc} c & & d \\ i \downarrow & & \vdots\, j \\ c' & & d' \end{array}$$

$$(c', d') \in R$$

and symmetrically

Bisimulation of functors

Bisimulation between $F : C \longrightarrow \mathbf{Ab}$ and $G : D \longrightarrow \mathbf{Ab}$
$=$ set R of triples (c, η, d) such that :

- c is an object of C
- d is an object of D
- $\eta : F(c) \longrightarrow G(d)$ is an isomorphism of groups

satisfying :

- for every object c of C, there exists d and η such that $(c, \eta, d) \in R$
-

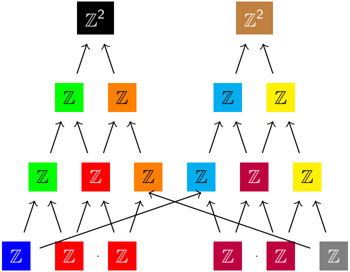
$$(c, \eta, d) \in R$$

$$\begin{array}{cccccc} c & Fc & \xrightarrow{\eta} & Gd & d \\ i \downarrow & Fi \downarrow & & \vdots Gj & \vdots j \\ c' & Fc' & \xrightarrow[\eta']{\cdots\cdots} & Gd' & d' \end{array}$$

$$(c', \eta', d') \in R$$

and symmetrically

Example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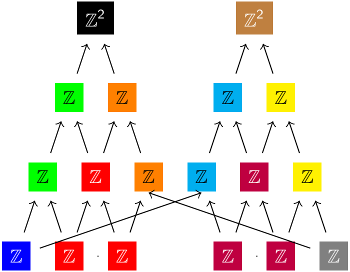
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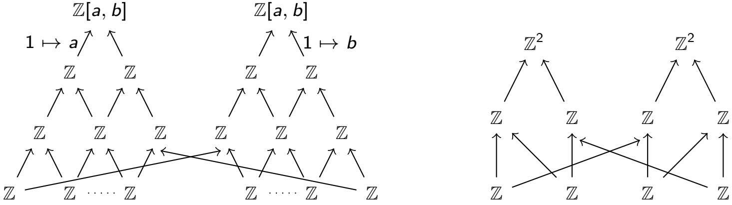
IV.

Computability, invariance by subdivision

Bisimilarity type of natural homology



Natural homology = uncountable, not computable

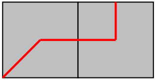
Our contribution (3/4) :

When X is finitely presented, we can compute a finite natural system bisimilar to $\overrightarrow{H}_n(X)$.

Discrete natural homology

X presented by a finite cubical complex
$\simeq$ glueing of cubes of unit length in $\mathbb{R}^n$

discrete trace = trace which is a glueing of segments joining center of cubes



f_X = category of discrete traces and extensions by discrete traces

Discrete natural homology $\overrightarrow{h}_n(X)$:

functor $\overrightarrow{h}_n(X) : \boxed{f_X} \longrightarrow \mathbf{Ab}$

$$(a \xrightarrow{\gamma} b) \longmapsto H_{n-1}(T(X)(a, b))$$

Computability

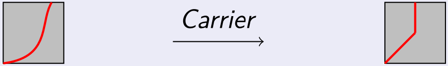
Theorem :

Given a finite cubical complex X, $\overrightarrow{h}_n(X)$ is :

- computable
- bisimilar to $\overrightarrow{H}_n(X)$

Proof :

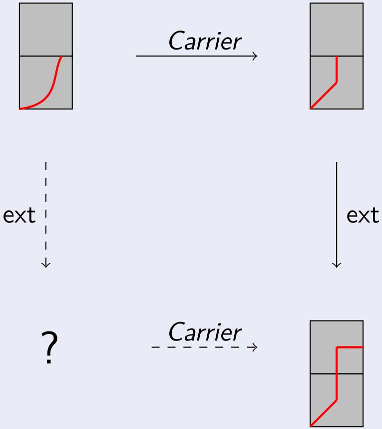
- *Carrier* **[Fajstrup05]** $(a \xrightarrow{\gamma} b) \mapsto$ trace drawn by joining centers of cubes traversed by γ



- We exhibit a **bisimulation** between γ and $Carrier(\gamma)$
- ...compatible with morphisms η, $F(i)$, $F(j)$ in **Ab** (not shown here)

Proof

Proof (bisimilarity) :



Proof

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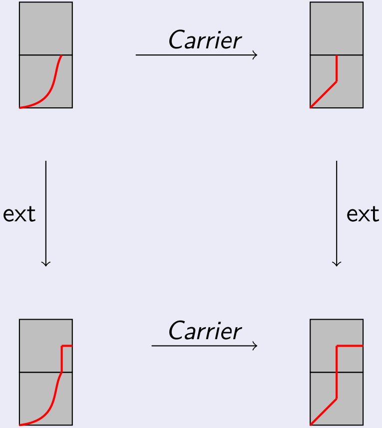

Invariance by subdivision

Our contribution (4/4) — Corollary :

$\overrightarrow{H}$ and $\overrightarrow{h}$ are invariant by subdivision.

$\simeq$ invariance by action refinement **[Goltz, van Glabbeek 89]**

Conclusion

Our contributions :

- definition of the first satisfactory directed homology, **natural homology**
- comparison of natural systems by **bisimilarity**
- definition of a **computable** natural system bisimilar to natural homology, **discrete natural homology**
- invariance by **subdivision**

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