# Justified sequences in string diagrams

## A comparison between two approaches to concurrent game semantics

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# Game semantics for concurrent languages

Game semantics (Hyland-Ong '00, Nickau '94):

- Types $\rightarrow$ games
- Programs $\rightarrow$ strategies

Game semantics for concurrent languages via sheaves:

- Hirschowitz et al.: CCS, $\pi$-calculus
- Tsukada and Ong: non-deterministic $\lambda$-calculus

# Sheaf-theoretic approaches to game semantics

## Both approaches

Innocent strategies = sheaves for a Grothendieck topology induced by the embedding of views into plays

Different notions of plays:

- Hirschowitz et al.: string diagrams
- Tsukada and Ong: justified sequences

## This work

- Design a string diagrammatic model of HON games
- Show a strong relationship between plays in both models
- Deduce equivalence of both notions of innocent strategies

# The level of plays

justified sequences: $\mathbb{V}_{A,B} \xhookrightarrow{\mathsf{i}_{HON}} \mathbb{P}_{A,B}$

string diagrams: $\mathbb{E}^{\mathbb{V}}(A \vdash B) \xhookrightarrow[\mathsf{i}]{} \mathbb{E}(A \vdash B)$

# The level of plays

$$
\begin{array}{lccc}
\text{justified sequences:} & \mathbb{V}_{A,B} & \xhookrightarrow{\mathsf{i}_{HON}} & \mathbb{P}_{A,B} \\
 & F^{\mathbb{V}} \downarrow \wr & & F \downarrow \ \text{full and faithful} \\
\text{string diagrams:} & \mathbb{E}^{\mathbb{V}}(A \vdash B) & \xhookrightarrow[\mathsf{i}]{} & \mathbb{E}(A \vdash B)
\end{array}
$$

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\text{justified sequences:} & \mathbb{V}_{A,B} & \xhookrightarrow{\mathrm{i}_{HON}} & \mathbb{P}_{A,B} \\
 & \downarrow\wr & & \downarrow \text{full and faithful} \\
\text{proof trees:} & \mathbb{B}(A \vdash B) & \hookrightarrow & \mathbb{T}(A \vdash B) \\
 & \downarrow\wr & & \downarrow\wr \\
\text{string diagrams:} & \mathbb{E}^{\mathbb{V}}(A \vdash B) & \xhookrightarrow[\mathrm{i}]{} & \mathbb{E}(A \vdash B)
\end{array}
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# The level of strategies

The square

$$
\begin{array}{ccc}
\mathbb{V}_{A,B} & \xhookrightarrow{\mathsf{i}_{HON}} & \mathbb{P}_{A,B} \\
{\scriptstyle F^{\mathbb{V}}}\downarrow & & \downarrow{\scriptstyle F} \\
\mathbb{E}^{\mathbb{V}}(A \vdash B) & \xhookrightarrow[\mathsf{i}]{} & \mathbb{E}(A \vdash B)
\end{array}
$$

is exact (Guitart, 1980), so

$$
\begin{array}{ccc}
\widehat{\mathbb{V}_{A,B}} & \xhookrightarrow{\prod_{\mathsf{i}_{HON}}} & \widehat{\mathbb{P}_{A,B}} \\
{\scriptstyle \Delta_{F^{\mathbb{V}}}}\uparrow & & \uparrow{\scriptstyle \Delta_F} \\
\mathbb{E}^{\mathbb{V}}\widehat{(A \vdash B)} & \xhookrightarrow[\prod_{\mathsf{i}}]{} & \mathbb{E}\widehat{(A \vdash B)}
\end{array}
$$

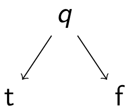
commutes up to isomorphism.

# Overview

# The level of plays

# Arenas

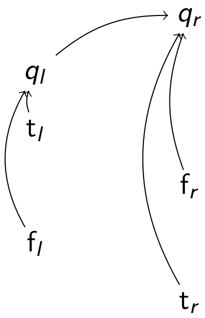
HON game semantics is based on arenas.
Arena = forest of moves
Example: boolean arena $\mathbb{B}$:



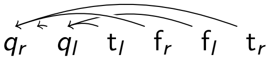

Residual of an arena $A \cdot m$: forest below $m$

# Plays in HON games

- Play on $(A, B)$ = justified sequence of moves in $A$ or $B$
  Example on $(\mathbb{B}, \mathbb{B})$:




Compact representation:



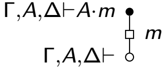

- View on $(A, B)$ = particular kind of play (inductive definition)

# Plays as string diagrams

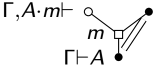
- Position $\approx$ set of players,
  - positive $\circ$ (labelled by a positive sequent of arenas $(\Gamma \vdash)$) or
  - negative $\bullet$ (labelled by a negative sequent of arenas $(\Gamma \vdash A)$)

# Plays as string diagrams

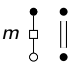
- Position ≈ set of players,
  - positive ∘ (labelled by a positive sequent of arenas $(\Gamma \vdash)$) or
  - negative • (labelled by a negative sequent of arenas $(\Gamma \vdash A)$)
- Two kinds of moves:




and




, in context, e.g.:



# Plays as string diagrams

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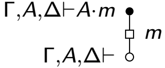


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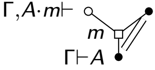


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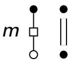
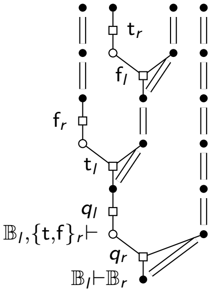
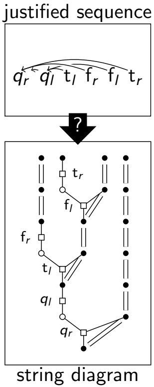



- Play = vertical pasting of moves



# The big picture



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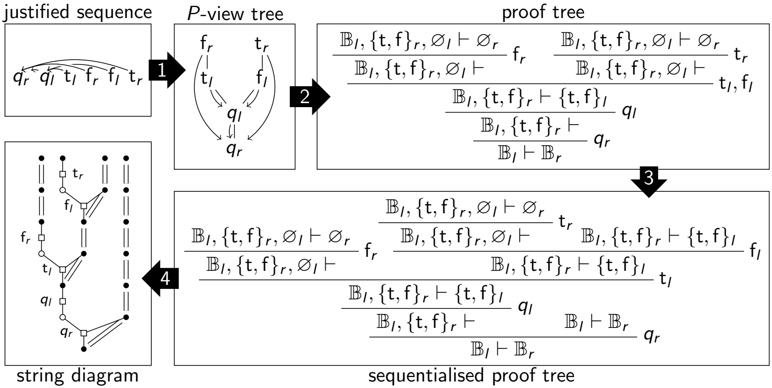
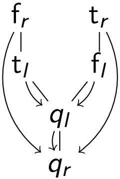


# $P$-view trees

Justified sequence $\rightarrow$ tree whose branches are views
Example:

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Justified sequence → tree whose branches are views
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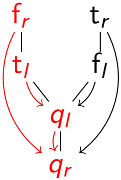

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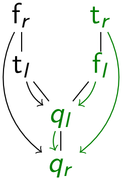


# Proof trees

Partial trees for:

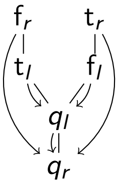
RIGHT
$$\frac{\ldots \quad \Gamma, A\cdot m(i) \vdash \quad \ldots \quad (\forall i \in n)}{\Gamma \vdash A}$$

LEFT
$$\frac{\Gamma, A, \Delta \vdash A\cdot m}{\Gamma, A, \Delta \vdash}$$

Example:

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathbb{B}_l, \{\mathsf{t},\mathsf{f}\}_r, \varnothing_l \vdash \varnothing_r}{\mathbb{B}_l, \{\mathsf{t},\mathsf{f}\}_r, \varnothing_l \vdash}\ \mathsf{f}_r \qquad \dfrac{\mathbb{B}_l, \{\mathsf{t},\mathsf{f}\}_r, \varnothing_l \vdash \varnothing_r}{\mathbb{B}_l, \{\mathsf{t},\mathsf{f}\}_r, \varnothing_l \vdash}\ \mathsf{t}_r}{\mathbb{B}_l, \{\mathsf{t},\mathsf{f}\}_r \vdash \{\mathsf{t},\mathsf{f}\}_l}\ \mathsf{t}_l, \mathsf{f}_l}{\mathbb{B}_l, \{\mathsf{t},\mathsf{f}\}_r \vdash}\ q_l}{\mathbb{B}_l \vdash \mathbb{B}_r}\ q_r$$

# $P$-view trees versus proof trees




$$\dfrac{\dfrac{\dfrac{\dfrac{\mathbb{B}_l, \{\mathsf{t},\mathsf{f}\}_r, \varnothing_l \vdash \varnothing_r}{\mathbb{B}_l, \{\mathsf{t},\mathsf{f}\}_r, \varnothing_l \vdash}\ \mathsf{f}_r \qquad \dfrac{\mathbb{B}_l, \{\mathsf{t},\mathsf{f}\}_r, \varnothing_l \vdash \varnothing_r}{\mathbb{B}_l, \{\mathsf{t},\mathsf{f}\}_r, \varnothing_l \vdash}\ \mathsf{t}_r}{\mathbb{B}_l, \{\mathsf{t},\mathsf{f}\}_r \vdash \{\mathsf{t},\mathsf{f}\}_l}\ \mathsf{t}_l, \mathsf{f}_l}{\mathbb{B}_l, \{\mathsf{t},\mathsf{f}\}_r \vdash}\ q_l}{\mathbb{B}_l \vdash \mathbb{B}_r}\ q_r$$

Differences:

- Presence of pointers
- Labelling

# From proof trees to string diagrams (part 1)

$$
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$$

arbitrarily branching tree ↓ binary tree

sequentialisation

*n*-ary node ↓ comb

$$
\dfrac{\dfrac{\dfrac{\dfrac{\mathbb{B}_l, \{\mathrm{t},\mathrm{f}\}_r, \varnothing_l \vdash \varnothing_r}{\mathbb{B}_l, \{\mathrm{t},\mathrm{f}\}_r, \varnothing_l \vdash}\;\mathrm{f}_r \qquad \dfrac{\dfrac{\mathbb{B}_l, \{\mathrm{t},\mathrm{f}\}_r, \varnothing_l \vdash \varnothing_r}{\mathbb{B}_l, \{\mathrm{t},\mathrm{f}\}_r, \varnothing_l \vdash}\;\mathrm{t}_r \quad \mathbb{B}_l, \{\mathrm{t},\mathrm{f}\}_r \vdash \{\mathrm{t},\mathrm{f}\}_l}{\mathbb{B}_l, \{\mathrm{t},\mathrm{f}\}_r \vdash \{\mathrm{t},\mathrm{f}\}_l}\;\mathrm{f}_l}{\mathbb{B}_l, \{\mathrm{t},\mathrm{f}\}_r \vdash \{\mathrm{t},\mathrm{f}\}_l}\;\mathrm{t}_l}{\mathbb{B}_l, \{\mathrm{t},\mathrm{f}\}_r \vdash}\;q_l \qquad \mathbb{B}_l \vdash \mathbb{B}_r}{\mathbb{B}_l \vdash \mathbb{B}_r}\;q_r
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# From proof trees to string diagrams (part 2)

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    \dfrac{\mathbb{B}_l, \{\mathrm{t},\mathrm{f}\}_r, \varnothing_l \vdash \varnothing_r}{\mathbb{B}_l, \{\mathrm{t},\mathrm{f}\}_r, \varnothing_l \vdash}\ \mathrm{f}_r
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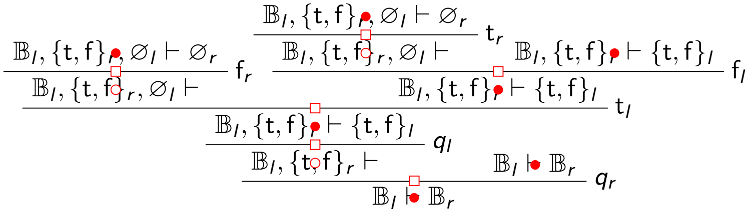
# From proof trees to string diagrams (part 2)



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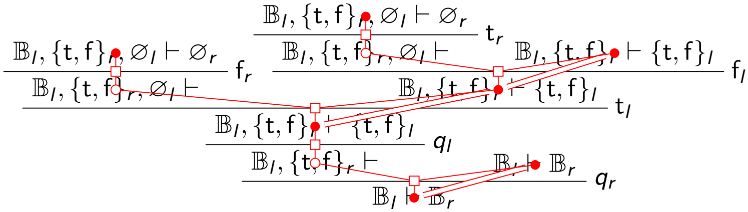

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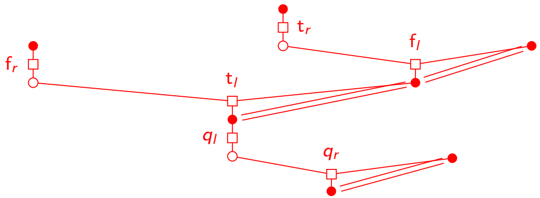

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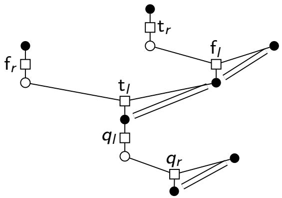
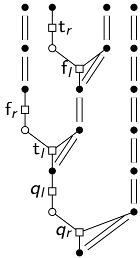


# The level of strategies

1. The level of plays
2. The level of strategies

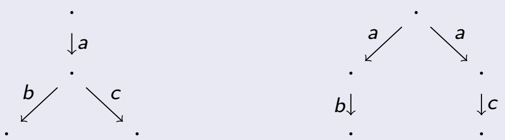
# Deterministic strategies

## Two notions of strategies

- Behaviours = prefix-closed set of views = $[\mathbb{V}^{op}, 2]$
- Innocent strategies = prefix-closed set of plays + innocence
  = some functors $[\mathbb{P}^{op}, 2]$

## Problem

Milner's coffee machines



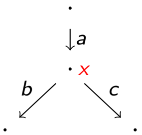

accept the same traces: $\varepsilon$, $a$, $ab$, and $ac$.

# Non-deterministic strategies

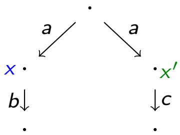
## Solution

~~Accept trace or not~~ $\rightarrow$ set of possible states after accepting trace

- Behaviours $= [\mathbb{V}^{op}, \mathrm{Set}] = \widehat{\mathbb{V}}$
- Innocent strategies $=$ some functors $[\mathbb{P}^{op}, \mathrm{Set}]$
  $=$ some presheaves in $\widehat{\mathbb{P}}$:
  essential image of $\prod_i$




$S(a) = \{x\}$




$S(a) = \{x, x'\}$

# Categories of innocent strategies

The square

$$\begin{array}{ccc} \widehat{\mathbb{V}_{A,B}} & \xhookrightarrow{\prod_{\mathrm{i}_{HON}}} & \widehat{\mathbb{P}_{A,B}} \\ \Delta_{F^{\mathbb{V}}} \uparrow & & \uparrow \Delta_F \\ \widehat{\mathbb{E}^{\mathbb{V}}(A \vdash B)} & \xhookrightarrow[\prod_{\mathrm{i}}]{} & \widehat{\mathbb{E}(A \vdash B)} \end{array}$$

commutes up to isomorphism:

- Behaviours are equivalent
- Innocent strategies are equivalent
- compatible with innocentisation
- (Non-innocent strategies are not)

# Conclusion

Done: link between two models of game semantics:

- At the level of plays:
  - Full embedding of justified sequences into string diagrams
  - Equivalence of categories of views
- At the level of strategies:
  - Equivalent categories of behaviours and innocent strategies
  - Compatible with innocentisation

To do: composition of strategies in our setting.