Template games, simple games, and Day convolution

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Introduction

Two abstract approaches to game semantics:

- game settings (E. and Hirschowitz, 2018):
  - rather complex (sheaves, polynomial functors. . .),
  - general (several instances: HON, AJM, Tsukada-Ong. . .),
- template games (Melliès, 2019):
  - very simple and elegant,
  - maybe less general, differs from classic models.

Here:

- retelling of template games,
- extended to cover simple games,
- brings formal connection to Day convolution.

Overview

- Template games:
  - games,
  - strategies,
  - copycats, composition.
- Weak double categories:
  - internal monads,
  - template games.
- Recovering simple game semantics:
  - games,
  - strategies,
  - copycats, composition.

Template games

Definition

Let $\pitchfork_g$ be the category freely generated by the graph

$$P \rightleftarrows O.$$

Definition

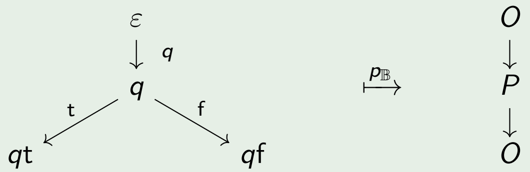
A template game is a category A equipped with a functor $p\colon A \to \pitchfork_g$.

Simple games $\to$ template games

Simple game A:

- rooted tree $\rightsquigarrow$ poset $\rightsquigarrow$ category,
- projection $p_A : A \to \pm_g$:
  - nodes at even depth to O,
  - nodes at odd depth to P.

Example: booleans $\mathbb{B}$



Template strategies

Definition

Let $\pitchfork_s$ be the category freely generated by

$$PP \rightleftarrows OP \rightleftarrows OO.$$

Definition

A template strategy from A to B is a category S equipped with projections

$$\begin{array}{ccccc} A & \xleftarrow{s'} & S & \xrightarrow{t'} & B \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle q} \\ \pitchfork_g & \xleftarrow[s]{} & \pitchfork_s & \xrightarrow[t]{} & \pitchfork_g . \end{array}$$

Template strategies

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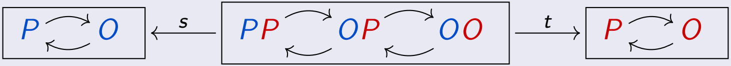

Definition

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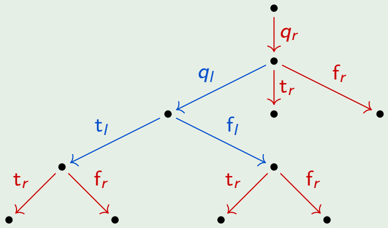
Simple strategies: the arrow game

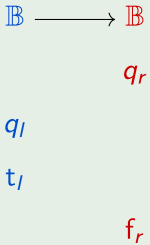
Definition

For A and B simple games, $A \to B$:

- polarity of moves in A reversed,
- O starts in B,
- only P switches sides.

Example: $\mathbb{B} \to \mathbb{B}$



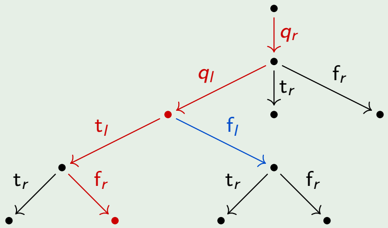


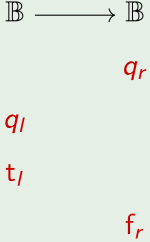
Boolean simple strategies

Definition

A boolean simple strategy from A to B is a non-empty, prefix-closed set of even-length plays of $A \to B$.

Example





Can be encoded by red nodes.

Simple strategies

Definition

Let $(A \to B)^{P^*}$ be the full subcategory of $A \to B$ spanning non-empty, even-length plays.

Definition

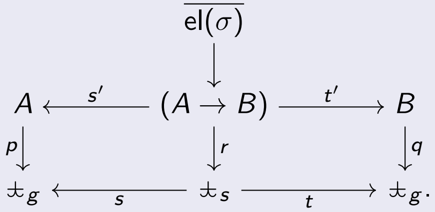
Simple strategies from A to B: $\widehat{(A \to B)^{P^*}}$.

Simple strategies $\to$ template strategies

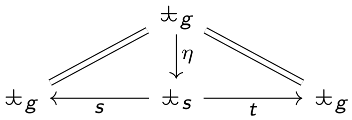
Simple strategy $\sigma \in \widehat{(A \to B)^{P*}}$:

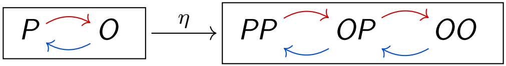
- $\mathrm{el}(\sigma)$: all possible "states" of the strategy after playing a non-empty, even-length play,
- $\overline{\mathrm{el}(\sigma)}$: all possible states, adding initial state and receptivity.

Translation

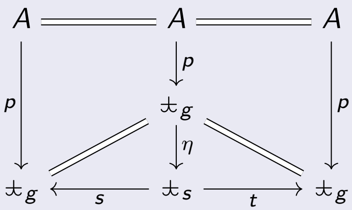


Template strategies: identities (copycats)

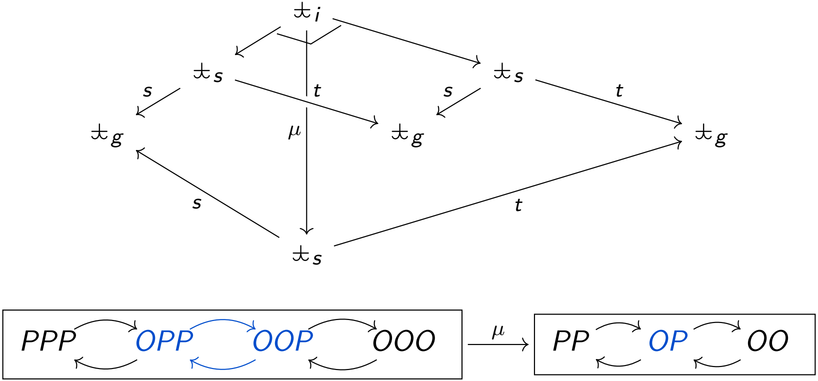




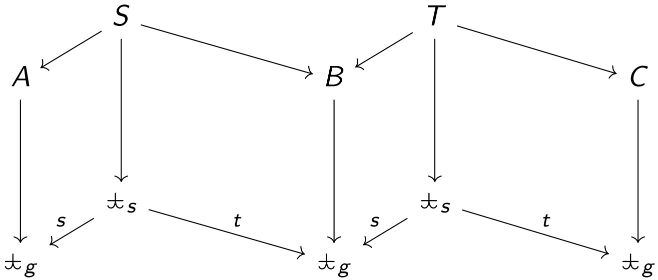
Copycat



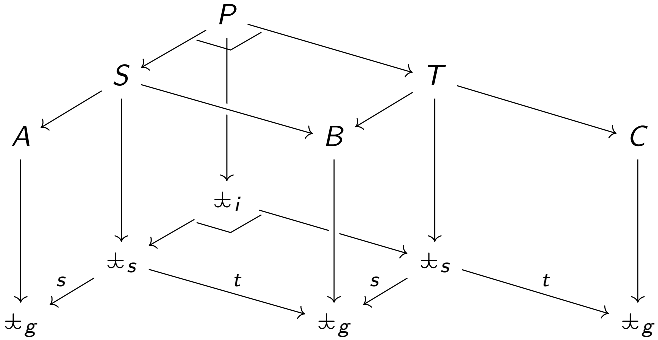
The template of interactions



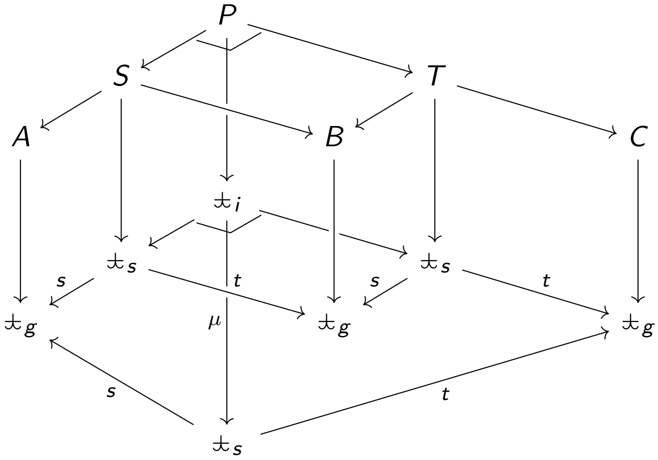
Template strategies: composition



Template strategies: composition



Template strategies: composition



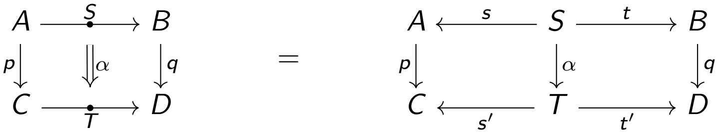
Weak double categories

$$\begin{array}{ccc} A & \overset{s}{-\!\!\bullet\!\!\to} & B \\ f\downarrow & \Downarrow\alpha & \downarrow g \\ C & \underset{t}{-\!\!\bullet\!\!\to} & D \\ f'\downarrow & \Downarrow\beta & \downarrow g' \\ E & \underset{u}{-\!\!\bullet\!\!\to} & F \end{array} \quad \mapsto \quad \begin{array}{ccc} A & \overset{s}{-\!\!\bullet\!\!\to} & C \\ f'\circ f\downarrow & \Downarrow\beta\circ\alpha & \downarrow g'\circ g \\ D & \underset{u}{-\!\!\bullet\!\!\to} & F \end{array} \qquad \begin{array}{ccc} A & \overset{s}{-\!\!\bullet\!\!\to} & B \\ id_A\downarrow & \Downarrow id_s & \downarrow id_B \\ A & \underset{s}{-\!\!\bullet\!\!\to} & B \end{array}$$

$$\begin{array}{ccccc} A & \overset{s}{-\!\!\bullet\!\!\to} & B & \overset{s'}{-\!\!\bullet\!\!\to} & C \\ f\downarrow & \Downarrow\alpha & \downarrow g & \Downarrow\beta & \downarrow h \\ D & \underset{t}{-\!\!\bullet\!\!\to} & E & \underset{t'}{-\!\!\bullet\!\!\to} & F \end{array} \quad \mapsto \quad \begin{array}{ccc} A & \overset{s'\bullet s}{-\!\!\bullet\!\!\to} & C \\ f\downarrow & \Downarrow\beta\bullet\alpha & \downarrow h \\ D & \underset{t'\bullet t}{-\!\!\bullet\!\!\to} & F \end{array} \qquad \begin{array}{ccc} A & \overset{id^\bullet_A}{-\!\!\bullet\!\!\to} & A \\ f\downarrow & \Downarrow id^\bullet_f & \downarrow f \\ B & \underset{id^\bullet_B}{-\!\!\bullet\!\!\to} & B \end{array}$$

$$\begin{array}{ccccc} A & -\!\!\bullet\!\!\to & B & -\!\!\bullet\!\!\to & C \\ \downarrow & \Downarrow\alpha & \downarrow & \Downarrow\gamma & \downarrow \\ D & -\!\!\bullet\!\!\to & E & -\!\!\bullet\!\!\to & F \\ \downarrow & \Downarrow\beta & \downarrow & \Downarrow\delta & \downarrow \\ G & -\!\!\bullet\!\!\to & H & -\!\!\bullet\!\!\to & I \end{array}$$

Span($\mathbb{C}$)



Horizontally:

- composition of spans by pullback,
- composition of cells by universal property,
- choice of pullbacks $\rightsquigarrow$ bicategory.

Template games

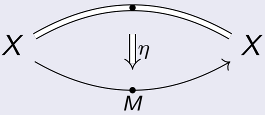
In Span(Cat):

- game: $A \to \pitchfork_g$,
- strategy: cell over $\pitchfork_s$.

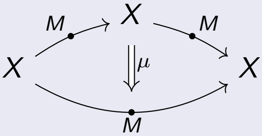
Internal monads

Definition

Monad in double category: horizontal morphism $M\colon X \mathrel{\circ\!\!\to} X$ with



and



satisfying the obvious generalisation of the usual monad laws.

Example

$\mp_s\colon \mp_g \mathrel{\circ\!\!\to} \mp_g$ is a monad in Span(Cat):

- $\eta\colon id^{\bullet}_{\mp_g} \Rightarrow \mp_s$: synchronous copycat schedule,
- $\mu\colon \mp_i \Rightarrow \mp_s$: parallel composition + hiding.

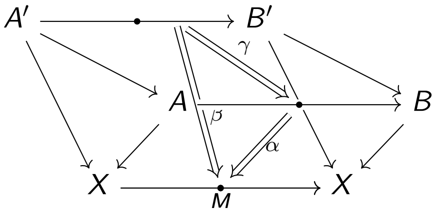
$\mathbb{D}/M$

Candidate weak double category:

- objects: vertical morphisms $A \to X$,
- vertical morphisms from $A \to X$ to $B \to X$: commuting triangles of vertical morphisms,
- horizontal morphisms from $A \to X$ to $B \to X$: cells over M,

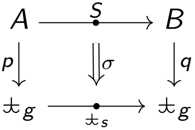
$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle p}\downarrow & \Downarrow{\scriptstyle \alpha} & \downarrow{\scriptstyle q} \\ X & \xrightarrow[M]{} & X, \end{array}$$

- cells: commuting "triangles" of cells.

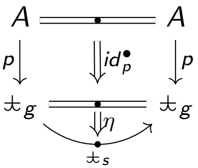


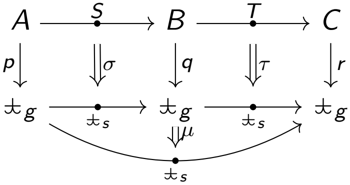
Template game semantics: $\mathrm{Span}(\mathrm{Cat})/オ_s$

Template strategy from A to B:



Copycats and composition:





Theorem

For any monad $M\colon X \to X$ internal to a weak double category $\mathbb{D}$, $\mathbb{D}/M$ is again a weak double category.

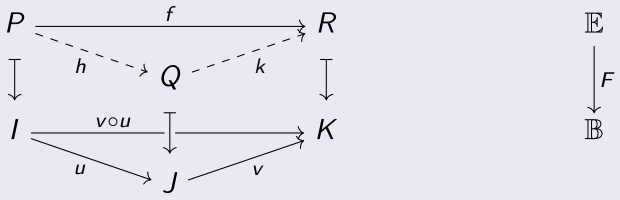
Template games vs. simple games: first discrepancy

First discrepancy

- Simple games: initial object, polarity O (root).
- Template games:
  - $\mathbb{Z} \to \pm_g$: no initial object,
  - $\ulcorner P \urcorner : 1 \to \pm_g$: polarity P.

Second discrepancy

Simple games $A \to \pm_g$ are discrete Conduché fibrations:



Recovering simple games

Change the template: $\mathbb{T}_g = O/\pitchfork_g$.

Definition

Refined template game: discrete fibration to $\mathbb{T}_g$.

Remark

$\mathbb{T}_g \cong \omega \rightsquigarrow$ discrete fibration into $\mathbb{T}_g$ = forest.

Lemma

Refined template game is isomorphic to simple game iff definite (fibre over id_O singleton).

Naively refined template strategies

Same discrepancies as games $\rightsquigarrow$ choose $\mathbb{T}_s = OO/\mathrel{\top\!\!\!\!\smile}_s$?

Definition

Naively refined template strategy from A to B:

$$
\begin{array}{ccccc}
A & \xleftarrow{s'} & S & \xrightarrow{t'} & B \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle q} \\
\mathbb{T}_g & \xleftarrow[s]{} & \mathbb{T}_s & \xrightarrow[t]{} & \mathbb{T}_g ,
\end{array}
$$

with r a discrete fibration.

Problem

$$
\begin{array}{ccccc}
A & \longleftarrow & (A \to B) & \longrightarrow & B \\
{\scriptstyle p}\downarrow & & \downarrow & & \downarrow{\scriptstyle q} \\
\mathbb{T}_g & \xleftarrow[s]{} & \mathbb{T}_s & \xrightarrow[t]{} & \mathbb{T}_g
\end{array}
$$

Refined template strategies

We want strategies to be presheaves over $(A \to B)^{P^*} \rightsquigarrow \mathbb{T}_s$ spans even-length plays of $OO/\pm_s$.

Definition

Refined template strategy from A to B: category S equipped with

$$\begin{array}{ccccc} A & \xleftarrow{s'} & S & \xrightarrow{t'} & B \\ \downarrow p & & \downarrow r & & \downarrow q \\ \mathbb{T}_g & \xleftarrow[s]{} & \mathbb{T}_s & \xrightarrow[t]{} & \mathbb{T}_g, \end{array}$$

with r a discrete fibration.

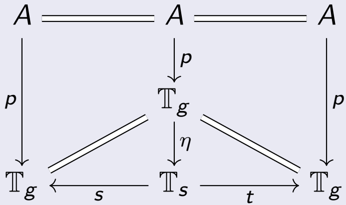
Lemma

Refined template strategy is isomorphic to simple strategy iff definite.

Copycats

$\mathbb{T}_s \colon \mathbb{T}_g \to\!\!\!\bullet\!\!\to \mathbb{T}_g$ is still a monad.

Copycat



Problem

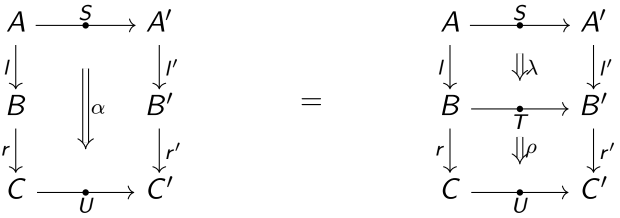
ηp is not always a discrete fibration.

Interlude: double factorisation systems

Given double category $\mathbb{D}$, pair of factorisation systems:

- $(\mathcal{L}_v, \mathcal{R}_v)$ on $\mathbb{D}_v$,
- $(\mathcal{L}_V, \mathcal{R}_V)$ on $\mathbb{D}_V$,

such that $s, t \colon \mathbb{D}_V \to \mathbb{D}_v$ map $\mathcal{L}_V$ (resp. $\mathcal{R}_V$) to $\mathcal{L}_v$ (resp. $\mathcal{R}_v$) and

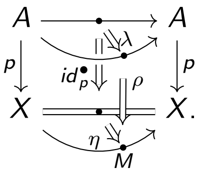


Example

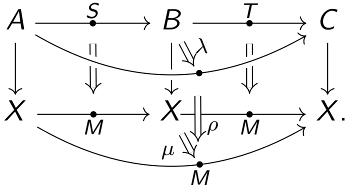
On Span($\mathbb{C}$): $(\mathcal{L}, \mathcal{R})$ and componentwise $(\mathcal{L}, \mathcal{R})$.

Simple games: $\mathsf{Span}(\mathsf{Cat})/_{\mathsf{DFib}}\mathbb{T}$

Horizontal identities:



Horizontal composition:



Theorem

If the double factorisation system is nice enough, then $\mathbb{D}/_{\mathcal{R}}M$ is a weak double category.

Day convolution

- Weak double category $\mathcal{W}$:
  - vertically trivial, i.e., essentially a monoidal category,
  - $\mathcal{W}_V = \mathsf{Cat}$,
  - horizontal composition $=$ cartesian product in Cat.
- Monad $\mathbb{C}$ $=$ strict monoidal category.
- $(\uparrow, \mathsf{DFib})$ forms a double factorisation system.

Corollary

- $\mathcal{W}/_{\mathsf{DFib}}\mathbb{C}$ vertically trivial, i.e., a monoidal category.
- $\mathcal{W}/_{\mathsf{DFib}}\mathbb{C} \simeq \widehat{\mathbb{C}}$.
- Horizontal composition is Day convolution.

Conclusion

Done:

- characterisation of simple games among template games,
- composition of simple strategies as refinement of composition of template strategies,
- general slicing construction,
- link with Day convolution.

To-do:

- more structure,
- get exactly simple games (unique initial state),
- get Day convolution for general monoidal categories.