What's in a game?

A theory of game models

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Game Semantics

Between operational and denotational:

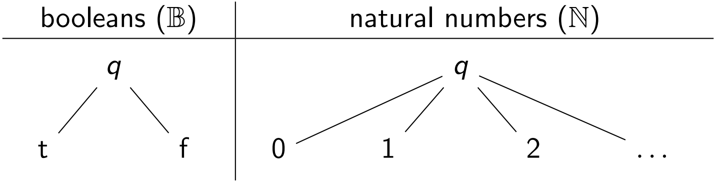
- types → games
- programs → strategies
- dynamic
- related to syntax (equivalence of programs is intensional, rather than extensional)

Game semantics:

- significant part of denotational semantics
- "solved" full abstraction for PCF
- many variants characterise different programming features
- applications: model checking, hardware synthesis...

HON Game Semantics: Games

Structures possible moves.



HON Game Semantics: Plays

Basically:

- sequence of moves
- interaction between program and environment

Example: `f = fun n -> 2 * n`

$$\begin{array}{ccc} \mathbb{N} & \longrightarrow & \mathbb{N} \\ & & q_r \\ q_l & & \\ 6_l & & \\ & & 12_r \end{array}$$

HON Game Semantics: Plays

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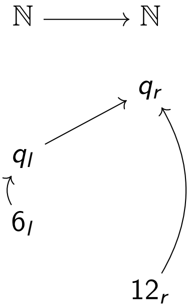
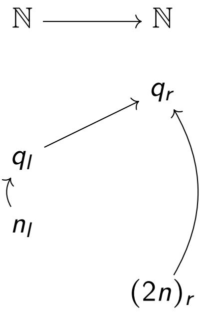


HON Game Semantics: Strategies

Strategy = prefix-closed set of accepted plays.

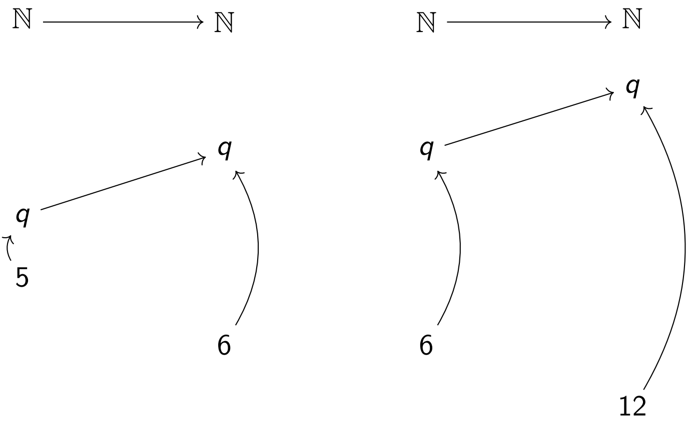
Example: for `f = fun n -> 2 * n`, (roughly) all plays of the form



HON Game Semantics: Composition

Parallel composition + hiding.

```
f = fun n -> n + 1        f = fun n -> 2 * n
```



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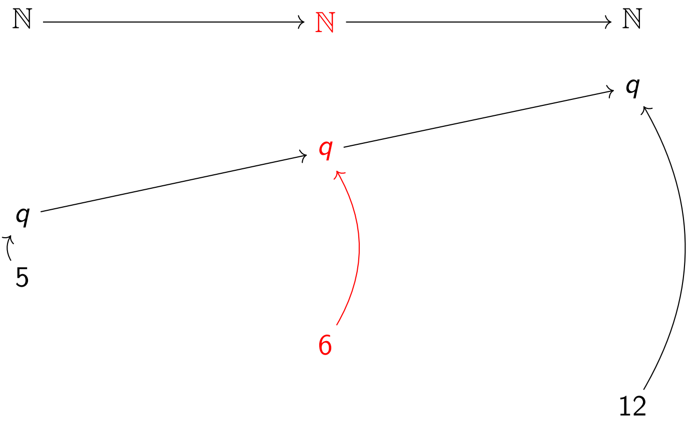
Parallel composition + hiding.

`f = fun n -> n + 1` `f = fun n -> 2 * n`



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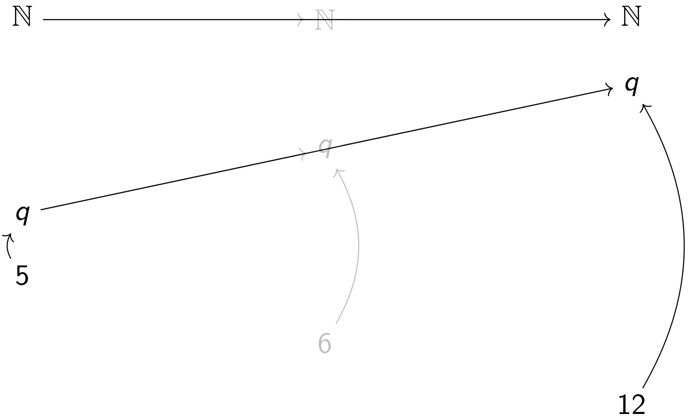

Strategies as Presheaves

Presheaf over $\mathbb{C}$: functor $\mathbb{C}^{op} \to \mathrm{Set}$.

Boolean presheaf over $\mathbb{C}$: functor $\mathbb{C}^{op} \to 2$ $(2 = 0 \to 1)$.

Two notions of strategies:

- prefix-closed sets of plays: boolean presheaves $\widetilde{\mathbb{P}_{A,B}}$
- concurrent strategies: presheaves $\widehat{\mathbb{P}_{A,B}}$

$$\begin{tikzcd}[column sep=small] & \cdot \arrow[d, "a"] & \\ & \cdot \arrow[dl, "b"'] \arrow[dr, "c"] & \\ \cdot & & \cdot \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & \cdot \arrow[dl, "a"'] \arrow[dr, "a"] & \\ \cdot \arrow[d, "b"'] & & \cdot \arrow[d, "c"] \\ \cdot & & \cdot \end{tikzcd}$$

$$\sigma(a) = \{x\} \qquad \sigma(a) = \{x, x'\}$$

With traditional strategies:

$$\sigma = \{\varepsilon, a, ab, ac\}$$

Motivation

Game models:

- HON (justified sequences)
- AJM (sequences)
- simple games, Blass (trees)
- concurrent (event structures)
- string diagrams
- variants
- ...

Definitions and proofs are similar...but tricky!

Goal: define a framework that

- encompasses many models
- factors out similar proofs

Recurring Pattern

- define games A, B, C, ...
- define categories of plays $\mathbb{P}_{A,B}$
- define strategies $A \to B$ as prefix-closed sets of plays in $\mathbb{P}_{A,B}$
- composition = parallel composition + hiding
- identities = copycat strategies
- prove that this defines a category of games and strategies

Recurring Pattern

- define games A, B, C, ... **assumed**
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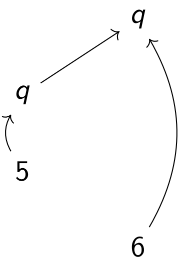
- composition = parallel composition + hiding
- identities = copycat strategies
- prove that this defines a category of games and strategies

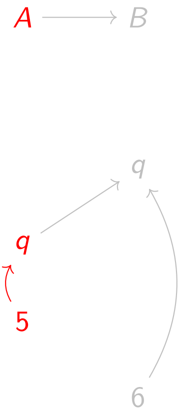
Categories of Plays

$$A \longrightarrow B$$

- games A, B, $C \dots$
- categories of plays $\mathbb{P}_A$, $\mathbb{P}_{A,B}$, $\mathbb{P}_{A,B,C} \cdots$



Categories of Plays



- games A, B, $C \dots$
- categories of plays $\mathbb{P}_A$, $\mathbb{P}_{A,B}$, $\mathbb{P}_{A,B,C} \cdots$
- projections $\mathbb{P}_{A,B} \to \mathbb{P}_A$

Categories of Plays

$$
\begin{tikzcd}
A \arrow[r] & B \\
 & q \\
q \arrow[ur] & \\
5 \arrow[u, bend left] & \\
 & 6 \arrow[uuu, bend right]
\end{tikzcd}
$$

- games A, B, $C \ldots$
- categories of plays $\mathbb{P}_A$, $\mathbb{P}_{A,B}$, $\mathbb{P}_{A,B,C} \cdots$
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Categories of Plays

- games A, B, C...
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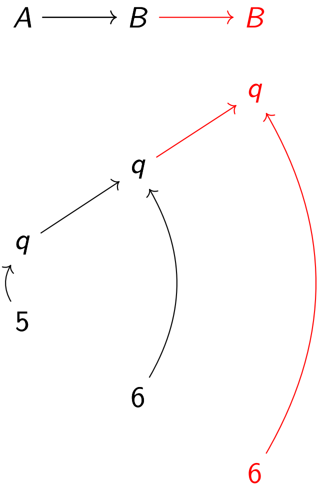
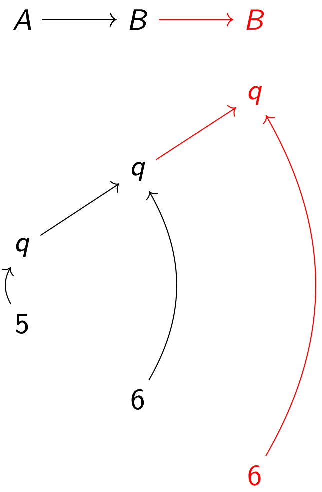


Categories of Plays

- games A, B, $C\ldots$
- categories of plays $\mathbb{P}_A$, $\mathbb{P}_{A,B}$, $\mathbb{P}_{A,B,C}\cdots$
- projections $\mathbb{P}_{A,B} \to \mathbb{P}_A$
- insertions $\mathbb{P}_{A,B} \to \mathbb{P}_{A,B,B}$
- compatibility between projections and insertions



Describing Categories of Plays Simplicially

Game setting:

- set $\mathbb{A}$ of games
- functor $\mathbb{P}\colon (\Delta/\mathbb{A})^{op} \to \mathrm{Cat}$

$\Delta/\mathbb{A}$:

- objects: lists $L = A_1, \ldots, A_n$ of games
- morphisms: insertions $(A, C \to A, B, C)$ and fusions $(A, A, B \to A, B)$

Strategies $A \to B$: $\widehat{\mathbb{P}_{A,B}}$.

Polynomial Functors

If $F\colon \mathbb{C} \to \mathbb{D}$:

$$\widehat{\mathbb{C}} \xleftarrow{\quad \Delta_F \quad} \widehat{\mathbb{D}}$$

Polynomial Functors

If $F : \mathbb{C} \to \mathbb{D}$:

$$\begin{tikzcd}
\widehat{\mathbb{C}} \arrow[rr, bend left=40, "\Sigma_F"] \arrow[rr, bend right=40, "\Pi_F"'] \arrow[rr, phantom, bend left=20, "\perp"] \arrow[rr, phantom, bend right=20, "\perp"] & & \widehat{\mathbb{D}} \arrow[ll, "\Delta_F" description]
\end{tikzcd}$$

Polynomial Functors

If $F : \mathbb{C} \to \mathbb{D}$:

$$\begin{tikzcd}
\widehat{\mathbb{C}} \arrow[rr, bend left, "\sum_F"] \arrow[rr, bend right, "\prod_F"'] & \arrow[d, phantom, "\perp"] \arrow[u, phantom, "\perp"] & \widehat{\mathbb{D}} \arrow[ll, "\Delta_F" description]
\end{tikzcd}$$

Polynomial functor: composite of Δ's, $\prod$'s, and $\sum$'s.

Composition

Idea: parallel composition + hiding.

$$\widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,C}} \xrightarrow{\Delta_{\delta_2+\delta_0}} \widehat{\mathbb{P}_{A,B,C} + \mathbb{P}_{A,B,C}} \xrightarrow{\Pi_\nabla} \widehat{\mathbb{P}_{A,B,C}} \xrightarrow{\Sigma_{\delta_1}} \widehat{\mathbb{P}_{A,C}}$$

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Justification:

$\mathrm{m}_{A,B,C}(\sigma,\tau)$ accepts p

iff

iff

iff

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Justification:

$\mathrm{m}_{A,B,C}(\sigma,\tau)$ accepts p

iff there exists an interaction sequence $u \in \mathbb{P}_{A,B,C}$ that is accepted and projects to p

iff

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iff both $\mathrm{inl}\, u$ and $\mathrm{inr}\, u$ are accepted

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Justification:

	$\mathrm{m}_{A,B,C}(\sigma,\tau)$ accepts p
iff	there exists an interaction sequence $u \in \mathbb{P}_{A,B,C}$ that is accepted and projects to p
iff	both $\mathrm{inl}\, u$ and $\mathrm{inr}\, u$ are accepted
iff	σ accepts $\delta_2(u)$ and τ accepts $\delta_0(u)$.

Copycat strategies

$$1 \cong \widehat{\varnothing} \xrightarrow{\Pi_!} \widehat{\mathbb{P}_A} \xrightarrow{\Sigma_{\iota_0}} \widehat{\mathbb{P}_{A,A}}$$

Justification:

œ_A accepts p

iff

iff

Copycat strategies

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iff	

Copycat strategies

$$1 \cong \widehat{\varnothing} \xrightarrow{\Pi_!} \widehat{\mathbb{P}_A} \xrightarrow{\Sigma_{\iota_0}} \widehat{\mathbb{P}_{A,A}}$$

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iff	there exists a sequence $s \in \mathbb{P}_A$ that is accepted and mapped to p
iff	there is an s that is mapped to p.

Game Settings

- set $\mathbb{A}$ of games
- functor $\mathbb{P}\colon (\Delta/\mathbb{A})^{op} \to \mathrm{Cat}$
- $\delta_1\colon \mathbb{P}_{A,B,C} \to \mathbb{P}_{A,C}$ and $\iota_0\colon \mathbb{P}_A \to \mathbb{P}_{A,A}$ discrete fibrations
- ...

Associativity of composition

Theorem (Composition is associative):

$$\begin{tikzcd}
\widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,C} + \mathbb{P}_{C,D}} \arrow[r, "\mathsf{m}_{A,B,C}+\mathbb{P}_{C,D}"] \arrow[d, "\mathbb{P}_{A,B}+\mathsf{m}_{B,C,D}"'] & \widehat{\mathbb{P}_{A,C} + \mathbb{P}_{C,D}} \arrow[d, "\mathsf{m}_{A,C,D}"] \\
\widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,D}} \arrow[r, "\mathsf{m}_{A,B,D}"'] & \widehat{\mathbb{P}_{A,D}}
\end{tikzcd}$$

commutes if

$$\begin{tikzcd}
\mathbb{P}_{A,B,C,D} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{P}_{A,B,D} \arrow[d] \\
\mathbb{P}_{B,C,D} \arrow[r] & \mathbb{P}_{B,D}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathbb{P}_{A,B,C,D} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{P}_{A,C,D} \arrow[d] \\
\mathbb{P}_{A,B,C} \arrow[r] & \mathbb{P}_{A,C}
\end{tikzcd}$$

are pullbacks (*zipping lemma*).

Unitality

Theorem (Copycat strategies are units):

$$
\begin{tikzcd}
\widehat{\varnothing + \mathbb{P}_{A,B}} \arrow[d, "\mathrm{cc}_A + \mathbb{P}_{A,B}"'] \arrow[dr, "\sim"] & & \widehat{\mathbb{P}_{A,B} + \varnothing} \arrow[d, "\mathbb{P}_{A,B} + \mathrm{cc}_B"] \arrow[dl, "\sim"'] \\
\widehat{\mathbb{P}_{A,A} + \mathbb{P}_{A,B}} \arrow[r, "\mathsf{m}_{A,A,B}"'] & \widehat{\mathbb{P}_{A,B}} & \widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,B}} \arrow[l, "\mathsf{m}_{A,B,B}"]
\end{tikzcd}
$$

commutes if

$$
\begin{tikzcd}
\mathbb{P}_{A,B} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{P}_A \arrow[d] \\
\mathbb{P}_{A,A,B} \arrow[r] & \mathbb{P}_{A,A}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathbb{P}_{A,B} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{P}_B \arrow[d] \\
\mathbb{P}_{A,B,B} \arrow[r] & \mathbb{P}_{B,B}
\end{tikzcd}
$$

are pullbacks.

Applications

Applications:

- HON
- variants
- AJM
- TO

May all be expressed as game settings, abstract composition agrees with traditional composition.

Conclusion

Results:

- conceptual definitions of composition of strategies
- abstract proof this forms a category
- applications to different game models

Not discussed here:

- conceptual definition of innocence
- proof that innocent strategies form a subcategory

Perspectives:

- more models
- more structure

Thank you.

Result Transfer

Adjunction:

$$\begin{tikzcd} \mathsf{Set} \arrow[r, bend left, "\mathsf{l}"] \arrow[r, phantom, "\perp"] & \mathbf{2} \arrow[l, bend left, "\mathsf{r}"] \end{tikzcd}$$

Result transfer:

$$\begin{tikzcd} \widehat{\mathbb{P}_{A,B}} \arrow[r, bend left, "\mathsf{l}_!"] \arrow[r, phantom, "\perp"] & \widetilde{\mathbb{P}_{A,B}} \arrow[l, bend left, "\mathsf{r}_!"] \end{tikzcd}$$

$$\begin{tikzcd} \widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,C}} \arrow[r, "\mathsf{m}_{A,B,C}"] \arrow[d, "\mathsf{l}_!"'] & \widehat{\mathbb{P}_{A,C}} \arrow[d, "\mathsf{l}_!"] \\ \widetilde{\mathbb{P}_{A,B} + \mathbb{P}_{B,C}} \arrow[r, "\overline{\mathsf{m}_{A,B,C}}"'] & \widetilde{\mathbb{P}_{A,C}} \end{tikzcd}$$

Exact Squares

$$\begin{tikzcd} \mathbb{A} \arrow[r, "T"] \arrow[d, "S"'] \arrow[dr, phantom, "\overset{\varphi}{\Longrightarrow}"] & \mathbb{C} \arrow[d, "V"] \\ \mathbb{B} \arrow[r, "U"'] & \mathbb{D} \end{tikzcd}$$

Mates:

$$\begin{tikzcd} \widehat{\mathbb{A}} \arrow[d, "\Sigma_S"'] \arrow[dr, phantom, "\Sigma_\varphi \Downarrow"] & \widehat{\mathbb{C}} \arrow[l, "\Delta_T"'] \arrow[d, "\Sigma_V"] \\ \widehat{\mathbb{B}} & \widehat{\mathbb{D}} \arrow[l, "\Delta_U"] \end{tikzcd} \qquad \begin{tikzcd} \widehat{\mathbb{A}} \arrow[r, "\Pi_T"] \arrow[dr, phantom, "\overset{\Pi_\varphi}{\Longleftarrow}"] & \widehat{\mathbb{C}} \\ \widehat{\mathbb{B}} \arrow[u, "\Delta_S"] \arrow[r, "\Pi_U"'] & \widehat{\mathbb{D}} \arrow[u, "\Delta_V"'] \end{tikzcd}$$

Exact square: the mates are isomorphisms.

Guitart: conditions for square to be exact.

Distributive Squares

Conditions for

$$\begin{tikzcd} \widehat{\mathbb{A}} \arrow[r, "\Pi_T"] \arrow[d, "\Sigma_S"'] \arrow[dr, phantom, "\overset{\tilde{\varphi}}{\Longrightarrow}"] & \widehat{\mathbb{C}} \arrow[d, "\Sigma_V"] \\ \widehat{\mathbb{B}} \arrow[r, "\Pi_U"'] & \widehat{\mathbb{D}} \end{tikzcd}$$

to commute.

Composition

Associativity of composition:

$$\begin{tikzcd}
\widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,C} + \mathbb{P}_{C,D}} \arrow[r, "\mathsf{m}_{A,B,C}+\mathbb{P}_{C,D}"] \arrow[d, "\mathbb{P}_{A,B}+\mathsf{m}_{B,C,D}"'] & \widehat{\mathbb{P}_{A,C} + \mathbb{P}_{C,D}} \arrow[d, "\mathsf{m}_{A,C,D}"] \\
\widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,D}} \arrow[r, "\mathsf{m}_{A,B,D}"'] & \widehat{\mathbb{P}_{A,D}}
\end{tikzcd}$$

Associativity of Composition

$$\begin{tikzcd}
\widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,C} + \mathbb{P}_{C,D}} \arrow[r, "\Pi"] \arrow[d, "\Pi"'] & \widehat{\mathbb{P}_{(A,B),(B,C)} + \mathbb{P}_{C,D}} \arrow[r, "\Delta"] & \widehat{\mathbb{P}_{A,B,C} + \mathbb{P}_{C,D}} \arrow[r, "\Sigma"] & \widehat{\mathbb{P}_{A,C} + \mathbb{P}_{C,D}} \arrow[d, "\Pi"] \\
\widehat{\mathbb{P}_{A,B} + \mathbb{P}_{(B,C),(C,D)}} \arrow[d, "\Delta"'] & & & \widehat{\mathbb{P}_{(A,C),(C,D)}} \arrow[d, "\Delta"] \\
\widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,C,D}} \arrow[d, "\Sigma"'] & & & \widehat{\mathbb{P}_{A,C,D}} \arrow[d, "\Sigma"] \\
\widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,D}} \arrow[r, "\Pi"'] & \widehat{\mathbb{P}_{(A,B),(B,D)}} \arrow[r, "\Delta"'] & \widehat{\mathbb{P}_{A,B,D}} \arrow[r, "\Sigma"'] & \widehat{\mathbb{P}_{A,D}}
\end{tikzcd}$$

Proof: Associativity of Composition

$$\begin{tikzcd}
\mathbb{P}_{A,B}+\mathbb{P}_{B,C}+\mathbb{P}_{C,D} \arrow[r, "\Pi"] \arrow[d, "\Pi"'] & \mathbb{P}_{(A,B),(B,C)}+\mathbb{P}_{C,D} & \mathbb{P}_{A,B,C}+\mathbb{P}_{C,D} \arrow[l, "\Delta"'] \arrow[r, "\Sigma"] & \mathbb{P}_{A,C}+\mathbb{P}_{C,D} \arrow[d, "\Pi"] \\
\mathbb{P}_{A,B}+\mathbb{P}_{(B,C),(C,D)} & & & \mathbb{P}_{(A,C),(C,D)} \\
\mathbb{P}_{A,B}+\mathbb{P}_{B,C,D} \arrow[u, "\Delta"] \arrow[d, "\Sigma"'] & & & \mathbb{P}_{A,C,D} \arrow[u, "\Delta"'] \arrow[d, "\Sigma"] \\
\mathbb{P}_{A,B}+\mathbb{P}_{B,D} \arrow[r, "\Pi"'] & \mathbb{P}_{(A,B),(B,D)} & \mathbb{P}_{A,B,D} \arrow[l, "\Delta"] \arrow[r, "\Sigma"'] & \mathbb{P}_{A,D}
\end{tikzcd}$$

Proof: Associativity of Composition

$$
\begin{tikzcd}
\mathbb{P}_{A,B} + \mathbb{P}_{B,C} + \mathbb{P}_{C,D} \arrow[r, "\Pi"] \arrow[d, "\Pi"'] & \mathbb{P}_{(A,B),(B,C)} + \mathbb{P}_{C,D} \arrow[d, "\Pi"] & \mathbb{P}_{A,B,C} + \mathbb{P}_{C,D} \arrow[l, "\Delta"'] \arrow[r, "\Sigma"] & \mathbb{P}_{A,C} + \mathbb{P}_{C,D} \arrow[d, "\Pi"] \\
\mathbb{P}_{A,B} + \mathbb{P}_{(B,C),(C,D)} \arrow[r, "\Pi"'] & \mathbb{P}_{(A,B),(B,C),(C,D)} & & \mathbb{P}_{(A,C),(C,D)} \\
\mathbb{P}_{A,B} + \mathbb{P}_{B,C,D} \arrow[u, "\Delta"] \arrow[d, "\Sigma"'] & & \mathbb{P}_{A,B,C,D} \arrow[ul, "\Delta"'] \arrow[r, "\Sigma"] \arrow[d, "\Sigma"'] & \mathbb{P}_{A,C,D} \arrow[u, "\Delta"'] \arrow[d, "\Sigma"] \\
\mathbb{P}_{A,B} + \mathbb{P}_{B,D} \arrow[r, "\Pi"'] & \mathbb{P}_{(A,B),(B,D)} & \mathbb{P}_{A,B,D} \arrow[l, "\Delta"] \arrow[r, "\Sigma"'] & \mathbb{P}_{A,D}
\end{tikzcd}
$$

Proof: Associativity of Composition (cont.)

$$
\begin{tikzcd}
\mathbb{P}_{A,B} + \mathbb{P}_{(B,C),(C,D)} \arrow[rr, "\Pi"] & & \mathbb{P}_{(A,B),(B,C),(C,D)} \\
\mathbb{P}_{A,B} + \mathbb{P}_{B,C,D} \arrow[u, "\Delta"] \arrow[r, "\Pi"] \arrow[dr, "\Pi"'] \arrow[dd, "\Sigma"'] & \mathbb{P}_{(A,B),(B,C),(C,D)} & \mathbb{P}_{A,B,C,D} \arrow[l, "\Delta"'] \arrow[u, "\Delta"'] \arrow[dl, "\Delta"] \arrow[dd, "\Sigma"] \\
& \mathbb{P}_{(A,B),(B,C,D)} \arrow[u, "\Delta"] \arrow[d, "\Sigma"] & \\
\mathbb{P}_{A,B} + \mathbb{P}_{B,D} \arrow[r, "\Pi"'] & \mathbb{P}_{(A,B),(B,D)} & \mathbb{P}_{A,B,D} \arrow[l, "\Delta"]
\end{tikzcd}
$$

Applications

Applications: HON, variants, AJM, TO.

May all be expressed as game settings, abstract composition agrees with traditional composition.

Subtleties:

- HON: liberal definition of $\mathbb{P}_A$ (for projections $\mathbb{P}_{A,B} \to \mathbb{P}_A$ to exist)
- AJM: slightly different definition of $\mathbb{P}_{A,B,C}$ (projection $\mathbb{P}_{A,B,C} \to \mathbb{P}_{A,C}$ should be a discrete fibration)

Blass games: composition known to be non-associative. Cannot be expressed as a game setting (zipping fails).

Innocence

Idea: characterise purely functional programs.

Innocent strategy: can only change its behaviour based on its view.

View: certain type of play.

- strategy of counter:

$$q \quad 0 \qquad\qquad q \quad 1$$

- strategy of successor function:

$$q_r \quad q_l \quad 5_l \quad 6_r \qquad\qquad q_r \quad q_l \quad 5_l \quad 6_r$$

Innocence: the strategy accepts a play iff it accepts all its views.

Innocence

Idea: characterise purely functional programs.

Innocent strategy: can only change its behaviour based on its view.

View: certain type of play.

- strategy of counter:

$$q \quad 0 \qquad\qquad q \quad 1$$

- strategy of successor function:

$$q_r \quad q_l \quad 5_l \quad 6_r \qquad\qquad q_r \quad q_l \quad 5_l \quad 6_r$$

Innocence: the strategy accepts a play iff it accepts all its views.

Innocence

Idea: characterise purely functional programs.

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- strategy of counter:

$$q \quad 0 \qquad\qquad q \quad 1$$

- strategy of successor function:

$$q_r \quad q_l \quad 5_l \quad 6_r \qquad\qquad q_r \quad q_l \quad 5_l \quad 6_r$$

Innocence: the strategy accepts a play iff it accepts all its views.

Innocent Game Settings

Add $\mathbb{V}_{A,B} \xrightarrow{\mathsf{i}_{A,B}} \mathbb{P}_{A,B}$ to the setting.

Innocent strategy: $\widehat{\mathbb{V}_{A,B}} \xrightarrow{\prod_{\mathsf{i}_{A,B}}} \widehat{\mathbb{P}_{A,B}}$.

Composition of innocent strategies:

$$\widehat{\mathbb{V}_{A,B} + \mathbb{V}_{B,C}} \xrightarrow{\prod_{\mathsf{i}_{A,B}+\mathsf{i}_{B,C}}} \widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,C}} \xrightarrow{\mathsf{m}_{A,B,C}} \widehat{\mathbb{P}_{A,C}}$$

Preservation of innocence: composition of innocent strategies is again innocent (in the image of $\prod_{\mathsf{i}_{A,C}}$).

Preservation of Innocence

Preservation of innocence:

$$\begin{tikzcd}
\widehat{\mathbb{V}_{A,B} + \mathbb{V}_{B,C}} \arrow[r, "\Pi_{\mathsf{i}_{A,B}+\mathsf{i}_{B,C}}"] \arrow[d, equal] & \widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,C}} \arrow[r, "\mathsf{m}_{A,B,C}"] & \widehat{\mathbb{P}_{A,C}} \arrow[d, "\Delta_{\mathsf{i}_{A,C}}"] \\
\widehat{\mathbb{V}_{A,B} + \mathbb{V}_{B,C}} \arrow[d, equal] & & \widehat{\mathbb{V}_{A,C}} \arrow[d, "\Pi_{\mathsf{i}_{A,C}}"] \\
\widehat{\mathbb{V}_{A,B} + \mathbb{V}_{B,C}} \arrow[r, "\Pi_{\mathsf{i}_{A,B}+\mathsf{i}_{B,C}}"'] & \widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,C}} \arrow[r, "\mathsf{m}_{A,B,C}"'] & \widehat{\mathbb{P}_{A,C}}
\end{tikzcd}$$

Proof: Preservation of Innocence

Alternative definition of composition:

$$\widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,C}} \xrightarrow{\Pi} \widehat{\mathbb{P}_{(A,B),(B,C)}} \xrightarrow{\Delta} \widehat{\mathbb{P}_{A,B,C}} \xrightarrow{\Sigma} \widehat{\mathbb{P}_{A,C}}$$

$$\begin{tikzcd}
\widehat{\mathbb{V}_{A,B} + \mathbb{V}_{B,C}} \arrow[r, "\Pi"] \arrow[d, equal] & \widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,C}} \arrow[r, "\Pi"] & \widehat{\mathbb{P}_{(A,B),(B,C)}} \arrow[r, "\Delta"] & \widehat{\mathbb{P}_{A,B,C}} \arrow[r, "\Sigma"] & \widehat{\mathbb{P}_{A,C}} \arrow[d, "\Delta"] \\
\widehat{\mathbb{V}_{A,B} + \mathbb{V}_{B,C}} \arrow[d, equal] & & & & \widehat{\mathbb{V}_{A,C}} \arrow[d, "\Pi"] \\
\widehat{\mathbb{V}_{A,B} + \mathbb{V}_{B,C}} \arrow[r, "\Pi"'] & \widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,C}} \arrow[r, "\Pi"'] & \widehat{\mathbb{P}_{(A,B),(B,C)}} \arrow[r, "\Delta"'] & \widehat{\mathbb{P}_{A,B,C}} \arrow[r, "\Sigma"'] & \widehat{\mathbb{P}_{A,C}}
\end{tikzcd}$$

Proof: Preservation of Innocence

$$\begin{tikzcd}
\mathbb{V}_{A,B} + \mathbb{V}_{B,C} \arrow[r, "\Pi"] \arrow[d, equal] & \mathbb{P}_{A,B} + \mathbb{P}_{B,C} \arrow[r, "\Pi"] & \mathbb{P}_{(A,B),(B,C)} & \mathbb{P}_{A,B,C} \arrow[l, "\Delta"'] \arrow[r, "\Sigma"] & \mathbb{P}_{A,C} \\
\mathbb{V}_{A,B} + \mathbb{V}_{B,C} \arrow[d, equal] & & & & \mathbb{V}_{A,C} \arrow[u, "\Delta"'] \arrow[d, "\Pi"] \\
\mathbb{V}_{A,B} + \mathbb{V}_{B,C} \arrow[r, "\Pi"'] & \mathbb{P}_{A,B} + \mathbb{P}_{B,C} \arrow[r, "\Pi"'] & \mathbb{P}_{(A,B),(B,C)} & \mathbb{P}_{A,B,C} \arrow[l, "\Delta"] \arrow[r, "\Sigma"'] & \mathbb{P}_{A,C}
\end{tikzcd}$$

Proof: Preservation of Innocence

$$\begin{tikzcd}
\mathbb{V}_{A,B} + \mathbb{V}_{B,C} \arrow[r, "\Pi"] \arrow[d, equal] & \mathbb{P}_{A,B} + \mathbb{P}_{B,C} \arrow[r, "\Pi"] & \mathbb{P}_{(A,B),(B,C)} & \mathbb{P}_{A,B,C} \arrow[l, "\Delta"'] \arrow[r, "\Sigma"] & \mathbb{P}_{A,C} \\
\mathbb{V}_{A,B} + \mathbb{V}_{B,C} \arrow[rr, "\Pi"] \arrow[d, equal] & & \mathbb{V}_{(A,B),(B,C)} \arrow[u, "\Delta"] \arrow[d, "\Pi"] & \mathbb{V}_{A,B,C} \arrow[l, "\Delta"'] \arrow[r, "\Sigma"] \arrow[u, "\Delta"] \arrow[d, "\Pi"] & \mathbb{V}_{A,C} \arrow[u, "\Delta"'] \arrow[d, "\Pi"'] \\
\mathbb{V}_{A,B} + \mathbb{V}_{B,C} \arrow[r, "\Pi"'] & \mathbb{P}_{A,B} + \mathbb{P}_{B,C} \arrow[r, "\Pi"'] & \mathbb{P}_{(A,B),(B,C)} & \mathbb{P}_{A,B,C} \arrow[l, "\Delta"] \arrow[r, "\Sigma"'] & \mathbb{P}_{A,C}
\end{tikzcd}$$

Proof: Preservation of Innocence

$$\begin{tikzcd}
\mathbb{V}_{A,B}+\mathbb{V}_{B,C} \arrow[r, "\Pi"] \arrow[d, equal] & \mathbb{P}_{A,B}+\mathbb{P}_{B,C} \arrow[r, "\Pi"] & \mathbb{P}_{(A,B),(B,C)} & \mathbb{P}_{A,B,C} \arrow[l, "\Delta"'] \arrow[r, "\Sigma"] & \mathbb{P}_{A,C} \\
\mathbb{V}_{A,B}+\mathbb{V}_{B,C} \arrow[rr, "\Pi"] \arrow[d, equal] & & \mathbb{V}_{(A,B),(B,C)} \arrow[u, "\Delta"] \arrow[d, "\Pi"'] & \mathbb{V}_{A,B,C} \arrow[l, "\Delta"'] \arrow[r, "\Sigma"] \arrow[u, "\Delta"] \arrow[d, "\Pi"'] & \mathbb{V}_{A,C} \arrow[u, "\Delta"'] \arrow[d, "\Pi"] \\
\mathbb{V}_{A,B}+\mathbb{V}_{B,C} \arrow[r, "\Pi"'] & \mathbb{P}_{A,B}+\mathbb{P}_{B,C} \arrow[r, "\Pi"'] & \mathbb{P}_{(A,B),(B,C)} & \mathbb{P}_{A,B,C} \arrow[l, "\Delta"] \arrow[r, "\Sigma"'] & \mathbb{P}_{A,C}
\end{tikzcd}$$

- simple commutation

Proof: Preservation of Innocence

$$
\begin{tikzcd}
\mathbb{V}_{A,B} + \mathbb{V}_{B,C} \arrow[r, "\Pi"] \arrow[d, equal] & \mathbb{P}_{A,B} + \mathbb{P}_{B,C} \arrow[r, "\Pi"] & \mathbb{P}_{(A,B),(B,C)} & \mathbb{P}_{A,B,C} \arrow[l, "\Delta"'] \arrow[r, "\Sigma"] & \mathbb{P}_{A,C} \\
\mathbb{V}_{A,B} + \mathbb{V}_{B,C} \arrow[rr, "\Pi"] \arrow[d, equal] & & \mathbb{V}_{(A,B),(B,C)} \arrow[u, "\Delta"] \arrow[d, "\Pi"'] & \mathbb{V}_{A,B,C} \arrow[l, "\Delta"'] \arrow[r, "\Sigma"] \arrow[u, "\Delta"] \arrow[d, "\Pi"'] & \mathbb{V}_{A,C} \arrow[u, "\Delta"'] \arrow[d, "\Pi"] \\
\mathbb{V}_{A,B} + \mathbb{V}_{B,C} \arrow[r, "\Pi"'] & \mathbb{P}_{A,B} + \mathbb{P}_{B,C} \arrow[r, "\Pi"'] & \mathbb{P}_{(A,B),(B,C)} & \mathbb{P}_{A,B,C} \arrow[l, "\Delta"] \arrow[r, "\Sigma"'] & \mathbb{P}_{A,C}
\end{tikzcd}
$$

- simple commutation
- exact squares (Guitart)

Proof: Preservation of Innocence

$$\begin{tikzcd}
\mathbb{V}_{A,B} + \mathbb{V}_{B,C} \arrow[r, "\Pi"] \arrow[d, equal] & \mathbb{P}_{A,B} + \mathbb{P}_{B,C} \arrow[r, "\Pi"] & \mathbb{P}_{(A,B),(B,C)} & \mathbb{P}_{A,B,C} \arrow[l, "\Delta"'] \arrow[r, "\Sigma"] & \mathbb{P}_{A,C} \\
\mathbb{V}_{A,B} + \mathbb{V}_{B,C} \arrow[rr, "\Pi"] \arrow[d, equal] & & \mathbb{V}_{(A,B),(B,C)} \arrow[u, "\Delta"] \arrow[d, "\Pi"'] & \color{red}{\mathbb{V}_{A,B,C}} \arrow[l, "\Delta"'] \arrow[u, "\Delta"'] \arrow[r, red, "\Sigma"] \arrow[d, red, "\Pi"'] & \color{red}{\mathbb{V}_{A,C}} \arrow[u, "\Delta"'] \arrow[d, red, "\Pi"] \\
\mathbb{V}_{A,B} + \mathbb{V}_{B,C} \arrow[r, "\Pi"'] & \mathbb{P}_{A,B} + \mathbb{P}_{B,C} \arrow[r, "\Pi"'] & \mathbb{P}_{(A,B),(B,C)} & \color{red}{\mathbb{P}_{A,B,C}} \arrow[l, "\Delta"] \arrow[r, red, "\Sigma"'] & \color{red}{\mathbb{P}_{A,C}}
\end{tikzcd}$$

- simple commutation
- exact squares (Guitart)
- distributive squares

Conditions to Preserve Innocence

Locality: $\delta_1 \colon \mathbb{P}_{A,B,C} \to \mathbb{P}_{A,C}$ and $\iota_0 \colon \mathbb{P}_A \to \mathbb{P}_{A,A}$ are sheaves.

$$\begin{tikzcd} & \mathbb{P}_{A,B,C} \arrow[d, "\delta_1"] \\ \mathbb{V}_{A,C} \arrow[r, "\mathrm{i}_{A,C}"'] & \mathbb{P}_{A,C} \end{tikzcd}$$

View-analyticity:

Conditions to Preserve Innocence

Locality: $\delta_1 \colon \mathbb{P}_{A,B,C} \to \mathbb{P}_{A,C}$ and $\iota_0 \colon \mathbb{P}_A \to \mathbb{P}_{A,A}$ are sheaves.

$$\begin{tikzcd}
\mathbb{V}_{A,B,C} \arrow[r, "\mathrm{i}_{A,B,C}"] \arrow[d, "\delta_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{P}_{A,B,C} \arrow[d, "\delta_1"] \\
\mathbb{V}_{A,C} \arrow[r, "\mathrm{i}_{A,C}"'] & \mathbb{P}_{A,C}
\end{tikzcd}$$

View-analyticity:

Conditions to Preserve Innocence

Locality: $\delta_1 : \mathbb{P}_{A,B,C} \to \mathbb{P}_{A,C}$ and $\iota_0 : \mathbb{P}_A \to \mathbb{P}_{A,A}$ are sheaves.

$$\begin{tikzcd}
\mathbb{V}_{A,B,C} \arrow[r, "\mathsf{i}_{A,B,C}"] \arrow[d, "\delta_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{P}_{A,B,C} \arrow[d, "\delta_1"] \\
\mathbb{V}_{A,C} \arrow[r, "\mathsf{i}_{A,C}"'] & \mathbb{P}_{A,C}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & u_1 \arrow[dd, dashed] & \\
u_n \arrow[dd, dashed] & & \\
 & v_1 \arrow[dr, "f_1"] & \\
v_n \arrow[rr, "f_n"'] & & p
\end{tikzcd}$$

View-analyticity:

Conditions to Preserve Innocence

Locality: $\delta_1 \colon \mathbb{P}_{A,B,C} \to \mathbb{P}_{A,C}$ and $\iota_0 \colon \mathbb{P}_A \to \mathbb{P}_{A,A}$ are sheaves.

$$\begin{tikzcd}
\mathbb{V}_{A,B,C} \arrow[r, "\mathsf{i}_{A,B,C}"] \arrow[d, "\delta_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{P}_{A,B,C} \arrow[d, "\delta_1"] \\
\mathbb{V}_{A,C} \arrow[r, "\mathsf{i}_{A,C}"'] & \mathbb{P}_{A,C}
\end{tikzcd}$$

$$\begin{tikzcd}
 & u_1 \arrow[dr, red] \arrow[dd, dashed] & \\
u_n \arrow[rr, red] \arrow[dd, dashed] & & \color{red}{u} \arrow[dd, dashed, red] \\
 & v_1 \arrow[dr, "f_1"] & \\
v_n \arrow[rr, "f_n"'] & & p
\end{tikzcd}$$

View-analyticity:

Conditions to Preserve Innocence

Locality: $\delta_1 \colon \mathbb{P}_{A,B,C} \to \mathbb{P}_{A,C}$ and $\iota_0 \colon \mathbb{P}_A \to \mathbb{P}_{A,A}$ are sheaves.

$$
\begin{tikzcd}
\mathbb{V}_{A,B,C} \arrow[r, "\mathrm{i}_{A,B,C}"] \arrow[d, "\delta_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{P}_{A,B,C} \arrow[d, "\delta_1"] \\
\mathbb{V}_{A,C} \arrow[r, "\mathrm{i}_{A,C}"'] & \mathbb{P}_{A,C}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & u_1 \arrow[dr, red] \arrow[dd, dashed] & \\
u_n \arrow[rr, red] \arrow[dd, dashed] & & {\color{red} u} \arrow[dd, dashed, red] \\
 & v_1 \arrow[dr, "f_1"] & \\
v_n \arrow[rr, "f_n"'] & & p
\end{tikzcd}
$$

View-analyticity:

$$
\mathrm{i}_{A,B}(v) \xrightarrow{\quad f \quad} \delta_2(u)
$$

Conditions to Preserve Innocence

Locality: $\delta_1 : \mathbb{P}_{A,B,C} \to \mathbb{P}_{A,C}$ and $\iota_0 : \mathbb{P}_A \to \mathbb{P}_{A,A}$ are sheaves.

$$\begin{tikzcd} \mathbb{V}_{A,B,C} \arrow[r, "\mathrm{i}_{A,B,C}"] \arrow[d, "\delta_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{P}_{A,B,C} \arrow[d, "\delta_1"] \\ \mathbb{V}_{A,C} \arrow[r, "\mathrm{i}_{A,C}"'] & \mathbb{P}_{A,C} \end{tikzcd}$$

$$\begin{tikzcd} & u_1 \arrow[rd, red] \arrow[d, dashed] & \\ u_n \arrow[rr, red] \arrow[dd, dashed] & v_1 \arrow[rd, "f_1"] & \color{red}{u} \arrow[d, dashed, red] \\ & & p \\ v_n \arrow[rru, "f_n"'] & & \end{tikzcd}$$

View-analyticity:

$$\begin{tikzcd} \mathrm{i}_{A,B}(v) \arrow[rr, "f"] \arrow[rd, "g"'] & & \delta_2(u) \\ & \delta_2(\mathrm{i}_{A,B,C}(w)) \arrow[ru, "\delta_2(h)"'] & \end{tikzcd}$$

Boolean Innocent Strategies

But (non-deterministic) innocent strategies should not compose!

Answer:

$$\begin{tikzcd}
\widehat{\mathbb{V}_{A,B} + \mathbb{V}_{B,C}} \arrow[r] & \widehat{\mathbb{P}_{A,C}} \arrow[d, "\mathsf{l}_!"] \\
\mathbb{V}_{A,\widetilde{B} + \mathbb{V}}_{B,C} \arrow[u, "\mathsf{r}_!"] \arrow[r] & \widetilde{\mathbb{P}_{A,C}}
\end{tikzcd}$$

does not commute.

- *concurrent* innocent strategies compose
- traditional innocent strategies do not