Categories and String Diagrams for Concurrent Game Semantics

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Semantics of Programming Languages

Semantics:

- programs = text (syntax)
- no meaning *per se*
- programs → meaning/math (semantics)

Why?

- prove properties (of programs)
- prove properties (of programming languages)
- understand logics (*Curry-Howard correspondence*)

Operational Semantics

Intuition: computing machine.

Types:

syntactic labelled transition systems	abstract machines
$\dfrac{}{(\lambda x.M)N \to M[N/x]}$ $\dfrac{M \to M'}{MN \to M'N}$ $\dfrac{N \to N'}{MN \to MN'}$	$MN \star \pi \to M \star N :: \pi$ $\lambda x.M \star N :: \pi \to M[N/x] \star \pi$

Characteristics:

- syntactic
- dynamic

Denotational Semantics

Idea:

- types → spaces
- programs → functions

Example: Scott domains:

- spaces = partial orders
- functions = monotone maps

Characteristics:

- syntax-free
- static
- compositional

Features of Game Semantics

Between operational and denotational:

- types → games
- programs → strategies
- dynamic
- related to syntax (equivalence of programs is intensional, rather than extensional)

Game semantics:

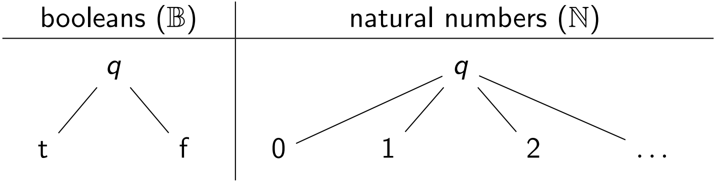
- significant part of denotational semantics
- "solved" full abstraction for PCF
- many variants characterise different programming features
- applications: model checking, hardware synthesis...

Game Semantics: a Brief (Pre)history

- Lorenz-Lorenzen (70s): *dialogical logic*
- Joyal (77): category of strategies
- Blass (92): links to *linear logic*
- Coquand (92): links to execution of programs and innocence
- Hyland-Ong/Nickau, Abramsky-Jagadeesan-Malacaria (90s): models of PCF

HON Game Semantics: Games

Structures possible moves.



HON Game Semantics: Plays

Basically:

- sequence of moves
- interaction between program and environment

Example: `f = fun n -> 2 * n`

$$
\begin{array}{ccc}
\mathbb{N} & \longrightarrow & \mathbb{N} \\
 & & q_r \\
q_l & & \\
6_l & & \\
 & & 12_r
\end{array}
$$

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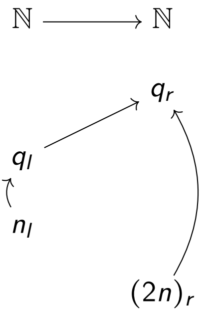
$$
\begin{tikzcd}
\mathbb{N} \arrow[r] & \mathbb{N} \\
& q_r \\
q_l \arrow[ur] & \\
6_l \arrow[u, bend left] & \\
& 12_r \arrow[uuu, bend right]
\end{tikzcd}
$$

HON Game Semantics: Strategies

Strategy = prefix-closed set of accepted plays.

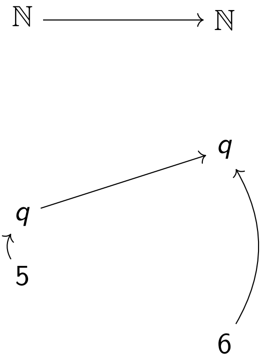
Example: for `f = fun n -> 2 * n`, (roughly) all plays of the form



HON Game Semantics: Composition

Parallel composition + hiding.

`f = fun n -> n + 1`



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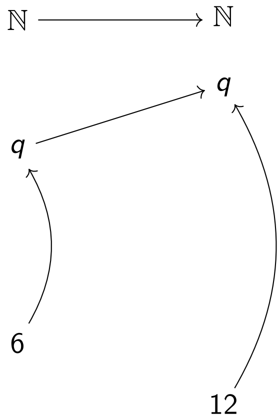
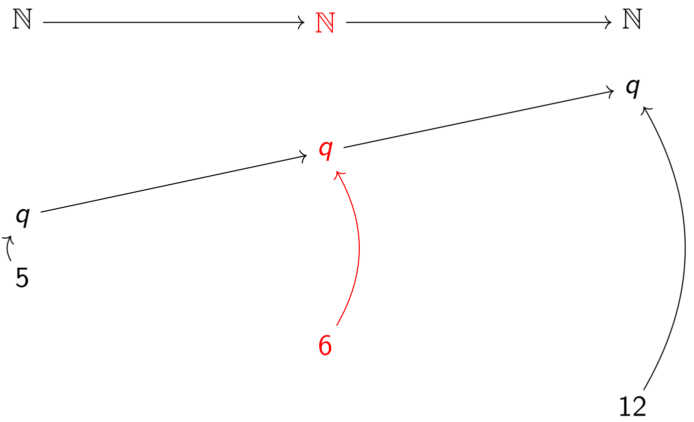


HON Game Semantics: Composition

Parallel composition + hiding.

```
f = fun n -> n + 1
```

```
f = fun n -> 2 * n
```



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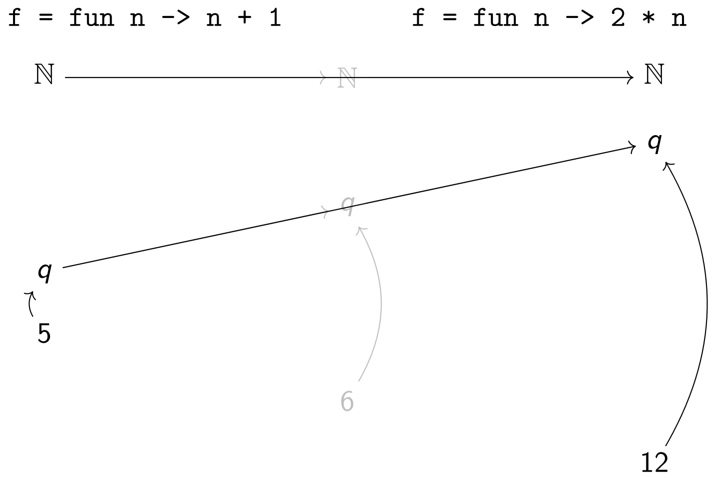

Innocence

Idea: characterise purely functional programs.

Innocent strategy: can only change its behaviour based on its view.

View: certain type of play.

- strategy of counter:

$$q \quad 0 \qquad\qquad q \quad 1$$

- strategy of successor function:

$$q_r \quad q_l \quad 5_l \quad 6_r \qquad\qquad q_r \quad q_l \quad 5_l \quad 6_r$$

Innocence: the strategy accepts a play iff it accepts all its views.

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Strategies as Presheaves

Presheaf over $\mathbb{C}$: functor $\mathbb{C}^{op} \to \mathrm{Set}$.

Boolean presheaf over $\mathbb{C}$: functor $\mathbb{C}^{op} \to 2$ $(2 = 0 \to 1)$.

Two notions of strategies:

- prefix-closed sets of plays: boolean presheaves $\widetilde{\mathbb{P}_{A,B}}$
- concurrent strategies: presheaves $\widehat{\mathbb{P}_{A,B}}$

$$\begin{tikzcd} & \cdot \arrow[d, "a"] & \\ & \cdot \arrow[dl, "b"'] \arrow[dr, "c"] & \\ \cdot & & \cdot \end{tikzcd} \qquad \begin{tikzcd} & \cdot \arrow[dl, "a"'] \arrow[dr, "a"] & \\ \cdot \arrow[d, "b"'] & & \cdot \arrow[d, "c"] \\ \cdot & & \cdot \end{tikzcd}$$

$$\sigma(a) = \{x\} \qquad \sigma(a) = \{x, x'\}$$

With traditional strategies:

$$\sigma = \{\varepsilon, a, ab, ac\}$$

Outline

1. Fibred Approach to Game Semantics
   - Building Categories of Plays Abstractly
   - Application: Justified Sequences in String Diagrams

2. A Theory of Game Models
   - Game Settings, Composition, and Identities
   - A Category of Strategies
   - Innocence

Motivation

Recurring construction in the string diagrammatic approach to game semantics:

- start from an operational description of a language
- design string diagrams for this language
- derive a pseudo double category $\mathbb{D}$ describing the game
- derive categories $\mathbb{E}(X)$ of plays
- strategies $= \widehat{\mathbb{E}(X)}$

Here: delineate hypotheses for $\mathbb{E}(X)$ to be a category.

String Diagrams

Intuition: graphs = presheaves over

$$\begin{tikzcd} & {[1]} \\ & \star \arrow[u, bend left, "s"] \arrow[u, bend right, "t"'] \end{tikzcd}$$

$$\begin{tikzcd} y & & b \arrow[dr, bend left, "\gamma"] \arrow[dr, bend right, "\delta"'] & \\ x \arrow[u, "\alpha"] & a \arrow[ur, "\beta"] & & c \arrow[ll, "\varepsilon"] \end{tikzcd}$$

String diagram = presheaf over $\mathbb{C}$

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\begin{tikzcd}
{[1]} \\
\star \arrow[u, bend left, "s"] \arrow[u, bend right, "t"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
E \arrow[d, bend right, "\sigma"'] \arrow[d, bend left, "\tau"] \\
V
\end{tikzcd}
$$

$$
\begin{tikzcd}
y & & b \arrow[dr, bend left, "\gamma"] \arrow[dr, "\delta"'] & \\
& & & \\
x \arrow[uu, "\alpha"] & a \arrow[uur, "\beta"] & & c \arrow[ll, "\varepsilon"]
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$$

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V
\end{tikzcd}
$$

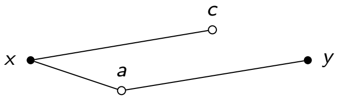
$$
\begin{tikzcd}
\overset{y}{\cdot} & & \overset{b}{\cdot} \arrow[dr, bend left, "\gamma"] \arrow[dr, "\delta"'] & \\
\underset{x}{\cdot} \arrow[u, "\alpha"] & \underset{a}{\cdot} \arrow[ur, "\beta"] & & \underset{c}{\cdot} \arrow[ll, "\varepsilon"]
\end{tikzcd}
$$

- $V = \{a, b, c, x, y\}$
- $E = \{\alpha, \beta, \gamma, \delta, \varepsilon\}$
- $\alpha \cdot s = x,\ \alpha \cdot t = y \ldots$

String diagram = presheaf over $\mathbb{C}$

Positions for the π-calculus

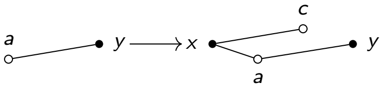
positions ~ graphs



presheaf X over $\mathbb{C}_{|\leq 1}$

- $X(\star) = \{a, c\}$,
  $X([1]) = \{y\} \ldots$
- $y \cdot s_1 = a \ldots$

Morphisms ~ embeddings of graphs



Natural transformations

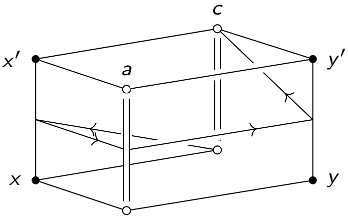
$$h \colon X' \to X$$

Moves for the π-calculus

move ~ step of computation

Example: $\bar{a}\langle c\rangle.P \mid a(b).Q \quad \to \quad P \mid Q[b := c]$

~ higher-dimensional graphs

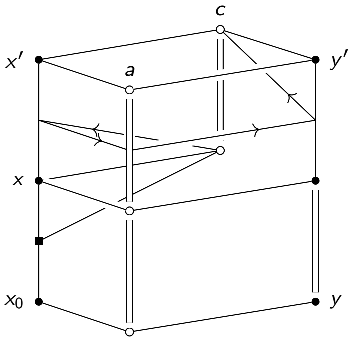


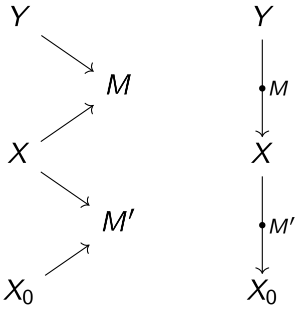
cospans of presheaves

$$\begin{array}{ccc} Y & & Y \\ \downarrow s & & \mid \\ M & & \bullet M \\ \uparrow t & & \downarrow \\ X & & X \end{array}$$

Plays for the π-calculus

play ~ composition (pasting) of moves





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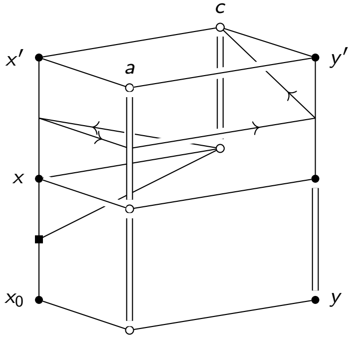

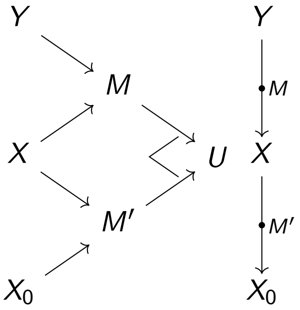

Pseudo Double Category

$\mathbb{D}$: describes the whole game.

- objects (positions)

$$\begin{array}{cc} Y & Y' \\ X & X' \end{array}$$

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- horizontal morphisms (embeddings)

$$\begin{array}{ccc} Y & & Y' \\ & & \\ X & \xrightarrow[h]{} & X' \end{array}$$

Pseudo Double Category

$\mathbb{D}$: describes the whole game.

- objects (positions)
- horizontal morphisms (embeddings)
- vertical morphisms (plays)

$$\begin{tikzcd} Y & Y' \\ X \arrow[from=1-1, "U"', "\bullet" marking] \arrow[r, "h"'] & X' \end{tikzcd}$$

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- horizontal morphisms (embeddings)
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$$\begin{tikzcd} Y \arrow[r, "k"] \arrow[d, "U"'] & Y' \arrow[d, "U'"] \\ X \arrow[r, "h"'] & X' \end{tikzcd}$$

Pseudo Double Category

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- objects (positions)
- horizontal morphisms (embeddings)
- vertical morphisms (plays)
- cells (embeddings of plays)

$$\begin{tikzcd} Y \arrow[r, "k"] \arrow[d, "U"'] \arrow[dr, phantom, "\xRightarrow{\alpha}"] & Y' \arrow[d, "U'"] \\ X \arrow[r, "h"'] & X' \end{tikzcd}$$

Pseudo Double Category

$\mathbb{D}$: describes the whole game.

- objects (positions)
- horizontal morphisms (embeddings)
- vertical morphisms (plays)
- cells (embeddings of plays)

$$
\begin{tikzcd}
Y \arrow[r, "k"] \arrow[d, "U"'] \arrow[dr, phantom, "\xRightarrow{\alpha}"] & Y' \arrow[d, "U'"] \\
X \arrow[r, "h"'] & X'
\end{tikzcd}
$$

$\mathbb{E}(X)$: describes plays over a fixed X.

objects:

$$
\begin{tikzcd}
Y \arrow[d, "U"] \\
X
\end{tikzcd}
$$

morphisms:

$$
\begin{tikzcd}
Z \arrow[r, "h"] \arrow[d, "W"'] & Y' \arrow[dd, "U'"] \\
Y \arrow[d, "U"'] \arrow[r, phantom, "\xRightarrow{\alpha}"] & \\
X \arrow[r, equal] & X
\end{tikzcd}
$$

Composition in $\mathbb{E}(X)$

$$
\begin{tikzcd}
 & Z' \arrow[r, "h'"] \arrow[d, "W'"] & Y'' \arrow[ddd, "U''"] \\
Z \arrow[r, "h"] \arrow[d, "W"'] & Y' \arrow[dd, "U'"] \arrow[r, phantom, "\overset{\alpha'}{\Longrightarrow}"] & {} \\
Y \arrow[d, "U"'] \arrow[r, phantom, "\overset{\alpha}{\Longrightarrow}"] & {} & \\
X \arrow[r, equal] & X \arrow[r, equal] & X
\end{tikzcd}
$$

Composition in $\mathbb{E}(X)$

$$
\begin{tikzcd}
Z'' \arrow[r] \arrow[d] \arrow[dr, phantom, "\Longrightarrow"] & Z' \arrow[r, "h'"] \arrow[d, "W'"] & Y'' \arrow[ddd, "U''"] \\
Z \arrow[r, "h"] \arrow[d, "W"'] & Y' \arrow[dd, "U'"] \arrow[r, phantom, "\overset{\alpha'}{\Longrightarrow}"] & \\
Y \arrow[d, "U"'] \arrow[r, phantom, "\overset{\alpha}{\Longrightarrow}"] & & \\
X \arrow[r, equal] & X \arrow[r, equal] & X
\end{tikzcd}
$$

Fibredness

$$\begin{tikzcd} & Y \arrow[d, "U"] \\ X' \arrow[r, "h"'] & X \end{tikzcd}$$

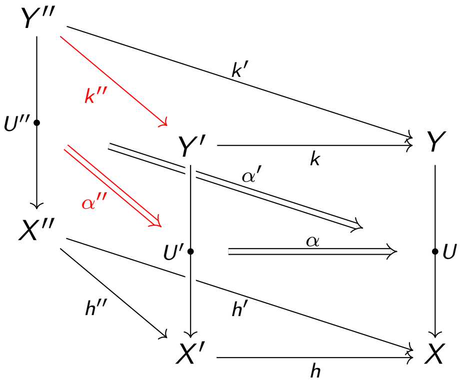
Fibredness

$$
\begin{tikzcd}
{\color{red}Y'} \arrow[r, red, "k"'] \arrow[d, red, "U'"'] \arrow[dr, phantom, "\Rightarrow\alpha"] & Y \arrow[d, "U"] \\
X' \arrow[r, "h"'] & X
\end{tikzcd}
$$

Fibredness

$$
\begin{tikzcd}
Y'' \arrow[dd, "U''"'] \arrow[drr, "k'"] & & \\
& Y' \arrow[r, "k"'] \arrow[dd, "U'"'] & Y \arrow[dd, "U"] \\
X'' \arrow[dr, "h''"'] \arrow[drr, "h'"] & & \\
& X' \arrow[r, "h"'] & X
\arrow[from=1-1, to=4-3, phantom, "\Rightarrow \alpha'"]
\arrow[from=2-2, to=4-3, phantom, "\Rightarrow \alpha"]
\end{tikzcd}
$$

Fibredness



Fibredness

$$
\begin{tikzcd}
Y'' \arrow[dr, red, "k''"'] \arrow[drr, "k'"] \arrow[dd, "U''"'] & & \\
\arrow[r, phantom, red, "\Rightarrow \alpha''"] \arrow[rr, phantom, "\Rightarrow \alpha'", shift left=2] & Y' \arrow[r, "k"'] \arrow[dd, "U'"'] & Y \arrow[dd, "U"] \\
X'' \arrow[dr, "h''"'] \arrow[drr, "h'"] & \arrow[r, phantom, "\Rightarrow \alpha"] & \\
& X' \arrow[r, "h"'] & X
\end{tikzcd}
$$

Theorem: If the data we start from is "nice enough", then $\mathbb{D}$ is fibred, so $\mathbb{E}(X)$ is a category.

Factorisation Systems

Orthogonality: $f \perp g$:

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "v"'] & D \end{tikzcd}$$

Extends to classes: $\mathcal{L} \perp \mathcal{R}$, $\mathcal{L}^{\perp}$, ${}^{\perp}\mathcal{R}$.

Factorisation system on $\mathbb{C}$: $(\mathcal{L}, \mathcal{R})$

- $A \xrightarrow{f} B = A \xrightarrow{l} X \xrightarrow{r} B$
- $\mathcal{L} \perp \mathcal{R}$

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Fibredness Through Factorisation Systems

$$\begin{tikzcd}
 & Y \arrow[d, "f"] \\
 & U \\
X' \arrow[r, "h"'] & X \arrow[u, "l"]
\end{tikzcd}$$

Fibredness Through Factorisation Systems

$$
\begin{tikzcd}
 & Y \arrow[d, "f"] \\
U' \arrow[r, "h'"'] & U \\
X' \arrow[u, "l'"] \arrow[r, "h"'] & X \arrow[u, "l"']
\end{tikzcd}
$$

Fibredness Through Factorisation Systems

$$\begin{tikzcd}
Y' \arrow[r, "h''"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "f"] \\
U' \arrow[r, "h'"'] & U \\
X' \arrow[u, "l'"] \arrow[r, "h"'] & X \arrow[u, "l"']
\end{tikzcd}$$

Fibredness Through Factorisation Systems

$$
\begin{tikzcd}
Y'' \arrow[dd, "f''"'] \arrow[drr, bend left, "q''"] & & \\
& Y' \arrow[r, "h''"] \arrow[dd, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[dd, "f"] \\
U'' \arrow[drr, bend left, "q'"'] & & \\
& U' \arrow[r, "h'"'] & U \\
X'' \arrow[uu, "l''"] \arrow[dr, "s"'] \arrow[drr, bend left, "q"'] & & \\
& X' \arrow[uu, "l'"] \arrow[r, "h"'] & X \arrow[uu, "l"']
\end{tikzcd}
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Fibredness Through Factorisation Systems

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Y'' \arrow[d, "f''"'] \arrow[rrd, bend left, "q''"] & & \\
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U'' \arrow[dr, red, "s'"'] \arrow[rrd, bend left, "q'"] & & \\
 & U' \arrow[r, "h'"'] & U \\
X'' \arrow[uu, "l''"] \arrow[dr, "s"'] \arrow[rrd, bend left, "q"] & & \\
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 & Y' \arrow[r, "h''"] \arrow[dd, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[dd, "f"] \\
U'' \arrow[dr, red, "s'"'] \arrow[drr, bend left, "q'"] & & \\
 & U' \arrow[r, "h'"'] & U \\
X'' \arrow[uu, "l''"] \arrow[dr, "s"'] \arrow[drr, bend left, "q"] & & \\
 & X' \arrow[uu, "l'"] \arrow[r, "h"'] & X \arrow[uu, "l"']
\end{tikzcd}
$$

Motivation

Two approaches to concurrent game semantics:

- Hirschowitz et al.: CCS (2011), π-calculus (2015)
- Tsukada and Ong: non-deterministic λ-calculus (2015)

Similar notions of strategies: sheaves for the "same" topology.

Different notions of plays:

- string diagrams
- justified sequences

Goal: prove formal link between the two approaches

- plays: full embedding of categories
- innocent strategies: equivalence of categories

The Level of Plays

Design string diagrams for HON games.

String diagrams are nice: position $A \vdash B \rightsquigarrow$ plays $\mathbb{E}(A \vdash B)$.

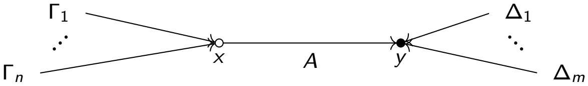
- full embedding $F{:}\,\mathbb{P}_{A,B} \to \mathbb{E}(A \vdash B)$
- restricts to equivalence $F^{\mathbb{V}}{:}\,\mathbb{V}_{A,B} \to \mathbb{E}^{\mathbb{V}}(A \vdash B)$

justified sequences:

string diagrams:

$$\begin{tikzcd}
\mathbb{V}_{A,B} \arrow[r, hook, "\mathrm{i}_{TO}"] \arrow[d, "F^{\mathbb{V}}"', "\wr"] & \mathbb{P}_{A,B} \arrow[d, "F"] \\
\mathbb{E}^{\mathbb{V}}(A \vdash B) \arrow[r, hook, "\mathrm{i}"'] & \mathbb{E}(A \vdash B)
\end{tikzcd}$$

String Diagrams for HON Games



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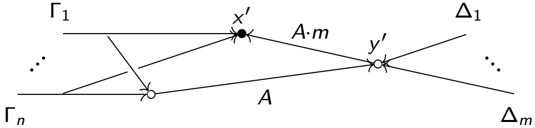

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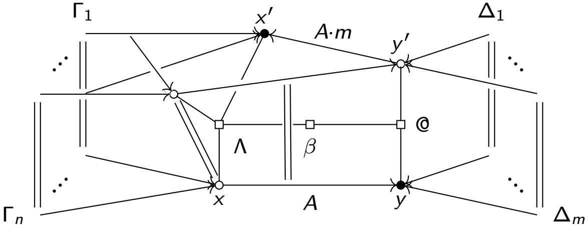
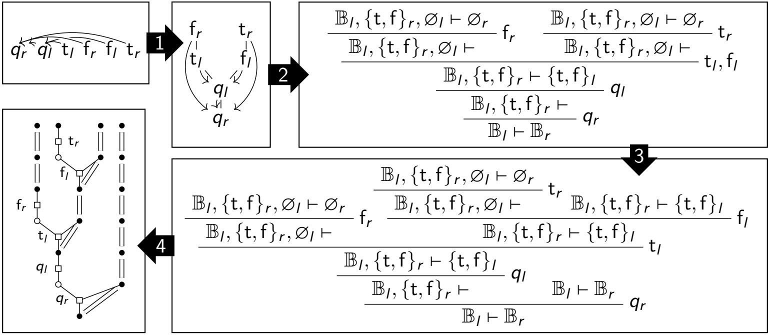


The Big Picture



Concurrent Innocence

If $F : \mathbb{C} \to \mathbb{D}$:

$$\begin{tikzcd} \widehat{\mathbb{C}} \arrow[rr, bend right, "\prod_F"'] \arrow[rr, phantom, "\perp"] & & \widehat{\mathbb{D}} \arrow[ll, bend right, "\Delta_F"'] \end{tikzcd}$$

$$\prod_F(X)(d) \cong \int_{c \in \mathbb{C}} [[F(c), d], X(c)]$$

Categories of views and plays: $\mathbb{E}^{\mathbb{V}}(A \vdash B) \xrightarrow{\mathrm{i}_{A,B}} \mathbb{E}(A \vdash B)$.

Functor: $\widehat{\mathbb{E}^{\mathbb{V}}(A \vdash B)} \xrightarrow{\prod_{\mathrm{i}_{A,B}}} \widehat{\mathbb{E}(A \vdash B)}$.

Innocent strategy: in the (essential) image of $\prod_{\mathrm{i}_{A,B}}$.

$$\prod_{\mathrm{i}_{A,B}}(\sigma)(p) \cong \int_{v \in \mathbb{V}_{A,B}} [[\mathrm{i}_{A,B}(v), p], \sigma(v)]$$

The Level of Strategies

$$\begin{tikzcd}
\widehat{\mathbb{V}_{A,B}} \arrow[r, hook, "\Pi_{\mathrm{i}_{TO}}"] & \widehat{\mathbb{P}_{A,B}} \\
\widehat{\mathbb{E}^{\mathbb{V}}(A \vdash B)} \arrow[u, "\Delta_{F^{\mathbb{V}}}"] \arrow[r, hook, "\Pi_{\mathrm{i}}"'] & \widehat{\mathbb{E}(A \vdash B)} \arrow[u, "\Delta_F"']
\end{tikzcd}$$

Theory of exact squares: square commutes (up to isomorphism).

Theorem: we get

- equivalent categories of innocent strategies
- compatible with innocentisation

Motivation

Game models:

- HON (justified sequences)
- AJM (sequences)
- simple games, Blass (trees)
- concurrent (event structures)
- string diagrams
- variants
- ...

Definitions and proofs are similar... but tricky!

Goal: define a framework that

- encompasses many models
- factors out similar proofs

Recurring Pattern

- define games A, B, C, ...
- define categories of plays $\mathbb{P}_{A,B}$
- define strategies $A \to B$ as prefix-closed sets of plays in $\mathbb{P}_{A,B}$
- composition $=$ parallel composition $+$ hiding
- identities $=$ copycat strategies
- prove that this defines a category of games and strategies
- define a notion of innocence
- prove that innocent strategies form a subcategory

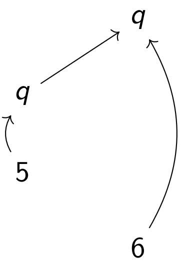
Recurring Pattern

- define games A, B, C, ... **assumed**
- define categories of plays $\mathbb{P}_{A,B}$ **assumed**
- define strategies $A \to B$ as prefix-closed sets of plays in $\mathbb{P}_{A,B}$
- composition = parallel composition + hiding
- identities = copycat strategies
- prove that this defines a category of games and strategies
- define a notion of innocence **assumed**
- prove that innocent strategies form a subcategory

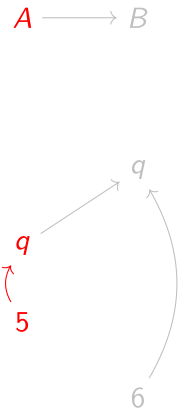
Categories of Plays

$$A \longrightarrow B$$

- games A, B, $C \ldots$
- categories of plays $\mathbb{P}_A$, $\mathbb{P}_{A,B}$, $\mathbb{P}_{A,B,C} \ldots$



Categories of Plays

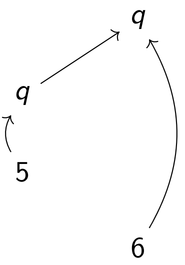


- games A, B, $C\dots$
- categories of plays $\mathbb{P}_A$, $\mathbb{P}_{A,B}$, $\mathbb{P}_{A,B,C}\dots$
- projections $\mathbb{P}_{A,B} \to \mathbb{P}_A$

Categories of Plays

$$A \longrightarrow B$$

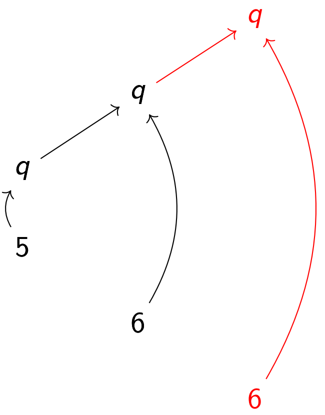
- games A, B, $C\ldots$
- categories of plays $\mathbb{P}_A$, $\mathbb{P}_{A,B}$, $\mathbb{P}_{A,B,C}\ldots$
- projections $\mathbb{P}_{A,B} \to \mathbb{P}_A$



Categories of Plays

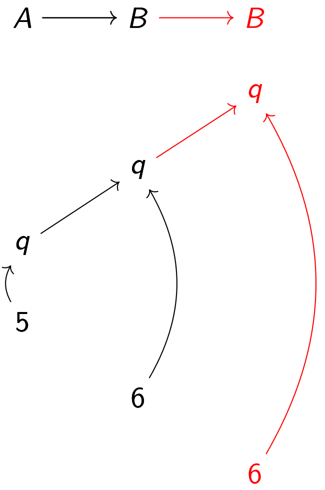
- games A, B, $C\ldots$
- categories of plays $\mathbb{P}_A$, $\mathbb{P}_{A,B}$, $\mathbb{P}_{A,B,C}\ldots$
- projections $\mathbb{P}_{A,B} \to \mathbb{P}_A$
- insertions $\mathbb{P}_{A,B} \to \mathbb{P}_{A,B,B}$

$$A \longrightarrow B \longrightarrow B$$



Categories of Plays

- games A, B, $C\ldots$
- categories of plays $\mathbb{P}_A$, $\mathbb{P}_{A,B}$, $\mathbb{P}_{A,B,C}\ldots$
- projections $\mathbb{P}_{A,B} \to \mathbb{P}_A$
- insertions $\mathbb{P}_{A,B} \to \mathbb{P}_{A,B,B}$
- compatibility between projections and insertions



Describing Categories of Plays Simplicially

Game setting:

- set $\mathbb{A}$ of games
- functor $\mathbb{P}\colon (\Delta/\mathbb{A})^{op} \to \mathrm{Cat}$

$\Delta/\mathbb{A}$:

- objects: lists $L = A_1, \ldots, A_n$ of games
- morphisms: insertions $(A, C \to A, B, C)$ and fusions $(A, A, B \to A, B)$

Strategies $A \to B$: $\widehat{\mathbb{P}_{A,B}}$.

Polynomial Functors

If $F : \mathbb{C} \to \mathbb{D}$:

$$\begin{tikzcd} \widehat{\mathbb{C}} \arrow[rr, bend right, "\Pi_F"'] \arrow[rr, phantom, "\perp"] & & \widehat{\mathbb{D}} \arrow[ll, "\Delta_F"'] \end{tikzcd}$$

Polynomial Functors

If $F\colon \mathbb{C} \to \mathbb{D}$:

$$\begin{tikzcd}[column sep=huge] \widehat{\mathbb{C}} \arrow[r, bend left=40, "\Sigma_F"] \arrow[r, bend right=40, "\Pi_F"'] & \widehat{\mathbb{D}} \arrow[l, "\Delta_F" description] \end{tikzcd}$$

Polynomial Functors

If $F{:}\,\mathbb{C} \to \mathbb{D}$:

$$\begin{tikzcd}
\widehat{\mathbb{C}} \arrow[rr, bend left, "\sum_F"] \arrow[rr, bend right, "\prod_F"'] & & \widehat{\mathbb{D}} \arrow[ll, "\Delta_F" description] \arrow[ll, phantom, shift right=4, "\perp"] \arrow[ll, phantom, shift left=4, "\perp"]
\end{tikzcd}$$

Polynomial functor: composite of Δ's, $\prod$'s, and $\sum$'s.

Composition

Idea: parallel composition + hiding.

$$\widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,C}} \xrightarrow{\Delta_{\delta_2+\delta_0}} \widehat{\mathbb{P}_{A,B,C} + \mathbb{P}_{A,B,C}} \xrightarrow{\Pi_{\nabla}} \widehat{\mathbb{P}_{A,B,C}} \xrightarrow{\Sigma_{\delta_1}} \widehat{\mathbb{P}_{A,C}}$$

Composition

Idea: parallel composition + hiding.

$$\widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,C}} \xrightarrow{\Delta_{\delta_2+\delta_0}} \widehat{\mathbb{P}_{A,B,C} + \mathbb{P}_{A,B,C}} \xrightarrow{\Pi_{\nabla}} \widehat{\mathbb{P}_{A,B,C}} \xrightarrow{\Sigma_{\delta_1}} \widehat{\mathbb{P}_{A,C}}$$

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Justification:

$\mathrm{m}_{A,B,C}(\sigma,\tau)$ accepts p

iff

iff

iff

Composition

Idea: parallel composition + hiding.

$$\widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,C}} \xrightarrow{\Delta_{\delta_2+\delta_0}} \widehat{\mathbb{P}_{A,B,C} + \mathbb{P}_{A,B,C}} \xrightarrow{\Pi_{\nabla}} \widehat{\mathbb{P}_{A,B,C}} \xrightarrow{\Sigma_{\delta_1}} \widehat{\mathbb{P}_{A,C}}$$

Justification:

$\mathrm{m}_{A,B,C}(\sigma,\tau)$ accepts p

iff there exists an interaction sequence $u \in \mathbb{P}_{A,B,C}$ that is accepted and projects to p

iff

iff

Composition

Idea: parallel composition + hiding.

$$\widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,C}} \xrightarrow{\Delta_{\delta_2+\delta_0}} \widehat{\mathbb{P}_{A,B,C} + \mathbb{P}_{A,B,C}} \xrightarrow{\Pi_\nabla} \widehat{\mathbb{P}_{A,B,C}} \xrightarrow{\Sigma_{\delta_1}} \widehat{\mathbb{P}_{A,C}}$$

Justification:

$\mathrm{m}_{A,B,C}(\sigma,\tau)$ accepts p

iff there exists an interaction sequence $u \in \mathbb{P}_{A,B,C}$ that is accepted and projects to p

iff both $\mathrm{inl}\, u$ and $\mathrm{inr}\, u$ are accepted

iff

Composition

Idea: parallel composition + hiding.

$$\widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,C}} \xrightarrow{\Delta_{\delta_2+\delta_0}} \widehat{\mathbb{P}_{A,B,C} + \mathbb{P}_{A,B,C}} \xrightarrow{\Pi_{\nabla}} \widehat{\mathbb{P}_{A,B,C}} \xrightarrow{\Sigma_{\delta_1}} \widehat{\mathbb{P}_{A,C}}$$

Justification:

	$\mathrm{m}_{A,B,C}(\sigma,\tau)$ accepts p
iff	there exists an interaction sequence $u \in \mathbb{P}_{A,B,C}$ that is accepted and projects to p
iff	both $\mathrm{inl}\, u$ and $\mathrm{inr}\, u$ are accepted
iff	σ accepts $\delta_2(u)$ and τ accepts $\delta_0(u)$.

Copycat strategies

$$1 \cong \widehat{\varnothing} \xrightarrow{\Pi_!} \widehat{\mathbb{P}_A} \xrightarrow{\Sigma_{\iota_0}} \widehat{\mathbb{P}_{A,A}}$$

Justification:

œ_A accepts p

iff

iff

Copycat strategies

$$1 \cong \widehat{\varnothing} \xrightarrow{\Pi_!} \widehat{\mathbb{P}_A} \xrightarrow{\Sigma_{\iota_0}} \widehat{\mathbb{P}_{A,A}}$$

Justification:

	$\mathfrak{c}_A$ accepts p
iff	there exists a sequence $s \in \mathbb{P}_A$ that is accepted and mapped to p
iff	

Copycat strategies

$$1 \cong \widehat{\varnothing} \xrightarrow{\color{red}{\Pi_!}} \widehat{\mathbb{P}_A} \xrightarrow{\Sigma_{\iota_0}} \widehat{\mathbb{P}_{A,A}}$$

Justification:

	$\text{\ae}_A$ accepts p
iff	there exists a sequence $s \in \mathbb{P}_A$ that is accepted and mapped to p
iff	there is an s that is mapped to p.

Game Settings

- set $\mathbb{A}$ of games
- functor $\mathbb{P}: (\Delta/\mathbb{A})^{op} \to \mathsf{Cat}$
- $\delta_1: \mathbb{P}_{A,B,C} \to \mathbb{P}_{A,C}$ and $\iota_0: \mathbb{P}_A \to \mathbb{P}_{A,A}$ discrete fibrations
- ...

Associativity of composition

Theorem (Composition is associative):

$$
\begin{tikzcd}
\widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,C} + \mathbb{P}_{C,D}} \arrow[r, "\mathsf{m}_{A,B,C}+\mathbb{P}_{C,D}"] \arrow[d, "\mathbb{P}_{A,B}+\mathsf{m}_{B,C,D}"'] & \widehat{\mathbb{P}_{A,C} + \mathbb{P}_{C,D}} \arrow[d, "\mathsf{m}_{A,C,D}"] \\
\widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,D}} \arrow[r, "\mathsf{m}_{A,B,D}"'] & \widehat{\mathbb{P}_{A,D}}
\end{tikzcd}
$$

commutes if

$$
\begin{tikzcd}
\mathbb{P}_{A,B,C,D} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{P}_{A,B,D} \arrow[d] \\
\mathbb{P}_{B,C,D} \arrow[r] & \mathbb{P}_{B,D}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathbb{P}_{A,B,C,D} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{P}_{A,C,D} \arrow[d] \\
\mathbb{P}_{A,B,C} \arrow[r] & \mathbb{P}_{A,C}
\end{tikzcd}
$$

are pullbacks (*zipping lemma*).

Unitality

Theorem (Copycat strategies are units):

$$
\begin{tikzcd}
\widehat{\varnothing + \mathbb{P}_{A,B}} \arrow[d, "\mathsf{cc}_A + \mathbb{P}_{A,B}"'] \arrow[dr, "\sim"] & & \widehat{\mathbb{P}_{A,B} + \varnothing} \arrow[d, "\mathbb{P}_{A,B} + \mathsf{cc}_B"] \arrow[dl, "\sim"'] \\
\widehat{\mathbb{P}_{A,A} + \mathbb{P}_{A,B}} \arrow[r, "\mathsf{m}_{A,A,B}"'] & \widehat{\mathbb{P}_{A,B}} & \widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,B}} \arrow[l, "\mathsf{m}_{A,B,B}"]
\end{tikzcd}
$$

commutes if

$$
\begin{tikzcd}
\mathbb{P}_{A,B} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{P}_A \arrow[d] \\
\mathbb{P}_{A,A,B} \arrow[r] & \mathbb{P}_{A,A}
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
\mathbb{P}_{A,B} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{P}_B \arrow[d] \\
\mathbb{P}_{A,B,B} \arrow[r] & \mathbb{P}_{B,B}
\end{tikzcd}
$$

are pullbacks.

Applications

Applications:

- HON
- variants
- AJM
- TO

May all be expressed as game settings, abstract composition agrees with traditional composition.

Innocent Game Settings

Add $\mathbb{V}_{A,B} \xrightarrow{\mathsf{i}_{A,B}} \mathbb{P}_{A,B}$ to the setting.

Innocent strategy: $\widehat{\mathbb{V}_{A,B}} \xrightarrow{\prod_{\mathsf{i}_{A,B}}} \widehat{\mathbb{P}_{A,B}}$.

Composition of innocent strategies:

$$\widehat{\mathbb{V}_{A,B} + \mathbb{V}_{B,C}} \xrightarrow{\prod_{\mathsf{i}_{A,B}+\mathsf{i}_{B,C}}} \widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,C}} \xrightarrow{\mathsf{m}_{A,B,C}} \widehat{\mathbb{P}_{A,C}}$$

Preservation of innocence: composition of innocent strategies is again innocent (in the image of $\prod_{\mathsf{i}_{A,C}}$).

Preservation of Innocence

Preservation of innocence:

$$
\begin{tikzcd}
\widehat{\mathbb{V}_{A,B} + \mathbb{V}_{B,C}} \arrow[r, "\Pi_{\mathsf{i}_{A,B}+\mathsf{i}_{B,C}}"] \arrow[d, equal] & \widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,C}} \arrow[r, "\mathsf{m}_{A,B,C}"] & \widehat{\mathbb{P}_{A,C}} \arrow[d, "\Delta_{\mathsf{i}_{A,C}}"] \\
\widehat{\mathbb{V}_{A,B} + \mathbb{V}_{B,C}} \arrow[d, equal] & & \widehat{\mathbb{V}_{A,C}} \arrow[d, "\Pi_{\mathsf{i}_{A,C}}"] \\
\widehat{\mathbb{V}_{A,B} + \mathbb{V}_{B,C}} \arrow[r, "\Pi_{\mathsf{i}_{A,B}+\mathsf{i}_{B,C}}"'] & \widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,C}} \arrow[r, "\mathsf{m}_{A,B,C}"'] & \widehat{\mathbb{P}_{A,C}}
\end{tikzcd}
$$

Proof: Preservation of Innocence

Alternative definition of composition:

$$\widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,C}} \xrightarrow{\Pi} \widehat{\mathbb{P}_{(A,B),(B,C)}} \xrightarrow{\Delta} \widehat{\mathbb{P}_{A,B,C}} \xrightarrow{\Sigma} \widehat{\mathbb{P}_{A,C}}$$

$$\begin{tikzcd}
\widehat{\mathbb{V}_{A,B} + \mathbb{V}_{B,C}} \arrow[r, "\Pi"] \arrow[d, equal] & \widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,C}} \arrow[r, "\Pi"] & \widehat{\mathbb{P}_{(A,B),(B,C)}} \arrow[r, "\Delta"] & \widehat{\mathbb{P}_{A,B,C}} \arrow[r, "\Sigma"] & \widehat{\mathbb{P}_{A,C}} \arrow[d, "\Delta"] \\
\widehat{\mathbb{V}_{A,B} + \mathbb{V}_{B,C}} \arrow[d, equal] & & & & \widehat{\mathbb{V}_{A,C}} \arrow[d, "\Pi"] \\
\widehat{\mathbb{V}_{A,B} + \mathbb{V}_{B,C}} \arrow[r, "\Pi"'] & \widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,C}} \arrow[r, "\Pi"'] & \widehat{\mathbb{P}_{(A,B),(B,C)}} \arrow[r, "\Delta"'] & \widehat{\mathbb{P}_{A,B,C}} \arrow[r, "\Sigma"'] & \widehat{\mathbb{P}_{A,C}}
\end{tikzcd}$$

Proof: Preservation of Innocence

$$
\begin{tikzcd}
\mathbb{V}_{A,B} + \mathbb{V}_{B,C} \arrow[r, "\Pi"] \arrow[d, equal] & \mathbb{P}_{A,B} + \mathbb{P}_{B,C} \arrow[r, "\Pi"] & \mathbb{P}_{(A,B),(B,C)} & \mathbb{P}_{A,B,C} \arrow[l, "\Delta"'] \arrow[r, "\Sigma"] & \mathbb{P}_{A,C} \\
\mathbb{V}_{A,B} + \mathbb{V}_{B,C} \arrow[d, equal] & & & & \mathbb{V}_{A,C} \arrow[u, "\Delta"'] \arrow[d, "\Pi"] \\
\mathbb{V}_{A,B} + \mathbb{V}_{B,C} \arrow[r, "\Pi"'] & \mathbb{P}_{A,B} + \mathbb{P}_{B,C} \arrow[r, "\Pi"'] & \mathbb{P}_{(A,B),(B,C)} & \mathbb{P}_{A,B,C} \arrow[l, "\Delta"] \arrow[r, "\Sigma"'] & \mathbb{P}_{A,C}
\end{tikzcd}
$$

Proof: Preservation of Innocence

$$
\begin{tikzcd}
\mathbb{V}_{A,B} + \mathbb{V}_{B,C} \arrow[r, "\Pi"] \arrow[d, equal] & \mathbb{P}_{A,B} + \mathbb{P}_{B,C} \arrow[r, "\Pi"] & \mathbb{P}_{(A,B),(B,C)} & \mathbb{P}_{A,B,C} \arrow[l, "\Delta"'] \arrow[r, "\Sigma"] & \mathbb{P}_{A,C} \\
\mathbb{V}_{A,B} + \mathbb{V}_{B,C} \arrow[rr, "\Pi"] \arrow[d, equal] & & \mathbb{V}_{(A,B),(B,C)} \arrow[u, "\Delta"] \arrow[d, "\Pi"] & \mathbb{V}_{A,B,C} \arrow[l, "\Delta"'] \arrow[r, "\Sigma"] \arrow[u, "\Delta"] \arrow[d, "\Pi"] & \mathbb{V}_{A,C} \arrow[u, "\Delta"'] \arrow[d, "\Pi"'] \\
\mathbb{V}_{A,B} + \mathbb{V}_{B,C} \arrow[r, "\Pi"'] & \mathbb{P}_{A,B} + \mathbb{P}_{B,C} \arrow[r, "\Pi"'] & \mathbb{P}_{(A,B),(B,C)} & \mathbb{P}_{A,B,C} \arrow[l, "\Delta"] \arrow[r, "\Sigma"'] & \mathbb{P}_{A,C}
\end{tikzcd}
$$

Proof: Preservation of Innocence

$$\begin{tikzcd}
\mathbb{V}_{A,B} + \mathbb{V}_{B,C} \arrow[r, "\Pi"] \arrow[d, equal] & \mathbb{P}_{A,B} + \mathbb{P}_{B,C} \arrow[r, "\Pi"] & \mathbb{P}_{(A,B),(B,C)} & \mathbb{P}_{A,B,C} \arrow[l, "\Delta"'] \arrow[r, "\Sigma"] & \mathbb{P}_{A,C} \\
\mathbb{V}_{A,B} + \mathbb{V}_{B,C} \arrow[rr, "\Pi"] \arrow[d, equal] & & \mathbb{V}_{(A,B),(B,C)} \arrow[u, "\Delta"] \arrow[d, "\Pi"'] & \mathbb{V}_{A,B,C} \arrow[l, "\Delta"'] \arrow[u, "\Delta"] \arrow[d, "\Pi"'] \arrow[r, "\Sigma"] & \mathbb{V}_{A,C} \arrow[u, "\Delta"'] \arrow[d, "\Pi"] \\
\mathbb{V}_{A,B} + \mathbb{V}_{B,C} \arrow[r, "\Pi"'] & \mathbb{P}_{A,B} + \mathbb{P}_{B,C} \arrow[r, "\Pi"'] & \mathbb{P}_{(A,B),(B,C)} & \mathbb{P}_{A,B,C} \arrow[l, "\Delta"] \arrow[r, "\Sigma"'] & \mathbb{P}_{A,C}
\end{tikzcd}$$

- simple commutation

Proof: Preservation of Innocence

$$\begin{tikzcd}
\mathbb{V}_{A,B} + \mathbb{V}_{B,C} \arrow[r, "\Pi"] \arrow[d, equal] & \mathbb{P}_{A,B} + \mathbb{P}_{B,C} \arrow[r, "\Pi"] & \mathbb{P}_{(A,B),(B,C)} & \mathbb{P}_{A,B,C} \arrow[l, "\Delta"'] \arrow[r, "\Sigma"] & \mathbb{P}_{A,C} \\
\mathbb{V}_{A,B} + \mathbb{V}_{B,C} \arrow[rr, "\Pi"] \arrow[d, equal] & & \mathbb{V}_{(A,B),(B,C)} \arrow[u, "\Delta"] \arrow[d, "\Pi"'] & \mathbb{V}_{A,B,C} \arrow[l, "\Delta"'] \arrow[r, "\Sigma"] \arrow[u, "\Delta"] \arrow[d, "\Pi"'] & \mathbb{V}_{A,C} \arrow[u, "\Delta"'] \arrow[d, "\Pi"] \\
\mathbb{V}_{A,B} + \mathbb{V}_{B,C} \arrow[r, "\Pi"'] & \mathbb{P}_{A,B} + \mathbb{P}_{B,C} \arrow[r, "\Pi"'] & \mathbb{P}_{(A,B),(B,C)} & \mathbb{P}_{A,B,C} \arrow[l, "\Delta"] \arrow[r, "\Sigma"'] & \mathbb{P}_{A,C}
\end{tikzcd}$$

- simple commutation
- exact squares (Guitart)

Proof: Preservation of Innocence

$$\begin{tikzcd}
\mathbb{V}_{A,B}+\mathbb{V}_{B,C} \arrow[r, "\Pi"] \arrow[d, equal] & \mathbb{P}_{A,B}+\mathbb{P}_{B,C} \arrow[r, "\Pi"] & \mathbb{P}_{(A,B),(B,C)} & \mathbb{P}_{A,B,C} \arrow[l, "\Delta"'] \arrow[r, "\Sigma"] & \mathbb{P}_{A,C} \\
\mathbb{V}_{A,B}+\mathbb{V}_{B,C} \arrow[rr, "\Pi"] \arrow[d, equal] & & \mathbb{V}_{(A,B),(B,C)} \arrow[u, "\Delta"] \arrow[d, "\Pi"'] & \color{red}{\mathbb{V}_{A,B,C}} \arrow[l, "\Delta"'] \arrow[r, red, "\Sigma"] \arrow[u, "\Delta"] \arrow[d, red, "\Pi"'] & \color{red}{\mathbb{V}_{A,C}} \arrow[u, "\Delta"'] \arrow[d, red, "\Pi"] \\
\mathbb{V}_{A,B}+\mathbb{V}_{B,C} \arrow[r, "\Pi"'] & \mathbb{P}_{A,B}+\mathbb{P}_{B,C} \arrow[r, "\Pi"'] & \mathbb{P}_{(A,B),(B,C)} & \color{red}{\mathbb{P}_{A,B,C}} \arrow[l, "\Delta"] \arrow[r, red, "\Sigma"'] & \color{red}{\mathbb{P}_{A,C}}
\end{tikzcd}$$

- simple commutation
- exact squares (Guitart)
- distributive squares

Conclusion

Results:

- abstract construction of categories of plays based on string diagrams
- application: standard plays versus string diagrams in HON games
- abstract construction of categories of games and (innocent) strategies

Not discussed here:

- a model of the π-calculus
- interpretation terms $\rightarrow$ strategies as a singular functor
- modelling concurrent rewriting traces as string diagrams

Perspectives

String diagrams:

- fibred approach + composition of strategies

Composition of strategies:

- more models
- more structure
- interpreting languages
- categorical implications?

Thank you.

- full embedding $F\colon \mathbb{P}_{A,B} \to \mathbb{E}(A \vdash B)$
- restricts to equivalence $F^{\mathbb{V}}\colon \mathbb{V}_{A,B} \to \mathbb{E}^{\mathbb{V}}(A \vdash B)$

$$\begin{tikzcd}
\mathbb{V}_{A,B} \arrow[r, hook, "\mathsf{i}_{TO}"] \arrow[d, "F^{\mathbb{V}}"'] & \mathbb{P}_{A,B} \arrow[d, "F"] \\
\mathbb{E}^{\mathbb{V}}(A \vdash B) \arrow[r, hook, "\mathsf{i}"'] & \mathbb{E}(A \vdash B)
\end{tikzcd}$$

Result Transfer

Adjunction:

$$
\begin{tikzcd}
\mathsf{Set} \arrow[r, bend left, "\mathsf{l}"] \arrow[r, phantom, "\perp"] & 2 \arrow[l, bend left, "\mathsf{r}"]
\end{tikzcd}
$$

Result transfer:

$$
\begin{tikzcd}
\widehat{\mathbb{P}_{A,B}} \arrow[r, bend left, "\mathsf{l}_!"] \arrow[r, phantom, "\perp"] & \widetilde{\mathbb{P}_{A,B}} \arrow[l, bend left, "\mathsf{r}_!"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,C}} \arrow[r, "\mathsf{m}_{A,B,C}"] \arrow[d, "\mathsf{l}_!"'] & \widehat{\mathbb{P}_{A,C}} \arrow[d, "\mathsf{l}_!"] \\
\widetilde{\mathbb{P}_{A,B} + \mathbb{P}_{B,C}} \arrow[r, "\overline{\mathsf{m}_{A,B,C}}"'] & \widetilde{\mathbb{P}_{A,C}}
\end{tikzcd}
$$

Exact Squares

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "T"] \arrow[d, "S"'] \arrow[dr, phantom, "\xRightarrow{\varphi}"] & \mathbb{C} \arrow[d, "V"] \\
\mathbb{B} \arrow[r, "U"'] & \mathbb{D}
\end{tikzcd}$$

Mates:

$$\begin{tikzcd}
\widehat{\mathbb{A}} \arrow[d, "\Sigma_S"'] \arrow[dr, phantom, "\Sigma_\varphi \Downarrow"] & \widehat{\mathbb{C}} \arrow[l, "\Delta_T"'] \arrow[d, "\Sigma_V"] \\
\widehat{\mathbb{B}} & \widehat{\mathbb{D}} \arrow[l, "\Delta_U"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\widehat{\mathbb{A}} \arrow[r, "\Pi_T"] \arrow[dr, phantom, "\xLeftarrow{\Pi_\varphi}"] & \widehat{\mathbb{C}} \\
\widehat{\mathbb{B}} \arrow[u, "\Delta_S"] \arrow[r, "\Pi_U"'] & \widehat{\mathbb{D}} \arrow[u, "\Delta_V"']
\end{tikzcd}$$

Exact square: the mates are isomorphisms.

Guitart: conditions for square to be exact.

Distributive Squares

Conditions for

$$\begin{tikzcd}
\widehat{\mathbb{A}} \arrow[d, "\Sigma_S"'] \arrow[r, phantom, "\xRightarrow{\tilde{\varphi}}"] & \widehat{\mathbb{C}} \arrow[l, "\Pi_T"'] \arrow[d, "\Sigma_V"] \\
\widehat{\mathbb{B}} & \widehat{\mathbb{D}} \arrow[l, "\Pi_U"]
\end{tikzcd}$$

to commute.

Composition

Associativity of composition:

$$\begin{tikzcd}
\widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,C} + \mathbb{P}_{C,D}} \arrow[r, "\mathsf{m}_{A,B,C}+\mathbb{P}_{C,D}"] \arrow[d, "\mathbb{P}_{A,B}+\mathsf{m}_{B,C,D}"'] & \widehat{\mathbb{P}_{A,C} + \mathbb{P}_{C,D}} \arrow[d, "\mathsf{m}_{A,C,D}"] \\
\widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,D}} \arrow[r, "\mathsf{m}_{A,B,D}"'] & \widehat{\mathbb{P}_{A,D}}
\end{tikzcd}$$

Associativity of Composition

$$
\begin{tikzcd}
\widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,C} + \mathbb{P}_{C,D}} \arrow[r, "\Pi"] \arrow[d, "\Pi"'] & \widehat{\mathbb{P}_{(A,B),(B,C)} + \mathbb{P}_{C,D}} \arrow[r, "\Delta"] & \widehat{\mathbb{P}_{A,B,C} + \mathbb{P}_{C,D}} \arrow[r, "\Sigma"] & \widehat{\mathbb{P}_{A,C} + \mathbb{P}_{C,D}} \arrow[d, "\Pi"] \\
\widehat{\mathbb{P}_{A,B} + \mathbb{P}_{(B,C),(C,D)}} \arrow[d, "\Delta"'] & & & \widehat{\mathbb{P}_{(A,C),(C,D)}} \arrow[d, "\Delta"] \\
\widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,C,D}} \arrow[d, "\Sigma"'] & & & \widehat{\mathbb{P}_{A,C,D}} \arrow[d, "\Sigma"] \\
\widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,D}} \arrow[r, "\Pi"'] & \widehat{\mathbb{P}_{(A,B),(B,D)}} \arrow[r, "\Delta"'] & \widehat{\mathbb{P}_{A,B,D}} \arrow[r, "\Sigma"'] & \widehat{\mathbb{P}_{A,D}}
\end{tikzcd}
$$

Proof: Associativity of Composition

$$
\begin{tikzcd}
\mathbb{P}_{A,B} + \mathbb{P}_{B,C} + \mathbb{P}_{C,D} \arrow[r, "\Pi"] \arrow[d, "\Pi"'] & \mathbb{P}_{(A,B),(B,C)} + \mathbb{P}_{C,D} & \mathbb{P}_{A,B,C} + \mathbb{P}_{C,D} \arrow[l, "\Delta"'] \arrow[r, "\Sigma"] & \mathbb{P}_{A,C} + \mathbb{P}_{C,D} \arrow[d, "\Pi"] \\
\mathbb{P}_{A,B} + \mathbb{P}_{(B,C),(C,D)} & & & \mathbb{P}_{(A,C),(C,D)} \\
\mathbb{P}_{A,B} + \mathbb{P}_{B,C,D} \arrow[u, "\Delta"] \arrow[d, "\Sigma"'] & & & \mathbb{P}_{A,C,D} \arrow[u, "\Delta"'] \arrow[d, "\Sigma"] \\
\mathbb{P}_{A,B} + \mathbb{P}_{B,D} \arrow[r, "\Pi"'] & \mathbb{P}_{(A,B),(B,D)} & \mathbb{P}_{A,B,D} \arrow[l, "\Delta"] \arrow[r, "\Sigma"'] & \mathbb{P}_{A,D}
\end{tikzcd}
$$

Proof: Associativity of Composition

$$
\begin{tikzcd}
\mathbb{P}_{A,B} + \mathbb{P}_{B,C} + \mathbb{P}_{C,D} \arrow[r, "\Pi"] \arrow[d, "\Pi"'] & \mathbb{P}_{(A,B),(B,C)} + \mathbb{P}_{C,D} \arrow[d, "\Pi"] & \mathbb{P}_{A,B,C} + \mathbb{P}_{C,D} \arrow[l, "\Delta"'] \arrow[r, "\Sigma"] & \mathbb{P}_{A,C} + \mathbb{P}_{C,D} \arrow[d, "\Pi"] \\
\mathbb{P}_{A,B} + \mathbb{P}_{(B,C),(C,D)} \arrow[r, "\Pi"'] & \mathbb{P}_{(A,B),(B,C),(C,D)} & & \mathbb{P}_{(A,C),(C,D)} \\
\mathbb{P}_{A,B} + \mathbb{P}_{B,C,D} \arrow[u, "\Delta"] \arrow[d, "\Sigma"'] & & \mathbb{P}_{A,B,C,D} \arrow[ul, "\Delta"'] \arrow[r, "\Sigma"] \arrow[d, "\Sigma"'] & \mathbb{P}_{A,C,D} \arrow[u, "\Delta"'] \arrow[d, "\Sigma"] \\
\mathbb{P}_{A,B} + \mathbb{P}_{B,D} \arrow[r, "\Pi"'] & \mathbb{P}_{(A,B),(B,D)} & \mathbb{P}_{A,B,D} \arrow[l, "\Delta"] \arrow[r, "\Sigma"'] & \mathbb{P}_{A,D}
\end{tikzcd}
$$

Proof: Associativity of Composition (cont.)

$$
\begin{tikzcd}
\mathbb{P}_{A,B} + \mathbb{P}_{(B,C),(C,D)} \arrow[rr, "\Pi"] & & \mathbb{P}_{(A,B),(B,C),(C,D)} \\
\mathbb{P}_{A,B} + \mathbb{P}_{B,C,D} \arrow[u, "\Delta"] \arrow[r, "\Pi"] \arrow[dr, "\Pi"'] \arrow[dd, "\Sigma"'] & \mathbb{P}_{(A,B),(B,C),(C,D)} & \mathbb{P}_{A,B,C,D} \arrow[u, "\Delta"'] \arrow[l, "\Delta"'] \arrow[dl, "\Delta"] \arrow[dd, "\Sigma"] \\
& \mathbb{P}_{(A,B),(B,C,D)} \arrow[u, "\Delta"] \arrow[d, "\Sigma"] & \\
\mathbb{P}_{A,B} + \mathbb{P}_{B,D} \arrow[r, "\Pi"'] & \mathbb{P}_{(A,B),(B,D)} & \mathbb{P}_{A,B,D} \arrow[l, "\Delta"]
\end{tikzcd}
$$

Applications

Applications: HON, variants, AJM, TO.

May all be expressed as game settings, abstract composition agrees with traditional composition.

Subtleties:

- HON: liberal definition of $\mathbb{P}_A$ (for projections $\mathbb{P}_{A,B} \to \mathbb{P}_A$ to exist)
- AJM: slightly different definition of $\mathbb{P}_{A,B,C}$ (projection $\mathbb{P}_{A,B,C} \to \mathbb{P}_{A,C}$ should be a discrete fibration)

Blass games: composition known to be non-associative. Cannot be expressed as a game setting (zipping fails).

Conditions to Preserve Innocence

Locality: $\delta_1 : \mathbb{P}_{A,B,C} \to \mathbb{P}_{A,C}$ and $\iota_0 : \mathbb{P}_A \to \mathbb{P}_{A,A}$ are sheaves.

$$\begin{tikzcd} & \mathbb{P}_{A,B,C} \arrow[d, "\delta_1"] \\ \mathbb{V}_{A,C} \arrow[r, "\mathsf{i}_{A,C}"'] & \mathbb{P}_{A,C} \end{tikzcd}$$

View-analyticity:

Conditions to Preserve Innocence

Locality: $\delta_1 \colon \mathbb{P}_{A,B,C} \to \mathbb{P}_{A,C}$ and $\iota_0 \colon \mathbb{P}_A \to \mathbb{P}_{A,A}$ are sheaves.

$$\begin{tikzcd}
\mathbb{V}_{A,B,C} \arrow[r, "\mathsf{i}_{A,B,C}"] \arrow[d, "\delta_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{P}_{A,B,C} \arrow[d, "\delta_1"] \\
\mathbb{V}_{A,C} \arrow[r, "\mathsf{i}_{A,C}"'] & \mathbb{P}_{A,C}
\end{tikzcd}$$

View-analyticity:

Conditions to Preserve Innocence

Locality: $\delta_1 \colon \mathbb{P}_{A,B,C} \to \mathbb{P}_{A,C}$ and $\iota_0 \colon \mathbb{P}_A \to \mathbb{P}_{A,A}$ are sheaves.

$$\begin{tikzcd}
\mathbb{V}_{A,B,C} \arrow[r, "\mathsf{i}_{A,B,C}"] \arrow[d, "\delta_1"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathbb{P}_{A,B,C} \arrow[d, "\delta_1"] \\
\mathbb{V}_{A,C} \arrow[r, "\mathsf{i}_{A,C}"'] & \mathbb{P}_{A,C}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & u_1 \arrow[d, dashed] & \\
u_n \arrow[dd, dashed] & v_1 \arrow[dr, "f_1"] & \\
 & & \\
v_n \arrow[rr, "f_n"'] & & p
\end{tikzcd}$$

View-analyticity:

Conditions to Preserve Innocence

Locality: $\delta_1 \colon \mathbb{P}_{A,B,C} \to \mathbb{P}_{A,C}$ and $\iota_0 \colon \mathbb{P}_A \to \mathbb{P}_{A,A}$ are sheaves.

$$\begin{tikzcd}
\mathbb{V}_{A,B,C} \arrow[r, "\mathsf{i}_{A,B,C}"] \arrow[d, "\delta_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{P}_{A,B,C} \arrow[d, "\delta_1"] \\
\mathbb{V}_{A,C} \arrow[r, "\mathsf{i}_{A,C}"'] & \mathbb{P}_{A,C}
\end{tikzcd}
\qquad
\begin{tikzcd}
& u_1 \arrow[dr] \arrow[dd, dashed] & \\
u_n \arrow[rr] \arrow[dd, dashed] & & u \arrow[dd, dashed] \\
& v_1 \arrow[dr, "f_1"] & \\
v_n \arrow[rr, "f_n"'] & & p
\end{tikzcd}$$

View-analyticity:

Conditions to Preserve Innocence

Locality: $\delta_1 \colon \mathbb{P}_{A,B,C} \to \mathbb{P}_{A,C}$ and $\iota_0 \colon \mathbb{P}_A \to \mathbb{P}_{A,A}$ are sheaves.

$$
\begin{tikzcd}
\mathbb{V}_{A,B,C} \arrow[r, "\mathrm{i}_{A,B,C}"] \arrow[d, "\delta_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{P}_{A,B,C} \arrow[d, "\delta_1"] \\
\mathbb{V}_{A,C} \arrow[r, "\mathrm{i}_{A,C}"'] & \mathbb{P}_{A,C}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & u_1 \arrow[drr, red] \arrow[dd, dashed] & & \\
u_n \arrow[rrr, red] \arrow[ddd, dashed] & & & \color{red}{u} \arrow[ddd, dashed, red] \\
 & & & \\
 & v_1 \arrow[drr, "f_1"] & & \\
v_n \arrow[rrr, "f_n"'] & & & p
\end{tikzcd}
$$

View-analyticity:

$$\mathrm{i}_{A,B}(v) \xrightarrow{\quad f \quad} \delta_2(u)$$

Conditions to Preserve Innocence

Locality: $\delta_1 \colon \mathbb{P}_{A,B,C} \to \mathbb{P}_{A,C}$ and $\iota_0 \colon \mathbb{P}_A \to \mathbb{P}_{A,A}$ are sheaves.

$$\begin{tikzcd}
\mathbb{V}_{A,B,C} \arrow[r, "\mathsf{i}_{A,B,C}"] \arrow[d, "\delta_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{P}_{A,B,C} \arrow[d, "\delta_1"] \\
\mathbb{V}_{A,C} \arrow[r, "\mathsf{i}_{A,C}"'] & \mathbb{P}_{A,C}
\end{tikzcd}
\qquad
\begin{tikzcd}
& u_1 \arrow[dr] \arrow[dd, dashed] & \\
u_n \arrow[rr] \arrow[dd, dashed] & & u \arrow[dd, dashed] \\
& v_1 \arrow[dr, "f_1"] & \\
v_n \arrow[rr, "f_n"'] & & p
\end{tikzcd}$$

View-analyticity:

$$\begin{tikzcd}
\mathsf{i}_{A,B}(v) \arrow[rr, "f"] \arrow[dr, "g"'] & & \delta_2(u) \\
& \delta_2(\mathsf{i}_{A,B,C}(w)) \arrow[ur, "\delta_2(h)"'] &
\end{tikzcd}$$

Boolean Innocent Strategies

But (non-deterministic) innocent strategies should not compose!

Answer:

$$
\begin{tikzcd}
\widehat{\mathbb{V}_{A,B} + \mathbb{V}_{B,C}} \arrow[r] & \widehat{\mathbb{P}_{A,C}} \arrow[d, "\mathsf{l}_!"] \\
\widetilde{\mathbb{V}_{A,B} + \mathbb{V}_{B,C}} \arrow[r] \arrow[u, "\mathsf{r}_!"] & \widetilde{\mathbb{P}_{A,C}}
\end{tikzcd}
$$

does not commute.

- *concurrent* innocent strategies compose
- traditional innocent strategies do not