Mathematical Semantics of Computer Systems, *MSCS* (4810-1168)
Handout for Lecture 10 (2016/12/12)

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1 Algebraic Semantics as a Precursor of Categorical Semantics

This section is essentially a brief recap of [2, Chap. 2], aimed also at the audience not familiar with formal logic.

1.1 The Word Problem

Consider the following "syntactic system."

- *Terms* are defined by the following BNF notation:

$$\textbf{Terms} \ni t, t_1, t_2 \quad ::= \quad \mathtt{x} \in \textbf{Var} \mid \mathtt{e} \mid t \cdot t \mid t^{-1} \ .$$

- The relation $\sim$ between terms is defined inductively by the following rules.

$$\frac{}{(t_1 \cdot t_2) \cdot t_3 \sim t_1 \cdot (t_2 \cdot t_3)} \ (\textsc{Associativity})$$

$$\frac{}{\mathtt{e} \cdot t \sim t} \ (\textsc{Unit-Left}) \qquad \frac{}{t \cdot \mathtt{e} \sim t} \ (\textsc{Unit-Right})$$

$$\frac{}{t^{-1} \cdot t \sim \mathtt{e}} \ (\textsc{Inverse-Left}) \qquad \frac{}{t \cdot t^{-1} \sim \mathtt{e}} \ (\textsc{Inverse-Right})$$

$$\frac{}{t \sim t} \ (\textsc{Reflexivity}) \qquad \frac{t \sim s}{s \sim t} \ (\textsc{Symmetry}) \qquad \frac{t \sim s \quad s \sim u}{t \sim u} \ (\textsc{Transitivity})$$

$$\frac{t_1 \sim s_1 \quad t_2 \sim s_2}{t_1 \cdot t_2 \sim s_1 \cdot s_2} \ (\cdot\text{-}\textsc{Congruence}) \qquad \frac{t \sim s}{t^{-1} \sim s^{-1}} \ ((\_)^{-1}\text{-}\textsc{Congruence})$$

Remark 1. (For those who are *not* familiar with formal logic) The "inductive definition of $\sim$ by the rules" means that we have $t \sim s$ if and only if we can draw a (finite-height) *proof tree* using the rules, for example

$$\dfrac{\dfrac{}{((xy)^{-1}x)y \sim (xy)^{-1}(xy)} \ (\textsc{Associativity}) \quad \dfrac{}{(xy)^{-1}(xy) \sim e} \ (\textsc{Inverse-Left})}{((xy)^{-1}x)y \sim e} \ (\textsc{Transitivity})$$

Remark 2. (For those who *are* familiar with formal logic) The above is an equational theory of groups, formulated as usual in equational logic.

Now the question is: given terms s and t, can we know if $s \sim t$ holds? How? This problem is known as the *word problem for groups*.

Theorem (Novikov, 1955)**.** *The word problem for groups is undecidable.*

Therefore there is no generic algorithm that decides the problem.

1.2 Use of Algebraic Semantics

For those of you who are familiar with abstract algebra or group theory, the following fact will come as trivial.

> (†) If there is a group G in which the terms s and t are not equal, then we know that $s \sim t$ does not hold.

Implicit here is the use of *algebraic semantics*.

Definition. Let G be a group and $V\colon \mathbf{Var} \to |G|$ be a function (here $|G|$ denotes the underlying set of G; we call the function V a *valuation*). The *denotation* $[\![t]\!]_V$ of a term t under V is an element of the group G defined in the obvious inductive way; namely

$$\begin{aligned} [\![x]\!]_V &:= V(x) & [\![\mathsf{e}]\!]_V &:= e_G \\ [\![t_1 \cdot t_2]\!]_V &:= [\![t_1]\!]_V \cdot_G [\![t_2]\!]_V & [\![t^{-1}]\!]_V &:= \big([\![t]\!]_V\big)^{-1} \ . \end{aligned}$$

Note here that the unit, the multiplication operator and the inverse operator on the left-hand sides are syntactic symbols; those on the right-hand sides are mathematical/semantical operators in the group G.

Now it is possible to "investigate" whether $s \sim t$ holds by looking at their semantics.

Theorem (soundness)**.** *If $s \sim t$ holds, then $[\![s]\!]_V = [\![t]\!]_V$ for any group G and any valuation $V\colon \mathbf{Var} \to |G|$.*

Proof. Straightforward, by structural induction on the construction of proof trees. □

You see that the quotation (†) in the above is the (sloppily stated version of the) contraposition of the theorem. Therefore, to *refute* $s \sim t$, it suffices to find convenient G and V such that $[\![s]\!]_V \neq [\![t]\!]_V$.

1.3 Completeness and the Term Model

The obvious question that remains is: is the above "investigation method" complete, too? The answer is positive:

Theorem (completeness)**.** *Assume that $[\![s]\!]_V = [\![t]\!]_V$ for any group G and any valuation $V\colon \mathbf{Var} \to |G|$. Then $s \sim t$ holds.*

Proof. We can in fact construct a special group G_0 by syntactic means—and a special valuation $V_0\colon \mathbf{Var} \to |G_0|$ that accompanies—such that $[\![s]\!]_{V_0} = [\![t]\!]_{V_0}$ if and only if $s \sim t$ holds.

Concretely:

- $|G_0| = \big\{\, [s]_\sim \,\big|\, s \text{ is a term} \,\big\}$, where $[s]_\sim$ is the $\sim$-equivalence class of the term s
- Operations are defined syntactically, that is for example,

$$[s]_\sim \cdot_{G_0} [t]_\sim \ = \ [s \cdot t]_\sim \tag{1}$$

  and so on. Note here that $\cdot_{G_0}$ on the left-hand side is a semantical/mathematical entity (a group multiplication); in contrast $\cdot$ on the right-hand side is a syntactic entity (an operation symbol).

We have to check the following. These are all straightforward.

- $\sim$ is an equivalence relation of terms. (This follows from the rules that define $\sim$)
- The operations in (1) are well-defined. (Follows from the CONGRUENCE rules)

- The set $|G_0|$, together with the operations defined as in (1), forms a group. (Easy)

We define the valuation V_0 by

$$V_0(x) \;:=\; [x]_\sim \;. \tag{2}$$

Then it is straightforward by induction to show that $[\![s]\!]_{V_0} = [s]_\sim$. This establishes: $[\![s]\!]_{V_0} = [\![t]\!]_{V_0}$ if and only if $s \sim t$. □

The group G_0 that we constructed is often called a *term model*, since it consists of (equivalence classes of) terms. A term model is a complete model—in the sense that $[\![s]\!]_{V_0} = [\![t]\!]_{V_0}$ if and only if $s \sim t$—but a common problem with it is that equality in the term model is complicated (deciding it is as hard as deciding $\sim$ itself!).

The term model G_0, in the current setting of an algebraic theory for groups, turns out to be isomorphic to the *free group* over the set **Var** of generators. It is called a *free* group since it satisfies the minimal set of equalities for it to be a group, in the sense that

$[\![s]\!]_{V_0} = [\![t]\!]_{V_0}$ if and only if $s \sim t$.

2 Cartesian Closed Categories as Models of Typed λ-Calculus

We continue and follow slides by Samson Abramsky (Oxford) found at `www.math.helsinki.fi/logic/sellc-2010/course/LectureIII.pdf`. See [3] for further details. There is a big body of literature on the λ-calculus, including [1, 4, 5].

- On conversion in λ-calculus
- Categorical Semantics, as a typed λ-calculus and Cartesian closed categories as examples

Definition. Type judgment. Type derivation tree.

NB: we use the term calculus *a la Church* (where bound variables have explicit types).

Lemma 1. *Each derivable type judgment has a unique derivation tree.*

Definition. Cartesian closed category: a category with finite products and exponentials.

Definition. Interpretation $[\![\_]\!]$ of typed λ-calculus. Interpreting types, type derivation trees, type judgments, and terms.

Definition. Substitution lemma: interpretation of $s[t/x]$ is given by composition of arrows.

Definition. Conversion rules, including congruence rules.

Theorem. *Soundness of categorical semantics: if $s =_{\beta\eta} t$, then $[\![s]\!] = [\![t]\!]$.*

If we have time:

- The Curry-Howard correspondence; terms as proofs; conversion as proof normalization

References

[1] H.P. Barendregt. *The Lambda Calculus. Its Syntax and Semantics.* North-Holland, Amsterdam, 2nd rev. edn., 1984.

[2] I. Hasuo. Introduction to logic and computability. Course material for the undergraduate course *Information Logic*, 2014. Available on the web, `www-mmm.is.s.u-tokyo.ac.jp/~ichiro/courseNotes/textbookInfLogic.pdf` (restricted access from inside UTokyo).

[3] J. Lambek and P.J. Scott. *Introduction to higher order Categorical Logic*. No. 7 in Cambridge Studies in Advanced Mathematics. Cambridge Univ. Press, 1986.

[4] M.H. Sørensen and P. Urzyczyn. *Lectures on the Curry-Howard Isomorphism*, vol. 149 of *Studies in Logic and the Foundations of Mathematics*. Elsevier Science Inc., New York, NY, USA, 2006.

[5] G. Winskel. *The Formal Semantics of Programming Languages*. MIT Press, 1993.