Mathematical Semantics of Computer Systems, *MSCS* (4810-1168)
Handout for Lecture 14 (2017/1/30)

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Video recording of the lectures is available at: http://www-mmm.is.s.u-tokyo.ac.jp/videos/mscs2016

1 Categorical Modeling of State-Based Dynamics by Coalgebras

Our principal reference is:

> [Jacobs 2012] Bart Jacobs, Introduction to Coalgebra: Towards Mathematics of States and Observations. Draft of a book, available online. 2012

Definition. F-coalgebra, F-coalgebra morphism, the category $\mathbf{Coalg}(F)$ of F-coalgebras.

Example. Exhibit coalgebras, coalgebra morphisms, for:
$F = A \times (\_)$; $F = 1 + A \times (\_)$; $F = A \times (\_) \times (\_)$; $F = 2 \times (\_)^A$ (deterministic automata); $F = 2 \times (\mathcal{P}\_)^A \cong \mathcal{P}(1 + A \times \_)$ (nondeterministic automata); and polynomial functors in general.

Definition. Final coalgebra.

Example. Final coalgebras for: $F = A \times (\_)$; $F = 1 + A \times (\_)$; $F = A \times (\_) \times (\_)$;

Lemma. *Lambek's lemma. It follows that final coalgebras need not exist.*

Compare with: algebra, initial algebra (understood as *datatype*). Consider $F = 1 + (\_)$ (induction). This leads to our interests in *coinduction*.

Example. *Coinduction as a definition principle.* $A = \{0, 1\}$, $F = A \times (\_)$, for which a final coalgebra is $\langle \mathsf{hd}, \mathsf{tl}\rangle \colon A^\omega \to A \times A^\omega$. Consider various *recursive definitions*:

$$\mathsf{evens}(\sigma) = \mathsf{hd}(\sigma) : \big(\mathsf{evens}(\mathsf{tl}(\mathsf{tl}(\sigma)))\big) \tag{1}$$
$$\mathsf{odds}(\sigma) = \mathsf{hd}(\mathsf{tl}(\sigma)) : \big(\mathsf{odds}(\mathsf{tl}(\mathsf{tl}(\sigma)))\big) \tag{2}$$
$$\mathsf{ones} = 1 : \mathsf{ones} \tag{3}$$
$$\mathsf{bad} = \mathsf{bad} \tag{4}$$

Well-definedness is ensured once you express the definition as a coalgebra.
How about $\mathsf{merge}(a_0 : a_1 : \dots, b_0 : b_1 : \dots) = a_0 : b_0 : a_1 : b_1 : \dots$?

Example. *Coinduction as a proof principle.* To see $\mathsf{odds} = \mathsf{evens} \circ \mathsf{tail}$, we check that $\mathsf{evens} \circ \mathsf{tail}$ makes the finality diagram for odds commute.

The examples might look easy... but imagine you want to *implement* the definitions and proofs on proof assistants like Coq!
A structure result (cf. [Prop. 5.2, Jacobs 2012]):

Theorem. *Let $F \colon \mathbb{C} \to \mathbb{C}$. The forgetful functor $U \colon \mathbf{Coalg}(F) \to \mathbb{C}$ creates colimits.* □

2 Final Report Assignment

Due: 23:59 JST, Friday 10 February, 2017

Submit to the lecturer's mailbox (on the corridor), to the (official) report box of the department, or by email to `ichiro@is.s.u-tokyo.ac.jp`

You can choose questions to answer. Each question is assigned points; and you are expected to answer 100 points worth. However, in case you have not submitted some of the previous report assignments, you can make it up by answering more.

1. (40) Prove the substitution lemma.

2. (30) Let $f\colon A \to B$ be an arrow in a Cartesian closed category $\mathbb{C}$. It induces natural transformations: $A \times (\_) \Rightarrow B \times (\_)$, and $(\_)^B \Rightarrow (\_)^A$. Show that the adjunctions $A \times (\_) \dashv (\_)^A$ and $B \times (\_) \dashv (\_)^B$ are "compatible" with those natural transformations. (You first have to formalize what "compatibility" means.)

3. We are interested in the CCC structure of a presheaf category $[\mathbb{C}^{\mathrm{op}}, \mathbf{Sets}]$. Here $\mathbb{C}$ is a small category.

   (a) (30) Prove that $[\mathbb{C}^{\mathrm{op}}, \mathbf{Sets}]$ has all small limits and colimits.
   (Hint: they are computed "pointwise.")

   (b) (40) The category $[\mathbb{C}^{\mathrm{op}}, \mathbf{Sets}]$ has exponentials, too. Let $P, Q\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Sets}$ be presheaves; Q^P be an exponential; and $C \in \mathbb{C}$. Describe the set $(Q^P)(C)$.
   (Hint: this is an interesting one. Use the Yoneda lemma!)

   (c) (30) Specialize the above answer to the case when $\mathbb{C}$ is the following category with two objects and four arrows.

   $$V \overset{d}{\underset{c}{\rightrightarrows}} E$$

   Here the identity arrows are implicit. Discuss the relationship with the notion of graph homomorphism.

4. *End* is a notion that generalizes that of limit.

   (a) (30) Describe its definition, and show that limits are special cases of ends.

   (b) (40) Let $\mathbb{C}$ be a small category; and $F, G\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Sets}$ be presheaves. Describe the set $\mathbf{Nat}(F, G)$ of natural transformations as a limit.

5. (50) Let $\mathbb{C}$ be a small category and consider the presheaf category $[\mathbb{C}^{\mathrm{op}}, \mathbf{Sets}]$. Prove that any presheaf $P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Sets}$ is a colimit of a certain diagram that consists solely of representable presheaves (i.e. those presheaves of the form $\mathbf{y}C = \mathbb{C}(\_, C)$).

   (Hint: You have to find a suitable diagram. It is given by so-called the *category of elements.*)

6. (40) Let evens, odds and merge be the stream functions earlier in this handout. Use the coinduction proof principle to prove, for each $\sigma \in A^\omega$,

   $$\mathsf{merge}(\mathsf{evens}(\sigma), \mathsf{odds}(\sigma)) = \sigma \ .$$

   (Draw a suitable finality diagram)