MSCS

CT in CS

- λ-calculus & CCCs
- coalgebras.
  mostly in $\mathbf{Sets}$

Ref. Bart Jacobs, a book from CUP

Def. Let $\mathbb{C}$ be a category. (In most examples $\mathbb{C} = \mathbf{Sets}$)

$F : \mathbb{C} \longrightarrow \mathbb{C}$ be a functor.
("endofunctor")

- An <u>F-coalgebra</u> is a pair

$$\left(X \in \mathbb{C},\quad \begin{array}{c} FX \\ c\uparrow \\ X \end{array} \text{ in } \mathbb{C} \right).$$

"the coalgebra structure"

We'll just write
$$\begin{array}{c} FX \\ \uparrow c \\ X \end{array}$$

- An <u>F-coalgebra homomorphism</u> (morphism) from

$$\left(\begin{array}{c} FX \\ \uparrow c \\ X \end{array} \right) \text{ to } \left(\begin{array}{c} FY \\ \uparrow d \\ Y \end{array} \right)$$

is a $\mathbb{C}$-arrow $X \xrightarrow{f} Y$ in $\mathbb{C}$ s.t.

$$\begin{tikzcd} FX \arrow[r, "Ff"] & FY \\ X \arrow[u, "c"] \arrow[r, "f"'] \arrow[ur, phantom, "="] & Y \arrow[u, "d"'] \end{tikzcd}$$

commutes.

- The category $\mathbf{Coalg}(F)$ has
  - <u>obj.</u> F-coalg.
  - <u>arr.</u> homomorphisms betw. F-coalg.

Examples

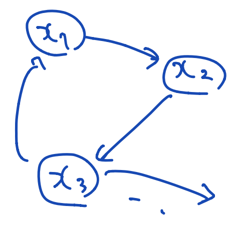
$$\begin{array}{l} FX \\ \uparrow c \\ X \end{array}$$

F — specification of the "transition type"

c — dynamics

X — the set of states

a C-alg. = state-based dynamics



1) $F = A \times (\_) : \mathbf{Sets} \longrightarrow \mathbf{Sets}$

A — a fixed set.

on arrows :

$$\frac{f : X \longrightarrow Y \text{ in } \mathbf{Sets}}{\begin{array}{rcl} A \times f : A \times X & \longrightarrow & A \times Y \\ (a, x) & \longmapsto & (a, f(x)) \end{array}}$$

$A \times f = id_A \times f$

For this F, an F-coalgebra is a function

$$\begin{array}{c} A \times X \\ c = \langle out, next \rangle \uparrow \\ X \end{array} \quad \boxed{\mathbf{Sets}}$$

$$//$$

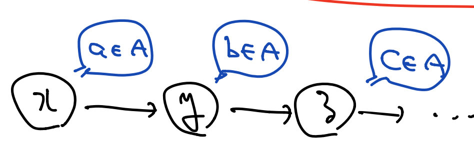
$$X \xrightarrow{out} A \qquad \boxed{\mathbf{Sets}}$$

What's the output of the current state?

$$X \xrightarrow{next} X$$

What's the next state after a transition?

$$//$$



Def. A <u>stream automaton</u> over A (a set) is a triple

$$(X, \quad out : X \to A, \quad next : X \to X)$$

X — a set, "the state space"

Prop. For $F = A \times (-)$,

$$\frac{\text{stream automata over } A}{\overline{\quad F\text{-coalgebras} \quad}}$$

Now: what are coalg. hom. for $F = A \times (-)$?

Prop. $F = A \times (-)$

$$\begin{pmatrix} FX \\ c\uparrow \\ X \end{pmatrix} \xrightarrow{f} \begin{pmatrix} FY \\ d\uparrow \\ Y \end{pmatrix} \text{ in } \mathbf{Coalg}(F)$$

$$\overline{\overline{\qquad\qquad\qquad\qquad\qquad\qquad}}$$

$f : X \to Y$, a function s.t.

$$\begin{tikzcd} FX \arrow[r, "Ff"] & FY \\ X \arrow[u, "c"] \arrow[r, "f"] & Y \arrow[u, "d"'] \end{tikzcd}$$

$$\begin{tikzcd} A \times X \arrow[r, "id_A \times f"] & A \times Y \\ X \arrow[u, "{(out, next)}"] \arrow[r, "f"] & Y \arrow[u, "{(out', next')}"'] \end{tikzcd}$$

$f : X \to Y$ s.t. for each $x \in X$,

- $out(x) = out'(f(x))$
- $f(next(x)) = next'(f(x))$

f is a behavior-preserving map

$$\begin{tikzcd} x \arrow[r, mapsto, "f"] \arrow[d, "c"'] & y \arrow[d] \\ x' \arrow[r, mapsto, "f"] & y' \end{tikzcd}$$

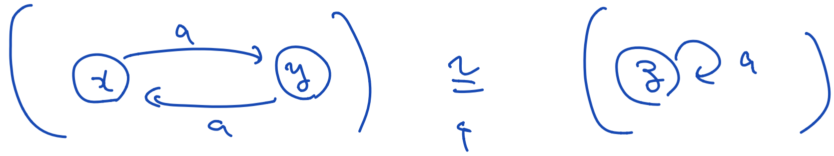
$f(x)$ mimicks x

and this mimicking relationship (witnessed by f) continues forever

The "black-box view" on system dynamics

- Don't care about internal states
- but do care about observations/outputs

E.g.



observationally equivalent

The def. of "behavior" follows this black-box view.

2) $F = \underbrace{1}_{\text{a one-element set, say } 1 = \{\checkmark\}} + A \times (\_)$

coproduct (in Sets), i.e. disjoint union $\sqcup$.

$$\begin{tikzcd} 1 + A \times X = \{\checkmark\} \sqcup A \times X \\ X \arrow[u, "c"] \end{tikzcd}$$

Given $x \in X$, $c(x)$ can be either

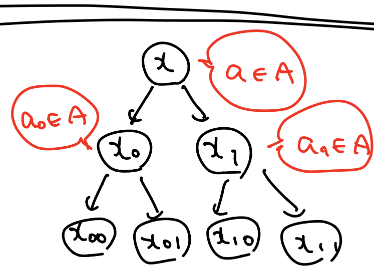
$$\begin{cases} c(x) = \checkmark, & \text{termination, or} \\ c(x) = \underbrace{(a, x')}_{(out(x),\ next(x))} \end{cases}$$

stream automata w/ explicit termination.

3) $F = A \times (\_) \times (\_)$

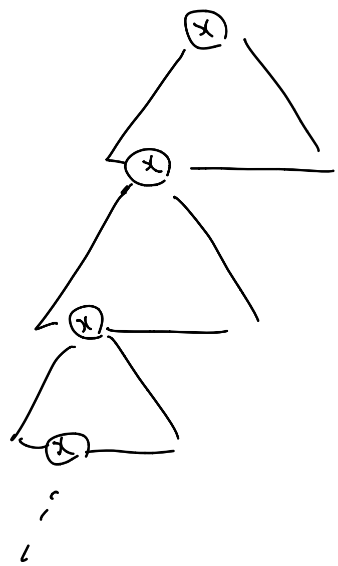
$$\begin{tikzcd} A \times X \times X \\ X \arrow[u, "c"] \end{tikzcd} \quad c = \langle out : X \to A,\ left : X \to X,\ right : X \to X \rangle$$

Deterministic generative binary tree automaton over A



... ↓ continue

Generate a complete binary tree whose nodes are A-labeled



rational trees

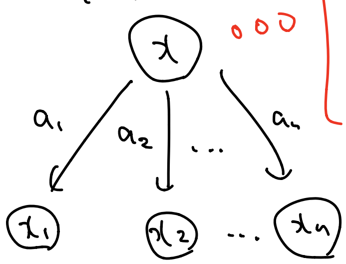
4) $F = 2 \times (-)^A$, where $2 = \{ tt, ff \}$

$$\begin{array}{c} 2 \times X^A \\ \uparrow c \\ X \end{array} = \left\langle \begin{array}{l} X \xrightarrow{\text{accept?}} 2 , \\ X \xrightarrow{\text{next}} X^A \end{array} \right\rangle$$

$$\left[\frac{X \longrightarrow X^A}{X \times A \longrightarrow X} \right] \qquad (-)\times A \dashv (-)^A$$

current state, input letter, next state

Assume $A = \{a_1, \dots, a_n\}$



$\left[x \ \text{ or } \ \text{⊚}x \right]$

$\left[\text{if } accept?(x) = tt \right]$

To summarize:

$$\frac{F\text{- coalgebras}}{\text{deterministic } \cancel{\text{finite}} \text{ automata.}}$$

locally finitely presentable categories

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J. Rosicky

⋮

5) $F = 2 \times (\mathcal{P}(-))^A$

the powerset construction

$$\begin{array}{c} 2 \times (\mathcal{P}X)^A \\ \uparrow c \\ X \end{array} \quad // \quad \begin{array}{l} X \xrightarrow{\text{accept?}} 2 = \{tt, ff\} \cong \{⊚, \circ\} \\ X \times A \xrightarrow{\text{next}} \mathcal{P}X \end{array}$$

$next(x, a) \subseteq X$

// nondeterministic ~~finite~~ automata.

Def. Let $F : \mathbb{C} \to \mathbb{C}$ be a functor.

A **final** F-coalg. is a final (or terminal) object in $\mathbf{Coalg}(F)$

- The limit of the empty diagram.
- **Def.** $X \in \mathbb{C}$ is **final** if for each $Y \in \mathbb{C}$, $Y \overset{\exists! f}{\dashrightarrow} X$ ($\mathbb{C}$)
- In **Sets**: a singleton is final

$$\begin{array}{ccc} X & \longrightarrow & \{*\} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ x & \longmapsto & * \end{array}$$

That is,

$\begin{pmatrix} FZ \\ \uparrow \zeta \\ Z \end{pmatrix}$ is final

for each c, $\qquad [\mathbf{Coalg}(F)]$

$$\begin{pmatrix} FX \\ c \uparrow \\ X \end{pmatrix} \overset{\exists!}{\dashrightarrow} \begin{pmatrix} FZ \\ \uparrow \zeta \\ Z \end{pmatrix}$$

for each c, $\exists!\, f$ s.t.

$$\begin{tikzcd} FX \arrow[r, dashed, "Ff"'] & FZ \\ X \arrow[u, "c"] \arrow[r, dashed, "f"'] & Z \arrow[u, "\zeta"'] \end{tikzcd}$$

Now: Z must be

- comprehensive (all the F-behaviors should be there) ← existence of f
- abstract (two different states never have the same behaviors) ← uniqueness of f

→ Z is "the set of F-behaviors" ??

Examples

- $F = A \times (\_)$,

$$\begin{array}{c} A \times A^{\omega} \\ \uparrow \scriptstyle \xi = (hd, tl) \\ A^{\omega} \end{array}$$

is final $\boxed{Q}$ Prove finality

$$\left[\begin{array}{c} (a_0, a_1 a_2 \ldots) \\ \uparrow \\ a_0 a_1 a_2 \ldots \end{array} \right]$$

∘∘∘ an F-behavior $=$ streams $a_0 a_1 a_2 \ldots \in \underbrace{A^{\omega}}_{= A^{\mathbb{N}}}$?? at least plausible ...

- $F = 1 + A \times (\_)$ (for str. automata w/ termination)

$$\begin{array}{c} 1 + A \times A^{\infty} \\ \uparrow \\ A^{\infty} \end{array}$$

is final.

$$\left[\begin{array}{cc} \checkmark & (a, \sigma) \\ \uparrow & \uparrow \\ \varepsilon & a \cdot \sigma \end{array} \right]$$

ε is the empty sequence (of length 0)

∘∘ behaviors ? $\underbrace{A^{*}}_{= \coprod_{n \in \mathbb{N}} A^n} \cup A^{\omega} =: A^{\infty}$

$\coprod_{n \in \mathbb{N}} A^n = \{$ all finite streams $\}$

- $F = 2 \times (\_)^A$ (for det. automata)

$$\begin{array}{c} 2 \times \left(2^{(A^*)}\right)^A \\ \uparrow \\ 2^{(A^*)} \end{array}$$

is final.

<u>Lem.</u> (Lambek)

Let $\begin{array}{c} FZ \\ \uparrow \zeta \\ Z \end{array}$ be a final coalgebra.

Then ζ is necessarily an iso.

[By this a final $2 \times (\mathcal{P}(-))^A$ - coalg. is prohibited]

<u>Proof.</u> [<u>aim</u> $FZ \underset{\zeta}{\overset{??}{\rightleftarrows}} Z$ $\cong$] ← use finality!

$\begin{array}{c} F(FZ) \\ \uparrow F\zeta \\ FZ \end{array}$ is an F-coalg. By finality of ζ,

we obtain $FZ \xrightarrow{f} Z$ s.t.

$$\begin{tikzcd}
F(FZ) \arrow[r, "Ff"] & FZ \\
FZ \arrow[u, "F\zeta"] \arrow[r, "f"'] \arrow[ur, phantom, "="] & Z \arrow[u, "\zeta"']
\end{tikzcd} \quad — \ 3$$

[<u>aim</u> $\zeta \circ f = id$, $f \circ \zeta = id$]

Now

$$\begin{tikzcd}
FZ \arrow[r, "F\zeta"] \arrow[rr, bend left, "F(f\circ\zeta)"] & F(FZ) \arrow[r, "Ff"] & FZ \\
Z \arrow[u, "\zeta"] \arrow[r, "\zeta"'] \arrow[rr, bend right, "f\circ\zeta"'] \arrow[ur, phantom, "= \text{(easy)}"] & FZ \arrow[u, "F\zeta"'] \arrow[r, "f"'] & Z \arrow[u, "\zeta"', "\text{final}"]
\end{tikzcd} \quad — \ 7$$

$$
\begin{tikzcd}
FZ \arrow[r, "F(id_Z) = id_{FZ}"] & FZ \\
Z \arrow[u, "\zeta"] \arrow[r, "id_Z"'] & Z \arrow[u, "\text{find}"', "\zeta"]
\end{tikzcd}
\qquad — (2)
$$

By (1), (2), by the def. of finality (uniqueness)

$$f \circ \zeta = id_Z \qquad — (4)$$

For the other one we have

$$
\begin{aligned}
\zeta \circ f &\underset{(3)}{=} Ff \circ F\zeta \\
&\underset{F:\text{ functor}}{=} F(f \circ \zeta) \\
&\underset{(4)}{=} F(id_Z) \underset{F:\text{ functor}}{=} id_{FZ}.
\end{aligned}
$$

$\square$