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Category Theory = language

- ↓
- computer science
- { - functional programming
- coalgebra (← automata)

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Example 2 (A preorder P)

cf. An equivalence relation:

$R \subseteq X \times X$ that is $\left\{ \begin{array}{l} \text{reflexive} \\ \text{transitive} \\ \text{symmetric} \end{array} \right\}$

- A (partial) order:

~~$R \subseteq X \times X$~~ that is $\left\{ \begin{array}{l} \text{refl.} \\ \text{trans.} \\ \text{antisymmetric.} \end{array} \right.$

$$\begin{array}{l} x \leq y \quad y \leq z \\ \hline x = z \end{array}$$

Def. A preorder is a pair

(P, \leq)

↖
a set

$\leq \subseteq P \times P$

that is

reflexive &
transitive

e.g. - $\leq = P \times P$

- $P = \{ \text{all polynomials} \}$

$f \leq g$

\iff
def. $\deg(f) \leq \deg(g)$

- Let $G = (V, E)$ be a directed
graph.
 $V \times V$

$x, y \in V$

$x \leq y$

\iff
def. y is reachable
from x .

Prop. Let (P, \leq) be a preorder.

Then the following data constitute a category \mathbb{P}

obj. elements of P

arr. $x \rightarrow y$ in \mathbb{P}
 $\underline{\underline{x \leq y}}$

if and only if
 \leftarrow in a 1-1 correspondence with

identities

$$\underline{\underline{x \xrightarrow{id_x} x}}$$

$x \leq x$ \leftarrow \in : reflexivity

composition

aim if then \Downarrow

$$\left[\begin{array}{c} x \xrightarrow{f} y \quad y \xrightarrow{g} z \\ \hline x \xrightarrow{g \circ f} z \end{array} \right]$$

by def. of arrows in \mathbb{P}

$$\begin{array}{c} \underline{\underline{x \xrightarrow{f} y}} \quad \underline{\underline{y \xrightarrow{g} z}} \\ \hline x \leq y \quad y \leq z \end{array}$$

transitivity of \leq

$$\underline{\underline{x \leq z}}$$

$$x \rightarrow z$$

Now we have to check equations:

Def. A monoid is a set M

equipped with

- opt.
 {
 - (unit) $e \in M$ $e: M^0 \rightarrow M$
 \xrightarrow{su}
1 0-ary opr.
 - (multiplication) $\cdot: M^2 \rightarrow M$ 2-ary opr.

subject to

- eqs.
 {
 - (the unit law) $e \cdot x = x$
 $x \cdot e = x$ for $\forall x \in M$
 - (the assoc. law) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$

NB. Groups = Monoids + x^{-1} for $\forall x, y, z \in M$

Monoids for actions

Groups for invertible actions \rightarrow symmetry

Ex.

- All matrices w/ multiplication
- $\Sigma^* = \bigsqcup_{n \in \mathbb{N}} \Sigma^n = \left\{ \begin{array}{l} \text{all words/} \\ \text{lists over} \\ \text{the alphabet } \Sigma \end{array} \right\}$
- $(\mathbb{N}, 0, +)$

Prop. Let (M, e, \cdot) be a monoid,

$$\begin{array}{c} \uparrow \quad \uparrow \\ e \in M \quad \cdot : M \times M \rightarrow M \end{array}$$

Then this induces a category \mathbb{M} whose

obj. We have only one object ~~*~~ \odot

arr. $\odot \rightarrow \odot$
 $\hline \hline$
 $x \in M$

id. $\odot \xrightarrow{\text{id}_{\odot}} \odot$
 $\hline \hline$
 $e \in M$

composition

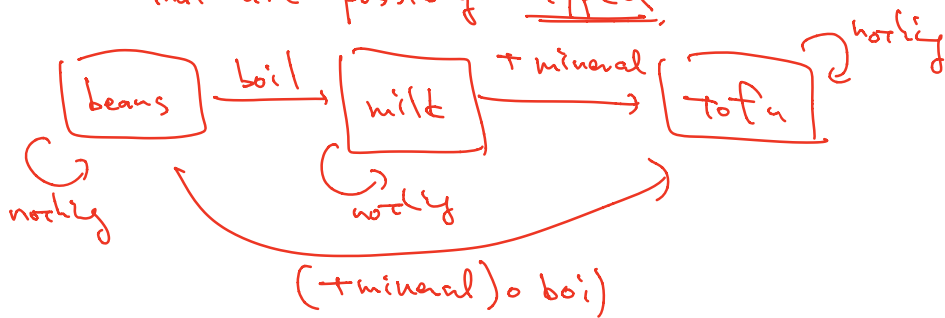
$\odot \xrightarrow{f} \odot \quad \odot \xrightarrow{g} \odot$
 $\hline \hline$
 $\odot \xrightarrow{g \circ f} \odot$
 \parallel
 $g \circ f$
 multip. of M

$\parallel : \mathbb{C} \mathbb{C} \mathbb{C}$

id. law.
assoc.

Follow from the monoid axioms.

NB A category
 = a collection of actions
 that are possibly typed



Example

The category Mon of monoids and monoid homomorphisms.

obj. an object in Mon

a monoid (M, e, \cdot)

arr. $(M, e, \cdot) \xrightarrow{f} (M', e', \cdot')$ in Mon

a monoid homomorphism f

$$f: M \rightarrow M'$$

such that $f(e) = e'$

$$f(x \cdot y) = f(x) \cdot' f(y)$$

Def. Let \mathcal{C} be a category.

Let $x, y \in \mathcal{C}$ be its objects.

A triple $(P, P \xrightarrow{\pi_1} x, P \xrightarrow{\pi_2} y)$

is a product of x and y if

- For any triple $(z, z \xrightarrow{f} x, z \xrightarrow{g} y)$,

- there uniquely exists an arrow

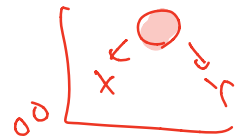
$$z \xrightarrow{h} P$$

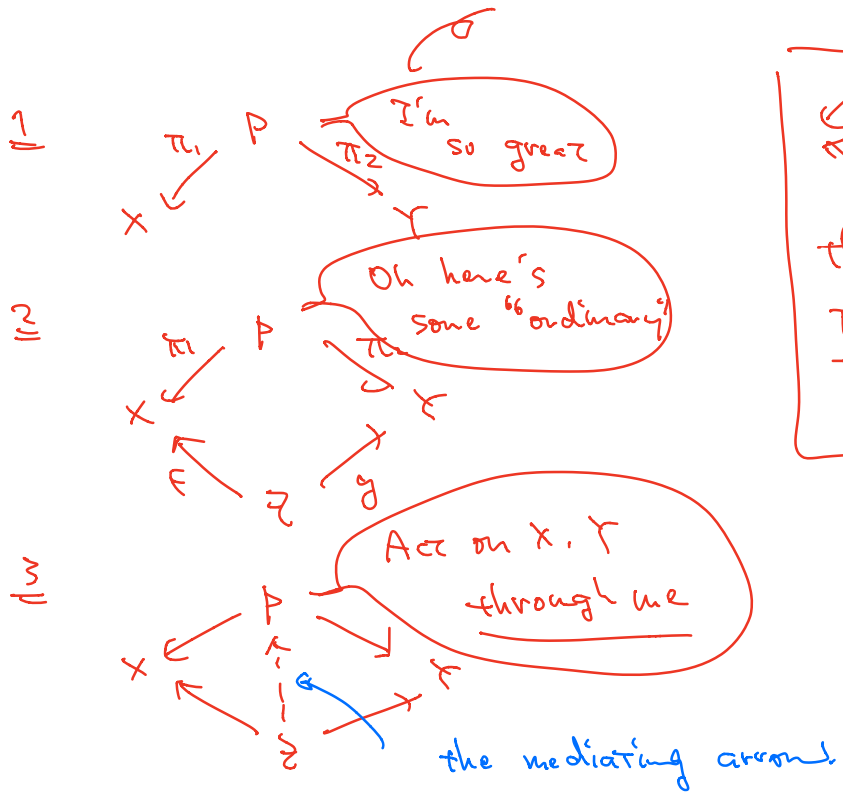
such that

$$\begin{array}{ccc} & P & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ x & \xrightarrow{h} & P & \xrightarrow{g} & y \\ \uparrow f & & \uparrow h & & \downarrow g \\ z & & z & & z \end{array}$$

Commutates.

φ





these two equalities

that is,

$$\pi_1 \circ h = f$$

$$\pi_2 \circ h = g$$