username :
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MSCS
3/10/2016

Category Theory = language

computer science
- functional programming
- coalgebra ($\leftrightarrow$ automata)

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Example 2 (A preorder P)

cf • An equivalence relation:

$R \subseteq X \times X$ that is reflexive, transitive, symmetric.

- A (partial) order:

$\cancel{R} \subseteq X \times X$ that is refl., trans., antisymmetric.

$\leq$

$x \leq x$

$\frac{x \leq y \quad y \leq x}{x = y}$

Def. A <u>preorder</u> is a pair

$$(P, \leq)$$

P: a set

$\leq$: $\leq \subseteq P \times P$ that is reflexive & transitive

e.g.
- $\leq \; = P \times P$
- $P = \{\text{all polynomials}\}$

  $f \leq g \underset{\text{def.}}{\iff} \deg(f) \leq \deg(g)$
- Let $G = (V, \underset{\substack{\cap\\ V \times V}}{E})$ be a directed graph.

  $x, y \in V$

  $x \leq y \underset{\text{def.}}{\iff} y$ is reachable from x.

Prop. Let $(P, \leq)$ be a preorder.

Then the following data constitute a category $\mathbb{P}$

obj. elements of P

arr.

$$\frac{x \longrightarrow y \text{ in } \mathbb{P}}{\overline{x \leq y}}$$

i.e. if and only if

i.e. in a 1-1 correspondence with

identities

$$\frac{x \xrightarrow{id_x} x}{\overline{x \leq x}} \longleftarrow \leq : \text{reflexivity}$$

composition

$$\left[\text{aim: if } \quad \frac{x \xrightarrow{f} y \qquad y \xrightarrow{g} z}{x \xrightarrow{g \circ f} z} \quad \Downarrow \text{ then} \right]$$

by def. of arrows in $\mathbb{P}$

$$\dfrac{\dfrac{\underset{\mathbb{P}}{x \xrightarrow{f} y}}{\overline{x \leq y}} \qquad \dfrac{\underset{\mathbb{P}}{y \xrightarrow{g} z}}{\overline{y \leq z}}}{\dfrac{x \leq z}{\overline{x \longrightarrow z}}} \quad \text{transitivity of } \leq$$

Now we have to check equations:

the (identity law)

By the def. of arrows in $\mathbb{P}$, the following holds:

[for each pair of objects $x, y \in \mathbb{P}$ we have at most one arrow of the type $x \to y$

diam

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[dr, "f"'] & y \arrow[d, "id_y"] \arrow[dr, "g"] \arrow[dl, phantom, "/\!/ @"] & \\ & y \arrow[r, "g"'] \arrow[ur, phantom, "/\!/ @"] & z \end{tikzcd}$$

$X \in \mathbb{C}$

"X is an object of a cat. $\mathbb{C}$"

(the associativity law)

(the same proof)

diam

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[dr, "g \circ f"'] & y \arrow[d, "g"] \arrow[dr, "h \circ g"] & \\ & z \arrow[r, "h"'] & u \end{tikzcd}$$

$$h \circ (g \circ f) \underset{@}{=} (h \circ g) \circ f$$

Def. A category $\mathbb{C}$ is called a preorder if, for any objects $X, Y \in \mathbb{C}$, there are at most 1 arrow $X \to Y$.

$|\underbrace{A(X,Y)}| \leq 1$

$\{\text{arrows } X \to Y\}$ in $\mathbb{C}$

$|\underbrace{\mathbb{C}(X,Y)}| \leq 1$

Def. A <u>monoid</u> is a set M

equipped with

opr.

- (unit) $e \in M$ $\quad e : M^0 \to M$ $\{$ 0-ary opr.
$\quad M^0 \cong 1$
- (multiplication) $\cdot : M^2 \longrightarrow M$ $\{$ 2-ary opr.

subject to

eq.

- (the unit law) $e \cdot x = x$, $x \cdot e = x$ $\}$ for $\forall x \in M$
- (the assoc. law) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for $\forall\, x, y, z \in M$

<u>NB.</u> Groups = Monoids + x^{-1}

Monoids for actions

Groups for invertible actions $\to$ symmetry

<u>Ex.</u>

- All matrices w/ multiplication
- $\Sigma^* = \coprod_{n \in \mathbb{N}} \Sigma^n = \{$ all |words / lists| over the alphabet Σ $\}$
- $(\mathbb{N}, 0, +)$

Prop. Let $(M, e, \cdot)$ be a monoid.

$e \in M \qquad \cdot : M \times M \to M$

Then this induces a category $\mathbb{M}$ whose

obj. We have only one object ~~$*$~~ ☺

arr.

$$\begin{array}{c} \text{☺} \longrightarrow \text{☺} \\ \hline\hline x \in M \end{array}$$

id.

$$\begin{array}{c} \text{☺} \xrightarrow{\ id_{\text{☺}}\ } \text{☺} \\ \hline\hline e \in M \end{array}$$

composition

$$\begin{array}{c} \text{☺} \xrightarrow{f} \text{☺} \qquad \text{☺} \xrightarrow{g} \text{☺} \\ \hline \text{☺} \xrightarrow{\ g \circ f\ } \text{☺} \end{array}$$

comp. of $\mathbb{M}$

$\quad\ \ \ \, \| $
$\quad\ \ g \cdot f$

multip. of M

11:47 —

id. law
assoc.
} Follow from the monoid axioms.

NB A category

= a collection of actions (= arrows)

that are possibly typed.

```tikzcd
\text{beans} \arrow[r, "\text{boil}"] \arrow[loop below, "\text{nothing}"] \arrow[rr, bend right, "(+\text{mineral}) \circ \text{boil}"'] & \text{milk} \arrow[r, "+\text{mineral}"] \arrow[loop below, "\text{nothing}"] & \text{tofu} \arrow[loop right, "\text{nothing}"]
```

Example

The category $\mathbb{M}\text{on}$ of monoids and monoid homomorphisms.

<u>obj.</u>

$$\frac{\text{an object in } \mathbb{M}\text{on}}{\text{a monoid } (M, e, \cdot)}$$

<u>arr.</u>

$$\frac{(M, e, \cdot) \xrightarrow{f} (M', e', \cdot') \text{ in } \mathbb{M}\text{on}}{\text{a monoid homomorphism } f}$$

$f : M \to M'$

such that $f(e) = e'$

$f(x \cdot y) = f(x) \cdot' f(y)$

<u>Def.</u> Let $\mathbb{C}$ be a category.

Let $X, Y \in \mathbb{C}$ be its objects.

$$\begin{tikzcd} & \bigcirc \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

A triple $\left(\underset{\in \mathbb{C}}{P},\ P \xrightarrow{\pi_1} X \in \mathbb{C},\ P \xrightarrow{\pi_2} Y \in \mathbb{C} \right)$

the "universal"

$$\begin{tikzcd} & \bigcirc \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

is a <u>product</u> of X and Y if

- For any triple $\left(\underset{\in \mathbb{C}}{Z},\ Z \xrightarrow{f} X \in \mathbb{C},\ Z \xrightarrow{g} Y \in \mathbb{C} \right)$,
- there uniquely exists an arrow

$$Z \xrightarrow{h} P$$

such that

$$\begin{tikzcd} & P \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & Y \\ & Z \arrow[ul, "f"] \arrow[uu, "h"] \arrow[ur, "g"'] & \end{tikzcd}$$

<u>Commutes.</u>

1.

$$\begin{tikzcd} & P \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & Y \end{tikzcd}$$

I'm so great

2.

$$\begin{tikzcd} & P \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & Y \\ & Z \arrow[ul, "f"] \arrow[ur, "g"'] & \end{tikzcd}$$

Oh here's some "ordinary"

3.

$$\begin{tikzcd} & P \arrow[dl] \arrow[dr] & \\ X & & Y \\ & Z \arrow[ul] \arrow[ur] \arrow[uu, dashed] & \end{tikzcd}$$

Act on X, Y through me

the mediating arrow.

$$\begin{tikzcd} & & \\ X & P \arrow[l] \arrow[r] & Y \\ & Z \arrow[ul] \arrow[u] \arrow[ur] & \end{tikzcd}$$

these two equalities

that is,

$\pi_1 \circ h = f$

$\pi_2 \circ h = g$