Mathematical Semantics of Computer Systems, *MSCS* (4810-1168)
Handout for Lecture 3 (2016/10/17)

Ichiro Hasuo, Dept. Computer Science, Univ. Tokyo
`http://www-mmm.is.s.u-tokyo.ac.jp/~ichiro`

Video recording of the lectures is available at:

`http://www-mmm.is.s.u-tokyo.ac.jp/videos/mscs2016`

(ask me for username, password)

We loosely follow [4], but it hardly serves as an introductory textbook. More beginner-friendly ones include [1, 5]; other classical textbooks include [6, 2]. nLab (`ncatlab.org`) is an excellent online information source.

1 Today's Goal, I

Come to understand and prove the following statement.

Proposition 1. *Let $X, Y \in \mathbb{C}$. A product of X and Y, if it exists, is unique up-to a canonical isomorphism.*

2 Today's Agenda I

2.1 Categories, Functors

Definition. Functor

Example. Monotone functions as functors (for preorders considered as categories); monoid homomorphisms as functors (for monoids).

Example. Graphs as functors. Monoid/group actions as functors.

2.2 Reasoning with Arrows

Definition. Epi, mono. Split epi, split mono.

Definition. Coproduct.

3 Today's Goal II

Identify the following framework of *abstract interpretation* [3] as an instance of adjunction. (Thanks are due to Kengo Kido for a nice introduction.)

Definition (Galois connection)**.** Let L and $\overline{L}$ be posets; and $\alpha\colon L \to \overline{L}$ and $\gamma\colon \overline{L} \to L$ be monotone functions. The pair (α, γ) is said to be a *Galois connection* if, for any $x \in L$ and $\overline{x} \in \overline{x}$,

$$\alpha(x) \leq_{\overline{L}} \overline{x} \quad \text{if and only if} \quad x \leq_L \gamma(\overline{x}) \enspace .$$

Example (interval domain)**.** Let

$$L \;:=\; \mathcal{P}(\mathbb{N}) \quad \text{and} \quad \overline{L} \;:=\; \{\emptyset\} \cup \bigl\{\, [l, r] \bigm| l, r \in \mathbb{N} \cup \{-\infty, \infty\},\, l \leq r \,\bigr\}$$

where each set is ordered by inclusion. Moreover,

$$\alpha(S) \;:=\; [\min S, \max X] \quad \text{and} \quad \gamma(\overline{S}) \;:=\; \{n \in \mathbb{N} \mid n \in \overline{S}\} \enspace .$$

Then the pair (α, γ) is a Galois connection.

4 Today's Agenda II

4.1 Natural Transformations

Definition. Natural transformation

Example. Natural transformations in graphs, and in monoid/group actions.
Natural transformations between monotone maps as functors.

Definition. Horizontal and vertical composition of natural transformation

4.2 Limits and Colimit

Definition. Diagram, cone, cocone

Definition. Limit, colimit

Proposition 2. *Limits from products and equalizers*

Corollary 1. *Concrete presentation of (co)limits in* **Sets**

4.3 Adjunction

Definition. Homset.

Definition. Adjunction.

Example. Free monoids.

Definition. Unit, counit.

Lemma 1. *Adjoint transposes by units and counits.*

Proposition 3. *Characterization of adjuction by: 1) the universality of η (Def. 3.2 of [Lambek & Scott], intuitive for free monoids); 2) the triangular equalities (Def. 3.1 of [Lambek & Scott]).*

Lemma 2.

1. *Adjoint functors determine each other uniquely up-to canonical natural isomorphisms.*
2. *Composition of adjoints.*

4.4 Limits as Adjoints

Definition. Functor category

Proposition 4. *Limits give rise to an adjunction.*

5 Exercises

1. Formulate and prove the following statement.

   A right adjoint preserves limits.

2. Prove the following: in an adjunction $F \dashv G$, G is faithful if and only if every component of the counit ε is an epi. [6, Thm. IV.3.1]

6 Report Problems

1. Prove that: in the category **Sets** of sets and functions, an arrow is a mono if and only if it is an injective function. Similarly, an arrow is an epi if and only if it is a surjective function.

2. Give a detailed proof of the *Today's Goal, I*:

    Let $X, Y \in \mathbb{C}$. A product of X and Y, if it exists, is unique up-to a canonical commuting isomorphism.

Reports are due at the beginning of the next lecture.

References

[1] S. Awodey. *Category Theory.* Oxford Logic Guides. Oxford Univ. Press, 2006.

[2] M. Barr and C. Wells. *Toposes, Triples and Theories.* Springer, Berlin, 1985. Available online.

[3] P. Cousot and R. Cousot. Abstract interpretation: A unified lattice model for static analysis of programs by construction or approximation of fixpoints. In R.M. Graham, M.A. Harrison and R. Sethi, editors, *Conference Record of the Fourth ACM Symposium on Principles of Programming Languages, Los Angeles, California, USA, January 1977*, pp. 238–252. ACM, 1977.

[4] J. Lambek and P.J. Scott. *Introduction to higher order Categorical Logic.* No. 7 in Cambridge Studies in Advanced Mathematics. Cambridge Univ. Press, 1986.

[5] T. Leinster. *Basic Category Theory.* Cambridge Univ. Press, 2014.

[6] S. Mac Lane. *Categories for the Working Mathematician.* Springer, Berlin, 2nd edn., 1998.