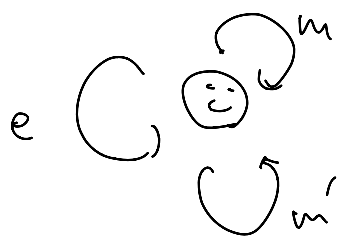
- A monoid as an example of categories



$e, m, m' \ldots \in M$

A mediating arrow

$$\begin{tikzcd}
& P \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[dd, bend left, "\times"] & \\
X & & Y \\
& Z \arrow[ul, "f"] \arrow[ur, "g"'] \arrow[uu, "h"] &
\end{tikzcd}$$

$f = \pi_1 \circ h$

$g = \pi_2 \circ h$

f is via π_1

g is via π_2

"f factors through π_1"

<u>NB</u> A mediating arrow is required to <u>exist</u> and be <u>unique</u>.

Examples

1) **Sets** $\left[\begin{array}{ll} \text{obj.} & \text{sets} \\ \text{arr.} & \text{functions} \end{array}\right]$

a categorical product is given by the set-th. notion of Cartesian products

$$\{(x,y) \mid x \in X,\ y \in Y\} = X \times Y$$

$$\begin{tikzcd} & X \times Y \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & Y \end{tikzcd}$$

projections

$$\begin{array}{rccc} \pi_1 : & X \times Y & \longrightarrow & X \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & (x,y) & \longmapsto & x \end{array}$$

Let's check its universality. Assume we have

$$\begin{tikzcd} X & & Y \\ & Z \arrow[ul, "f"] \arrow[ur, "g"'] & \end{tikzcd}$$

[aim: existence, uniqueness]

$$\begin{tikzcd} & X \times Y \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & Y \\ & Z \arrow[ul, "f"] \arrow[ur, "g"'] \arrow[uu, dashed] & \end{tikzcd}$$

existence

$h := \underline{\langle f, g \rangle} :$ The tupling of f and g

$$\begin{array}{ccc} Z & \longrightarrow & X \times Y \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ z & \longmapsto & \underbrace{(\underset{\in X}{f(z)},\ \underset{\in Y}{g(z)})}_{\in X \times Y} \end{array}$$

- This $\langle f, g \rangle$ makes the diagram commute.

$$\begin{tikzcd}
 & (f(z), g(z)) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
f(z) \arrow[d, phantom, "\parallel"] & & g(z) \arrow[d, phantom, "\parallel"] \\
f(z) & & g(z) \\
 & z \arrow[ul, "f"] \arrow[ur, "g"'] \arrow[uuu, "\langle f, g \rangle"] &
\end{tikzcd}$$

the required equalities

commutativity

uniqueness

Let k be another mediating arrow, that is,

$$\begin{tikzcd}
 & X \times Y \arrow[dl] \arrow[dr] & \\
X & & Y \\
 & Z \arrow[ul] \arrow[ur] \arrow[uu, "k"] &
\end{tikzcd}$$

commutes.

$\left(\underline{\text{aim}} \quad k = \langle f, g \rangle \right)$

$\boxed{Q}$ (Advanced) T: a monad over $\mathcal{C}$

$$\begin{tikzcd}
\mathcal{C}^T \arrow[d, bend right, "F"'] \arrow[d, phantom, "\dashv"] \\
\mathcal{C} \arrow[u, bend right, "U"']
\end{tikzcd}$$

U creates products ---> ncatlab.org

Now the function $k: Z \longrightarrow X \times Y$ can be written in the form

$$k(\underset{\in Z}{z}) = (\underbrace{k_1(z)}_{\in X}, \underbrace{k_2(z)}_{\in Y}).$$

(Due to the def. of $X \times Y$)

By the commutativity we must have

$$k_1(z) = f(z)$$
$$k_2(z) = g(z)$$

Therefore

$$k(z) = (f(z), g(z)), \quad \forall z \in Z$$

This proves $k = (f, g)$ $\square$

Example

Let $(P, \leq)$ be a preorder {refl. trans.}, considered a category.

Let $x, y \in P$ (i.e. objects)

If a product p of x and y exists, then

- $x \leftarrow p \rightarrow y$ // $p \le x$, $p \le y$

- $\dfrac{x \leftarrow z \rightarrow y}{}$ } then

$$\begin{tikzcd}
 & p \arrow[dl] \arrow[dr] & \\
x & & y \\
 & z \arrow[ul] \arrow[ur] \arrow[uu, dashed] &
\end{tikzcd}$$

// $\dfrac{z \le x \quad z \le y}{z \le p}$

p is the infimum $x \wedge y$ of x and y

Therefore products need not exist, e.g. in a preorder

x $\geq$ $\leq$ y

$\vdots$
$\geq$
2
$\geq$
1
$\geq$
0

all nat. number

Notations

- A product of X, Y is often denoted by

$$\begin{tikzcd} & X \times Y \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & Y \end{tikzcd}$$

Example

- $\mathbb{M}on$ obj   monoids
  arr.   monoid homomorphisms

Products of X and Y, if they exist, they are essentially unique:

Prop. Let $X \xleftarrow{\pi_1} P \xrightarrow{\pi_2} Y$ ($\in \mathbb{C}$) and $X \xleftarrow{\pi_1'} P' \xrightarrow{\pi_2'} Y$ ($\in \mathbb{C}$) be products of X, Y.

Then there exists a unique isomorphism $P \underset{\cong}{\xrightarrow{f}} P'$ s.t.

$$\begin{tikzcd} & P \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[dd, "f", "\cong"'] & \\ X & & Y \\ & P' \arrow[ul] \arrow[ur] & \end{tikzcd}$$

commutes.

uniqueness up-to a unique coherent isomorphism

BTW **Def.** An arrow $X \xrightarrow[\cong]{f} Y$ is an <u>isomorphism</u>

if there exists $Y \xrightarrow{g} X$ s.t.

$$g \circ f = id_X \qquad f \circ g = id_Y$$

$\therefore$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, bend right, "id"'] & Y \arrow[d, "g"] \arrow[dr, "id"] \arrow[dl, phantom, "="] \arrow[d, phantom, shift left=4, "="] & \\ & X \arrow[r, "f"'] & Y \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[loop left, "id"] \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \arrow[loop right, "id"] \end{tikzcd}$$

Proof. $\boxed{\underline{\text{aim}} \quad \begin{tikzcd} P \arrow[loop left, "id"] \arrow[r, bend left, "f"] & P' \arrow[l, bend left, "g"] \arrow[loop right, "id"] \end{tikzcd}}$

We are in the situation

$$\begin{tikzcd} & P' \arrow[dl, "\pi_1'"'] \arrow[dr, "\pi_2'"] & \\ X & & Y \\ & P \arrow[ul, "\pi_1"] \arrow[ur, "\pi_2"'] \arrow[uu, dashed, "(\pi_1', \pi_2') =: f"] & \end{tikzcd}$$

By the universality of (P', π_1', π_2') we get

$$f := (\pi_1', \pi_2')$$

Similarly

$$\begin{tikzcd} & P \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & Y \\ & P' \arrow[ul, "\pi_1'"] \arrow[ur, "\pi_2'"'] \arrow[uu, dashed, "(\pi_1, \pi_2) =: g"] & \end{tikzcd}$$

We get

$$g := (\pi_1, \pi_2)$$

Now $\left(\underline{\text{aim}} \quad g \circ f = id_P \right)$

we have $\therefore$

$$\begin{tikzcd} & \cdot \arrow[dl] \arrow[dr] & \\ \cdot & & \cdot \\ & \cdot \arrow[ul] \arrow[ur] \arrow[uu, dashed, "g \circ f"] & \end{tikzcd}$$

$$
\begin{tikzcd}
 & P \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
X & P' \arrow[l, "\pi_1'"'] \arrow[r, "\pi_2'"] \arrow[u, "g"'] & Y \\
 & P \arrow[ul, "\pi_1"] \arrow[ur, "\pi_2"'] \arrow[u, "f"'] &
\end{tikzcd}
$$

The whole diagram commutes.

Therefore

$$
\begin{tikzcd}
 & P \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
X & & Y \\
 & P \arrow[ul, "\pi_1"] \arrow[ur, "\pi_2"'] \arrow[uu, "g \circ f"'] &
\end{tikzcd}
\quad (1)
$$

$$
\left(
\begin{aligned}
\because\ & \pi_1 \circ (g \circ f) \\
&\underset{\text{assoc.}}{=} (\pi_1 \circ g) \circ f \\
&\underset{(1)}{=} \pi_1' \circ f \\
&\underset{(2)}{=} \pi_1
\end{aligned}
\right)
$$

But we also have

$$
\begin{tikzcd}
 & P \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
X & & Y \\
 & P \arrow[ul, "\pi_1"] \arrow[ur, "\pi_2"'] \arrow[uu, "id_P"'] &
\end{tikzcd}
\quad (2)
$$

{ uniqueness
exist.

top

By (1) and (2), by the universality of P

$$g \circ f = id_P$$

(... the rest is an exercise) ∎

The dual notion of product $\Longrightarrow$ Coproduct

reversing all the arrows

Def. A coproduct of $X, Y \in \mathcal{C}$ is

$$\left(C,\ X \xrightarrow{\kappa_1} C,\ Y \xrightarrow{\kappa_2} C \right)$$

coprojections

that is universal, in the sense that

$$\begin{tikzcd}
 & C \arrow[dd, dashed, "\exists!"'] & \\
X \arrow[ur, "\kappa_1"] \arrow[dr, "f"'] & & Y \arrow[ul, "\kappa_2"'] \arrow[dl, "g"] \\
 & Z &
\end{tikzcd}$$

$\dashrightarrow$ means "uniquely exists"

cotupling, $[f, g]$.

Ex.

- In Sets?

$$x \longmapsto (0, x) \qquad (1, y) \longmapsfrom y$$

$$X \xrightarrow{\kappa_1} X \sqcup Y \xleftarrow{\kappa_2} Y$$

$$X \sqcup Y = \{(0, x) \mid x \in X\} \cup \{(1, y) \mid y \in Y\}$$

labels to enforce disjointness

$$\begin{tikzcd}
 & X \sqcup Y \arrow[dd, "{[f,g]}"] & \\
X \arrow[ur, "\kappa_1"] \arrow[dr, "f"'] & & Y \arrow[ul, "\kappa_2"'] \arrow[dl, "g"] \\
 & Z &
\end{tikzcd}$$

$$\begin{array}{cccc}
X \sqcup Y & \ni & (0, x) & (1, y) \\
\Big\downarrow [f, g] & & \mapsto & \mapsto \\
Z & \ni & f(x) & g(y)
\end{array}$$

- In a preorder? Supremum!!

Notation A coproduct is often denoted by $X + Y$.

Def. An arrow $m : X \to Y$ in $\mathcal{C}$

is a mono / monic / left-cancelable if,

for any $Z \in \mathcal{C}$, and $Z \underset{g}{\overset{f}{\rightrightarrows}} X$,

we have

$$\left[\ m \circ f = m \circ g \ \text{ implies } \ f = g.\ \right]$$

$\forall Z \underset{\forall g}{\overset{\forall f}{\rightrightarrows}} X \xrightarrow{m} Y$

Prop. In Sets,

an arrow f is a mono — element-free formulation

$\Leftrightarrow$ the fuction f is injective — based on elements of sets

$$\boxed{f(x) = f(x') \implies x = x'}$$

Proof. Exercise.

Hint.
$$\frac{x \in X}{\dfrac{}{1 \longrightarrow X \text{ in Sets}}}$$
$$* \longmapsto x$$

where $1 = \{*\}$ is a singleton

Def.

- $e: X \to Y$ is an epi (epic)

$\overset{def}{\Longleftrightarrow}$ $\left[X \xrightarrow{e} Y \underset{\forall g}{\overset{\forall f}{\rightrightarrows}} \forall Z \right.$

$\left. f \circ e = g \circ e \implies f = g \right]$

Notations

$m: X \rightarrowtail Y$ mono

$e: X \twoheadrightarrow Y$ epi

right-cancelable

- $m: X \to Y$ is a split mono if it has a left inverse $f: Y \to X$ s.t. $f \circ m = id$

- $e: X \to Y$ is a split epi if it has a right inverse.

Prop.

- In Sets,

epi $\Longleftrightarrow$ surjective.

Prop.

- iso $\Rightarrow$ split mono $\Rightarrow$ mono

  iso $\Rightarrow$ split epi $\Rightarrow$ epi

in Sets,

$0 \dashrightarrow 1$ is a mono but not a split mono.

$\boxed{0 = \emptyset}$ $\boxed{1 = \{*\}}$

- $\begin{pmatrix} \text{split mono} \\ \text{epi} \end{pmatrix} \Rightarrow$ iso

nice exercise :)