Mathematical Semantics of Computer Systems, *MSCS* (4810-1168)
Handout for Lecture 4 (2016/10/24)

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Video recording of the lectures is available at:

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1 Today's Goal

Identify the following framework of *abstract interpretation* [?] as an instance of adjunction. (Thanks are due to Kengo Kido for a nice introduction.)

Definition (Galois connection)**.** Let L and $\overline{L}$ be posets; and $\alpha\colon L \to \overline{L}$ and $\gamma\colon \overline{L} \to L$ be monotone functions. The pair (α, γ) is said to be a *Galois connection* if, for any $x \in L$ and $\overline{x} \in \overline{x}$,

$$\alpha(x) \leq_{\overline{L}} \overline{x} \quad \text{if and only if} \quad x \leq_L \gamma(\overline{x}) \ .$$

Example (interval domain)**.** Let

$$L \ := \ \mathcal{P}(\mathbb{N}) \quad \text{and} \quad \overline{L} \ := \ \{\emptyset\} \cup \left\{ [l, r] \;\middle|\; l, r \in \mathbb{N} \cup \{-\infty, \infty\}, \, l \leq r \right\}$$

where each set is ordered by inclusion. Moreover,

$$\alpha(S) \ := \ [\min S, \max X] \quad \text{and} \quad \gamma(\overline{S}) \ := \ \{n \in \mathbb{N} \mid n \in \overline{S}\} \ .$$

Then the pair (α, γ) is a Galois connection.

2 Today's Agenda

2.1 From the Last Lecture

Proposition 1. *If an arrow is at the same time a split mono and an epi, then it is an isomorphism.*

2.2 Functors

Definition. Functor

Example. Monotone functions as functors (for preorders considered as categories); monoid homomorphisms as functors (for monoids).

Example. Graphs as functors. Monoid/group actions as functors.

2.3 Natural Transformations

Definition. Natural transformation

Example. Natural transformations in graphs, and in monoid/group actions.
Natural transformations between monotone maps as functors.

Definition. Horizontal and vertical composition of natural transformation

2.4 Limits and Colimit

Definition. Diagram, cone, cocone

Definition. Limit, colimit

Proposition 2. *Limits from products and equalizers*

Corollary 1. *Concrete presentation of (co)limits in* **Sets**

2.5 Adjunction

Definition. Homset.

Definition. Adjunction.

Example. Free monoids.

Definition. Unit, counit.

Lemma 1. *Adjoint transposes by units and counits.*

Proposition 3. *Characterization of adjuction by: 1) the universality of η (Def. 3.2 of [Lambek & Scott], intuitive for free monoids); 2) the triangular equalities (Def. 3.1 of [Lambek & Scott]).*

Lemma 2.

1. *Adjoint functors determine each other uniquely up-to canonical natural isomorphisms.*
2. *Composition of adjoints.*

2.6 Limits as Adjoints

Definition. Functor category

Proposition 4. *A limit gives rise to an adjunction.*

3 Exercises

1. Formulate and prove the following statement.

   A right adjoint preserves limits.

2. Prove the following: in an adjunction $F \dashv G$, G is faithful if and only if every component of the counit ε is an epi. [**?**, Thm. IV.3.1]