- (co) products
- (split) mono / epi.

injective
$\uparrow$ in Sets
left-cancellable
$\uparrow$

mono $\wedge$ epi
$\not\Downarrow$ isomorphism
$\uparrow$
invertible

Prop. $f : X \to Y$

If f is both

($\underset{\text{a}}{\wedge}$ split mono and $\underset{\text{an}}{\wedge}$ epi)

then f is an isomorphism.

Proof. Since f is a split mono it has a left inverse g.

$$\begin{tikzcd} X \arrow[loop left, "id_X"] \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd}$$

$g \circ f = id_X$ — ①

(aim $f \circ g = id_Y$)

$\Uparrow$ f: an epi (right-cancelable)

(aim $(f \circ g) \circ f = id_Y \circ f$)

Now $(f \circ g) \circ f \underset{\text{assoc. of } \circ}{=} f \circ (g \circ f)$

$\underset{\text{① above}}{=} f \circ id_X = f = id_Y \circ f$

Since f is an epi we have

$f \circ g = id_Y$ — ②

By ①, ②, g is an inverse of f. ▨

Adjunction

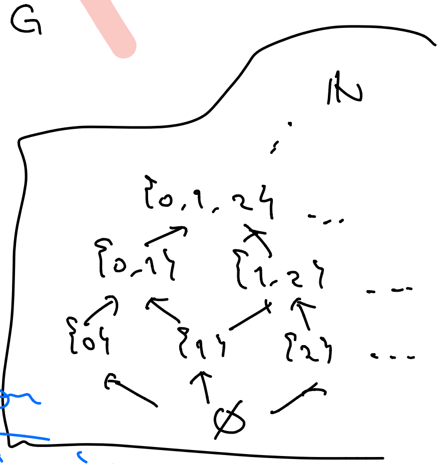
$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbb{D} \arrow[l, bend left, "G"] \end{tikzcd}$$

Today's goal

Cousot-Cousot

Example "fine" $L = \mathcal{P}(\mathbb{N})$

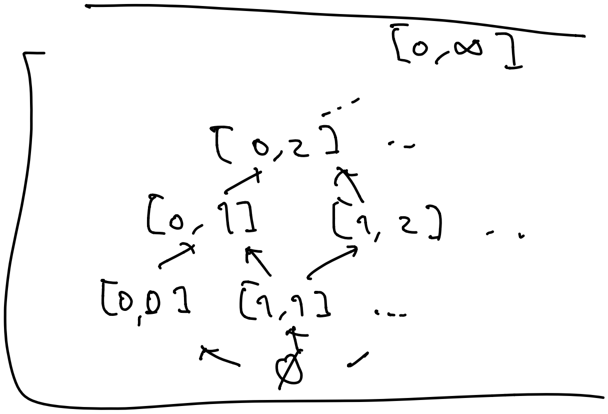
uncountable.



abstract interpretation is about (over) approximation

"coarser" countable

$$I = \{\emptyset\} \cup \{[l,r] \mid l, r \in \mathbb{N} \cup \{\infty\}, l \le r\}$$



Now consider

abstraction map

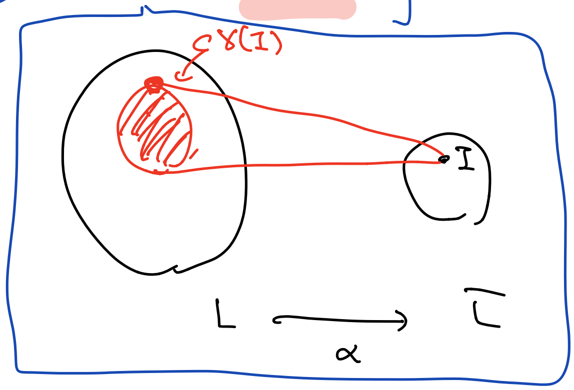
$$\begin{array}{ccc} \mathcal{P}(\mathbb{N}) & \longrightarrow & \bar{L} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ S & \longmapsto & [\min S, \sup S] \\ \{0, 1, 10^{10}\} & \longmapsto & [0, 10^{10}] \end{array}$$

$$\begin{tikzcd} L \arrow[rr, bend left, "\alpha"] \arrow[rr, phantom, "\perp"] & & \bar{L} \arrow[ll, bend left, "\gamma"] \end{tikzcd}$$

- $\begin{cases} \gamma \circ \alpha \supseteq id_L \\ \alpha \circ \gamma = id_{\bar{L}} \end{cases}$

- For each $I \in \bar{L}$,

  $\left[\begin{array}{l} \alpha(S) \subseteq I \\ \iff S \subseteq \gamma(I) \end{array} \right]$



Concretization map

$$\begin{array}{ccc} \bar{L} & \longrightarrow & \mathcal{P}(\mathbb{N}) \\ \rotatebox{90}{$\in$} & & \\ [l, r] & \longmapsto & \{ n \in \mathbb{N} \mid l \leq n \leq r \} \end{array}$$

An adjunction will look like

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbb{D} \arrow[l, bend left, "G"] \end{tikzcd}$$

functors
関手

homomorphisms of categories

Def. (functor)

Let $\mathbb{C}$, $\mathbb{D}$ be categories.

A <u>functor</u> $F : \mathbb{C} \to \mathbb{D}$ consists of

$$(F_O, F_A)$$

where

$$\begin{array}{c} X \xrightarrow{f} Y \quad \mathbb{C} \\ \hline FX \xrightarrow{Ff} FY \quad \mathbb{D} \end{array}$$

- $F_O : O_{\mathbb{C}} \longrightarrow O_{\mathbb{D}}$, F

  a correspondence. ("function" modulo the size issue)

- $\left(F_A(X, Y) : A_{\mathbb{C}}(X, Y) \longrightarrow A_{\mathbb{D}}(F_O X, F_O Y) \right)_{X, Y \in \mathbb{C}}$ F F

  a family of correspondences

subject to

$$- \ F_A\left(X \xrightarrow{id_X} X\right)^{\mathbb{C}} = \left(F_o X \xrightarrow{id_{F_o X}} F_o X\right)^{\mathbb{D}}$$

$$- \ F\left(X \xrightarrow{f} Y \xrightarrow{g} Z\right)^{\mathbb{C}} = \left(FX \xrightarrow{Ff} FY \xrightarrow{Fg} FZ\right)^{\mathbb{D}}$$

- $F(id) = id$
- $F(g \underset{\text{in } \mathbb{C}}{\circ} f) = (Fg) \underset{\text{in } \mathbb{D}}{\circ} (Ff)$

Examples

- P, P' : preorders

$\boxed{Q}$ What's a functor from $\underline{P}$ to $\underline{P'}$?

categories

$\boxed{A}$

$$\begin{tikzcd} F : P \arrow[rr] & & P' \\ P] \quad x_1 \arrow[d] \arrow[rr, mapsto] & & P'] \quad F(x_1) \arrow[d] \\ x_2 & & F(x_2) \end{tikzcd}$$

$\Rightarrow$

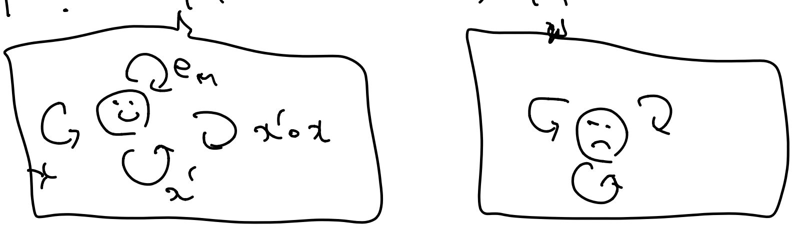
$$\text{then } \frac{x_1 \leq x_2 \text{ in } P}{F(x_1) \leq F(x_2) \text{ in } P'}$$

monotonicity !

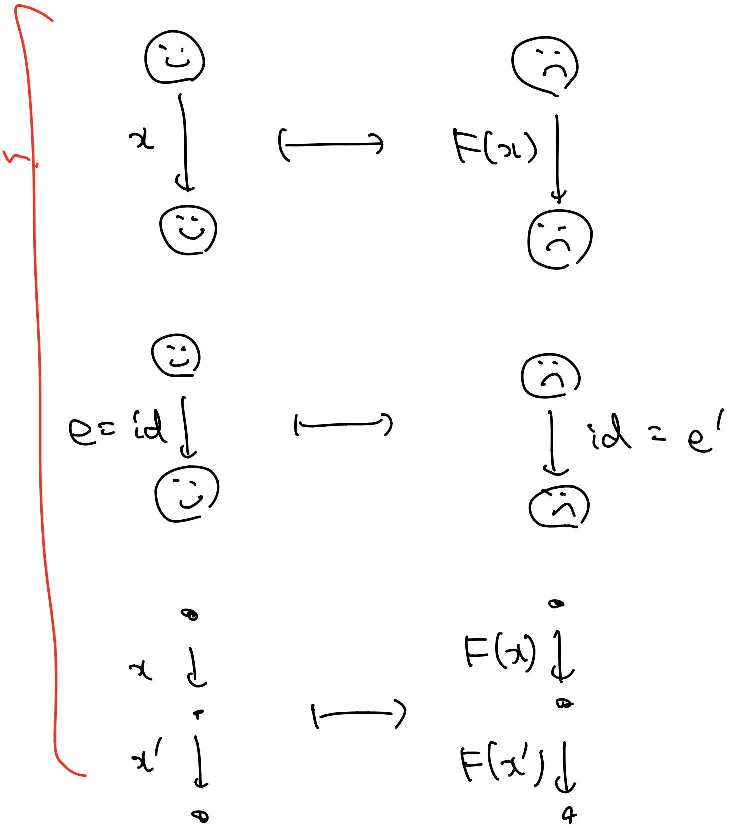
- M, M' : monoids (as categories)

$\boxed{Q}$ What's a functor $M \to M'$?

$\boxed{A}$ $F : M \longrightarrow M'$



monoid homomorphism



Examples

- Def A graph is $\cancel{\left(\underset{\{\text{vertices}\}}{\underset{\|}{V}},\ \underset{\{\text{edges}\}}{\underset{\|}{\overset{\overset{V\times V}{\cup\!\mid}}{E}}}\right)}$.

Sets

$$\begin{tikzcd} E \arrow[r, shift left, "\text{source}"] \arrow[r, shift right, "\text{target}"'] & V \end{tikzcd}$$

Prop.

A graph is identified with a functor

two objects

two non-identity arrows.

$$\overset{\mathbb{G}}{\boxed{\left(\underset{☺}{0} \overset{f}{\underset{g}{\rightrightarrows}} \underset{☹}{1} \right)}} \xrightarrow{F} \text{Sets.}$$

by $F(0) = \{\text{edges}\}$

$F(1) = \{\text{vertices}\}$

$Ff = \text{source} / \text{domain}$

$Fg = \text{target} / \text{codomain}.$

$$\begin{tikzcd} \mathbb{G} \arrow[r, "F"] & \text{Sets} \end{tikzcd}$$

$\mathbb{G})$

$$\begin{tikzcd} 0 \arrow[dd, shift right, "f"'] \arrow[dd, shift left, "g"] & & E = F0 \arrow[dd, shift right, "\substack{Ff \\ \| \\ \text{source}}"'] \arrow[dd, shift left, "\substack{Fg \\ \| \\ \text{target.}}"] \\ {} \arrow[rr, mapsto] & & {} \\ 1 & & V = F1 \end{tikzcd}$$

Natural transformations

$$\begin{tikzcd}
\mathbb{C} \arrow[r, bend left=40, "F"] \arrow[r, bend right=40, "G"'] \arrow[r, phantom, "\Downarrow \alpha"] & \mathbb{D}
\end{tikzcd}$$

a natural transf.

Def. $F, G : \mathbb{C} \to \mathbb{D}$, functors.

A <u>natural transformation</u> $\alpha : F \Rightarrow G$

is given by

$$\left(\alpha_X : FX \longrightarrow GX \quad \boxed{\mathbb{D}} \right)_{X \in \mathbb{C}}$$

... A bunch of arrows

subject to the following naturality condition.

$$\left[\begin{array}{l} \text{For each } X \xrightarrow{f} Y \text{ in } \mathbb{C}, \\ \text{we have} \qquad (Gf) \circ \alpha_X = \alpha_Y \circ (Ff) \end{array} \right]$$

That is,

$$\begin{tikzcd}
\mathbb{C} & & \mathbb{D} & \\
X \arrow[d, "f"'] & & FX \arrow[r, "\alpha_X"] \arrow[d, "Ff"'] \arrow[dr, phantom, "="] & GX \arrow[d, "Gf"] \\
Y & & FY \arrow[r, "\alpha_Y"'] & GY
\end{tikzcd}$$

Examples

- $$\begin{tikzcd} \mathbb{G} \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\Downarrow \alpha"] & \mathrm{Sets} \end{tikzcd}$$

$$/\!/$$

$$\boxed{\begin{tikzcd} 0 \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & 1 & \boxed{\mathbb{G}} \end{tikzcd}}$$

$\boxed{Q}$ What is α, in concrete terms?

α consists of

$$\begin{tikzcd} \alpha_0 : & F0 \arrow[r] \arrow[d, equal] & F'0 \arrow[d, equal] & \lfloor \mathrm{Sets} \\ & E & E' & \\ \alpha_1 : & F1 \arrow[r] \arrow[d, equal] & F'1 \arrow[d, equal] & \lfloor \mathrm{Sets} \\ & V & V' & \end{tikzcd}$$

and the naturality condition requires:

$$\begin{tikzcd} 0 \arrow[d, shift right, "f"'] \arrow[d, shift left, "g."] \\ 1 \end{tikzcd} \qquad \begin{tikzcd} E \arrow[r, "\alpha_0"] \arrow[d, "src"] \arrow[dr, phantom, "/\!/"] & E' \arrow[d, "src'"] \\ V \arrow[r, "\alpha_1"'] & V' \end{tikzcd} \qquad \begin{tikzcd} E \arrow[r, "\alpha_0"] \arrow[d, "tgt"] \arrow[dr, phantom, "/\!/"] & E' \arrow[d, "tgt'"] \\ V \arrow[r, "\alpha_1"'] & V' \end{tikzcd}$$

$\Longrightarrow$ α is a graph homomorphism!

Horizontal
Vertical | Composition of Nat. Trans.

$$\begin{tikzcd}
\mathbb{C} \arrow[r, bend left=40, "F"] \arrow[r, bend right=40, "F'"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathbb{D} \arrow[r, bend left=40, "G"] \arrow[r, bend right=40, "G'"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathbb{E}
\end{tikzcd}
\quad / \quad
\begin{tikzcd}
\mathbb{C} \arrow[rr, bend left=40, "G\circ F"] \arrow[rr, bend right=40, "G'\circ F'"'] \arrow[rr, phantom, "\Downarrow\beta\circ\alpha"] & & \mathbb{E}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{C} \arrow[rr, bend left=50, "F"] \arrow[rr, "F'" near end] \arrow[rr, bend right=50, "F''"'] \arrow[rr, phantom, shift left=3ex, "\Downarrow\alpha"] \arrow[rr, phantom, shift right=3ex, "\Downarrow\beta"] & & \mathbb{D}
\end{tikzcd}
\quad / \quad
\begin{tikzcd}
\mathbb{C} \arrow[rr, bend left=40, "F"] \arrow[rr, bend right=40, "F''"'] \arrow[rr, phantom, "\Downarrow\beta\bullet\alpha"] & & \mathbb{D}
\end{tikzcd}$$

Concretely

- $\beta\bullet\alpha : F \Rightarrow F''$ is given by

$$(\beta\bullet\alpha)_X : \begin{tikzcd}
FX \arrow[rr] \arrow[dr, "\alpha_X"'] & & F''X \\
& F'X \arrow[ur, "\beta_X"'] &
\end{tikzcd} \quad \boxed{\mathbb{D}}$$

the X component of $\beta\bullet\alpha$

(That is, $(\beta\bullet\alpha)_X := \beta_X \underset{\uparrow \text{ in } \mathbb{D}}{\circ} \alpha_X$)

Ex. Check naturality. (easy)

- $\beta\circ\alpha : G\circ F \Rightarrow G'\circ F'$ is given by $\boxed{\mathbb{E}}$

$$(\beta\circ\alpha)_X : \begin{tikzcd}
G(FX) \arrow[rr] \arrow[dr, "\beta_{FX}"] \arrow[dd, "\underline{G\alpha_X}"'] & & G'(F'X) \\
& G'(FX) \arrow[ur, "G'\alpha_X"'] & \\
G(F'X) \arrow[uurr, bend right, "\beta_{F'X}"'] \arrow[ur, phantom, "/\!/ \ \text{☺}"] & &
\end{tikzcd}$$

$$\boxed{\begin{array}{l} \alpha_X : FX \to F'X \quad \boxed{\mathbb{D}} \\ \hline G\alpha_X : GFX \to GF'X \end{array}}$$

Lem. The square ☺ commutes.

Proof. By the following naturality of β.

$$\boxed{\mathbb{D}} \quad \begin{tikzcd}
FX \arrow[d, "\alpha_X"] \\
F'X
\end{tikzcd} \qquad \boxed{\mathbb{E}} \quad \begin{tikzcd}
G(FX) \arrow[r, "\beta_{FX}"] \arrow[d, "G\alpha_X"'] \arrow[dr, phantom, "/\!/"] & G'(FX) \arrow[d, "G'\alpha_X"'] \\
G(F'X) \arrow[r, "\beta_{F'X}"'] & G'(F'X)
\end{tikzcd}$$

Same as ☺

Prop. We have so-called the <u>interchange law</u>. In the situation

$$\begin{tikzcd}[column sep=large] \bullet \arrow[r, bend left=50] \arrow[r] \arrow[r, bend right=50] \arrow[r, phantom, shift left=2.5ex, "\Downarrow\alpha"] \arrow[r, phantom, shift right=2.5ex, "\Downarrow\alpha'"] & \bullet \arrow[r, bend left=50] \arrow[r] \arrow[r, bend right=50] \arrow[r, phantom, shift left=2.5ex, "\Downarrow\beta"] \arrow[r, phantom, shift right=2.5ex, "\Downarrow\beta'"] & \bullet \end{tikzcd}$$

we have

$$(\beta' \bullet \beta) \circ (\alpha' \bullet \alpha) = (\beta' \circ \alpha') \bullet (\beta \circ \alpha)$$

[Proof. Exercise.]

Let's go back to adjunctions

$$\begin{tikzcd} L \arrow[r, bend left, "\alpha"] \arrow[r, phantom, "\perp"] & C \arrow[l, bend left, "\gamma"] \end{tikzcd} \overset{}{\Longrightarrow} \begin{tikzcd} \mathbb{C} \arrow[loop left, "id_{\mathbb{C}}"] \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbb{D} \arrow[l, bend left, "G"] \end{tikzcd}$$

categorically

$$id_L \le \gamma \circ \alpha \qquad\qquad id_{\mathbb{C}} \xRightarrow{\eta} G \circ F$$

$(\mathbb{C} \to \mathbb{C})$

Def. An adjunction

$$F \dashv G : \mathbb{C} \longrightarrow \mathbb{D}$$

consists of two functors

$F : \mathbb{C} \longrightarrow \mathbb{D}$

$G : \mathbb{D} \longrightarrow \mathbb{C}$

and two nat. trans.

$$\eta : \mathrm{id}_{\mathbb{C}} \Longrightarrow G \circ F$$

$$\varepsilon : F \circ G \Longrightarrow \mathrm{id}_{\mathbb{D}}$$

such that

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "F"] \arrow[dr, "\mathrm{id}"'] \arrow[dr, phantom, bend left, "\Uparrow \eta"] & \mathbb{D} \arrow[d, "G"] \arrow[dr, "\mathrm{id}_{\mathbb{D}}"] \arrow[dr, phantom, bend right, "\Uparrow \varepsilon"'] & \\
& \mathbb{C} \arrow[r, "F"'] & \mathbb{D}
\end{tikzcd}
=
\begin{tikzcd}
\mathbb{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\Uparrow \mathrm{id}_F"] & \overset{\mathbb{D}}{\cancel{\mathbb{C}}}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{D} \arrow[r, "G"] \arrow[dr, "\mathrm{id}_{\mathbb{D}}"'] \arrow[dr, phantom, bend left, "\Downarrow \varepsilon"] & \mathbb{C} \arrow[d, "F"] \arrow[dr, "\mathrm{id}_{\mathbb{C}}"] \arrow[dr, phantom, bend right, "\Downarrow \eta"'] & \\
& \mathbb{D} \arrow[r, "G"'] & \mathbb{C}
\end{tikzcd}
=
\begin{tikzcd}
\mathbb{D} \arrow[r, bend left, "G"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow \mathrm{id}_G"] & \overset{\mathbb{C}}{\cancel{\mathbb{D}}}
\end{tikzcd}$$