Report Assignments

(Due: the beginning of the lect. Mon. 2016.11.7)

$\boxed{1a}$ In the last proof, in the construction of

$$\eta : \mathrm{id} \Rightarrow GF$$

$$\varepsilon : FG \Rightarrow \mathrm{id}$$

from

$$\left(\mathbb{C}(X, GA) \xrightarrow[\varphi_{X,A}]{\cong} \mathbb{D}(FX, A) \right)_{X \in \mathbb{C}, A \in \mathbb{A}},$$

prove that the family

$$\left(\eta_X : X \longrightarrow GFX \right)_{X \in \mathbb{C}}$$

$\llcorner \mathbb{C}$

indeed satisfies the naturality requirement, and hence constitutes a natural transformation.

Proof.) We have to show the following commutativity.

$$\mathbb{C}\Big| \quad \begin{tikzcd} X \arrow[d, "f"] \\ X' \end{tikzcd} \qquad \mathbb{C}\Big| \quad \begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] \arrow[dr, phantom, "/\!/ \ \text{aim}"] & GFX \arrow[d, "GFf"] \\ X' \arrow[r, "\eta_{X'}"'] & GFX' \end{tikzcd}$$

Towards this goal we use the following naturality of the bijective correspondence.

$$\mathbb{C}\Big| \ \begin{tikzcd} X \\ X \arrow[u, "id_X"'] \end{tikzcd} \quad \mathbb{D}\Big| \ \begin{tikzcd} FX \arrow[d, "Ff"] \\ FX' \end{tikzcd} \quad \mathbf{Sets}\Big| \ \begin{tikzcd} \mathbb{C}(X, GFX) \arrow[r, "\cong", "\varphi_{X,FX}"'] \arrow[d, "\mathbb{C}(id_X, GFf)"] \arrow[dr, phantom, "/\!/ \ \text{naturality of } \varphi"] & \mathbb{D}(FX, FX) \arrow[d, "\mathbb{D}(id_{FX}, Ff)"] \\ \mathbb{C}(X, GFX') \arrow[r, "\cong", "\varphi_{X,FX'}"'] & \mathbb{D}(FX, FX') \end{tikzcd}$$

$\Downarrow$

Therefore

$$\begin{tikzcd} \varphi^{-1}_{X,FX}(id_{FX}) \arrow[d, phantom, "\| \text{ by def. of } \eta"] & id_{FX} \arrow[l, mapsto, "\varphi^{-1}"', "\cong"] \arrow[ddd, mapsto, "\mathbb{D}(id, Ff)"] \\ \eta_X \arrow[d, mapsto, "\mathbb{C}(id_X, GFf)"] & \\ (GFf)\circ \eta_X \arrow[d, phantom, "\|"] & \\ \varphi^{-1}_{X,FX'}(Ff) & Ff \arrow[l, mapsto, "\varphi^{-1}"] \end{tikzcd}$$

$$\mathbf{Sets}\Big| \quad \begin{array}{ccc} \mathbb{C}(X, GFX) & \xleftarrow{\varphi^{-1}_{X,FX}} & \mathbb{D}(FX, FX) \\ \cup\!\!\!\!\shortmid & & \cup\!\!\!\!\shortmid \\ \underline{\varphi^{-1}_{X,FX}(id_{FX})} & \longleftarrow\!\shortmid & \left(id_{FX} : FX \to FX \ \ \mathbb{D}\right) \\ \| & & \\ \eta_X & & \end{array}$$

Thus we've shown

$$(GFf)\circ\eta_X = \varphi^{-1}_{X,FX'}(Ff). \quad (1)$$

Here we invoke on another naturality square:

$$\begin{tikzcd} \underline{\mathbb{C}} & \underline{\mathbb{D}} & \underline{\mathbf{Sets}} & \\ X' & FX' \arrow[d, "id_{FX'}"] & \mathbb{C}(X', GFX') \arrow[r, "\cong", "\varphi"'] \arrow[d, "\mathbb{C}(f, id)"] & \mathbb{D}(FX', FX') \arrow[d, "\mathbb{C}(Ff, id)"] \\ X \arrow[u, "f"] & FX' & \mathbb{C}(X, GFX') \arrow[r, "\cong", "\varphi_{X,FX'}"'] & \mathbb{D}(FX, FX') \end{tikzcd}$$

Hence

$$\begin{tikzcd} \eta_{X'} \arrow[d, mapsto, "\mathbb{C}(f, id)"] & id_{FX'} \arrow[l, mapsto, "\varphi^{-1}"', "\cong"] \arrow[dd, mapsto, "\mathbb{C}(Ff, id)"] \\ \eta_{X'}\circ f \arrow[d, equal] & \\ \varphi^{-1}_{X,FX'}(Ff) & Ff \arrow[l, mapsto, "\varphi^{-1}"', "\cong"] \end{tikzcd}$$

Thus we have

$$\eta_{X'}\circ f = \varphi^{-1}_{X,FX'}(Ff) \quad (2)$$

Combining ①, ② yield

$$(GFf)\circ\eta_X = \eta_{X'}\circ f$$

that is the desired commutativity. $\square$

$\boxed{1b}$ Show that the same η, ε as above in $\boxed{1a}$ satisfy the equational axioms in the def. of adjunction.

Proof. We shall first show

$$\begin{tikzcd} \mathbb{C} \arrow[r, "F"] \arrow[dr, "id"'] \arrow[dr, phantom, shift left=3, "\Uparrow\eta"] & \mathbb{D} \arrow[d, "G"] \arrow[dr, "id"] \arrow[dr, phantom, shift right=3, "\Uparrow\varepsilon"] & \\ & \mathbb{C} \arrow[r, "F"'] & \mathbb{D} \end{tikzcd} \quad \overset{\text{aim}}{=} \quad \begin{tikzcd} \mathbb{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\parallel"] & \mathbb{D} \end{tikzcd} \quad \text{—} \; (*)$$

that is, for each $X \in \mathbb{C}$,

$$\begin{tikzcd} FX \arrow[r, "F\eta_X"] \arrow[dr, "id"'] \arrow[dr, phantom, shift left=3, "/\!/\,\text{aim}"] & FGFX \arrow[d, "\varepsilon_{FX}"] \\ & FX \end{tikzcd}$$

Convince yourself that $\xrightarrow{F\eta}\downarrow\varepsilon_{F}$ is indeed the X-component of the LHS of (*)

for $\alpha : F \Rightarrow G$, a nat. trans. $\alpha_X : FX \to GX$ is its X-component

Since

$$\mathbb{D}(FX, FX) \xrightarrow[\cong]{\varphi^{-1}_{X,FX}} \mathbb{C}(X, GFX)$$

is an isomorphism in Sets (i.e. a bijection), it suffices to show that

$$\varphi^{-1}_{X,FX}(\mathrm{id}_{FX}) = \varphi^{-1}_{X,FX}(\varepsilon_{FX} \circ F\eta_X) \qquad (**)$$

Now

$$(\mathrm{LHS}) = \left(X \xrightarrow{\eta_X} GFX \right) \quad \left(\text{by def. of } \eta\right)$$

$$(\mathrm{RHS}) = \underbrace{\varphi^{-1}_{X,FX}}_{\mathbb{C}(X,GFX) \xleftarrow{\varphi^{-1}} \mathbb{D}(FX,FX)} \left(FX \xrightarrow{F\eta_X} FGFX \xrightarrow{\varepsilon_{FX}} FX \right)_{\mathbb{D}}$$

$$\varepsilon_{FX} = \varphi_{GFX,FX}(\mathrm{id}_{GFX}) \quad (\text{def. of } \varepsilon)$$

$$\underset{\varphi}{=} \eta_X$$

naturality of φ, that is,

$$\mathbb{C}) \qquad \begin{tikzcd} GFX \\ X \arrow[u, "\eta_X"] \end{tikzcd} \qquad \mathrm{Sets}) \qquad \begin{tikzcd} \mathbb{C}(GFX, GFX) \arrow[r, "\varphi_{GFX,FX}", "\cong"'] \arrow[d, "\mathbb{C}(\eta_X, \mathrm{id})"] & \mathbb{D}(FGFX, FX) \arrow[d, "\mathbb{C}(F\eta_X, \mathrm{id})"] \\ \mathbb{C}(X, GFX) \arrow[r, "\varphi_{X,FX}", "\cong"'] & \mathbb{D}(FX, FX) \end{tikzcd}$$

from which it follows that

$$\begin{tikzcd} \mathrm{id}_{GFX} \arrow[r, mapsto] \arrow[d, "\mathbb{C}(\eta_X, \mathrm{id})"] & \varphi(\mathrm{id}_{GFX}) \overset{\text{def. of } \varepsilon}{=} \varepsilon_{FX} \arrow[d, "\mathbb{C}(F\eta_X, \mathrm{id})"] \\ \underset{\text{claim}}{\eta_X = \varphi^{-1}(\varepsilon_{FX} \circ F\eta_X)} & \varepsilon_{FX} \circ F\eta_X \arrow[l, mapsto, "\varphi^{-1}_{X,FX}"'] \end{tikzcd}$$

Therefore we have (LHS) = (RHS) in (**).

The other eq. axiom is proved similarly.

$\boxed{2a}$ Let $(F, G, \eta, \varepsilon)$ form an adjunction and define

$$\varphi_{X,A} : \mathbb{C}(X, GA) \longrightarrow \mathbb{D}(FX, A)$$

by

$$\left(X \xrightarrow{h} GA \;\;\text{in } \mathbb{C}\right) \longmapsto \left(FX \xrightarrow{Fh} FGA \xrightarrow{\varepsilon_A} A \;\;\text{in } \mathbb{D}\right).$$

Then φ is natural in X and A.

Proof.

We have to show the following.

$$\begin{tikzcd}
\mathbb{C} & \mathbb{D} & \mathrm{Sets} & \\
X & A & \mathbb{C}(X, GA) \arrow[r, "\varphi"] \arrow[d] \arrow[dr, phantom, "= \text{ aim}"] & \mathbb{D}(FX, A) \arrow[d, "\mathbb{D}(Ff, g)"] \\
X' \arrow[u, "f"] & A' \arrow[from=u, "g"] & \mathbb{C}(X', GA') \arrow[r, "\varphi"] & \mathbb{D}(FX', A')
\end{tikzcd}$$

That is, for each $h : X \to GA$ in $\mathbb{C}$,

$$\begin{tikzcd}
h \arrow[r, mapsto] \arrow[d, mapsto] & \varepsilon_A \circ Fh \arrow[d, mapsto] \\
& g \circ \varepsilon_A \circ Fh \circ Ff \quad \text{aim} \\
(Gg \circ h \circ f) \arrow[r, mapsto] & \left(FX' \xrightarrow{F(Gg \circ h \circ f)} FGA' \xrightarrow{\varepsilon_{A'}} A'\right) \arrow[u, phantom, "\parallel"]
\end{tikzcd}$$

Now we have

$$
\begin{tikzcd}
FX' \arrow[r, "Ff"] & FX \arrow[r, "Fh"] & FGA \arrow[r, "\varepsilon_A"] \arrow[d, "FGg"] \arrow[dr, phantom, "/\!/"] & A \arrow[d, "g"] \\
 & & FGA' \arrow[r, "\varepsilon_{A'}"'] & A'
\end{tikzcd}
$$

naturality of ε

Therefore

$$g \circ \varepsilon_A \circ Fh \circ Ff = \varepsilon_{A'} \circ FGg \circ Fh \circ Ff$$

and this proves the claim. $\square$

$\boxed{2b}$ In the above setting of $\boxed{2a}$, $\varphi_{X,A}$ is an isomorphism.

$\boxed{\text{Proof.}}$ We define

$$\psi_{X,A} : \mathbb{D}(FX, A) \longrightarrow \mathbb{C}(X, GA)$$

$$(FX \xrightarrow{i} A) \longmapsto (X \xrightarrow{\eta_X} GFX \xrightarrow{Gi} GA).$$

We shall show that $\varphi_{X,A}$ and $\psi_{X,A}$ are inverse to each other.

$\boxed{\varphi \circ \psi = \mathrm{id}}$

$$(\varphi \circ \psi)(i) = \varphi(Gi \circ \eta_X)$$

$$= \left(\begin{tikzcd} FX \arrow[r, "F\eta_X"] \arrow[drr, bend right, "\mathrm{id}"'] & FGFX \arrow[r, "FGi"] \arrow[dr, "\varepsilon_{FX}"'] & FGA \arrow[r, "\varepsilon_A"] \arrow[d, phantom, "\text{nat. of } \varepsilon"] & A \\ & & FX \arrow[ur, "i"'] & \end{tikzcd} \right)$$

eq. axiom for η, ε

$$= i.$$

$\boxed{\psi \circ \varphi = \mathrm{id}}$ Similar. $\square$