__Lem.__

- $\mathbb{D} \xrightarrow{G} \mathbb{C}$, a functor.

Then its left adjoint is unique up-to canonical natural isomorphisms.

That is,

$$\begin{tikzcd} \mathbb{D} \arrow[r, bend left, "G"] \arrow[r, phantom, "\top"] & \mathbb{C} \arrow[l, bend left, "F"] \end{tikzcd} \qquad \begin{tikzcd} \mathbb{D} \arrow[r, bend left, "G"] \arrow[r, phantom, "\top"] & \mathbb{C} \arrow[l, bend left, "F'"] \end{tikzcd}$$

$\eta'_X : X \to GF'X$

$\Longrightarrow$ F and F' are "essentially the same"

$\exists\, \alpha : F \xrightarrow{\cong} F'$, a natural isomorphism

canonical

- $FX \neq F'X$ $|_{\mathbb{D}}$

$\cong$

α_X

"naturally arising"

canonicity $\longleftrightarrow$ coincidental

__Prop.__ V, V' : vector spaces $/ \mathbb{R}$ w/ the same dimension.

$\Longrightarrow \exists f : V \xrightarrow{\cong} V'$

But this f is __not__ canonical

& there is some arbitrariness here

Proof.) $\left[\begin{array}{c} \underline{\text{ain}} \quad \dfrac{\dfrac{FX \xrightarrow{\alpha_X} F'X \quad (\mathbb{D})}{}}{X \xrightarrow{\eta'_X} GF'X \quad (\mathbb{C})} \quad F \dashv G \end{array} \right]$

We define $\alpha : F \Rightarrow F'$ by

$$\alpha_X := \underline{\varphi_{X,F'X}}(\eta'_X) \quad (\in \mathbb{D}(FX, F'X))$$

$$\boxed{\varphi_{X,Y} : \mathbb{C}(X, GY) \xrightarrow{\cong} \mathbb{D}(FX, Y)}$$

We can show

- $(\alpha_X)_{X \in \mathbb{C}}$ form a natural transf., that is,

$$\underline{\mathbb{C}} \qquad \underline{\mathbb{D}}$$

$$\begin{tikzcd} X \arrow[d, "f"'] \\ X' \end{tikzcd} \qquad \begin{tikzcd} FX \arrow[r, "\alpha_X"] \arrow[d, "Ff"'] \arrow[dr, phantom, "="] & F'X \arrow[d, "F'f"] \\ FX' \arrow[r, "\alpha_{X'}"'] & F'X' \end{tikzcd}$$

- $\alpha_X : FX \longrightarrow F'X \ (\mathbb{D})$ is an isomorphism for each $X \in \mathbb{C}$

$$\boxed{\alpha \text{ is a nat. iso.}}$$

☑

Lem.

- Similarly,

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbb{D} \arrow[l, bend left, "G"] \end{tikzcd} \qquad \begin{tikzcd} \mathbb{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbb{D} \arrow[l, bend left, "G'"] \end{tikzcd}$$

$\Longrightarrow \quad G \cong G'$ (a canonical natural isomorphism)

- $$\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbb{D} \arrow[l, bend left, "G"] \arrow[r, bend left, "F'"] \arrow[r, phantom, "\perp"] & \mathbb{E} \arrow[l, bend left, "G'"] \end{tikzcd}$$

$$\Longrightarrow \quad \begin{tikzcd} \mathbb{C} \arrow[rr, bend left, "F' \circ F"] \arrow[rr, phantom, "\perp"] & & \mathbb{E} \arrow[ll, bend left, "\cancel{G' \circ G}\ G \circ G'"] \end{tikzcd}$$

Proof. [(strategy 1)

$$\frac{X \longrightarrow GG'Z \quad \mathbb{C}}{F'FX \longrightarrow Z \quad \mathbb{E}} \longleftarrow \text{aim}$$

]

[

$$\begin{array}{ll} X \longrightarrow GG'Z & \\ \hline\hline FX \longrightarrow G'Z & F \dashv G \\ \hline\hline F'FX \longrightarrow Z & F' \dashv G' \end{array}$$

and we check the naturality]

[(Strategy 2)

$$\begin{tikzcd} \mathbb{C} \arrow[r, "F' \circ F"] \arrow[dr, "id"'] & \mathbb{E} \arrow[d, "G \circ G'"] \arrow[dl, phantom, "\Nearrow \eta''"] \\ & \mathbb{C} \end{tikzcd}$$

We aim at this unit, and at a counit, and we prove the eq. axioms betw. them.]

$$
\left[\begin{tikzcd}
\mathbb{C} \arrow[r, "F"'] \arrow[rr, bend left, "F'\circ F"] \arrow[ddrr, bend right, "\mathrm{id}_{\mathbb{C}}"'] & \mathbb{D} \arrow[r, "F'"'] \arrow[dr, "\mathrm{id}_{\mathbb{D}}"'] \arrow[dr, phantom, bend left, "\Uparrow \eta'"] & \mathbb{E} \arrow[d, "G'"] \arrow[dd, bend left, "G\circ G'"] \\
 & \arrow[ur, phantom, "\Uparrow \eta"] & \mathbb{D} \arrow[d, "G"] \\
 & & \mathbb{C}
\end{tikzcd}\right]
$$

$G\circ F$

("the triangular equalities")

Concretely:

$$
X \xrightarrow{\eta_X} GFX = (G\circ \mathrm{id}_{\mathbb{D}}\circ F)X \xrightarrow{G\eta'_{FX}} GG'F'FX
$$

$$
\boxed{\eta'_{FX} : FX \longrightarrow G'F'FX}
$$

Limits and Colimits

Def. A product of $X, Y \in \mathbb{C}$ is

$$
\begin{tikzcd}
& X\times Y \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
X & & Y
\end{tikzcd}
$$

that is universal among

$$
\begin{tikzcd}
& @ \arrow[dl] \arrow[dr] & \\
X & & Y
\end{tikzcd}
$$

Def. Let $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ $\in \mathbb{C}$.

[two obj.
two arr.]

An __equalizer__ (of f, g)

is $\left(E \in \mathbb{C}, \quad E \xrightarrow{e} X \in \mathbb{C} \right)$

that satisfies

$E \xrightarrow{e} X \underset{g}{\overset{f}{\rightrightarrows}} Y$ $\in \mathbb{C}$

- $f \circ e = g \circ e$
- and universal among such data.

  that is,

  if $D \xrightarrow{h} X$ satisfies $f \circ h = g \circ h$

  then

$$
\begin{tikzcd}
E \arrow[r, "e"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \\
D \arrow[u, dashed, "\exists ! i"] \arrow[ur, "h"'] & &
\end{tikzcd}
$$

s.t. $e \circ i = h$

__Example__

- In Sets, an equalizer of f, g is given by

$$\{ x \in X \mid f(x) = g(x) \} \hookrightarrow X$$

$$\begin{array}{ccc} \cup & & \cup \\ x & \longmapsto & x. \end{array}$$

<u>Lem</u>

If $\left(E \xrightarrow{e} X \underset{g}{\overset{f}{\rightrightarrows}} Y \right)$

e is an equalizer of f, g

then e is a <u>mono</u>.

left-cancelable.

$$\forall A \underset{\forall b}{\overset{\forall a}{\rightrightarrows}} E \xrightarrow{e} X$$

$$e \circ a = e \circ b$$

$$\Rightarrow a = b$$

<u>Proof.</u> Let

$$A \underset{b}{\overset{a}{\rightrightarrows}} E \xrightarrow{e} X$$

$$e \circ a = e \circ b. \qquad \left(\frac{\text{aim}}{} \quad a = b \right)$$

Now consider

$$\begin{tikzcd} & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \\ A \arrow[ur, "e \circ a"'] & & \end{tikzcd}$$

$e \circ a$

$(= e \circ b)$

by assump.

- $f \circ (e \circ a) = g \circ (e \circ a)$

$$\left[\because \right. \quad \begin{array}{ccc} f \circ (e \circ a) & & g \circ (e \circ a) \\ \| \text{ by assoc. of } \circ & & \| \text{ by assoc.} \\ (f \circ e) \circ a & \overset{\text{by asmp.}}{=} & (g \circ e) \circ a \end{array}$$

By the universality of an equalizer,

$$\begin{tikzcd} E \arrow[r, "e"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \\ & A \arrow[ul, dashed, "\exists! h"] \arrow[u, "e \circ a"'] & \end{tikzcd}$$

Now

- a is such a mediating arrow.

  [$\because$]
$$\begin{tikzcd} \cdot \arrow[r, "e"] & \cdot \\ \cdot \arrow[u, "a"] \arrow[ur, "e \circ a"'] & \end{tikzcd}$$

- b is ________________

  [$\because$]
$$\begin{tikzcd} \cdot \arrow[r, "e"] & \cdot \\ \cdot \arrow[u, "b"] \arrow[ur, "e \circ a = e \circ b"'] & \end{tikzcd}$$

  by assump.

Therefore by uniqueness $h = a = b$. $\square$

<u>Def.</u> Given $X \overset{f}{\underset{g}{\rightrightarrows}} Y$, $\mathbb{C}$ a <u>Coequalizer</u> of f, g

is $X \overset{f}{\underset{g}{\rightrightarrows}} Y \xrightarrow{c} C$ $\mathbb{C}$

s.t. $\begin{cases} - \ c \circ f = c \circ g \\ - \ \text{universal among such, that is,} \end{cases}$

$$
\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "c"] \arrow[dr, "\forall d"'] & C \arrow[d, dashed, "\exists !"] \\
& & D
\end{tikzcd}
$$

$d \circ f = d \circ g$

factor through

Ex. In Sets,

$$
\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "\text{quotient}"] & Y/\sim
\end{tikzcd}
$$

$x \overset{f}{\longmapsto} f(x)$, $x \overset{g}{\longmapsto} g(x)$

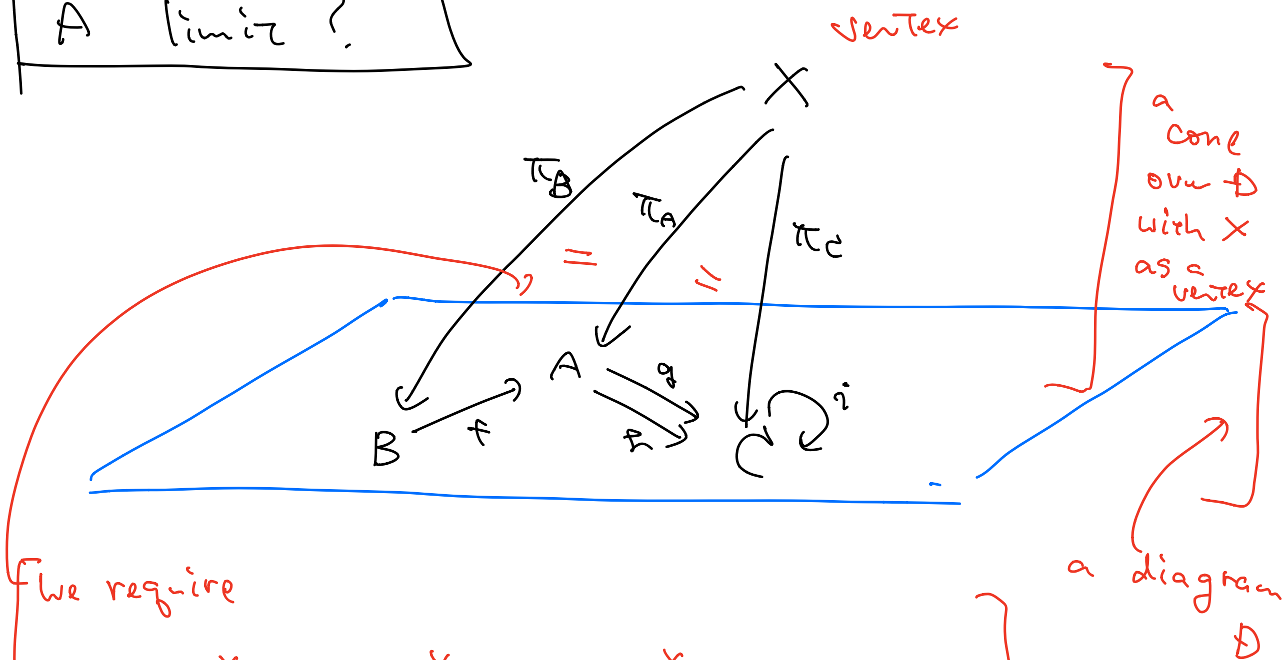
where $\sim$ is the smallest equivalence relation that contains

$$\{ (f(x), g(x)) \mid x \in X \}$$

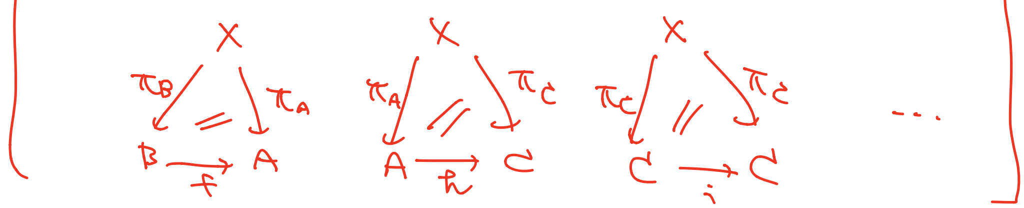
---

product	$\longleftrightarrow$ dual	Coprod.
equalizer		Coequal.
⋮		⋮
limits		colimits

A limit?



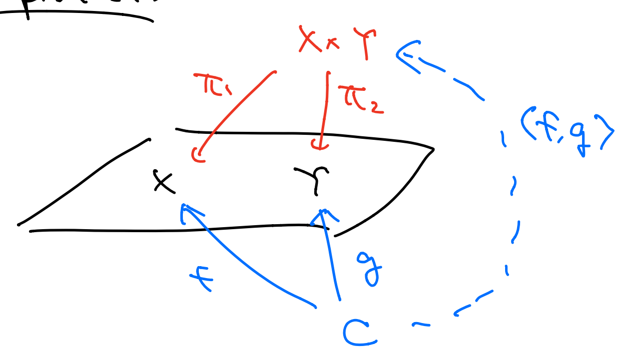
We require



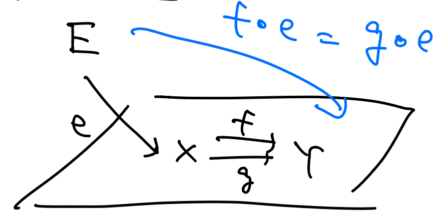
A limit is a universal cone.

e.g.

products



equalizers



The report assignments in the handout are ~~CANCELED~~