Mathematical Semantics of Computer Systems, *MSCS* (4810-1168)
Handout for Lecture 7 (2016/11/14)

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Video recording of the lectures is available at: `http://www-mmm.is.s.u-tokyo.ac.jp/videos/mscs2016`

Part I: Adjuction (ctn'd)

1 Today's Agenda

1.1 Limits and Colimit

Definition. Diagram, cone, cocone

Definition. Limit, colimit

Proposition 1. *Limits from products and equalizers*

Corollary 1. *Concrete presentation of (co)limits in* **Sets**

1.2 Limits as Adjoints

Definition. Functor category

Proposition 2. *A limit gives rise to an adjunction.*

2 Exercises

1. Formulate and prove the following statement.

   A right adjoint preserves limits.

2. Prove the following: in an adjunction $F \dashv G$, G is faithful if and only if every component of the counit ε is an epi. [5, Thm. IV.3.1]

Part II: the Yoneda Lemma

Remember: we loosely follow [3], but it hardly serves as an introductory textbook. More beginner-friendly ones include [1, 4]; other classical textbooks include [5, 2]. nLab (`ncatlab.org`) is an excellent online information source.

3 Today's Goal

Familiarize yourself with the *Yoneda lemma*. Identify it as a category theory analogue of the *Cayley representation theorem*:

Theorem (Cayley)**.** *Every group G is isomorphic to a subgroup of $\pi(|G|)$.*

4 Today's Agenda

4.1 Equivalence of Categories

Definition. Subcategory, faithful functor, full functor

Lemma 1. *Any functor preserves isomorphisms.*
A full and faithful functor reflects isomorphisms.

Definition. Equivalence of categories

Proposition 3. *Equivalence from a full, faithful and iso-dense functor.*

4.2 The Yoneda Lemma

Definition. Covariance, contravariance

Theorem (Yoneda)**.** *The Yoneda lemma, the Yoneda embedding as a full and faithful functor*

Definition. end, coend

Theorem. *The Yoneda lemma, the (co)end form*

Lemma 2. *Ends as limits [5, Prop. IX.5.1]*

Lemma 3. *Homfunctors preserve (co)limits, hence also (co)ends*

5 Exercises

1. Formulate the "naturality" of the Yoneda correspondence

$$\mathrm{Nat}\big(\mathbb{C}(\_\,, X),\, F\big) \;\cong\; FX$$

and prove it.

References

[1] S. Awodey. *Category Theory.* Oxford Logic Guides. Oxford Univ. Press, 2006.

[2] M. Barr and C. Wells. *Toposes, Triples and Theories.* Springer, Berlin, 1985. Available online.

[3] J. Lambek and P.J. Scott. *Introduction to higher order Categorical Logic.* No. 7 in Cambridge Studies in Advanced Mathematics. Cambridge Univ. Press, 1986.

[4] T. Leinster. *Basic Category Theory.* Cambridge Univ. Press, 2014.

[5] S. Mac Lane. *Categories for the Working Mathematician.* Springer, Berlin, 2nd edn., 1998.