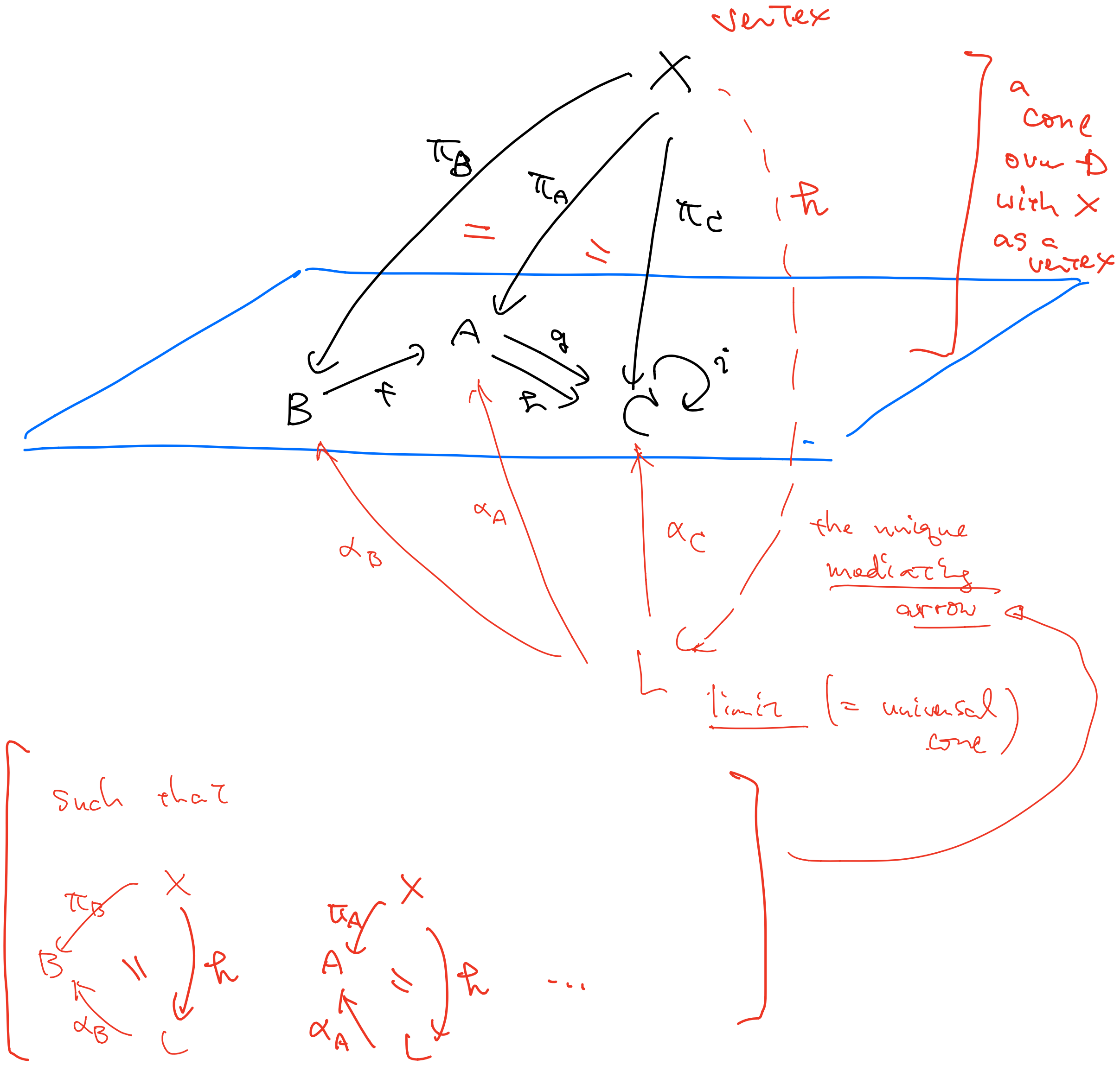



Def. $\mathbb{I}$ : a category (called the index category)

A diagram of the shape $\mathbb{I}$ in the category $\mathbb{C}$ is a functor

$$\mathbb{D} : \mathbb{I} \longrightarrow \mathbb{C}.$$

**Ex.** - products = limits of the shape

$$\boxed{I \qquad J}\ \mathbb{I}.$$

two objects
two (identity) arrows

A diagram of shape $\mathbb{I}$:

$$D : \mathbb{I} \longrightarrow \mathbb{C}$$

$$I \longmapsto DI \ (=: X)$$

$$J \longmapsto DJ \ (=: Y)$$

Possibly $X = Y$

$$\begin{array}{ccc} I & & X \\ \big\downarrow id_I & \longmapsto & \big\downarrow id_X \\ I & & X \end{array}$$

$$\boxed{X \qquad Y}\ \mathbb{C}$$

- equalizers = limits of the shape

$$\boxed{\begin{tikzcd} I \arrow[r, shift left, "i"] \arrow[r, shift right, "j"'] & J \end{tikzcd}}\ \mathbb{I}$$

two obj.
four arr.

$\boxed{i, j, id_I, id_J}$

$$D : \mathbb{I} \longrightarrow \mathbb{C} \ \circ \circ \circ$$

$$\boxed{\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}}\ \mathbb{C}$$

**Def.** Let $\mathbb{I}$, $\mathbb{C}$ be categories, and $X \in \mathbb{C}$.

The diagonal functor $\Delta X : \mathbb{I} \to \mathbb{C}$

is
$$\begin{aligned} \Delta X : \mathbb{I} &\longrightarrow \mathbb{C} \\ I &\longmapsto X \\ \begin{pmatrix} I \\ \downarrow i \\ J \end{pmatrix} &\longmapsto \begin{pmatrix} X \\ \downarrow \mathrm{id}_X \\ X \end{pmatrix} \end{aligned}$$

---

**Def.** Let $D : \mathbb{I} \to \mathbb{C}$ be a diagram and $X \in \mathbb{C}$.

$X$: vertex

$\Delta X : \mathbb{I} \to \mathbb{C}$

A cone from $\underset{\mathbb{C}}{\underset{\cap}{X}}$ to $D : \mathbb{I} \to \mathbb{C}$ is

a natural transformation

$$\begin{tikzcd} \mathbb{I} \arrow[r, bend left=40, "\Delta X"{name=U}] \arrow[r, bend right=40, "D"'{name=L}] & \mathbb{C} \arrow[from=U, to=L, Rightarrow, "\gamma"] \end{tikzcd}$$

Concretely:

$$\gamma = \left( \underbrace{(\Delta X)(I)}_{\substack{\parallel \text{ def. of } \Delta X \\ X}} \xrightarrow{\ \gamma_I\ } \underline{DI} \right)_{I \in \mathbb{I}} \qquad \text{in } \mathbb{C}$$

subject to

$$\begin{tikzcd}
\mathbb{I} & & \mathbb{C} & & X \arrow[d, bend right] \arrow[dd, bend left] \\
I \arrow[d, "i"] & & & & (\Delta X)(I) \arrow[r, "\gamma_I"] \arrow[d, "(\Delta X)(i)"'] \arrow[dr, phantom, "/\!/"] & DI \arrow[d, "Di"] \\
J & & & \mathrm{id}_X \arrow[ur, phantom] & (\Delta X)(J) \arrow[r, "\gamma_J"'] & DJ
\end{tikzcd}$$

$\mathrm{id}_X$

$(\Delta X) : \mathbb{I} \to \mathbb{C}$

$I \in \mathbb{I}$

$(\Delta X)(I) \in \mathbb{C}$

$DI \xrightarrow{Di} DJ \quad \text{in } \mathbb{C}$

That is,

$$\begin{tikzcd}
X \arrow[r, "\gamma_I"] \arrow[d, equal] & DI \arrow[d, "Di"] \\
X \arrow[r, "\gamma_J"'] & DJ
\end{tikzcd}$$

that is,

$$\begin{tikzcd}
& X \arrow[dl, "\gamma_I"'] \arrow[d, "\gamma_J"] & \mathbb{C} \\
DI \arrow[r, "Di"'] & DJ &
\end{tikzcd}$$

**Def.** (Lem.) Let $f : X \to Y \in \mathbb{C}$ be an arrow.
It induces a natural transformation

$$\begin{tikzcd} \mathbb{I} \arrow[r, bend left, "\Delta X"] \arrow[r, bend right, "\Delta Y"'] \arrow[r, phantom, "\Downarrow \Delta f"] & \mathbb{C} \end{tikzcd}$$

"$\Delta$" is a functor from $\mathbb{C}$ to $\mathbb{C}^{\mathbb{I}}$

later

$$\left[\begin{array}{ccc} \mathbb{I} & & \mathbb{C} \\ I & & X \xrightarrow{f} Y \\ \downarrow i & & \| \qquad \| \\ J & & X \xrightarrow{f} Y \end{array}\right]$$

$(\Delta X)(I)$

**Def.** Let $D : \mathbb{I} \to \mathbb{C}$ be a diagram.
A limit of $D$ is a pair

$$\left( L \in \mathbb{C}, \quad (\Delta L) \xRightarrow{\pi} D \right)$$

vertex

$$\begin{tikzcd} \mathbb{I} \arrow[r, bend left, "\Delta L"] \arrow[r, bend right, "D"'] \arrow[r, phantom, "\Downarrow \pi"] & \mathbb{C} \end{tikzcd}$$

a cone from $L$ to $D$

that is universal. This means: given another such pair

$$\left( X \in \mathbb{C}, \quad (\Delta X) \xRightarrow{\gamma} D \right)$$

we have

$$
\begin{tikzcd}
 & & L \arrow[dll, bend right, "\pi_I"'] \arrow[dl, "\pi_J"] \\
DI \arrow[r, "Di"] & DJ & \\
 & & X \arrow[ull, bend left, "\gamma_I"] \arrow[ul, "\gamma_J"'] \arrow[uu, bend right, dashed, "h"']
\end{tikzcd}
$$

a unique arrow $h : X \longrightarrow L$ such that

$\mathbb{C}$

$$
\begin{tikzcd}
\Delta X \arrow[r, Rightarrow, "\gamma"] \arrow[d, Rightarrow, "\Delta h"'] & D \\
\Delta L \arrow[ur, Rightarrow, "\pi"'] &
\end{tikzcd}
$$

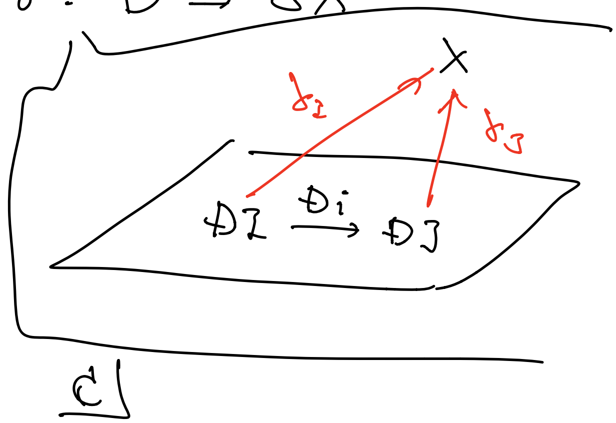
For each $I \in \mathbb{I}$,

$$
\begin{tikzcd}
X \arrow[r, "\gamma_I"] \arrow[d, "h"] & DI \\
L \arrow[ur, "\pi_I"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbb{I} \arrow[r, bend left, "\Delta X"] \arrow[r, bend right, "D"'] \arrow[r, phantom, "\Downarrow\gamma"] & \mathbb{C}
\end{tikzcd}
=
\begin{tikzcd}
\mathbb{I} \arrow[r, bend left=50, "\Delta X"] \arrow[r, "\Delta L"] \arrow[r, bend right=60, "D"'] \arrow[r, phantom, shift left=2, "\Downarrow\Delta h"] \arrow[r, phantom, shift right=3, "\pi\Downarrow"] & \mathbb{C}
\end{tikzcd}
$$

**Def.** Let $D : \mathbb{I} \to \mathbb{C}$ be a diagram.

- A <u>cocone</u> from $D$ to $X \in \mathbb{C}$ is a nat. trans. $\gamma : D \Rightarrow \Delta X$




- A <u>colimit</u> is a universal cocone, that is,

colimit

$$\begin{tikzcd}
D \arrow[r, Rightarrow, "\kappa"] \arrow[dr, Rightarrow, "\gamma"'] & \Delta C \arrow[d, dashed, Rightarrow, "\Delta h"] \\
& \Delta X
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[d, dashed, "h"] \\
X
\end{tikzcd}$$

---

"An arbitrary limit is presented only using products and equalizers".

Let $D : \mathbb{I} \longrightarrow \mathbb{C}$ be a diagram.

$DI \quad DJ \dots \Rightarrow X \times Y \times \dots$

Consider

codomain

$$\prod_{I \in \mathbb{I}} DI \underset{g}{\overset{f}{\rightrightarrows}} \prod_{i:\text{ an arrow in } \mathbb{I}} D(\mathrm{cod}(i))$$

$J$ if $i : I \longrightarrow J$ in $\mathbb{I}$

where $f, g$ are defined as follows.

Recall:

$$\frac{X \to Y \times Z}{X \to Y \quad X \to Z}$$

$[\mathbb{I}]\ h$, $(\pi_1 \circ h, \pi_2 \circ h)$

$[\mathbb{1}]\ \langle f, g\rangle$, $(f, g)$

**BTU**

$$\frac{\prod_{I \in \mathbb{I}} \mathcal{D}I \xrightarrow{h} \prod_i \mathcal{D}(\mathrm{cod}(i))}{\prod_{I \in \mathbb{I}} \mathcal{D}I \to \mathcal{D}(\mathrm{cod}(i)) \text{ for each } i}$$

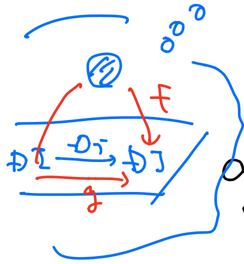
$\mathbb{C}$

$\prod_{I \in \mathbb{I}} \mathcal{D}I$, $\pi_I$, $\pi_{I'}$, $\pi_{I''}$, ..., $\mathcal{D}I\ \ \mathcal{D}I'\ \ \mathcal{D}I''$ ...

$$f := \left\langle \prod_{I \in \mathbb{I}} \mathcal{D}I \xrightarrow{\pi_{\mathcal{D}(\mathrm{cod}(i))}} \mathcal{D}(\mathrm{cod}(i)) \right\rangle_{i:\text{ an arr. in } \mathbb{I}}$$

(big) tupling

$$g := \left\langle \prod_{I \in \mathbb{I}} \mathcal{D}I \xrightarrow{\pi_{\mathcal{D}(\mathrm{dom}(i))}} \mathcal{D}(\mathrm{dom}(i)) \xrightarrow{\mathcal{D}i} \mathcal{D}(\mathrm{cod}(i)) \right\rangle_{i:\text{ arr. in } \mathbb{I}}$$




apply $\mathcal{D}$

$$\frac{i : \mathrm{dom}(i) \to \mathrm{cod}(i) \text{ in } \mathbb{I}}{\mathcal{D}i : \mathcal{D}(\mathrm{dom}(i)) \to \mathcal{D}(\mathrm{cod}(i)) \text{ in } \mathbb{C}}$$

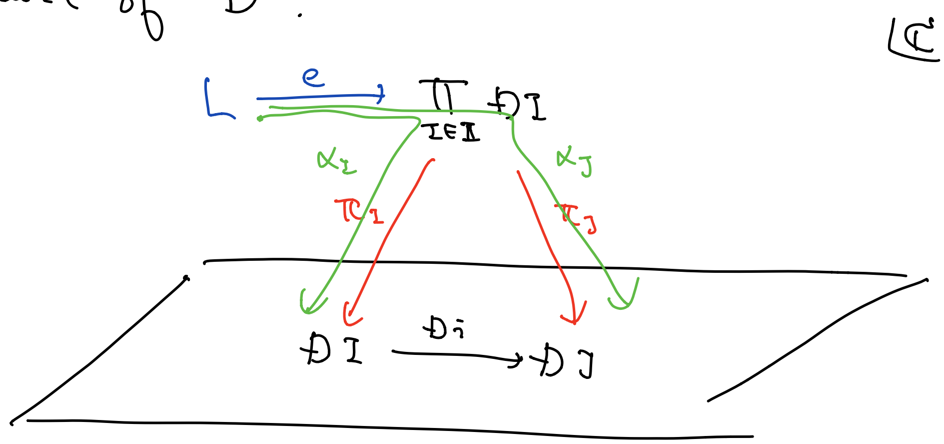
Take an equalizer

equalizer

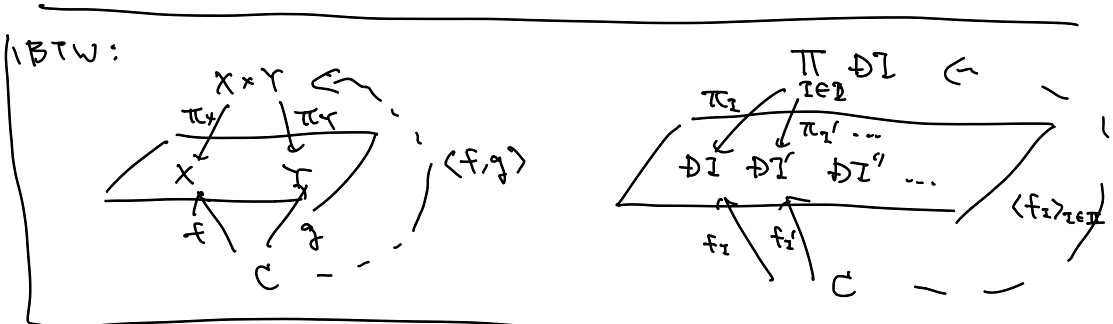
$$L \xrightarrow{e} \prod_{I \in \mathbb{I}} DI \overset{f}{\underset{g}{\rightrightarrows}} \prod_{i} D(\mathrm{cod}(i))$$

$i$: an arrow in $\mathbb{I}$

Then $L$ is the vertex of a limit of $D$.




BTW:




An exercise:

- Show that this indeed gives a cone w/ suitable commutativity
- Prove its universality

cf.

$$\begin{tikzcd} L_1 & L_2 \arrow[l] & L_3 \arrow[l] & \cdots \arrow[l] & L \arrow[llll, bend right] \arrow[lll, bend right] \arrow[ll, bend right] \arrow[l] \end{tikzcd}$$

projective limits
(= limits)

$$\begin{tikzcd} L_1 \arrow[r] \arrow[rrrr, bend right] & L_2 \arrow[r] \arrow[rrr, bend right] & L_3 \arrow[r] \arrow[rr, bend right] & \cdots \arrow[r] & L \end{tikzcd}$$

injective limits
(= colimits)

---

**Thm.** If $\mathbb{C}$ has arbitrary products and equalizers then $\mathbb{C}$ has arbitrary limits.

for any $\mathbb{I}$ and $D : \mathbb{I} \to \mathbb{C}$

---

In $\mathbf{Sets}$, a limit of a diagram $D : \mathbb{I} \longrightarrow \mathbf{Sets}$

$$\begin{array}{ccc} DI & \xrightarrow{Di} & DJ \\ \Psi & & \Psi \\ x_I & & x_J \end{array} \quad (\mathbf{Sets})$$

is given by

$$\underbrace{\left\{ (x_I)_{I \in \mathbb{I}} \;\middle|\; \begin{array}{l} x_I \in \overset{\mathbf{Sets}}{\overset{\Psi}{DI}}, \text{ and for each arrow } I \xrightarrow{i} J, \\ (Di)(x_I) = x_J \end{array} \right\}}_{\text{the set of "coherent elements"}}$$

Similarly, colimits are described using

coproducts and $\quad$ [in $\mathbf{Sets}$ : disjoint union

coequalizers $\quad$ in $\mathbf{Sets}$ : quotienting]

and this gives a descr. of colimits in $\mathbf{Sets}$ :

$$\left(\coprod_{I \in \mathbb{I}} DI\right) \Big/ \simeq$$

where $\simeq$ is the smallest equiv. rel. that contains

$$\underset{DI}{\underset{\cap}{x}} \simeq \underset{DJ}{\underset{\cap}{(Di)(x)}}$$

$$\frac{I \xrightarrow{i} J \quad \mathbb{I}}{DI \xrightarrow{Di} DJ \quad \mathbf{Sets}}$$

$x \in DI$

---

In **Sets** there are arbitrary small limits

in $D : \mathbb{I} \to \mathbb{C}$, $\mathbb{I}$ must be small ($\mathrm{obj}(\mathbb{I})$ and $\mathrm{arr}(\mathbb{I})$ are small sets)

Lem. If $\mathbb{C}$ has arbitrary products (not nec. small) then $\mathbb{C}$ is a preorder

very few arrows

Coq

Agda

effective toposes (Hyland)

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