**Mathematical Semantics of Computer Systems, *MSCS* (4810-1168)**
**Handout for Lecture 8 (2016/11/21)**

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Video recording of the lectures is available at: `http://www-mmm.is.s.u-tokyo.ac.jp/videos/mscs2016`

There's a report assignment that is due next week (see below). The next lecture: Mon 28 November. I'd said that the lecture on Mon 5 Dec would probably be canceled: my trip has been canceled instead, and there *will be* a lecture.

# Part I: Adjuction (ctn'd)

## 1 Today's Agenda

### 1.1 Limits as Adjoints

**Definition.** Functor category

**Proposition 1.** *A limit gives rise to an adjunction.*

## 2 Report Assignment

Deadline: at the beginning of the next lecture.

1. Formulate and prove the following statement.

   A right adjoint preserves limits.

2. Prove the following: in an adjunction $F \dashv G$, $G$ is faithful if and only if every component of the counit $\varepsilon$ is an epi. [5, Thm. IV.3.1]

# Part II: the Yoneda Lemma

Remember: we loosely follow [3], but it hardly serves as an introductory textbook. More beginner-friendly ones include [1, 4]; other classical textbooks include [5, 2]. nLab (`ncatlab.org`) is an excellent online information source.

## 3 Today's Goal

Familiarize yourself with the *Yoneda lemma.* Identify it as a category theory analogue of the *Cayley representation theorem*:

**Theorem** (Cayley)**.** *Every group $G$ is isomorphic to a subgroup of $\pi(|G|)$.*

## 4 Today's Agenda

### 4.1 Equivalence of Categories

**Definition.** Subcategory, faithful functor, full functor

**Lemma 1.** *Any functor preserves isomorphisms.*
*A full and faithful functor reflects isomorphisms.*

**Definition.** Equivalence of categories

**Proposition 2.** *Equivalence from a full, faithful and iso-dense functor.*

### 4.2 The Yoneda Lemma

**Definition.** Covariance, contravariance

**Theorem** (Yoneda)**.** *The Yoneda lemma, the Yoneda embedding as a full and faithful functor*

**Definition.** end, coend

**Theorem.** *The Yoneda lemma, the (co)end form*

**Lemma 2.** *Ends as limits [5, Prop. IX.5.1]*

**Lemma 3.** *Homfunctors preserve (co)limits, hence also (co)ends*

## 5 Exercises

1. Formulate the "naturality" of the Yoneda correspondence

$$\mathrm{Nat}\big(\mathbb{C}(\_\,, X),\, F\big) \;\cong\; FX$$

   and prove it.

## References


[1] S. Awodey. *Category Theory.* Oxford Logic Guides. Oxford Univ. Press, 2006.

[2] M. Barr and C. Wells. *Toposes, Triples and Theories.* Springer, Berlin, 1985. Available online.

[3] J. Lambek and P.J. Scott. *Introduction to higher order Categorical Logic.* No. 7 in Cambridge Studies in Advanced Mathematics. Cambridge Univ. Press, 1986.

[4] T. Leinster. *Basic Category Theory.* Cambridge Univ. Press, 2014.

[5] S. Mac Lane. *Categories for the Working Mathematician.* Springer, Berlin, 2nd edn., 1998.