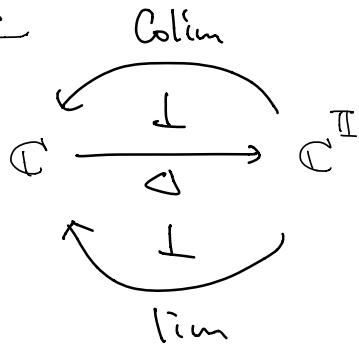


MSCS
Goal

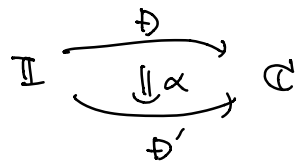
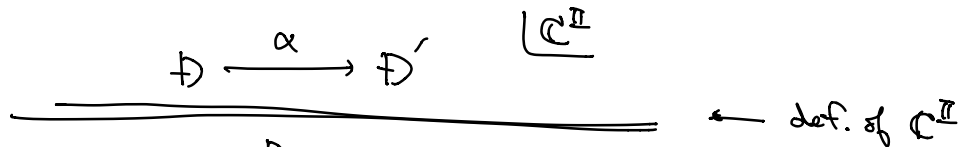


Def. \mathbb{I}, \mathcal{C} be categories.
Let

- The functor category $\mathcal{C}^{\mathbb{I}}$ has

obj. functors $\mathcal{D}: \mathbb{I} \rightarrow \mathcal{C}$

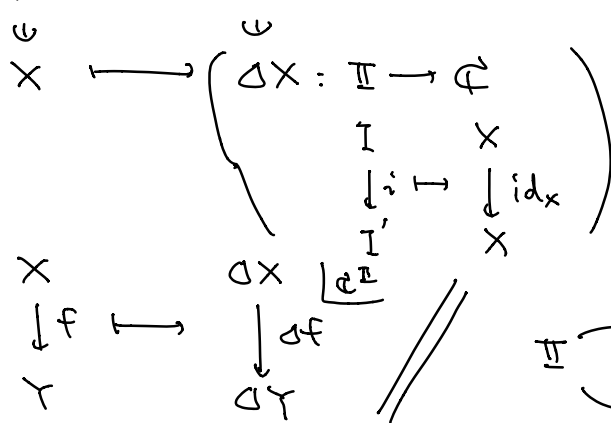
arr. natural transformations. That is,



so $\alpha: \mathcal{D} \Rightarrow \mathcal{D}': \mathbb{I} \rightarrow \mathcal{C}$

- There is ^{the} ~~x~~ (canonical) diagonal functor

$\Delta: \mathcal{C} \rightarrow \mathcal{C}^{\mathbb{I}}$



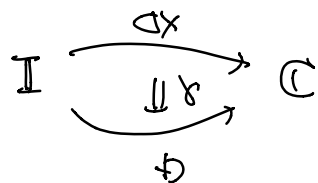
The component for $I \in \mathbb{I}$ is $(\Delta f)_I: (\Delta X)(I) \rightarrow (\Delta Y)(I)$

Now let's see $\mathbb{C} \begin{matrix} \xrightarrow{\Delta} \\ \perp \\ \xleftarrow{\Delta} \end{matrix} \mathbb{C}^{\mathbb{I}}$, that is,
 for each $X \in \mathbb{C}$ and $\mathcal{D} \in \mathbb{C}^{\mathbb{I}}$,

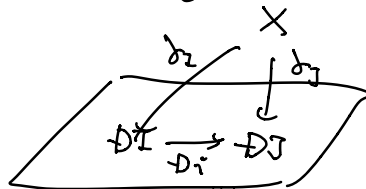
$$\frac{\Delta X \longrightarrow \mathcal{D} \text{ in } \mathbb{C}^{\mathbb{I}}}{X \longrightarrow \text{Lim } \mathcal{D} \text{ in } \mathbb{C}}$$

Indeed,

$$\frac{\Delta X \xrightarrow{\gamma} \mathcal{D} \text{ in } \mathbb{C}^{\mathbb{I}}}{\text{a nat. trans.}} \quad \text{By def. of } \mathbb{C}^{\mathbb{I}}$$



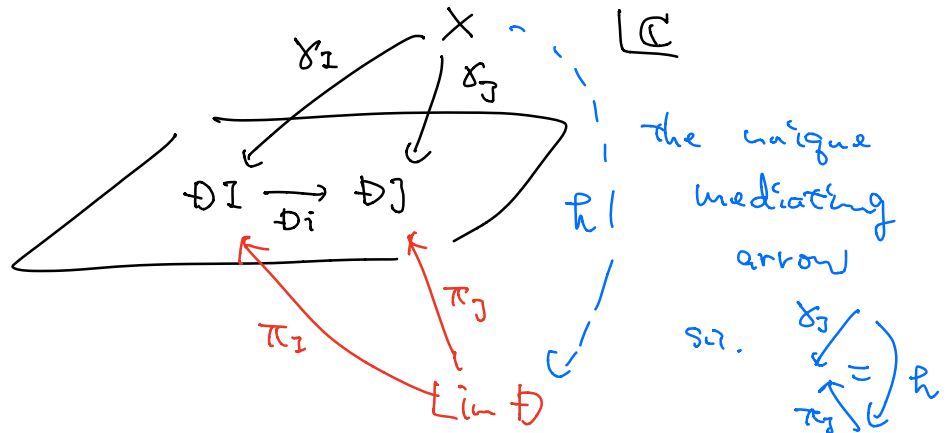
a cone from X to a diagram \mathcal{D}
 a vertex



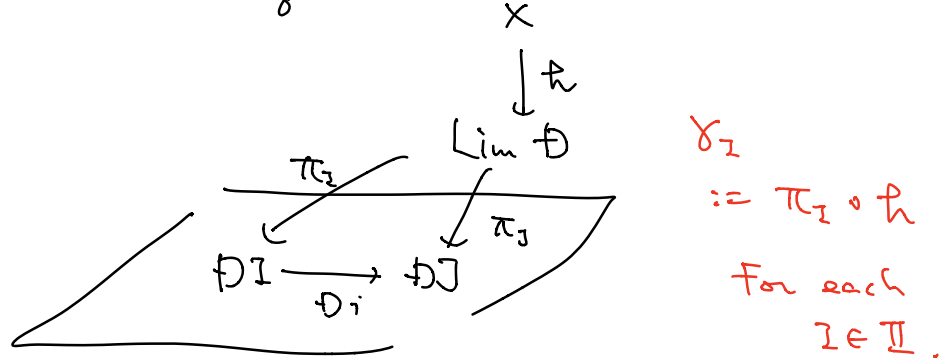
$$X \longrightarrow \text{Lim } \mathcal{D} \quad \text{☺}$$

where ☺ is given as follows.

[I] By the universality of $\text{Lim } \mathcal{D}$,

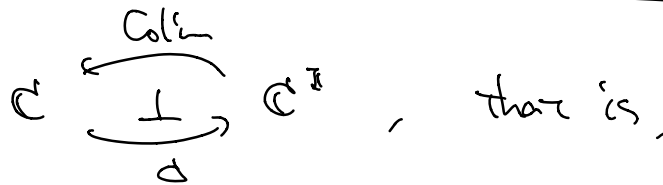


[↑] Given $h: X \rightarrow \text{Lim } \mathcal{D}$, we define a cone γ from X to \mathcal{D} by



Showing [I][↑] are mutually inverse is not hard ... we exploit the uniqueness of a mediating arrow.

Similarly



, that is,

$$\frac{\text{Colim } \mathcal{D} \longrightarrow X}{\mathcal{D} \Rightarrow \Delta X}$$

Hint for Assignment 7

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ \downarrow \perp & & \uparrow \mathcal{D} \\ \mathcal{C} & \xleftarrow{v} & \mathcal{D} \\ & & \mathbb{I} \end{array} \Rightarrow \mathbb{I} \xrightarrow{U \circ \mathcal{D}} \mathcal{C} \text{ is a diagram in } \mathcal{C}.$$

- aim $U(\text{Lim } \mathcal{D})$ is $\text{Lim } U\mathcal{D}$

that is,

$$\frac{X \longrightarrow U(\text{Lim } \mathcal{D})}{\Delta X \Rightarrow U\mathcal{D}}$$

characterized by

$$\frac{X \longrightarrow \text{Lim } U\mathcal{D}}{\Delta X \Rightarrow U\mathcal{D}}$$

(aim)

- You can also try using $\mathcal{C} \xleftarrow{\text{Lim}} \mathcal{C}^{\mathbb{I}}$

Hint: composition of adjunctions.

- $f: X \rightarrow Y$ is an epi

$$\begin{array}{ccc} \mathcal{C}(Y, Z) & \xrightarrow{(-) \circ f} & \mathcal{C}(X, Z) \\ g & \longmapsto & g \circ f \end{array}$$

is injective

right-cancellable,

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{h} & Z \\ & & & \searrow g & \\ & & & & \mathcal{C} \end{array}$$

$$hf = gf \Rightarrow h = g$$

Toneda Lemma

One of the (few) structural theorems in category theory

1 As a categorical analogue of Cayley's representation theorem (for groups)

2 As an incarnation of the intuition "in a category, an object is characterized in its relationship to the other objects" terms of

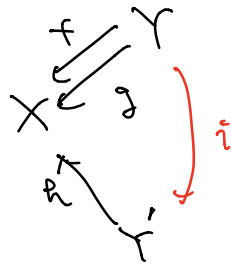
Thm. (Cayley)

Every gp G is a subgroup of $\pi(|G|)$

↑
invertible actions
Symmetry
permutation

the sym. gp. permutations
↑
the underlying set

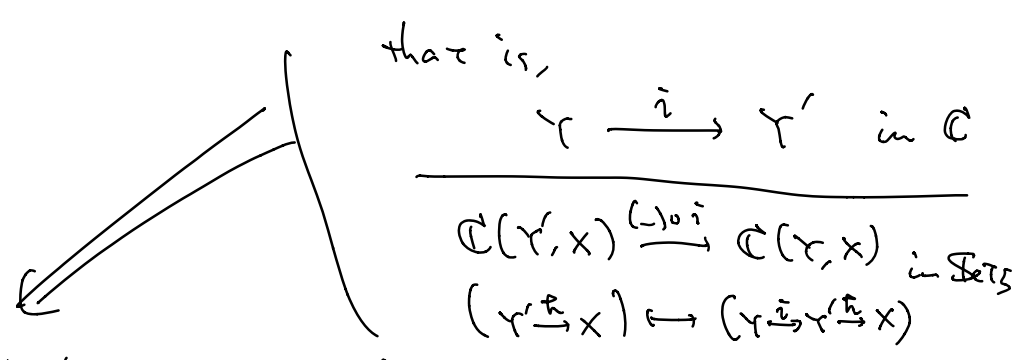
C



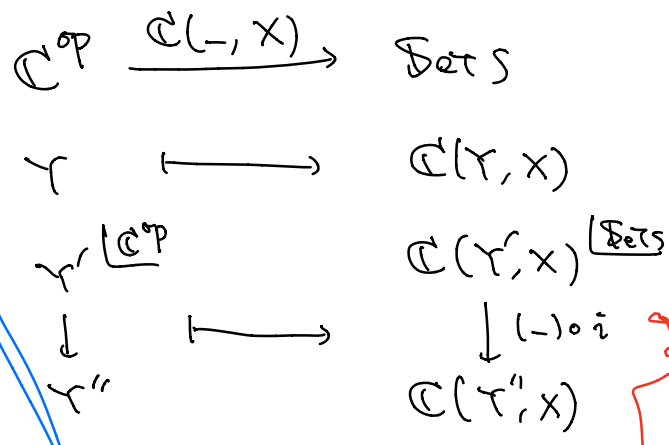
- Relation betw. X and other objects?

$\Rightarrow \left. \begin{matrix} \mathcal{C}(Y, X) \\ \mathcal{C}(Y', X) \\ \vdots \end{matrix} \right\} \text{homsets}$

- It'd also make sense to consider how these homsets are related in arrows in \mathcal{C} ,



We are looking at a functor



\mathcal{C}^{op} the opposite category of \mathcal{C}
 obj. $X \in \mathcal{C}$
 arr. $X \rightarrow Y \text{ in } \mathcal{C}^{\text{op}}$
 $\xrightarrow{\quad} Y \rightarrow X \text{ in } \mathcal{C}$

a usual functor (covariant) from \mathcal{C}^{op} to \mathbf{Sets} , that is, a contravariant functor from \mathcal{C} to \mathbf{Sets} .

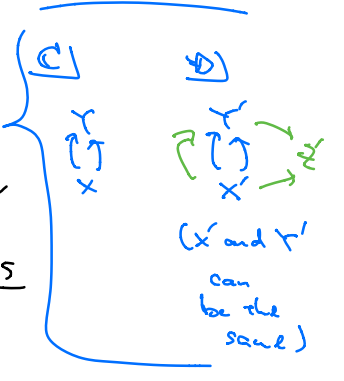
Claim

- $\mathcal{C}(-, X)$ is a concrete, set-theoretic representation of $X \in \mathcal{C}$
- This is formalized in the Yoneda Lem.

Def. $F: \mathcal{C} \rightarrow \mathcal{D}$

- It is faithful if, for each $x, y \in \mathcal{C}$, the action of F

$$F_{x,y}: \mathcal{C}(x, y) \xrightarrow{\quad} \mathcal{D}(F_x, F_y)$$



is injective.

$$\begin{array}{c} X \xrightarrow{F} Y \text{ in } \mathcal{C} \\ \hline FX \xrightarrow{FF} FY \text{ in } \mathcal{D} \end{array}$$

- F is full if $F_{X,Y}$ is surjective for $\forall X, Y$.
- If F 's action on obj. is injective too, then F is faithful and

\mathcal{C} is a subcategory of \mathcal{D} via F .

LEM. $F: \mathcal{C} \rightarrow \mathcal{D}$

1. If $X \cong X'$ in \mathcal{C} , then $FX \cong FX'$ in \mathcal{D} .

$$\begin{array}{l} X \cong X' \\ \Leftrightarrow \exists f, g, \text{ isomorphisms,} \\ \text{def.} \\ \text{st. } X \begin{array}{c} \xrightarrow{f} \\ \xleftarrow{g} \end{array} X' \end{array}$$

Proof. We get

$$FX \begin{array}{c} \xrightarrow{FF} \\ \xleftarrow{Fg} \end{array} FX'$$

Now

$$\begin{aligned} (Fg) \circ (Ff) &= F(g \circ f) \\ &\text{def. of a fun} \\ &= F(id_X) \\ &g \circ f = id \end{aligned}$$

Similarly

$$(Ff) \circ (Fg) = id_{FX'}$$

□

2. If F is full and faithful, then F reflects isomorphisms, that is, for each $X, X' \in \mathcal{C}$,

$$FX \cong FX' \implies X \cong X'.$$

Proof. By the assumption we have

$$FX \begin{array}{c} \xrightarrow{r} \\ \cong \\ \xleftarrow{k} \end{array} FX' \quad \square$$

Since F is full we get

$$X \begin{array}{c} \xrightarrow{f} \\ \xleftarrow{g} \end{array} X' \quad \square \quad \text{s.t.} \quad \begin{array}{l} r = Ff \\ k = Fg. \end{array}$$

We have to show that $g \circ f = \text{id}_X$.

Since F is faithful it suffices to show

$$\text{that } F(g \circ f) = F(\text{id}_X)$$

$$\begin{aligned} \text{Now } F(g \circ f) &= Fg \circ Ff \\ &= k \circ r = \text{id}_{FX} \end{aligned}$$

$$f \circ g = \text{id}_{X'} \text{ is similar. } \quad \square \quad \square$$