MSCS

Goal

$$\begin{tikzcd} \mathbb{C} \arrow[rr, "\Delta"] \arrow[rr, phantom, bend left=25, "\perp"] \arrow[rr, phantom, bend right=25, "\perp"] & & \mathbb{C}^{\mathbb{I}} \arrow[ll, bend right=50, "\mathrm{Colim}"'] \arrow[ll, bend left=50, "\lim"] \end{tikzcd}$$

Def. Let $\mathbb{I}$, $\mathbb{C}$ be categories.

- The functor category $\mathbb{C}^{\mathbb{I}}$ has

  obj. functors $D : \mathbb{I} \to \mathbb{C}$

  arr. natural transformations. That is,

$$D \xrightarrow{\ \alpha\ } D' \quad \text{in } \mathbb{C}^{\mathbb{I}}$$

$$=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!= \quad \leftarrow \text{def. of } \mathbb{C}^{\mathbb{I}}$$

$$\begin{tikzcd} \mathbb{I} \arrow[r, bend left, "D"] \arrow[r, bend right, "D'"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathbb{C} \end{tikzcd} \quad \therefore \quad \alpha : D \Rightarrow D' : \mathbb{I} \to \mathbb{C}$$

- There is ~~a~~ the (canonical) diagonal functor

$$\begin{aligned} \Delta : \mathbb{C} &\longrightarrow \mathbb{C}^{\mathbb{I}} \\ X &\longmapsto \left(\Delta X : \mathbb{I} \to \mathbb{C}, \quad \begin{array}{ccc} I & & X \\ \downarrow i & \mapsto & \downarrow id_X \\ I' & & X \end{array} \right) \end{aligned}$$

$$\begin{array}{ccc} X & & \Delta X \\ \downarrow f & \longmapsto & \downarrow \Delta f \\ Y & & \Delta Y \end{array} \quad \text{in } \mathbb{C}^{\mathbb{I}} \qquad // \qquad \begin{tikzcd} \mathbb{I} \arrow[r, bend left, "\Delta X"] \arrow[r, bend right, "\Delta Y"'] \arrow[r, phantom, "\Downarrow\Delta f"] & \mathbb{C} \end{tikzcd}$$

The component for $I \in \mathbb{I}$ $(\Delta f)_I$: is

$$\begin{array}{ccc} (\Delta X)(I) & \longrightarrow & (\Delta Y)(I) \\ \| & & \| \\ X & \xrightarrow{\ f\ } & Y \end{array} \quad \text{in } \mathbb{C}$$

Now let's see
$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "\Delta"] \arrow[r, phantom, "\perp"] & \mathbb{C}^{\mathbb{I}} \arrow[l, bend left, "\mathrm{Lim}"] \end{tikzcd},$$
that is,

for each $X \in \mathbb{C}$ and $D \in \mathbb{C}^{\mathbb{I}}$,

$$\frac{\Delta X \longrightarrow D \text{ in } \mathbb{C}^{\mathbb{I}}}{X \longrightarrow \mathrm{Lim}\, D \text{ in } \mathbb{C}}$$

Indeed,

$$\Delta X \xrightarrow{\gamma} D \text{ in } \mathbb{C}^{\mathbb{I}}$$

$=\!=\!=\!=\!=$ By def. of $\mathbb{C}^{\mathbb{I}}$

a nat. trans.

$$\begin{tikzcd} \mathbb{I} \arrow[r, bend left, "\Delta X"] \arrow[r, bend right, "D"'] \arrow[r, phantom, "\Downarrow\gamma"] & \mathbb{C} \end{tikzcd}$$

$=\!=\!=\!=\!=$

a cone from $\underbrace{X}_{\text{a vertex}}$ to a diagram D

$$\begin{tikzcd} & X \arrow[dl, "\delta_I"'] \arrow[d, "\delta_J"] \\ DI \arrow[r, "Di"'] & DJ \end{tikzcd}$$

$=\!=\!=\!=\!=$ ← ☺

$$X \longrightarrow \mathrm{Lim}\, D \quad \text{in } \mathbb{C}$$

where ☺ is given as follows.

[I] By the universality of $\mathrm{Lim}\, D$,

$$\begin{tikzcd}
& X \arrow[dl, "\gamma_I"'] \arrow[d, "\gamma_J"] \arrow[dd, dashed, bend left=60, "h"] \\
DI \arrow[r, "Di"'] & DJ \\
& \mathrm{Lim}\, D \arrow[ul, "\pi_I"] \arrow[u, "\pi_J"']
\end{tikzcd} \qquad \mathbb{C}$$

the unique mediating arrow s.t.
$$\begin{tikzcd}
X \arrow[d, "\gamma_J"'] \arrow[dd, bend left=60, "h"] \\
DJ \arrow[phantom, r, "="] & {} \\
\mathrm{Lim}\, D \arrow[u, "\pi_J"]
\end{tikzcd}$$

[$\hat{I}$] Given $h : X \longrightarrow \mathrm{Lim}\, D$, we define a cone γ from X to D by

$$\begin{tikzcd}
& X \arrow[d, "h"] \\
& \mathrm{Lim}\, D \arrow[dl, "\pi_I"'] \arrow[d, "\pi_J"] \\
DI \arrow[r, "Di"'] & DJ
\end{tikzcd}$$

$\gamma_I := \pi_I \circ h$ for each $I \in \mathbb{I}$.

Showing [I] [$\hat{I}$] are mutually inverse is not hard ... we exploit the uniqueness of a mediating arrow.

Similarly
$$\begin{tikzcd}
\mathbb{C} \arrow[r, bend right, "\Delta"'] \arrow[r, phantom, "\perp"] & \mathbb{C}^{\mathbb{I}} \arrow[l, bend right, "\mathrm{Colim}"']
\end{tikzcd}$$
, that is,

$$\frac{\mathrm{Colim}\, D \longrightarrow X}{D \Rightarrow \Delta X}$$

Hint for Assignment 7

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbb{D} \arrow[l, bend left, "U"] \\ & \mathbb{I} \arrow[u, "D"] \end{tikzcd} \quad \Longrightarrow \quad \mathbb{I} \xrightarrow{U \circ D} \mathbb{C} \text{ is a diagram in } \mathbb{C}.$$

- aim $U(\mathrm{Lim}\, D)$ is $\mathrm{Lim}\, UD$

characterized by

$$\frac{X \longrightarrow \mathrm{Lim}\, UD}{\Delta X \Rightarrow UD}$$

that is,

$$\frac{X \longrightarrow U(\mathrm{Lim}\, D)}{\Delta X \Rightarrow UD.}$$

aim

- You can also try using $\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "\Delta"] \arrow[r, phantom, "\perp"] & \mathbb{C}^{\mathbb{I}} \arrow[l, bend left, "\mathrm{Lim}"] \end{tikzcd}$.

Hint: composition of adjunctions.

- $f : X \to Y$ $\mathbb{C}$ is an epi

right-cancellable,

$X \xrightarrow{f} Y \overset{h}{\underset{g}{\rightrightarrows}} Z$ $\mathbb{C}$

$hf = gf \Rightarrow h = g$

$$\begin{array}{ccc} \mathbb{C}(Y, Z) & \xrightarrow{(-)\circ f} & \mathbb{C}(X, Z) \\ g & \longmapsto & g \circ f \end{array}$$

is injective

Yoneda Lemma ← One of the (few) structural theorems in category theory

1. As a categrical analogue of Cayley's representation theorem (for groups)

2. As an incarnation of the intuition
"in a category, an object is characterized in terms of its relationship to the other objects"

Thm. (Cayley)

Every gp G is a subgroup of $\overbrace{\mathfrak{S}}^{\text{the sym. gp.}}(\underline{|G|})$

invertible actions
symmetry
permutation

permutations

the underlying set

$\mathbb{C}$

$$\begin{tikzcd} & Y \arrow[dl, shift right, "f"'] \arrow[dl, shift left, "g"] \arrow[dd, bend left, "i"] \\ X & \\ & Y' \arrow[ul, "h"] \end{tikzcd}$$

- Relation betw. X and other objects?

$\Rightarrow$ $\left.\begin{array}{l} \mathbb{C}(Y, X) \\ \mathbb{C}(Y', X) \\ \vdots \end{array}\right\}$ homsets

- It'd also make sense to consider how these homsets are related in arrows in $\mathbb{C}$.

that is,

$$Y \xrightarrow{i} Y' \text{ in } \mathbb{C}$$

$$\mathbb{C}(Y',X) \xrightarrow{(-)\circ i} \mathbb{C}(Y,X) \text{ in } \mathbf{Sets}$$

$$(Y' \xrightarrow{h} X) \longmapsto (Y \xrightarrow{i} Y' \xrightarrow{h} X)$$

We are looking at a functor

$$\mathbb{C}^{op} \xrightarrow{\mathbb{C}(-,X)} \mathbf{Sets}$$

$$Y \longmapsto \mathbb{C}(Y,X)$$

$$\begin{array}{ccc} Y' \,\lfloor \mathbb{C}^{op} & & \mathbb{C}(Y',X) \,\lfloor \mathbf{Sets} \\ \downarrow & \longmapsto & \downarrow (-)\circ i \\ Y'' & & \mathbb{C}(Y'',X) \end{array}$$

$$\begin{array}{c} Y' \,\lfloor \mathbb{C} \\ \uparrow i \\ Y'' \end{array}$$

$\boxed{\mathbb{C}^{op}}$ the opposite category of $\mathbb{C}$

obj. $X \in \mathbb{C}$

arr. $\dfrac{X \longrightarrow Y \text{ in } \mathbb{C}^{op}}{Y \longrightarrow X \text{ in } \mathbb{C}}$

a usual functor (covariant) from $\mathbb{C}^{op}$ to $\mathbf{Sets}$, that is, a contravariant functor from $\mathbb{C}$ to $\mathbf{Sets}$.

Claim

- $\mathbb{C}(-,X)$ is a concrete, set-theoretic representation of $X \in \mathbb{C}$

- This is formalized in the Yoneda lem.

Def. $F : \mathbb{C} \longrightarrow \mathbb{D}$

- It is faithful if, for each $X, Y \in \mathbb{C}$, the action of F

$$F_{X,Y} : \mathbb{C}(X,Y) \longrightarrow \mathbb{D}(FX, FY) \,\lfloor \mathbf{Sets}$$

$\lfloor \mathbb{C}$: $\begin{array}{c} Y \\ \uparrow\uparrow \\ X \end{array}$ $\qquad$ $\lfloor \mathbb{D}$: $\begin{array}{c} Y' \\ \uparrow\uparrow \\ X' \end{array}$ $\not\Rightarrow$

(X' and Y' can be the same)

is injective.

$$\frac{X \xrightarrow{f} Y \text{ in } \mathbb{C}}{FX \xrightarrow{Ff} FY \text{ in } \mathbb{D}}$$

- F is <u>full</u> if $F_{X,Y}$ is surjective for $\forall X, Y$.
- If F's action on obj. is injective too, then F is faithful and

  $\mathbb{C}$ is a <u>subcategory</u> of $\mathbb{D}$ via F.

<u>Lem.</u> $F : \mathbb{C} \to \mathbb{D}$.

1. If $X \cong X'$ in $\mathbb{C}$, then

   $FX \cong FX'$ in $\mathbb{D}$.

$X \cong X'$

$\underset{\text{def.}}{\iff}$ $\exists f, g$, isomorphisms, s.t. $X \underset{g}{\overset{f}{\rightleftarrows}} X'$ ($\cong$)

<u>Proof.</u> We get

$$FX \underset{Fg}{\overset{Ff}{\rightleftarrows}} FX'$$

Now $(Fg) \circ (Ff) \underset{\substack{\text{def. of} \\ \text{a fnctr}}}{=} F(g \circ f)$

$\underset{g \circ f = id}{=} F(id_X)$

$= id_{FX}$.

Similarly $(Ff) \circ (Fg) = id_{FX'}$. ∎

2. If F is full and faithful, then F reflects isomorphisms, that is, for each $X, X' \in \mathbb{C}$,

$$FX \cong FX' \implies X \cong X'.$$

$\boxed{\text{Proof.}}$ By the assumption we have

$$FX \underset{k}{\overset{h}{\rightleftarrows}} FX' \quad \text{in } \mathbb{D}.$$

Since F is full we get

$$X \underset{g}{\overset{f}{\rightleftarrows}} X' \quad \text{in } \mathbb{C} \quad \text{s.t.} \quad \begin{aligned} h &= Ff \\ k &= Fg. \end{aligned}$$

We have to show that $g \circ f = id_X$.

Since F is faithful it suffices to show that $F(g \circ f) = F(id_X)$

Now
$$\begin{aligned} F(g \circ f) &= Fg \circ Ff \\ &= k \circ h = id_{FX} \\ &= F(id_X). \quad \text{OK} \quad \Box \end{aligned}$$

$f \circ g = id_{X'}$ is similar.