Mathematical Semantics of Computer Systems, *MSCS* (4810-1168)
Handout for Lecture 9 (2016/12/5)

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Video recording of the lectures is available at: `http://www-mmm.is.s.u-tokyo.ac.jp/videos/mscs2016`

I'd said that the lecture on Mon 5 Dec would probably be canceled: my trip has been canceled instead, and there *will be* a lecture.

Part I: the Yoneda Lemma

Remember: we loosely follow [3], but it hardly serves as an introductory textbook. More beginner-friendly ones include [1, 4]; other classical textbooks include [5, 2]. nLab (`ncatlab.org`) is an excellent online information source.

1 Today's Goal

Familiarize yourself with the *Yoneda lemma*. Identify it as a category theory analogue of the *Cayley representation theorem*:

Theorem (Cayley)**.** *Every group G is isomorphic to a subgroup of $\pi(|G|)$.*

2 Today's Agenda

2.1 Equivalence of Categories

Definition. Equivalence of categories

Proposition 1. *Equivalence from a full, faithful and iso-dense functor.*

2.2 The Yoneda Lemma

Definition. Covariance, contravariance

Theorem (Yoneda)**.** *The Yoneda lemma, the Yoneda embedding as a full and faithful functor*

Definition. end, coend

Theorem. *The Yoneda lemma, the (co)end form*

Lemma 1. *Ends as limits [5, Prop. IX.5.1]*

Lemma 2. *Homfunctors preserve (co)limits, hence also (co)ends*

3 Exercises

1. Formulate the "naturality" of the Yoneda correspondence

$$\mathrm{Nat}\big(\mathbb{C}(\_\,, X), F\big) \;\cong\; FX$$

and prove it.

Part II: Algebraic Semantics

4 Algebraic Semantics as a Precursor of Categorical Semantics

This section is essentially a brief recap of [**?**, Chap. 2], aimed also at the audience not familiar with formal logic.

4.1 The Word Problem

Consider the following "syntactic system."

- *Terms* are defined by the following BNF notation:

$$\mathbf{Terms} \ni t, t_1, t_2 \quad ::= \quad \mathtt{x} \in \mathbf{Var} \mid \mathtt{e} \mid t \cdot t \mid t^{-1} \ .$$

- The relation $\sim$ between terms is defined inductively by the following rules.

$$\frac{}{(t_1 \cdot t_2) \cdot t_3 \sim t_1 \cdot (t_2 \cdot t_3)} \ (\textsc{Associativity})$$

$$\frac{}{\mathtt{e} \cdot t \sim t} \ (\textsc{Unit-Left}) \qquad \frac{}{t \cdot \mathtt{e} \sim t} \ (\textsc{Unit-Right})$$

$$\frac{}{t^{-1} \cdot t \sim \mathtt{e}} \ (\textsc{Inverse-Left}) \qquad \frac{}{t \cdot t^{-1} \sim \mathtt{e}} \ (\textsc{Inverse-Right})$$

$$\frac{}{t \sim t} \ (\textsc{Reflexivity}) \qquad \frac{t \sim s}{s \sim t} \ (\textsc{Symmetry}) \qquad \frac{t \sim s \quad s \sim u}{t \sim u} \ (\textsc{Transitivity})$$

$$\frac{t_1 \sim s_1 \quad t_2 \sim s_2}{t_1 \cdot t_2 \sim s_1 \cdot s_2} \ (\cdot\text{-}\textsc{Congruence}) \qquad \frac{t \sim s}{t^{-1} \sim s^{-1}} \ ((\_)^{-1}\text{-}\textsc{Congruence})$$

Remark 1. (For those who are *not* familiar with formal logic) The "inductive definition of $\sim$ by the rules" means that we have $t \sim s$ if and only if we can draw a (finite-height) *proof tree* using the rules, for example

$$\frac{\dfrac{}{((xy)^{-1}x)y \sim (xy)^{-1}(xy)} \ (\textsc{Associativity}) \qquad \dfrac{}{(xy)^{-1}(xy) \sim e} \ (\textsc{Inverse-Left})}{((xy)^{-1}x)y \sim e} \ (\textsc{Transitivity})$$

Remark 2. (For those who *are* familiar with formal logic) The above is an equational theory of groups, formulated as usual in equational logic.

Now the question is: given terms s and t, can we know if $s \sim t$ holds? How? This problem is known as the *word problem for groups.*

Theorem (Novikov, 1955)**.** *The word problem for groups is undecidable.*

Therefore there is no generic algorithm that decides the problem.

4.2 Use of Algebraic Semantics

For those of you who are familiar with abstract algebra or group theory, the following fact will come as trivial.

> (†) If there is a group G in which the terms s and t are not equal, then we know that $s \sim t$ does not hold.

Implicit here is the use of *algebraic semantics*.

Definition. Let G be a group and $V\colon \mathbf{Var} \to |G|$ be a function (here $|G|$ denotes the underlying set of G; we call the function V a *valuation*). The *denotation* $[\![t]\!]_V$ of a term t under V is an element of the group G defined in the obvious inductive way; namely

$$[\![x]\!]_V := V(x) \qquad\qquad [\![\mathsf{e}]\!]_V := e_G$$
$$[\![t_1 \cdot t_2]\!]_V := [\![t_1]\!]_V \cdot_G [\![t_2]\!]_V \qquad\qquad [\![t^{-1}]\!]_V := \big([\![t]\!]_V\big)^{-1} \ .$$

Note here that the unit, the multiplication operator and the inverse operator on the left-hand sides are syntactic symbols; those on the right-hand sides are mathematical/semantical operators in the group G.

Now it is possible to "investigate" whether $s \sim t$ holds by looking at their semantics.

Theorem (soundness). *If $s \sim t$ holds, then $[\![s]\!]_V = [\![t]\!]_V$ for any group G and any valuation $V\colon \mathbf{Var} \to |G|$.*

Proof. Straightforward, by structural induction on the construction of proof trees. □

You see that the quotation (†) in the above is the (sloppily stated version of the) contraposition of the theorem. Therefore, to *refute* $s \sim t$, it suffices to find convenient G and V such that $[\![s]\!]_V \neq [\![t]\!]_V$.

4.3 Completeness and the Term Model

The obvious question that remains is: is the above "investigation method" complete, too? The answer is positive:

Theorem (completeness). *Assume that $[\![s]\!]_V = [\![t]\!]_V$ for any group G and any valuation $V\colon \mathbf{Var} \to |G|$. Then $s \sim t$ holds.*

Proof. We can in fact construct a special group G_0 by syntactic means—and a special valuation $V_0\colon \mathbf{Var} \to |G_0|$ that accompanies—such that $[\![s]\!]_{V_0} = [\![t]\!]_{V_0}$ if and only if $s \sim t$ holds.

Concretely:

- $|G_0| = \big\{\, [s]_\sim \,\big|\, s \text{ is a term} \,\big\}$, where $[s]_\sim$ is the $\sim$-equivalence class of the term s
- Operations are defined syntactically, that is for example,

  $$[s]_\sim \cdot_{G_0} [t]_\sim \;=\; [s \cdot t]_\sim \tag{1}$$

  and so on. Note here that $\cdot_{G_0}$ on the left-hand side is a semantical/mathematical entity (a group multiplication); in contrast $\cdot$ on the right-hand side is a syntactic entity (an operation symbol).

We have to check the following. These are all straightforward.

- $\sim$ is an equivalence relation of terms. (This follows from the rules that define $\sim$)
- The operations in (1) are well-defined. (Follows from the Congruence rules)
- The set $|G_0|$, together with the operations defined as in (1), forms a group. (Easy)

We define the valuation V_0 by

$$V_0(x) := [x]_\sim \ . \tag{2}$$

Then it is straightforward by induction to show that $[\![s]\!]_{V_0} = [s]_\sim$. This establishes: $[\![s]\!]_{V_0} = [\![t]\!]_{V_0}$ if and only if $s \sim t$. □

The group G_0 that we constructed is often called a *term model*, since it consists of (equivalence classes of) terms. A term model is a complete model—in the sense that $[\![s]\!]_{V_0} = [\![t]\!]_{V_0}$ if and only if $s \sim t$—but a common problem with it is that equality in the term model is complicated (deciding it is as hard as deciding $\sim$ itself!).

The term model G_0, in the current setting of an algebraic theory for groups, turns out to be isomorphic to the *free group* over the set **Var** of generators. It is called a *free* group since it satisfies the minimal set of equalities for it to be a group, in the sense that

$[\![s]\!]_{V_0} = [\![t]\!]_{V_0}$ if and only if $s \sim t$.

References

[1] S. Awodey. *Category Theory.* Oxford Logic Guides. Oxford Univ. Press, 2006.

[2] M. Barr and C. Wells. *Toposes, Triples and Theories.* Springer, Berlin, 1985. Available online.

[3] J. Lambek and P.J. Scott. *Introduction to higher order Categorical Logic.* No. 7 in Cambridge Studies in Advanced Mathematics. Cambridge Univ. Press, 1986.

[4] T. Leinster. *Basic Category Theory.* Cambridge Univ. Press, 2014.

[5] S. Mac Lane. *Categories for the Working Mathematician.* Springer, Berlin, 2nd edn., 1998.