MSCS

(plain) Categories

+ fin. prod.
+ exponentials → Cartesian closed categories ⇝ λ-calculus

+ fin. prod.
+ subobj. classifiers → (elementary) Toposes ⇝ (higher-order) logic

- A category

  = an alg. str. for

  typed, not necessarily invertible

  actions

$f : A \to B$ instead of

f, g

(a monoid as a category)

no $(-)^{-1}$

recap. ↑

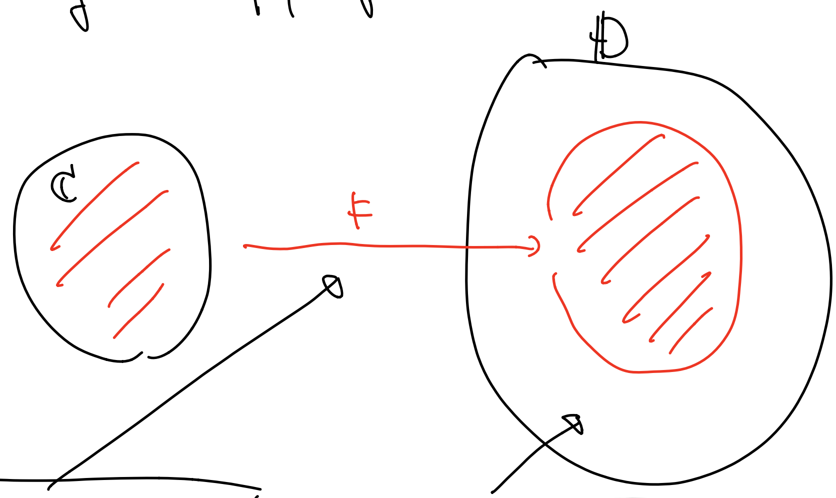
Rem. A full and faithful functor

$$F : \mathbb{C} \longrightarrow \mathbb{D}$$

can be thought of

"having a copy of $\mathbb{C}$ in $\mathbb{D}$"

embedding



It can be the case that

$X \neq X'$ and

$FX = FX'$

but in this case we always have

$X \cong X'$

- There can be objects in $\mathbb{D}$ outside $F(\mathbb{C})$
- But, for $X, X' \in \mathbb{C}$,

$$\mathbb{C}(X, X') \cong \mathbb{D}(FX, FX')$$

Thm. $\mathbb{C}$: a category.

also called the presheaf category (contravariant) of $\mathbb{C}$

The functor

$$\mathcal{Y} : \mathbb{C} \longrightarrow \mathbf{Sets}^{\mathbb{C}^{op}}$$
$$X \longmapsto \mathbb{C}(-, X)$$

the Yoneda embedding

a (contravariant) presheaf over $\mathbb{C}$

is full and faithful.

"embedding"

The functor category from $\mathbb{C}^{op}$ to $\mathbf{Sets}$

obj. a functor $F : \mathbb{C}^{op} \to \mathbf{Sets}$

arr. $\alpha : F \to G$ in $\mathbf{Sets}^{\mathbb{C}^{op}}$

a nat. trans.

$$\begin{tikzcd} \mathbb{C}^{op} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathbf{Sets} \end{tikzcd}$$

Lem. (The Yoneda lemma)

$\mathbb{C}$: a category

$F : \mathbb{C}^{op} \to \mathbf{Sets}$, a functor (a presheaf)

$X \in \mathbb{C}$ ($\xrightarrow{\text{induces}}$ $\mathbb{C}(-, X) : \mathbb{C}^{op} \to \mathbf{Sets}$,)

presheaf

Then we have

$$\underline{\mathrm{Nat}\big(\mathbb{C}(-,X),\ F\big)} \cong \overbrace{FX}^{\mathrm{Sets}}$$

Sets

the set of natural transf. from $\mathbb{C}(-,X)$ to F

$\mathbb{C}^{op}$, $\mathbb{C}^{op} \to \mathrm{Sets}$

that is,

$$\frac{\alpha : \mathbb{C}(-,X) \Rightarrow F : \mathbb{C}^{op} \to \mathrm{Sets}}{t \in FX}$$

<u>idea</u>

- $\underline{\alpha : \mathbb{C}(-,X) \Rightarrow F}$, a nat. trans.
  - given by
  $$\Big(\alpha_Y : \mathbb{C}(Y,X) \longrightarrow FY^{\,\mathrm{Sets}}\Big)_{Y \in \mathbb{C}}$$
  - subj. to the naturality cond.

  looks very complicated ...

- But the naturality condition allows us to <u>recover</u> all the data from a small crux

The naturality of $\alpha : \mathbb{C}(-,X) \Longrightarrow F$

$: \mathbb{C}^{op} \longrightarrow \mathbf{Sets}$

$$
\begin{array}{ccc}
\underline{\mathbb{C}} & \underline{\mathbb{C}^{op}} & \underline{\mathbf{Sets}} \\
\begin{tikzcd} Y \\ Y' \arrow[u, "f"'] \end{tikzcd}
&
\begin{tikzcd} Y \arrow[d, "f"] \\ Y' \end{tikzcd}
&
\begin{tikzcd}
\mathbb{C}(Y,X) \arrow[r, "\alpha_Y"] \arrow[d, "\mathbb{C}(f,X)"] & FY \arrow[d, "Ff"] \\
\mathbb{C}(Y',X) \arrow[r, "\alpha_{Y'}"] & FY'
\end{tikzcd}
\end{array}
$$

$$
\begin{array}{ccc}
\mathbb{C}(Y,X) & \longrightarrow & \mathbb{C}(Y',X) \\
\Psi & & \Psi \\
(Y \xrightarrow{g} X^{\mathbb{C}}) & \mapsto & (Y' \xrightarrow{f} Y \xrightarrow{g} X^{\mathbb{C}})
\end{array}
$$

$F : \mathbb{C}^{op} \to \mathbf{Sets}$

Now consider the following special case.

$$
\begin{array}{ccc}
\underline{\mathbb{C}} & \underline{\mathbb{C}^{op}} & \underline{\mathbf{Sets}} \\
\begin{tikzcd} X \\ Y \arrow[u, "f"] \end{tikzcd}
&
\begin{tikzcd} X \arrow[d, "f"'] \\ Y \end{tikzcd}
&
\begin{tikzcd}
\mathbb{C}(X,X) \arrow[r, "\alpha_X"] \arrow[d, "\mathbb{C}(f,X)"] & FX \arrow[d, "Ff"] \\
\mathbb{C}(Y,X) \arrow[r, "\alpha_Y"] & FY
\end{tikzcd}
\end{array}
$$

What if we start with $id_X \in \mathbb{C}(X,X)$?

$$
\begin{array}{ccc}
id_X & \longmapsto & \alpha_X(id_X) \\
\downarrow & \boxed{\text{Commutativity}} & \\
f & \longmapsto & \alpha_Y(f) \underset{★}{=} (Ff)(\alpha_X(id_X))
\end{array}
$$

$\Uparrow$

We recover

$$\alpha_Y : \mathbb{C}(Y, X) \longrightarrow FY$$

by

- $\alpha_X(\mathrm{id}_X) \in FX$, and
- the functor F !!

Proof. (of the lemma)

Sets

$$\mathrm{Nat}(\mathbb{C}(-, X), F) \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} FX \quad \cong$$

$$\Phi : \left(\alpha : \mathbb{C}(-, X) \Rightarrow F\right) \longmapsto \alpha_X(\underbrace{\mathrm{id}_X}_{\in \mathbb{C}(X,X)})$$

$\alpha_X :$
$\mathbb{C}(X, X) \to FX$

$$\Psi : (t \in FX) \longmapsto \left(\begin{array}{ccc} \mathbb{C}(Y,X) & \xrightarrow{(\Psi(t))_Y} & FY^{\;\boxed{\mathbf{Sets}}} \\ \Psi & & \Psi \\ (Y \xrightarrow{f} X) & \longmapsto & \underbrace{(Ff)}_{\boxed{FX \longrightarrow FY \;\; \boxed{\mathbf{Sets}}}}(t) \end{array} \right)_{Y \in \mathbb{C}}$$

We need to prove that these are mutually inverse.

- $\Phi \circ \Psi = \mathrm{id}$ (easy, skip)
- $\Psi \circ \Phi = \mathrm{id}$

$\therefore$ We need to show that

$$\Big(\Psi\big(\alpha_X(\mathrm{id}_X)\big)\Big)_Y = \alpha_Y$$

$$: \mathbb{C}(Y,X) \longrightarrow FY,$$

that is, for each $f : Y \to X$ in $\mathbb{C}$ (i.e. $f \in \mathbb{C}(Y,X)$),

$$\Big(\Psi\big(\alpha_X(\mathrm{id}_X)\big)\Big)_Y (f) = \alpha_Y(f)$$

$$(\mathrm{LHS}) \underset{\text{by def. of } \Psi}{=} (Ff)\big(\alpha_X(\mathrm{id}_X)\big)$$

$$\underset{\circledast}{=} (\mathrm{RHS}) \qquad \circledast \left(\text{naturality of } \alpha \right) \qquad \blacksquare$$

Def. A presheaf of the form

$$\mathbb{C}(-,X) : \mathbb{C}^{op} \longrightarrow \mathbf{Sets} \quad \left(\begin{array}{c}\text{for some}\\ X \in \mathbb{C}\end{array}\right)$$

is called <u>representable</u>.

Proof (of the theorem)

We need to show that

$$\mathcal{Y}_{X,Y} : \mathbb{C}(X,Y) \longrightarrow \mathbf{Sets}^{\mathbb{C}^{op}}\big(\mathbb{C}(-,X), \mathbb{C}(-,Y)\big)$$

is injective and surjective.

$$\mathbf{Sets}^{\mathbb{C}^{op}}\big(\mathbb{C}(-,X), \mathbb{C}(-,Y)\big) = \mathrm{Nat}\big(\mathbb{C}(-,X), \mathbb{C}(-,Y)\big) \xrightarrow[\text{the Yoneda lemma}]{\cong} \big(\mathbb{C}(-,Y)\big)(X) = \mathbb{C}(X,Y)$$

We need to show that

$\mathcal{Y}_{X,Y}$

is indeed the inverse of the function ☺ on the right. This is not hard. ∎

end $\xleftarrow[\text{generalize}]{}$ limit

Coend $\xleftarrow[\text{gen.}]{}$ colim.

<u>Def.</u> $D : \mathbb{C}^{op} \times \mathbb{C} \longrightarrow \mathbb{D}$

Like a cone

An <u>end</u> of D is

$$\left(L \in \mathbb{D},\ \left(L \xrightarrow{\pi_X} D(X,X) \right)_{X \in \mathbb{C}} \right)$$

that is subj. to the following Cond.:

- $\boxed{\mathbb{C}}$ $\quad X \xrightarrow{f} Y$

$$\begin{tikzcd} & L \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ D(X,X) \arrow[dr, "D(X,f)"'] & = & D(Y,Y) \arrow[dl, "D(f,Y)"] \\ & D(X,Y) & \end{tikzcd}$$

- $(L, (\pi_X)_{X \in \mathbb{C}})$ is universal among such data.

denoted by

$$\int_{X \in \mathbb{C}} D(X,X)$$

<u>Lem.</u> $F, G : \mathbb{C} \longrightarrow \mathbb{D}$

$$\mathrm{Nat}(F, G) \cong \int_{X \in \mathbb{C}} \mathbf{Sets}(FX, GX)$$

Lem. (the Yoneda lemma, end form)

$$\int_{Y \in \mathbb{C}} \mathbf{Sets}\big(\mathbb{C}(Y, X),\ FY\big) \cong FX$$

Lem. For $F : \mathbb{C} \longrightarrow \mathbf{Sets}$,

$$\underbrace{\int^{Y \in \mathbb{C}} \mathbb{C}(Y, X) \times FY}_{\text{coend.}} \cong FX$$