Implication and Functional Dependency in intensional Contexts

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Introduction

- **Formal concept**:
  - Mathematical notion proposed by R. Wille in 1970's
  - Made for formal context(binary relation)
  - The set of all formal concepts forms a complete lattice
  - Implies the features of a formal context
- **Formal concept analysis**:
  The method is used to discover hidden information, such as patterns and correlations between attributes.

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Example of Formal concept

	(1)	$(\bar{1})$	(2)	$(\bar{2})$	(3)	$(\bar{3})$
a	O			O	O	
b		O	O			O
c		O		O		O
d	O			O	O	
e	O		O		O	

Breakfast	(1)	No Breakfast	$(\bar{1})$
Sleepy	(2)	Not sleepy	$(\bar{2})$
Concentrate	(3)	Not Conc	$(\bar{3})$

Concept lattice

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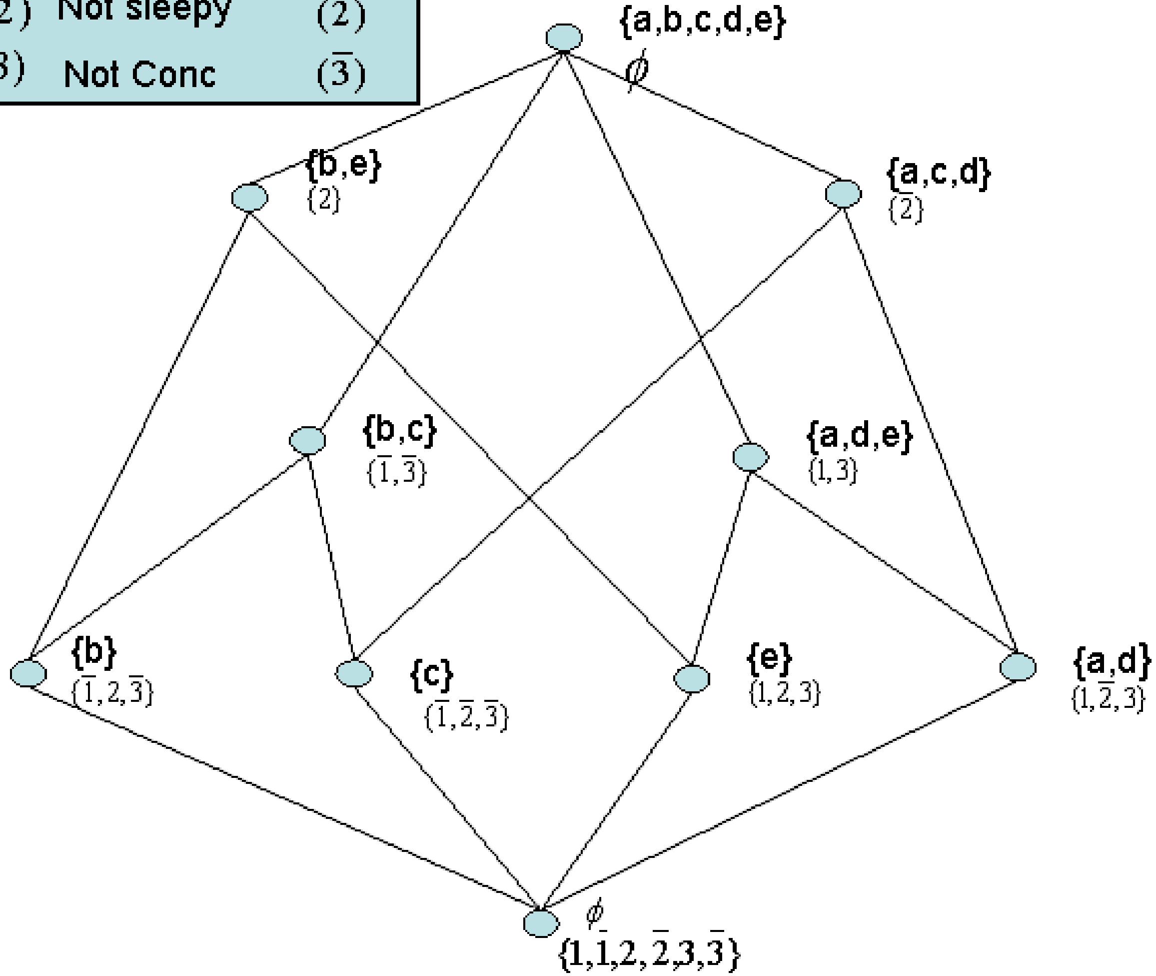

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An attribute in the database uniquely determines other attributes.

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  For relational database
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  For formal context
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  For formal context
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Implication and functional dependency are sound and complete for Armstrong's inference rules.
Further, to distinguish semantics and syntax, we give a comon proof.
We give an example which shows the difference between implication and functional dependency.

Intensional context

Intensional context

Let Y be a set of attributes and $\wp(Y)$ the power set of Y. A subset $\mathcal{T}$ of $\wp(Y)$ is called an *intensional context* on Y.

	y_0	y_1	y_2	$\cdots$	
x_0	1	1	0	$\cdots$	T_{x0}
x_1	1	1	1	$\cdots$	T_{x1}
x_2	0	1	0	$\cdots$	T_{x2}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

$\leftrightarrow \quad \mathcal{T} = \{T_{x0}, T_{x1}, T_{x2}, \ldots\} \subseteq \wp(Y$

$B^{\downarrow\uparrow} = \bigcap\{T \in \mathcal{T} \mid B \subseteq T\}$ for $B \subseteq Y$.
We define $\mathcal{T}^* = \{\cap\mathcal{A} \mid \mathcal{A} \subseteq \mathcal{T}\}$. Then $\mathcal{T}^*$ is the set of all formal concepts for an intensional context $\mathcal{T}$

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For constructing concept lattices, it is enough to treat with a family of subsets of attributes instead of a formal context.

We argue by using such framework.

Armstrong's Inference Rules

Armstrong's inference rules

Let A, B, C and D be subsets of attributes.
Armstrong's inference rules are

$$\text{[A0]}\ \frac{}{A \triangleright A} \qquad \text{[A1]}\ \frac{A \triangleright B}{A \cup C \triangleright B} \qquad \text{[A2]}\ \frac{A \triangleright B \quad B \cup C \triangleright D}{A \cup C \triangleright D}$$

Armstrong's inference rules are equivalent to

$$\text{[A0']}\ \frac{A \supseteq B}{A \triangleright B} \qquad \text{[A1']}\ \frac{A \triangleright B \quad C \supseteq D}{A \cup C \triangleright B \cup D} \qquad \text{[A2']}\ \frac{A \triangleright B \quad B \triangleright C}{A \triangleright C}$$

[A0'], [A1'], [A2'] $\rightarrow$ [A0], [A1], [A2]

[A0]

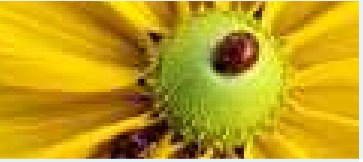
$$\text{[A0']}\ \frac{\overline{A \supseteq A}}{A \rhd A}$$

[A1]

$$\text{[A1']}\ \frac{A \rhd B \qquad \overline{C \supseteq \emptyset}}{A \cup C \rhd B}$$

[A2]

$$\text{[A2']}\ \frac{\text{[A1']}\ \dfrac{A \rhd B \qquad \overline{C \supseteq C}}{A \cup C \rhd B \cup C} \qquad B \cup C \rhd D}{A \cup C \rhd D}$$



[A0], [A1], [A2] $\rightarrow$ [A0'], [A1'], [A2']

[A0']

$$\dfrac{\text{[A1]}\,\dfrac{\text{[A0]}\,\dfrac{}{B \rhd B}}{B \cup A \rhd B} \qquad A \supseteq B}{A \rhd B}$$

[A1']

$$\text{[A2]}\,\dfrac{A \rhd B \qquad \text{[A0']}\,\dfrac{\dfrac{C \supseteq D}{B \cup C \supseteq B \cup D}}{B \cup C \rhd B \cup D}}{A \cup C \rhd B \cup D}$$

[A2']

$$\dfrac{\text{[A2]}\,\dfrac{A \rhd B \qquad \text{[A1']}\,\dfrac{B \rhd C}{B \cup A \rhd C}}{A \cup A \rhd C}}{A \rhd C}$$

Provability

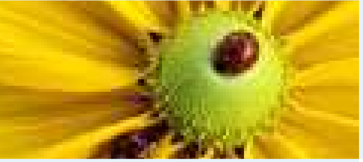
Let $\mathcal{L}$ be a set of dependencies. $\mathcal{L} \vdash A \triangleright B$ is defined Armstrong's inference rules. (A dependency $A \triangleright B$ is *provable* from $\mathcal{L}$).

Let Y be a set of all attributes, A a subset of Y. We define a subset $A_{\mathcal{L}}$ of Y by $A_{\mathcal{L}} = \{y \in Y \mid \mathcal{L} \vdash A \triangleright \{y\}\}$.

Lemma 1. *If B is a finite subset of Y then*

$$\mathcal{L} \vdash A \triangleright B \;\leftrightarrow\; B \subseteq A_{\mathcal{L}}.$$

Functional Dependency



Functional dependency

Let $\mathcal{T}$ be an intensional context on Y.

$$\begin{aligned} & \mathcal{T} \models_F A \rhd B \\ \leftrightarrow \quad & \forall S, T \in \mathcal{T}.\ (S \cap A = T \cap A \to S \cap B = T \cap B). \end{aligned}$$

If $\mathcal{T} \models_F A \rhd B$ then $A \rhd B$ is called a *functional dependency* on $\mathcal{T}$ and a dependency $A \rhd B$ is *valid* (as functional dependency) for an intensional context $\mathcal{T}$ on Y.

	(1)	($\bar{1}$)	(2)	($\bar{2}$)	(3)	($\bar{3}$)
a	○			○	○	
b		○	○			○
c		○		○		○
d	○			○	○	
e	○		○		○	

Breakfast	(1)	No Breakfast	($\bar{1}$)
Sleepy	(2)	Not sleepy	($\bar{2}$)
Concentrate	(3)	Not Conc	($\bar{3}$)

$\mathcal{T} = \{\{1, \bar{2}, 3\}, \{\bar{1}, 2, \bar{3}\}, \{\bar{1}, \bar{2}, \bar{3}\}, \{1, 2, 3\}\}$

functional dependency

$\mathcal{T} \models_F \{1\} \rhd \{3\}, \quad \mathcal{T} \not\models_F \{1\} \rhd \{2\}$

Proposition

- (soundness) Let $\mathcal{T}$ be an intensional context and $A \triangleright B$ a dependency on Y.
  (A0') If $A \supseteq B$ then $\mathcal{T} \models_F A \triangleright B$.
  (A1') If $\mathcal{T} \models_F A \triangleright B$ and $C \supseteq D$ then $\mathcal{T} \models_F A \cup C \triangleright B \cup D$.
  (A2') If $\mathcal{T} \models_F A \triangleright B$ and $\mathcal{T} \models_F B \triangleright C$ then $\mathcal{T} \models_F A \triangleright C$.

- Let A be a proper subset of Y. There exists a intensional context $\mathcal{T}_0$ such that $(\mathcal{T}_0 = \{A, Y\})$
  1. $C \subseteq A$ iff $\mathcal{T}_0 \models_F \emptyset \triangleright C$,
  2. $C \not\subseteq A$ iff $\mathcal{T}_0 \models_F C \triangleright Y$.

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Implication

Implication

Let $\mathcal{T}$ be an intensional context on Y, and $\mathcal{L}$ a set of dependencies.

$$\mathcal{T} \models_I A \rhd B \;\leftrightarrow\; \forall T \in \mathcal{T}.\, (A \subseteq T \rightarrow B \subseteq T).$$

If $\mathcal{T} \models_I A \rhd B$ then $A \rhd B$ is *valid* (as implication) for an intensional context $\mathcal{T}$ on Y or is called an *implication* on $\mathcal{T}$.

Example

	(1)	$(\bar{1})$	(2)	$(\bar{2})$	(3)	$(\bar{3})$
a	○			○	○	
b		○	○			○
c		○		○		○
d	○			○	○	
e	○		○		○	

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$$\mathcal{T} = \{\{1, \bar{2}, 3\}, \{\bar{1}, 2, \bar{3}\}, \{\bar{1}, \bar{2}, \bar{3}\}, \{1, 2, 3\}\}$$
$$\mathcal{T}^* = \{\varphi, \{1, \bar{2}, 3\}, \{\bar{1}, 2, \bar{3}\}, \{\bar{1}, \bar{2}, \bar{3}\}, \{1, 2, 3\}, \{1, 3\}, \{\bar{1}, \bar{3}\}, \{2\}, \{\bar{2}\}, Y\}$$

Implications

$$\mathcal{T} \models_I \{1\} \rhd \{3\}$$
$$\mathcal{T} \not\models_I \{\bar{1}\} \rhd \{\bar{3}\}$$

Proposition

Any intentional context $\mathcal{T}$ satisfies followings.

- (soundness)
  1. $\mathcal{T} \models_I A \triangleright A$, (A0)
  2. If $\mathcal{T} \models_I A \triangleright B$ then $\mathcal{T} \models_I A \cup C \triangleright B$, (A1)
  3. If $\mathcal{T} \models_I A \triangleright B$ and $\mathcal{T} \models_I B \cup C \triangleright D$ then $\mathcal{T} \models_I A \cup C \triangleright D$. (A2)
- Let A be a proper subset of Y.
  There exists a set $\mathcal{T}_0$ such that $(\mathcal{T}_0 = \{A\})$
  1. $C \subseteq A$ iff $\mathcal{T}_0 \models_I \emptyset \triangleright C$,
  2. $C \not\subseteq A$ iff $\mathcal{T}_0 \models_I C \triangleright Y$.
- Every dependency $A \triangleright B$ satisfies
  1. $\mathcal{T} \models_I A \triangleright B \iff \mathcal{T}^* \models_I A \triangleright B$.
  2. $\mathcal{T} \models_I A \triangleright B \iff B \subseteq A^{\downarrow\uparrow}$.

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Soundness and Completeness

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$\mathcal{T} \models_{\bullet} \mathcal{L} \;\leftrightarrow\; \forall A \rhd B \in \mathcal{L}.\, \mathcal{T} \models_{\bullet} A \rhd B.$

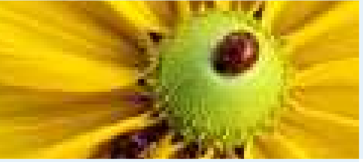
Theorem 1. *Let $A \rhd B$ be a dependency, and $\mathcal{L}$ be a set of dependencies on a finite set Y. Then*

$$\mathcal{L} \vdash A \rhd B \leftrightarrow \forall \mathcal{T} \subseteq \wp(Y).\, (\mathcal{T} \models_{\bullet} \mathcal{L} \rightarrow \mathcal{T} \models_{\bullet} A \rhd B).$$

($\bullet = F$ or I)

Functional dependency and implication are sound and complete for Armstrong's Inference rules.

Difference between Implication and Functional Dependency



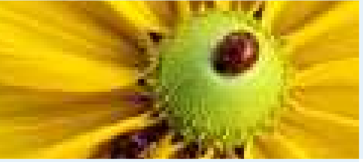
Example

Let $\mathcal{T} = \{\{a, b\}, \{b, c\}, \{c\}\}$.

	a	b	c
x	○	○	
y		○	○
z			○

The dependency $\{a\} \triangleright \{b\}$
Since $\{a\}^{\downarrow\uparrow} = \{a, b\}$. Therefore $\mathcal{T} \models_I \{a\} \triangleright \{b\}$.
For $\{b, c\}, \{c\} \in \mathcal{T}$, both $\{b, c\} \cap \{a\} = \{c\} \cap \{a\}$, and $\{b, c\} \cap \{b\} \neq \{c\} \cap \{b\}$ hold. Therefore $\mathcal{T} \not\models_F \{a\} \triangleright \{b\}$.

The dependency $\{a\} \triangleright \{c\}$
$\mathcal{T} \models_F \{a\} \triangleright \{c\}$,
$\mathcal{T} \not\models_I \{a\} \triangleright \{c\}$.



Proposition

Let $\mathcal{T}$ be an intensional context.

- For $\mathcal{T}$, there is not often exit $\mathcal{U} \subseteq \wp(Y)$ such that $\mathcal{T} \models_I A \rhd B \iff \mathcal{U} \models_F A \rhd B$.
- If $Y \in \mathcal{T}$ then $\mathcal{T} \models_F A \rhd B \to \mathcal{T} \models_I A \rhd B$.
- Define a set $\mathcal{T}'$ of subsets of Y by

$$\mathcal{T}' = \{(S^- \cup T) \cap (T^- \cup S) \mid S, T \in \mathcal{T}\}.$$

Then

1. $\mathcal{T}' \models_I A \rhd B \iff \mathcal{T} \models_F A \rhd B$.
2. If $\mathcal{T} \subseteq \mathcal{T}' \subseteq \mathcal{T}^*$, then $\mathcal{T} \models_F A \rhd B \iff \mathcal{T} \models_I A \rhd B$.

Summary and Outlook

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Summary

- functional dependency is sound and complete for Armstrong's inference rules.
- implication of formal concept is complete and sound for Armstrong's inference rules.
- an example which shows the difference between implication and functional dependency.

Future works

- the conditions in which implication and functional dependency are equivalent.