Axiom of Choice and Zorn's Lemma in Dedekind Categories

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It is a fundamental fact that Axiom of Choice and Zorn's Lemma are mutually equivalent in set theory.

This talk aims to give a relational proof of the equivanece of the axiom of choice and Zorn's lemma in Dedekind (Schröder) categories.

A Dedekind category is an algebraic system generalising the category of binary relations.

A morphism in a Dedekind category will be denoted by a half arrow $\alpha : X \rightharpoondown Y$, and the composite of a morphism $\alpha : X \rightharpoondown Y$ followed by a morphism $\beta : Y \rightharpoondown Z$ will be written as $\alpha\beta : X \rightharpoondown Z$. Also we will denote the identity morphism on X as id_X.

Definition 1. A Dedekind category $\mathcal{D}$ is a category such that

DC1. Every hom-set $\mathcal{D}(X,Y) = (\mathcal{D}(X,Y), \sqsubseteq, \sqcap, \sqcup, \Rightarrow, 0_{XY}, \nabla_{XY})$ is a complete Heyting algebra with the least morphism 0_{XY} and the greatest morphism ∇_{XY}.

DC2. A converse operation $^\sharp : \mathcal{D}(X,Y) \to \mathcal{D}(Y,X)$ satisfies (a) $(\alpha\beta)^\sharp = \beta^\sharp\alpha^\sharp$, (b) $(\alpha^\sharp)^\sharp = \alpha$, (c) If $\alpha \sqsubseteq \alpha'$, then $\alpha^\sharp \sqsubseteq \alpha'^\sharp$.

DC3. Dedekind formula $\alpha\beta \sqcap \gamma \sqsubseteq \alpha(\beta \sqcap \alpha^\sharp\gamma)$ holds.

DC4. Residual composition $\alpha \ominus \beta$ is a morphism such that $\alpha^\sharp\delta \sqsubseteq \beta$ if and only if $\delta \sqsubseteq \alpha \ominus \beta$. □

- $f^\sharp f \sqsubseteq \mathrm{id}_Y$ $\cdots$ *univalent*,
- $\mathrm{id}_X \sqsubseteq f f^\sharp$ $\cdots$ *total*,
- univalent and total $\cdots$ *function*,
- $f^\sharp f = \mathrm{id}_Y$ and $\mathrm{id}_X \sqsubseteq f f^\sharp$ $\cdots$ *surjection*,
- $f f^\sharp = \mathrm{id}_X$ and $f^\sharp f \sqsubseteq \mathrm{id}_Y$ $\cdots$ *injection*.

[Schröder Category] Every hom-set $\mathcal{D}(X, Y)$ is a complete Boolean algebras, that is, $\alpha = \neg\neg\alpha$ for all relations $\alpha : X \rightharpoonup Y$. (Where $\neg\alpha = \alpha \Rightarrow 0_{XY}$.)

[Subobjects] For all subidentities $u : X \rightharpoonup X$ (that is, $u \sqsubseteq \mathrm{id}_X$) there exists an injection $i : S \to X$ such that $u = i^\sharp i$.

[Power Objects] For all objects Y there exists a membership relation $\ni_Y : \wp(Y) \rightharpoonup Y$ such that $(\ni_Y \ominus \ni_Y{}^\sharp) \sqcap (\ni_Y \ominus \ni_Y{}^\sharp)^\sharp \sqsubseteq \mathrm{id}_{\wp(Y)}$ and for all relations $\alpha : X \rightharpoonup Y$ there is a unique function $\alpha^@ : X \to \wp(Y)$ such that $\alpha = \alpha^@ \ni_Y$.

The relation $\Xi_X = \ni_X \ominus \ni_X{}^\sharp : \wp(X) \rightharpoonup \wp(X)$ is called the *power order* on X.

There are three types of relational axiom of choice in Dedekind categories:

(AC1) For all total relations $\alpha : X \rightharpoonup Y$ there exists a function $f : X \to Y$ such that $f \sqsubseteq \alpha$.

(AC2) For all relations $\alpha : X \rightharpoonup Y$ there exists a partial function $f : X \to Y$ such that $f \sqsubseteq \alpha$ and $\mathrm{dom}(f) = \mathrm{dom}(\alpha)$.

(AC3) All surjections have sections, that is, for all surjections $f : X \to Y$ there exists a function $s : Y \to X$ such that $sf = \mathrm{id}_Y$.

(AC1), (AC2) and (AC3) are mutually equivalent in (rational or tabular) Dedekind categories.

Definition 2. A relation $\xi : X \rightharpoonup X$ is called a (partial) *order* if it is reflexive ($\mathrm{id}_X \sqsubseteq \xi$), transitive ($\xi\xi \sqsubseteq \xi$) and antisymmetric ($\xi \sqcap \xi^\sharp \sqsubseteq \mathrm{id}_X$).

Note that $\xi^\sharp \ominus \xi = \xi$ iff $\mathrm{id}_X \sqsubseteq \xi$ and $\xi\xi \sqsubseteq \xi$.

Definition 3. A relation $\sigma : V \rightharpoonup X$ is said to be a *ξ-chain* if $\sigma^\sharp\sigma \sqsubseteq \xi \sqcup \xi^\sharp$, or equivalently if $\sigma \sqsubseteq \sigma \ominus (\xi \sqcup \xi^\sharp)$.

Definition 4. Let $\xi : X \rightharpoondown X$ be a relation and V an object.

(a) ξ is *complete* on V if for all relations $\sigma : V \rightharpoondown X$ the supremum $\sup(\sigma, \xi) : V \rightharpoondown X$ is total.

(b) ξ is *chain-complete* on V if for all ξ-chains $\sigma : V \rightharpoondown X$ the supremum $\sup(\sigma, \xi) : V \rightharpoondown X$ is total.

(c) ξ is *inductive* on V if for all ξ-chains $\sigma : V \rightharpoondown X$ the upper bound $\sigma \ominus \xi : V \rightharpoondown X$ is total. □

Definition 5. Let $\xi : X \rightharpoondown X$ be a relation. A function $x : V \to X$ is called *ξ-maximal* if $x\xi = x$. □

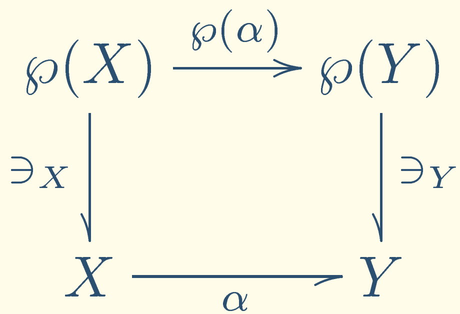
(Zorn's Lemma) (ZL) If an order $\xi : X \rightharpoondown X$ is inductive on V, then there exists a ξ-maximal function $x : V \to X$.

Our goal is to show the following

Theorem A. (AC) if and only if (ZL).

Lemma 6. Let $\xi : X \rightharpoondown X$ be an order and $\sigma : V \rightharpoondown X$ a relation. If $\sigma \ominus \xi : V \rightharpoondown X$ is total and $\sigma \ominus \xi \sqsubseteq \sigma$ holds, then $\sigma \ominus \xi$ is a ξ-maximal function. □

Let $\alpha : X \rightharpoondown Y$ be a relation. A function $\wp(\alpha) : \wp(X) \to \wp(Y)$ is a unique function such that $\wp(\alpha)\ni_Y = \ni_X \alpha$, that is, $\wp(\alpha) = (\ni_X \alpha)^@$.

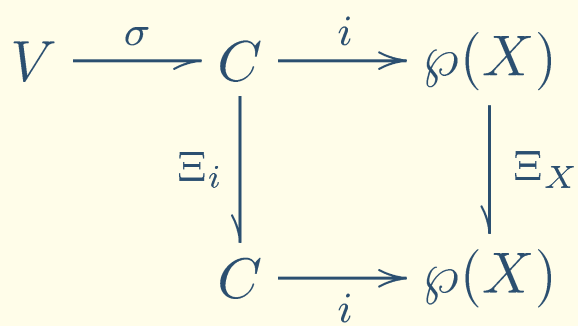


Proposition 7.

(a) $\sup(\ni_{\wp(X)}, \Xi_X) = \wp(\ni_X)$.
(b) Ξ_X is complete on all objects V,
(c) The power object $\wp(X)$ forms a Boolean algebra (if $\mathcal{D}$ is a Schröder category). $\square$

For an injection $i : C \to \wp(X)$ we define two relations $\ni_i : C \rightharpoonup X$ and $\Xi_i : \wp(X) \rightharpoonup \wp(X)$ by $\ni_i = i\ni_X$ and $\Xi_i = i\Xi_X i^\sharp$. It is obvious that $\Xi_i = \ni_i \ominus \ni_i{}^\sharp$.

Lemma 8. Let $\delta : X \rightharpoonup X$ be a relation. If $i : C \to \wp(X)$ is an injection with $i^\sharp i = \mathrm{id}_{\wp(X)} \sqcap [\ni_X \ominus (\ni_X \ominus \delta)^\sharp]$, then the induced power order Ξ_i is chain-complete on all objects V.



□

Proposition 9. Let $\xi : X \rightharpoonup X$ be a reflexive relation and $i : C \to \wp(X)$ an injection with $i^\sharp i = \mathrm{id}_{\wp(X)} \sqcap [\ni_X \ominus (\ni_X \ominus \delta)^\sharp]$, where $\delta = \xi \sqcup \xi^\sharp$.

(a) If $s : V \to C$ is a function, then $s\ni_i$ is a ξ-chain.

(b) If $f : C \to X$ is a partial function with $f \sqsubseteq \ni_i \ominus \xi$, then there exists a function $g : C \to C$ such that $g\ni_i = \ni_i \sqcup f$. □

Definition 10. Let $\xi : X \rightharpoonup X$ be a relation. A function $g : X \to X$ is called *ξ-adjunct* if $g \sqsubseteq \xi$ and $\xi \sqcap g\xi^\sharp \sqsubseteq \mathrm{id}_X \sqcup g$.

Note. [Schröder] $\quad \xi \sqcap g\xi^\sharp \sqsubseteq \mathrm{id}_X \sqcup g \leftrightarrow g \sqsubseteq \neg(\xi_< \xi_<)$.

Proposition 11. [Schröder] Let $\xi : C \rightharpoonup C$ be an order and $f : C \to X$ a partial function. If $\xi = \varepsilon \ominus \varepsilon^\sharp$ for a relation $\varepsilon : C \rightharpoonup X$, then every function $g : C \to C$ satisfying $g\varepsilon = \varepsilon \sqcup f$ is ξ-adjunct. $\square$

A relational version of the concept of towers will be defined as follows:

Definition 12. Let $\xi : X \rightharpoonup X$ be a relation and $g : X \rightarrow X$ a function. A relation $\tau : V \rightharpoonup X$ is called a (ξ, g)-*tower* if

(T1) $\sup(\sigma, \xi) \sqsubseteq \tau$ for all ξ-chains $\sigma : V \rightharpoonup X$ with $\sigma \sqsubseteq \tau$,

(T2) $\tau g \sqsubseteq \tau$. □

The universal relation ∇_{VX} is a (ξ, g)-tower and so (ξ, g)-towers do exist.

The meet of all (ξ, g)-towers $\tau : V \rightharpoonup X$ is also a (ξ, g)-tower and is the minimum (ξ, g)-tower.

Therefore for each object V the minimum (ξ, g)-tower $\tau_0 : V \rightharpoonup X$ exists.

The minimum (ξ, g)-tower τ_0 will play an important rôle to prove a fixpoint theorem below.

Now we will show a fixpoint theorem.

Theorem 13. [Schröder] Let $\xi : X \rightharpoonup X$ be an order, $g : X \to X$ a ξ-adjunct function and $\tau_0 : V \rightharpoonup X$ the minimum (ξ, g)-tower. Set $\mu = \tau_0 \ominus (\xi^\sharp \sqcup \xi)$ and $\nu = \tau_0 \sqcap [\mu \ominus (\xi^\sharp \sqcup g\xi)]$.

(a) ν is a (ξ, g)-tower,
(b) μ is a (ξ, g)-tower,
(c) τ_0 is a ξ-chain,
(d) $\sup(\tau_0, \xi)g \sqsubseteq \max(\tau_0, \xi)$,
(e) If ξ is chain-complete, then $\sup(\tau_0, \xi)g = \sup(\tau_0, \xi)$.
□

Theorem B. [Schröder] (AC) → (ZL).

Proof. Assume that an order $\xi : X \rightharpoonup X$ is inductive on V. By the virtue of Lemma 6 it suffices to find a ξ-chain $\sigma : V \rightharpoonup X$ with $\sigma \ominus \xi \sqsubseteq \sigma$. Using the existence of subobjects [Subobjects] we can construct an injection $i : C \to \wp(X)$ with

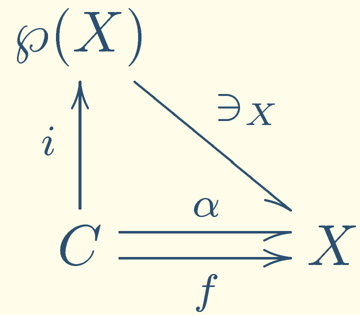
$$i^\sharp i = \mathrm{id}_{\wp(X)} \sqcap (\ni_X \ominus [\ni_X \ominus (\xi \sqcup \xi^\sharp)]^\sharp).$$

By Lemma 8 the induced order $\Xi_i = i \Xi_X i^\sharp : C \rightharpoonup C$ is chain-complete.

Define a relation $\alpha : C \rightharpoondown X$ by

$$\alpha = (\ni_i \ominus \xi) \sqcap \ni_i^{\smile}.$$

Then by (AC2) there is a partial function $f : C \to X$ such that $f \sqsubseteq \alpha$ and $\mathrm{dom}(f) = \mathrm{dom}(\alpha)$.



By Proposition 9(b) there exists a function $g : C \to C$ such that $g \ni_i = \ni_i \sqcup f$. Moreover, g is ξ-adjunct by Proposition 11.

Hence by Fixpoint Theorem 13(e) there exists a function $s : V \to C$ such that $sg = s$. It is easy to see $sf = 0_{VX}$ from

$$
\begin{aligned}
sf &\sqsubseteq sg\ni_i \sqcap s\ni_i{}^{-} && \{\ f \sqsubseteq g\ni_i \text{ and } f \sqsubseteq \alpha \sqsubseteq \ni_i{}^{-}\ \} \\
&= s\ni_i \sqcap s\ni_i{}^{-} && \{\ sg = s\ \} \\
&= 0_{VX}, && \{\ s\ni_i{}^{-} = (s\ni_i)^{-}\ \}
\end{aligned}
$$

and so $s\alpha = 0_{VX}$ from

$$
\begin{aligned}
s\alpha &= s\,\mathrm{dom}(\alpha)\alpha && \{\ \alpha = \mathrm{dom}(\alpha)\alpha\ \} \\
&= s\,\mathrm{dom}(f)\alpha && \{\ \mathrm{dom}(\alpha) = \mathrm{dom}(f)\ \} \\
&= s(ff^\sharp \sqcap \mathrm{id}_C)\alpha \\
&= 0_{VX}. && \{\ sf = 0_{VX}\ \}
\end{aligned}
$$

Therefore we have $s\alpha = (s\ni_i \ominus \xi) \sqcap (s\ni_i)^{-} = 0_{VC}$, that is, $s\ni_i \ominus \xi \sqsubseteq s\ni_i$. This completes the proof, because $\sigma = s\ni_i$ is a ξ-chain by Proposition 9(a). □

In set theory the proof is an application of Zorn's lemma. However in relational case there is a certian gap between the maximality and the totality for partial functions.

So the relational proof requires some other idea ....
An object I is called a *unit* iff $0_{II} \neq \mathrm{id}_I = \nabla_{II}$ and $\nabla_{XI}\nabla_{IY} = \nabla_{XY}$ for all objects X and Y.