Kripke Completeness of First-Order Constructive Logics with Strong Negation

ICHIRO HASUO and RYO KASHIMA, *Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, O-okayama, Meguro, Tokyo 152-8552, Japan.*
E-mail: {hasuo2, kashima}@is.titech.ac.jp

Abstract

This paper considers Kripke completeness of Nelson's constructive predicate logic $\mathsf{N3}$ and its several variants. $\mathsf{N3}$ is an extension of intuitionistic predicate logic Int by an contructive negation operator $\sim$ called *strong negation*. The variants of $\mathsf{N3}$ in consideration are by omitting the axiom $A \to (\sim A \to B)$, by adding the axiom of constant domain $\forall x(A(x) \vee B) \to \forall x A(x) \vee B$, by adding $(A \to B) \vee (B \to A)$, and by adding $\neg\neg(A \vee \sim A)$; the last one we would like to call *the axiom of potential omniscience* and can be interpreted that *we can eventually verify or falsify a statement, with proper additional information.* The proofs of completeness are by the widely-applicable *tree-sequent* method; however, for those logics with the axiom of potential omniscience we can hardly go through with a simple application of it. For them we present two different proofs: one is by an embedding of classical logic, and the other by the TSg method, which is an extension of the tree-sequent method.

Keywords: Strong Negation, Quantified Constructive Logic, Kripke-type Semantics, Intermediate Logics

1 Introduction

1.1 Strong negation

Strong (or *constructive*) *negation*, denoted by $\sim$ in this paper, is a negation operator in constructive logics introduced by Nelson [16], Markov [15], and von Kutschera [29].[1]

In intuitionistic logic, the negation operator $\neg A$, which we would like to call *Heyting's negation* to make it distinct from the strong one, is an abbreviation for $A \to \perp$, i.e. *A implies absurdity.* This kind of treatment of negation is justified by such a viewpoint as Grzegorczyk's one, "the compound sentences are not a product of experiment, they arise from reasoning. This concerns also negation: we see that the lemon is yellow, we do not see that it is not blue" [8].

However, such an example as Kracht cites yields an alternative definition of negation, especially for constructivists: "we can not only *verify* a simple proposition such as *This door is locked.* by direct inspection, but also *falsify* it" [13]. This motivates the idea of strong negation, that is, taking negative information as primitive as positive one.

[1] Markov refers to Nelson [16] in his short note from 1950, nevertheless it is claimed by many authors that they introduced strong negation independently.

Wansing [31] argues that constructive logics with strong negation and without some of structural rules (in Gentzen-style sequent systems) can be candidates for the logic of *information structure*, due to the character of strong negation stated above. And in recent years, constructive logics have been paid attention to in the field of logic programming, e.g. [10], [19], [20], [30] and [1].

Nelson's constructive logic $\mathsf{N3}$ is an extension of intuitionistic logic Int by the strong negation operator $\sim$, where the word *logic* designates a pair of a formal language and a set of its formulas which are admitted as theorems. $\sim$ is axiomatized in the Hilbert-style system as follows:

$$\begin{array}{l} A \to ({\sim}A \to B), \\ {\sim}(A \wedge B) \leftrightarrow {\sim}A \vee {\sim}B, \quad {\sim}(A \vee B) \leftrightarrow {\sim}A \wedge {\sim}B, \\ {\sim}(A \to B) \leftrightarrow A \wedge {\sim}B, \quad {\sim}{\sim}A \leftrightarrow A, \quad {\sim}\neg A \leftrightarrow A, \\ {\sim}\forall x A \leftrightarrow \exists x {\sim}A, \quad {\sim}\exists x A \leftrightarrow \forall x {\sim}A. \end{array}$$

Introduced by the axioms above, strong negation $\sim$ and Nelson's logic $\mathsf{N3}$ enjoys several properties reflecting its motivation. In what follows we shall see three examples.

One is the principle of *constructible falsity*,

$$\vdash {\sim}(A \wedge B) \quad \text{iff} \quad \vdash {\sim}A \quad \text{or} \quad \vdash {\sim}B,$$

which can be regarded as the negative counterpart of *disjunction property*,

$$\vdash A \vee B \quad \text{iff} \quad \vdash A \quad \text{or} \quad \vdash B.$$

While the former does not hold as to Heyting's negation (i.e. in Int), it holds as to strong negation (i.e. in $\mathsf{N3}$) by Kripke completeness of $\mathsf{N3}$. This is one of the facts that imply that positive and negative information have equal importance in $\mathsf{N3}$.

Another example is logical equivalence in $\mathsf{N3}$. As is shown in [28], if logical equivalence in $\mathsf{N3}$ (denoted by $A \cong_{\mathsf{N3}} B$ here) is defined by $\mathsf{N3} \vdash A \leftrightarrow B$ (just like in logics without $\sim$), then the equivalence theorem fails, i.e. $A \cong_{\mathsf{N3}} B$ does not necessarily imply $C[A] \cong_{\mathsf{N3}} C[B]$. However, if both $\mathsf{N3} \vdash A \leftrightarrow B$ and $\mathsf{N3} \vdash {\sim}A \leftrightarrow {\sim}B$ hold, that is, if A and B are equivalent in both positive and negative senses, then $\mathsf{N3} \vdash C[A] \leftrightarrow C[B]$.

The last is seen in the Kripke-type possible world semantics characterizing $\mathsf{N3}$, which is an extension of that for Int by an extra interpretation I^-. I^- is the *falsum* interpretation, designating which formula is *falsified* at each possible world. In a Kripke-model for $\mathsf{N3}$ (we would like to call it an $\mathsf{N3}$-model), the falsity of an atomic formula is not reducible to the *verum* interpretation I^+, but can only be shown by I^-. About $\mathsf{N3}$-models we shall see in detail later.

There have been many suggestions for an interpretation of intuitionistic logic Int: Kripke's one of *development of knowledge* [14], Grzegorczyk's one of *scientific research* [8], Brouwer-Heyting-Kolmogorov (BHK) interpretation in terms of *proofs* [11] [27], and so on. Those three cited above can be easily modified into interpretations of $\mathsf{N3}$, as Wansing shows in [31].

1.2 The axiom of potential omniscience

Since N3 is a conservative extension of Int, it is natural to consider extensions of intermediate logics (which include Int and are included by classical logic Cl) by the strong negation operator.[2] However, the study of extra axioms which are peculiar to logics with strong negation seems to have been paid less attention to. We will consider one of such axioms in this paper: $\neg\neg(A \vee \sim A)$, which we would like to call *the axiom of potential omniscience*. Intuitively it is interpreted that *we can eventually verify or falsify any statement, with proper additional information.* Now we shall see what can be implied by the axiom.

In the Kripke-type interpretation, the axiom of potential omniscience corresponds to the following statement: for every closed formula A and every state of information a, there is a state b which is reachable from a and where A is either verified or refuted. Hence the axiom, especially when combined with the axiom $(A \to B) \vee (B \to A)$ which can be interpreted that *the set of information states is linearly ordered*, seems useful to formalize such cases as some kind of games, e.g. cryptography; consider a game where one player (or a dealer, say Alice) knows the answer and the other player (say Bob) tries to find it. In such a game Bob can reach the correct answer if he obtains the information that Alice has.

The axiom may also be considered as one of the weaker versions of *the law of excluded middle*, $A \vee \sim A$.[3]

1.3 What is shown in this paper

This paper is to show Kripke completeness of several variants of Nelson's constructive logic N3, for the quantified case. The proofs are by the usage of a *tree-sequent*, which is a labelled tree each node of which is associated with a sequent. The tree-sequent method can be regarded as a kind of semantic tableaux, and since it aims at completeness as to Kripke-models of a tree-shape, a tree-sequent has its shape. Needless to say the method is also applicable to Kripke completeness of Int and some intermediate predicate logics; the results can be found in [12].

The logics whose Kripke completeness is proved in this paper are presented in Fig. 1, enclosed in a box.[4] For those four logics which are not enclosed, we shall see the difficulty therein in the last section.

Here the basic logics are N3 and N4: N4 is obtained from N3 by omitting the axiom $A \to (\sim A \to B)$, which is one of those which axiomatize $\sim$ and may be considered as the constructive version of *the ex falso rule* $\bot \to B$. In Kripke-type semantics for N3 the verum interpretation I^+ and the falsum one I^- are always disjoint, hence for each formula A and each possible world a we have one of the following three: A is *verified* at a, A is *falsified* at a, or A is *neither verified nor falsified* at a. For N4 where I^+ and I^- may intersect, we have another possibility: A is *both verified and falsified* at a. Hence the name of the logics; N4 is for *four-valued.* N4 is known to be

[2] For example, Goranko [6], Sendlewski [22], [23], [24], and Kracht [13] discuss unquantified cases using algebraic methods.

[3] $\neg A \vee \neg\neg A$ is said to be *the weak law of excluded middle* and characterizes intermediate logics. Corsi and Ghilardi [4] considers Kripke completeness of Int plus the axiom and its extensions, for the quantified case.

[4] Almukdad and Nelson [2] consider $\neg$-free fragments of N3d, N4 and N4d (denoted by N^+, N^- and N^{+-} respectively) and gives Gentzen-style sequent systems for them.

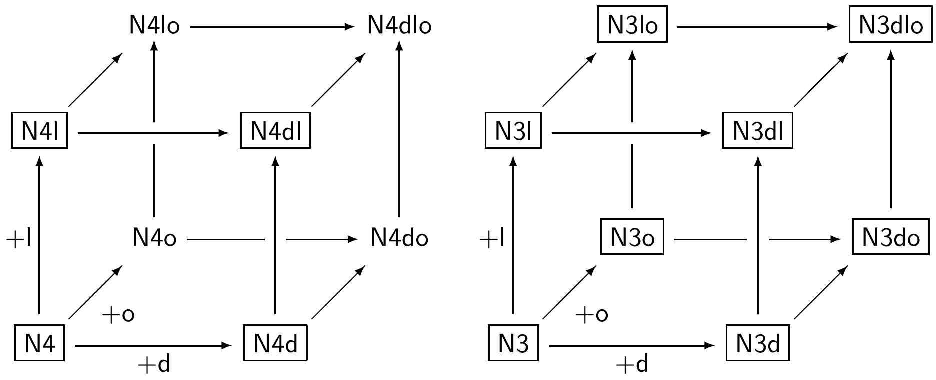


Fig. 1. The N-family

paraconsistent.[5]

Each of the additional letters d, l, and o designates what follows:

$\boxed{\mathsf{d}}$ Adding *the axiom of constant domain*, $\forall x(A(x) \vee B) \to \forall x A(x) \vee B$, where x have no free occurrences in B. The intermediate logic CD, which is Int plus this axiom, is characterized by the class of Kripke models whose domain is a constant map, as Görnemann [7] shows. d is for *Domain.*

$\boxed{\mathsf{l}}$ Adding the axiom $(A \to B) \vee (B \to A)$. Dummett's logic LC, which is Int plus this axiom, is characterized by the class of linearly ordered Kripke models, as Corsi [3] shows.[6] l is for *Linear.*

$\boxed{\mathsf{o}}$ Adding the axiom of potential omniscience, $\neg\neg(A \vee {\sim}A)$. o is for *Omniscience.*

The family of sixteen logics presented in the lattices above we would like to call the N-*family* in this paper.

While the completeness proofs for those without o are easy by simple applications of the tree-sequent method, the proofs for those with o are not, since the axiom of potential omniscience refers to, so to speak, upper-bounds of a Kripke model. For them we present two different proofs. One proof is applicable only to N3o and N3lo; it is by an embedding of Cl into N3o, and omniscient possible worlds (where every formula is either verified or falsified) are induced by Cl-models (or *structures*). The other is by an extension of the tree-sequent method which we would like to call the *tree-sequent with guardians* (abbreviated TSg) method.

There are some logics in the N-family whose Kripke completeness is already shown: N3 is by Gurevich [9] and by van Dalen [28], N3d is by Thomason [25], and N4 is by Odintsov and Wansing [18]. Nevertheless the authors do not take their own tree-sequent-based proofs as useless; the method used in them is applicable to other logics.

---

[5] Priest and Routley [21] give an introduction to paraconsistent logics. See also Odintsov [17], Odintsov and Wansing [18]

[6] Kripke completeness of CD or LC can be proved more easily using the tree-sequent method. The proof is just the same as that for N3d or N3l, and presented in [12].

1.4 Notations

In this paper we adopt Gentzen-style sequent systems for the formal presentation of logics;[7] a sequent system for logic L is denoted by $\mathbf{G}\mathsf{L}$. And later we will introduce tree-sequent systems and TSg systems for the proofs of Kripke completeness; the tree-sequent system for logic L is denoted by $\mathbf{T}\mathsf{L}$, and the TSg system is by $\mathbf{Tg}\mathsf{L}$.

We often denote logics by such a form as $\mathsf{N}^3_4[\mathsf{d}]$: this is for "$\mathsf{N3}$, $\mathsf{N3d}$, $\mathsf{N4}$ and $\mathsf{N4d}$". $\mathsf{N4d}[\mathsf{o}]$ is for "$\mathsf{N4d}$ and $\mathsf{N4do}$".

We do not consider constants or function symbols, which makes the argument simpler without essential loss of generality.

Syntactical equivalence is denoted by $\equiv$. For example, $A \wedge B$ and $B \wedge A$ are logically equivalent in those logics considered in this paper; however, $A \wedge B \not\equiv B \wedge A$.

$A[y/x]$ is a substitution, obtained by replacing every free occurrences of x in A by y. It is not preferable that by substitution new bound variables come to existence; we avoid such cases by taking variants, i.e. replacing bound variables.

As in Tarski-type semantics for classical logic Cl, in defining $\models$ and its variants we will introduce temporary constants each of which designates a certain individual u. This kind of constant is said to be the *name* of u and denoted by $\underline{u}$.

For a finite set of formulas $\Gamma = \{A_1, \ldots, A_m\}$, $\bigwedge \Gamma$ (or $\bigvee \Gamma$) is an abbreviation for $A_1 \wedge \ldots \wedge A_m$ (or $A_1 \vee \ldots \vee A_m$). If $\Gamma = \emptyset$, it is $\top$ (or $\bot$), which is an abbreviation for $A \to A$ (or $\neg(A \to A)$, respectively).

2 Syntax and semantics

2.1 Gentzen-style sequent systems $\mathbf{G}\mathsf{N}^3_4[\mathsf{d}][\mathsf{l}][\mathsf{o}]$

Here we introduce Gentzen-style sequent systems for logics in the N-family. They share one formal language, consisting of the following symbols: countably many variables, $x_1, x_2, \ldots$; countably many m-ary predicate symbols for each $m \in \mathbb{N}$, p^m_1, p^m_2, $\ldots$; and logical connectives, $\wedge$, $\neg$, $\to$, $\sim$ and $\forall$. $\vee$ and $\exists$ are introduced as defined symbols:

$$A \vee B :\equiv {\sim}({\sim}A \wedge {\sim}B), \qquad \exists x A :\equiv {\sim}\forall x {\sim} A.$$

$A \leftrightarrow B$ is an abbreviation of $(A \to B) \wedge (B \to A)$. Terms and formulas are composed in the same way as those of Cl, and note that $\sim$ is unary. The binding strength of the connectives are: $\forall, \neg, \sim \,\geq\, \wedge \,\geq\, \to$.

A *sequent* is defined as an ordered pair of finite sets of formulas separated by the symbol $\Rightarrow$, hence the rule of exchange or contraction can be omitted.

Now we present the initial sequents and derivation rules of a Gentzen-style sequent system $\mathbf{G}\mathsf{N3}$ for the logic $\mathsf{N3}$:

$$\frac{}{A \Rightarrow A}\ (\text{Identity, Id}) \qquad \frac{}{A, {\sim}A \Rightarrow}\ (\text{Ex Falso, Fal})$$

$$\frac{\Gamma \Rightarrow \Delta}{\Sigma, \Gamma \Rightarrow \Delta, \Pi}\ (\text{Weakening, W}) \qquad \frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (\text{Cut, C})$$

[7] The duality of positive and negative information, syntactically A and ${\sim}A$, motivates another choice of formal presentation of $\mathsf{N3}$: Wansing [32] introduces higher-arity Gentzen systems using four-place sequents, based on Belnap's display logic

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ (\wedge\mathrm{L}) \qquad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\ (\wedge\mathrm{R})$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \to B, \Gamma \Rightarrow \Delta}\ (\to\mathrm{L}) \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \to B}\ (\to\mathrm{R})_{\mathrm{S}}$$

$$\frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta}\ (\neg\mathrm{L}) \qquad \frac{A, \Gamma \Rightarrow}{\Gamma \Rightarrow \neg A}\ (\neg\mathrm{R})_{\mathrm{S}}$$

$$\frac{A[y/x], \Gamma \Rightarrow \Delta}{\forall x A, \Gamma \Rightarrow \Delta}\ (\forall\mathrm{L}) \qquad \frac{\Gamma \Rightarrow A[z/x]}{\Gamma \Rightarrow \forall x A}\ (\forall\mathrm{R})_{\mathrm{S,\ VC}}$$

$$\frac{\sim A, \Gamma \Rightarrow \Delta \quad \sim B, \Gamma \Rightarrow \Delta}{\sim(A \wedge B), \Gamma \Rightarrow \Delta}\ (\sim\wedge\mathrm{L}) \qquad \frac{\Gamma \Rightarrow \Delta, \sim A, \sim B}{\Gamma \Rightarrow \Delta, \sim(A \wedge B)}\ (\sim\wedge\mathrm{R})$$

$$\frac{A, \sim B, \Gamma \Rightarrow \Delta}{\sim(A \to B), \Gamma \Rightarrow \Delta}\ (\sim\to\mathrm{L}) \qquad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, \sim B}{\Gamma \Rightarrow \Delta, \sim(A \to B)}\ (\sim\to\mathrm{R})$$

$$\frac{A, \Gamma \Rightarrow \Delta}{\sim\neg A, \Gamma \Rightarrow \Delta}\ (\sim\neg\mathrm{L}) \qquad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \sim\neg A}\ (\sim\neg\mathrm{R})$$

$$\frac{A, \Gamma \Rightarrow \Delta}{\sim\sim A, \Gamma \Rightarrow \Delta}\ (\sim\sim\mathrm{L}) \qquad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \sim\sim A}\ (\sim\sim\mathrm{R})$$

$$\frac{\sim A[z/x], \Gamma \Rightarrow \Delta}{\sim\forall x A, \Gamma \Rightarrow \Delta}\ (\sim\forall\mathrm{L})_{\mathrm{VC}} \qquad \frac{\Gamma \Rightarrow \Delta, \sim A[y/x]}{\Gamma \Rightarrow \Delta, \sim\forall x A}\ (\sim\forall\mathrm{R})$$

Here the subscript S indicates the condition of no side formulas, that is, the succedent of the lower sequent must consist of only one formula, and VC the *eigenvariable condition*, i.e. the eigenvariable z must not have free occurrences in the lower sequent.

The initial sequents and derivation rules which do not involve $\sim$ precisely coincides with those of Maehara's $\mathbf{LJ}'$, which is equivalent to **LJ** and one of sequent systems for Int.

If a sequent $\Gamma \Rightarrow \Delta$ is derived using the initial sequents and rules above, $\Gamma \Rightarrow \Delta$ is *provable in* **GN3**, denoted by **GN3** $\vdash \Gamma \Rightarrow \Delta$. A formula A is *provable in* **GN3**, denoted **GN3** $\vdash A$, if **GN3** $\vdash\ \Rightarrow A$.

For variants of N3,[8] we can obtain sequent systems by the following modifications:

$\boxed{\text{N4}}$ Omit the initial sequents (Fal).

$\boxed{\text{d}}$ Allow side formulas in the rule $(\forall\mathrm{R})$, resulting in the new rule

$$\frac{\Gamma \Rightarrow \Delta, A[z/x]}{\Gamma \Rightarrow \Delta, \forall x A}\ (\forall\mathrm{R})_{\mathrm{VC}}$$

This is equivalent to adding $\Rightarrow \forall x(A(x) \vee B) \to \forall x A(x) \vee B$ as initial sequents, as is well-known about the intermediate logic CD.

$\boxed{\text{l}}$ Add the initial sequents

$$\frac{}{\Rightarrow (A \to B) \vee (B \to A)}\ (\mathrm{Li})$$

$\boxed{\text{o}}$ Add the initial sequents

$$\frac{}{\Rightarrow \neg\neg(A \vee \sim A)}\ (\text{Potential Omniscience, Om})$$

There are a couple of equivalent additional rules; we shall see them later.

[8] When we speak of *variants of* N3 they include N4[d][l][o].

2.2 Kripke-type possible world semantics

Kripke-type semantics for logics in the N-family are extensions of that for Int, as stated in the previous section. First we shall describe the precise definitions, and then proceed to explain how the definitions are justified by our intuition.

Let $(M, \leq)$ be a poset, W a non-empty set, and U be a map of M into $\mathcal{P}W$, satisfying: $U(a) \neq \emptyset$ for all $a \in M$; $a \leq b$ implies $U(a) \subseteq U(b)$.

For every predicate symbol p (we assume p is m-ary), we define two interpretations of p at $a \in M$, denoted by $p^{I^+(a)}$ and $p^{I^-(a)}$, as subsets of $U(a)^m$, satisfying:

1. $a \leq b$ implies $p^{I^+(a)} \subseteq p^{I^+(b)}$ and $p^{I^-(a)} \subseteq p^{I^-(b)}$;
2. $p^{I^+(a)} \cap p^{I^-(a)} = \emptyset$.

Then the quintuple $\mathcal{M} = (M, \leq, U, I^+, I^-)$ is said to be an N3-*model*.

Given an N3-model $\mathcal{M}$, we extend two interpretations I^+ and I^- into two relations between an element in M and a closed formula, $a \models^+ A$ and $a \models^- A$, inductively on the construction of a closed formula A:

$$
\begin{aligned}
&a \models^+ p(\underline{u_1}, \ldots, \underline{u_m}) \quad \text{iff} \quad (u_1, \ldots, u_m) \in p^{I^+(a)} \ ; \\
&a \models^- p(\underline{u_1}, \ldots, \underline{u_m}) \quad \text{iff} \quad (u_1, \ldots, u_m) \in p^{I^-(a)} \ ; \\
&a \models^+ A \wedge B \quad \text{iff} \quad a \models^+ A \quad \text{and} \quad a \models^+ B \ ; \\
&a \models^- A \wedge B \quad \text{iff} \quad a \models^- A \quad \text{or} \quad a \models^- B \ ; \\
&a \models^+ A \to B \quad \text{iff} \quad \text{for every } b \geq a, \quad b \models^+ A \quad \text{implies} \quad b \models^+ B \ ; \\
&a \models^- A \to B \quad \text{iff} \quad a \models^+ A \quad \text{and} \quad a \models^- B \ ; \\
&a \models^+ \neg A \quad \text{iff} \quad \text{for every } b \geq a, \quad b \not\models^+ A \ ; \\
&a \models^- \neg A \quad \text{iff} \quad a \models^+ A \ ; \\
&a \models^+ {\sim}A \quad \text{iff} \quad a \models^- A \ ; \\
&a \models^- {\sim}A \quad \text{iff} \quad a \models^+ A \ ; \\
&a \models^+ \forall x A \quad \text{iff} \quad \text{for every } b \geq a \text{ and every } u \in U(b), \quad b \models^+ A[\underline{u}/x] \ ; \\
&a \models^- \forall x A \quad \text{iff} \quad \text{for some } u \in U(a), \quad a \models^- A[\underline{u}/x] \ .
\end{aligned}
$$

We write $a \not\models^{+-} A$ if neither $a \models^+ A$ nor $a \models^- A$ holds.

A formula A of N3 is *valid in an* N3-*model* $\mathcal{M}$, denoted by $\mathcal{M} \models A$, if $a \models^+ \forall\vec{x}A$ for every $a \in M$, where $\forall\vec{x}A$ is a universal closure of A. A is *valid*, denoted by N3 $\models A$, if A is valid in every N3-model.

A sequent $\Gamma \Rightarrow \Delta$ of **G**N3 is *valid* (or *valid in* $\mathcal{M}$), denoted by N3 $\models \Gamma \Rightarrow \Delta$ (or $\mathcal{M} \models \Gamma \Rightarrow \Delta$), if the formula $(\bigwedge \Gamma) \to (\bigvee \Delta)$ is valid (or valid in $\mathcal{M}$, respectively).

Now we describe how the definitions above can be understood.

The elements of M can be considered as "points in time (or 'evidential situations'), at which we may have various pieces of information" [14], and are often called *possible worlds*. Then the relation $a \leq b$ can be interpreted that a possible world a can develop into b.

$U(a)$ is the domain of individuals at a, more precisely, individuals whose existence is recognized at a. $p^{I^+(a)}$ (or $p^{I^-(a)}$) designates atomic formulas which are verified (or falsified) at a, by direct inspections. If $a \leq b$, then it is natural to assume that

b inherits all the information available at a, hence $U(a) \subseteq U(b)$, $p^{I^+(a)} \subseteq p^{I^+(b)}$ and $p^{I^-(a)} \subseteq p^{I^-(b)}$.

$a \models^+ A$, $a \models^- A$ or $a \not\models^{+-} A$ is interpreted that a *verifies*, *falsifies* or *neither verifies nor falsifies* A. While whether Heyting's negation holds is reduced (through *reasoning*) to *verum* interpretation I^+, strong negation is reduced to *falsum* interpretation I^-, which is primitive and independent of I^+. We can see the constructive character of N3-models in I^- and $\models^-$, comparing with Int-models.

Note that $(M, \leq)$ need not be linearly-ordered; a may develop into different possible worlds.

Kripke-type semantics for the logics other than N3 is obtained by the following modifications:

$\boxed{\mathsf{N4}}$ Omit the condition $p^{I^+(a)} \cap p^{I^-(a)} = \emptyset$.

$\boxed{\mathsf{d}}$ Add the condition that the domain $U : M \to \mathcal{P}W$ is a constant map.

$\boxed{\mathsf{l}}$ Add the condition that $(M, \leq)$ is linearly ordered, that is, for every $a, b \in M$ either $a \leq b$ or $b \leq a$ holds.

$\boxed{\mathsf{o}}$ Add the condition that for every $a \in M$ and every closed formula A, there exists $b \geq a$ where either $b \models^+ A$ or $b \models^- A$ holds.

In fact completeness shown later is often for smaller class of models; first, we can assume a set of possible worlds $(M, \leq)$ to be a tree, since every model $\mathcal{M}$ can be transformed into an equivalent tree-shape model $\mathcal{M}^t$ by taking each path of $\mathcal{M}$ as a node of $\mathcal{M}^t$. Second, the additional condition for o shall be strengthened as follows: for every $a \in M$, there exists $a_g \geq a$ (we will often call a_g the *guardian* of a) which is *omniscient*, that is, for every predicate symbol p (which is m-ary), $p^{I^+(a_g)} \cup p^{I^-(a_g)} = U(a_g)^m$. Moreover, such a_g shall be restricted to be a node which is an immediate successor of a and maximal in $(M, \leq)$ for N3o and N3do, or restricted to be the maximum of $(M, \leq)$ for N3lo and N3dlo.

Several examples of models for logics in the N-family are presented in Fig. 2, where a circle is a possible world, an arrow from a to b denotes that $a \leq b$, the upper colored part of a circle designates the closed formulas verified there, and the lower one those fasified.

The following lemmas which are true to our intuition are easily obtained by induction.

Lemma 2.1 (Heredity)
Let A a closed formula of N3 (or equivalently N_4^3[d][l][o]), $\mathcal{M} = (M, \leq, U, I^+, I^-)$ be an N_4^3[d][l][o]-model, $a, b \in M$ and $a \leq b$. Then $a \models^+ A$ (or $a \models^- A$) implies $b \models^+ A$ (or $b \models^- A$, respectively).

Lemma 2.2
Let A a closed formula of N3, $\mathcal{M} = (M, \leq, U, I^+, I^-)$ be an N3[d][l][o]-model and $a \in M$. Then it is impossible that both $a \models^+ A$ and $a \models^- A$ hold.

Soundness of the sequent systems introduced in the previous section is again easily obtained by induction on the derivation.

Theorem 2.3 (Kripke soundness of $\mathbf{G}\mathsf{N}_4^3$[d][l][o])
If $\mathbf{G}\mathsf{N}_4^3$[d][l][o] $\vdash \Gamma \Rightarrow \Delta$, then N_4^3[d][l][o] $\models \Gamma \Rightarrow \Delta$, respectively.

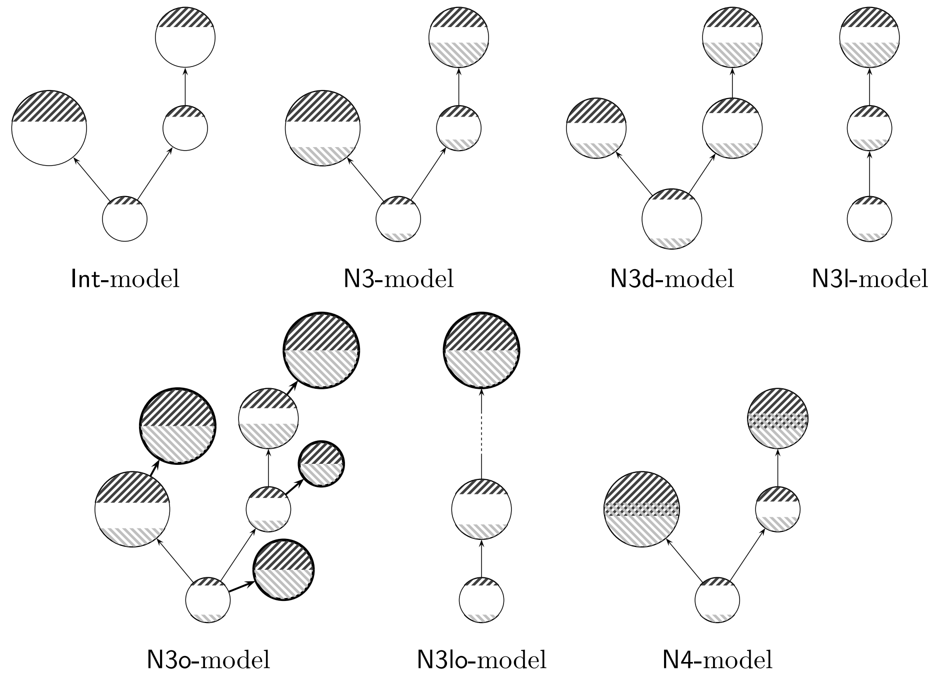


FIG. 2. Kripke models

3 Kripke completeness of $\mathbf{G}\mathsf{N}_4^3[\mathsf{d}][\mathsf{l}]$

3.1 Tree-sequent system $\mathbf{T}\mathsf{N}_4^3[\mathsf{d}][\mathsf{l}]$

In this section we present proofs of Kripke completeness of eight logics, $\mathsf{N}_4^3[\mathsf{d}][\mathsf{l}]$, which are by the usage of the tree-sequent method. In fact, the proofs are just the same as that for Int, CD or LC. In what follows, the word *tree-sequent* is often abbreviated as *TS*.

First we introduce the TS systems $\mathbf{T}\mathsf{N}_4^3[\mathsf{d}][\mathsf{l}]$. Recall that $\mathbf{T}\mathsf{N}_4^3$ is for "$\mathbf{T}\mathsf{N3}$ and $\mathbf{T}\mathsf{N4}$".

DEFINITION 3.1 (Tree-sequent of $\mathbf{T}\mathsf{N}_4^3$)
A *tree-sequent* $\mathcal{T}$ of $\mathbf{T}\mathsf{N}_4^3$ is a finite labelled tree, each node a of which is associated with a sequent $\Gamma_a \Rightarrow \Delta_a$ of $\mathbf{G}\mathsf{N3}$ (or equivalently of $\mathbf{G}\mathsf{N}_4^3[\mathsf{d}][\mathsf{l}]$) and a finite set of variables α_a, denoted by $(a : \Gamma_a \overset{\alpha_a}{\Rightarrow} \Delta_a)$, satisfying the following conditions:

1. Let 0 be the root of $\mathcal{T}$, and $a_0(= 0), a_1, \ldots, a_n$ be an arbitrary path in $\mathcal{T}$, and $(a_i : \Gamma_i \overset{\alpha_i}{\Rightarrow} \Delta_i)$ for each $i \in [0, n]$. Then $\alpha_0, \alpha_1, \ldots, \alpha_n$ are disjoint. The (disjoint) union $\alpha_0 \cup \alpha_1 \cup \cdots \cup \alpha_n$ is said to be the set of *available variables* at the node a_n.
2. Every variable which has free occurrences in the sequent associated to a is available at a.

In other words, for $(a : \Gamma_a \overset{\alpha_a}{\Rightarrow} \Delta_a)$ α_a is the set of variables which are available at a

for the first time in tracing from the root.

DEFINITION 3.2 (Pre-tree-sequent of $\mathbf{TN}_4^3$)
Omitting the condition 2, we obtain the definition of *pre-tree-sequent*, abbreviated as pTS, of $\mathbf{TN}_4^3$.

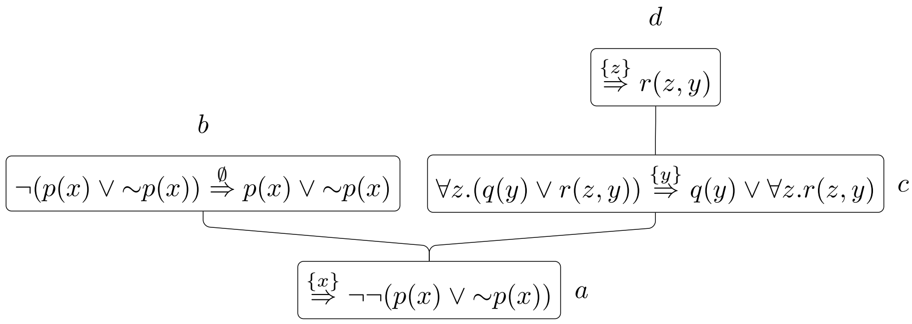
pTS's emerge as subtrees of TS's.

DEFINITION 3.3 (Tree-sequent of $\mathbf{TN}_4^3\mathsf{d}$)
A *tree-sequent* $\mathcal{T}$ of $\mathbf{TN}_4^3\mathsf{d}$ is simply a finite labelled tree each node of which is associated with a sequent, just like $(a : \Gamma_a \Rightarrow \Delta_a)$. The definition does not concern availability of variables.

DEFINITION 3.4 (Tree-sequent and pre-tree-sequent of $\mathbf{TN}_4^3[\mathsf{d}]\mathsf{l}$)
A *tree-sequent* (or *pre-tree-sequent*) of $\mathbf{TN}_4^3[\mathsf{d}]\mathsf{l}$ is that of $\mathbf{TN}_4^3[\mathsf{d}]$ which is linearly ordered, i.e. each node of which has at most one successor.

As seen in the following proof of compleness, the set of available variables works as a seed of the domain at the node; hence the concept is unnecessary for the case of constant domain models, i.e. logics with d.

Here is an example of a TS of $\mathbf{TN}_4^3$:

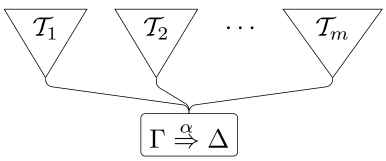


where the sets of available variables at the node a, b, c and d are $\{x\}$, $\{x\}$, $\{x,y\}$ and $\{x,y,z\}$, respectively.

In what follows, we adopt the notation below of a TS, for economy of space:

$$\Big[\ \overset{\{x\}}{\Rightarrow} \neg\neg(p(x) \vee {\sim}p(x)) \ \mid \ \big[\neg(p(x) \vee {\sim}p(x)) \overset{\emptyset}{\Rightarrow} p(x) \vee {\sim}p(x)\ \big] \\ \big[\,\forall z.(q(y) \vee r(z,y)) \overset{\{y\}}{\Rightarrow} q(y) \vee \forall z.r(z,y) \ \mid \overset{\{z\}}{\Rightarrow} r(z,y)\ \big] \ \Big],$$

that is,



is denoted by $[\Gamma \overset{\alpha}{\Rightarrow} \Delta \mid \mathcal{T}_1 \ldots \mathcal{T}_m]$.

However, this notation hardly clarifies the structure of a TS; it will be helpful for to rewrite it in the tree-style.

Since a TS of $\mathbf{TN}^3_4[\mathsf{d}]\mathsf{l}$ is a linearly ordered finite tree, which is nothing but a finite sequence, we often omit [and] in its notation. Hence it is denoted in the form

$$\Gamma_1 \overset{\alpha_1}{\Rightarrow} \Delta_1 \mid \Gamma_2 \overset{\alpha_2}{\Rightarrow} \Delta_2 \mid \cdots \mid \Gamma_k \overset{\alpha_k}{\Rightarrow} \Delta_k.$$

Definition 3.5 (Tree-sequent system $\mathbf{TN}$)
Initial tree-sequents and derivation rules of $\mathbf{TN}$ are presented below, followed by some remarks which describe what they designate.

$$\frac{}{\cdots[A \overset{\alpha}{\Rightarrow} A \mid \cdots}\ (\mathrm{Id}) \qquad \frac{}{\cdots[A, {\sim}A \overset{\alpha}{\Rightarrow} \mid \cdots}\ (\mathrm{Fal})$$

$$\frac{\cdots[\Gamma \overset{\alpha}{\Rightarrow} \Delta \mid \cdots}{\cdots[\Sigma, \Gamma \overset{\alpha}{\Rightarrow} \Delta, \Pi \mid \cdots}\ (\mathrm{W}) \qquad \frac{\cdots[\Gamma \overset{\alpha}{\Rightarrow} \Delta \mid \cdots[A, \Sigma \overset{\beta}{\Rightarrow} \Pi \mid \cdots]\cdots]\cdots}{\cdots[A, \Gamma \overset{\alpha}{\Rightarrow} \Delta \mid \cdots[\Sigma \overset{\beta}{\Rightarrow} \Pi \mid \cdots]\cdots]\cdots}\ (\mathrm{Drop, D})$$

$$\frac{\cdots[A, B, \Gamma \overset{\alpha}{\Rightarrow} \Delta \mid \cdots}{\cdots[A \wedge B, \Gamma \overset{\alpha}{\Rightarrow} \Delta \mid \cdots}\ (\wedge\mathrm{L}) \qquad \frac{\cdots[\Gamma \overset{\alpha}{\Rightarrow} \Delta, A \mid \cdots \quad \cdots[\Gamma \overset{\alpha}{\Rightarrow} \Delta, B \mid \cdots}{\cdots[\Gamma \overset{\alpha}{\Rightarrow} \Delta, A \wedge B \mid \cdots}\ (\wedge\mathrm{R})$$

$$\frac{\cdots[\Gamma \overset{\alpha}{\Rightarrow} \Delta, A \mid \cdots \quad \cdots[B, \Gamma \overset{\alpha}{\Rightarrow} \Delta \mid \cdots}{\cdots[A \to B, \Gamma \overset{\alpha}{\Rightarrow} \Delta \mid \cdots}\ (\to\mathrm{L}) \qquad \frac{\cdots[\Gamma \overset{\alpha}{\Rightarrow} \Delta \mid \cdots[A \overset{\emptyset}{\Rightarrow} B]\cdots]\cdots}{\cdots[\Gamma \overset{\alpha}{\Rightarrow} \Delta, A \to B \mid \cdots}\ (\to\mathrm{R})_{\mathrm{T}}$$

$$\frac{\cdots[\Gamma \overset{\alpha}{\Rightarrow} \Delta, A \mid \cdots}{\cdots[\neg A, \Gamma \overset{\alpha}{\Rightarrow} \Delta \mid \cdots}\ (\neg\mathrm{L}) \qquad \frac{\cdots[\Gamma \overset{\alpha}{\Rightarrow} \Delta \mid \cdots[A \overset{\emptyset}{\Rightarrow}\ ]\cdots]\cdots}{\cdots[\Gamma \overset{\alpha}{\Rightarrow} \Delta, \neg A \mid \cdots}\ (\neg\mathrm{R})_{\mathrm{T}}$$

$$\frac{\cdots[A[y/x], \Gamma \overset{\alpha}{\Rightarrow} \Delta \mid \cdots}{\cdots[\forall x A, \Gamma \overset{\alpha}{\Rightarrow} \Delta \mid \cdots}\ (\forall\mathrm{L}) \qquad \frac{\cdots[\Gamma \overset{\alpha}{\Rightarrow} \Delta \mid \cdots[\ \overset{\{z\}}{\Rightarrow} A[z/x]\,]\cdots]\cdots}{\cdots[\Gamma \overset{\alpha}{\Rightarrow} \Delta, \forall x A \mid \cdots}\ (\forall\mathrm{R})_{\mathrm{T}}$$

$$\frac{\cdots[{\sim}A, \Gamma \overset{\alpha}{\Rightarrow} \Delta \mid \cdots \quad \cdots[{\sim}B, \Gamma \overset{\alpha}{\Rightarrow} \Delta \mid \cdots}{\cdots[{\sim}(A \wedge B), \Gamma \overset{\alpha}{\Rightarrow} \Delta \mid \cdots}\ ({\sim}\wedge\mathrm{L}) \qquad \frac{\cdots[\Gamma \overset{\alpha}{\Rightarrow} \Delta, {\sim}A, {\sim}B \mid \cdots}{\cdots[\Gamma \overset{\alpha}{\Rightarrow} \Delta, {\sim}(A \wedge B) \mid \cdots}\ ({\sim}\wedge\mathrm{R})$$

$$\frac{\cdots[A, {\sim}B, \Gamma \overset{\alpha}{\Rightarrow} \Delta \mid \cdots}{\cdots[{\sim}(A \to B), \Gamma \overset{\alpha}{\Rightarrow} \Delta \mid \cdots}\ ({\sim}{\to}\mathrm{L}) \qquad \frac{\cdots[\Gamma \overset{\alpha}{\Rightarrow} \Delta, A \mid \cdots \quad \cdots[\Gamma \overset{\alpha}{\Rightarrow} \Delta, {\sim}B \mid \cdots}{\cdots[\Gamma \overset{\alpha}{\Rightarrow} \Delta, {\sim}(A \to B) \mid \cdots}\ ({\sim}{\to}\mathrm{R})$$

$$\frac{\cdots[A, \Gamma \overset{\alpha}{\Rightarrow} \Delta \mid \cdots}{\cdots[{\sim}\neg A, \Gamma \overset{\alpha}{\Rightarrow} \Delta \mid \cdots}\ ({\sim}\neg\mathrm{L}) \qquad \frac{\cdots[\Gamma \overset{\alpha}{\Rightarrow} \Delta, A \mid \cdots}{\cdots[\Gamma \overset{\alpha}{\Rightarrow} \Delta, {\sim}\neg A \mid \cdots}\ ({\sim}\neg\mathrm{R})$$

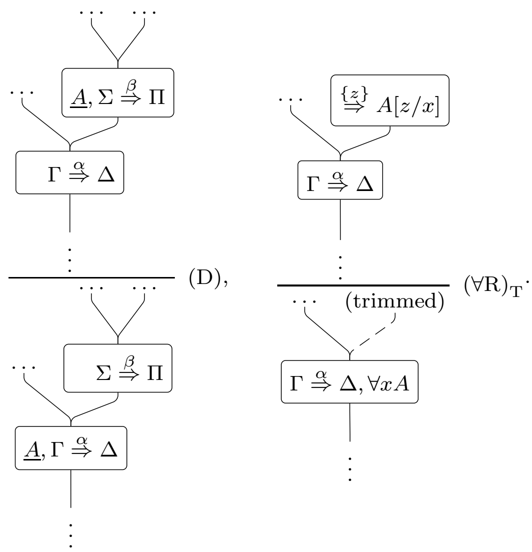
$$\frac{\cdots[A, \Gamma \overset{\alpha}{\Rightarrow} \Delta \mid \cdots}{\cdots[{\sim}{\sim}A, \Gamma \overset{\alpha}{\Rightarrow} \Delta \mid \cdots}\ ({\sim}{\sim}\mathrm{L}) \qquad \frac{\cdots[\Gamma \overset{\alpha}{\Rightarrow} \Delta, A \mid \cdots}{\cdots[\Gamma \overset{\alpha}{\Rightarrow} \Delta, {\sim}{\sim}A \mid \cdots}\ ({\sim}{\sim}\mathrm{R})$$

$$\frac{\cdots[{\sim}A[z/x], \Gamma \overset{\alpha\cup\{z\}}{\Rightarrow} \Delta \mid \cdots}{\cdots[{\sim}\forall x A, \Gamma \overset{\alpha}{\Rightarrow} \Delta \mid \cdots}\ ({\sim}\forall\mathrm{L})_{\mathrm{VC}} \qquad \frac{\cdots[\Gamma \overset{\alpha}{\Rightarrow} \Delta, {\sim}A[y/x] \mid \cdots}{\cdots[\Gamma \overset{\alpha}{\Rightarrow} \Delta, {\sim}\forall x A \mid \cdots}\ ({\sim}\forall\mathrm{R})$$

In the above, a tree-structure and undisplayed nodes are arbitrary; (Id) can be read as every TS which has a node of the form $A \overset{\alpha}{\Rightarrow} A$ is an initial TS.

In the derivation rules, tree-structures and undisplayed nodes are the same between the conclusion and the hypothesis (or hypotheses); (W) is adding formulas to one node of the upper TS.

The rules which are unique to TS's, namely the structural rule Drop (D) and the logical rules with the subscript T, are displayed in the tree-style:



In (D), a formula A in the antecedent of a node is dropped to that of its immedeate successor. In $(\forall R)_T$ a leaf (i.e. a maximal node) $\overset{\{z\}}{\Rightarrow} A[z/x]$ is *trimmed*. Note that the formula $\forall x A$ must obtain its occurrence in the immediate successor of the trimmed leaf, and that the variable z has no free occurrence in $\Gamma \overset{\alpha}{\Rightarrow} \Delta$ since z is available for the first time at $\overset{\{z\}}{\Rightarrow} A[z/x]$. The former remark is also true of the rules $(\neg R)_T$ and $(\to R)_T$.

In the rule $(\sim\forall L)_{VC}$, VC is for the condition that the eigenvariable z have no free occurrences in *any* node of the upper TS.

It is seen that except (D) and those rules with the subscript T (which correspond to the rules of **GN** with the subscript S), the rules are of the form that the corresponding rules of **GN** are applied to one node of the upper TS (or TS's).

$\mathbf{TN}_4^3$[d][l] enjoy one remarkable property: they are cut-free. Although it may imply some other proof-theoretical results, they are not considered in this paper.

Definition 3.6 (Tree-sequent system $\mathbf{TN}_4^3$[d][l])
Tree-sequent systems $\mathbf{TN}_4^3$[d][l] for the logics N_4^3[d][l] are obtained from **TN3** through the following modifications:

$\boxed{4}$ Omit the initial TS's (Fal).

$\boxed{d}$ TS's do not involve the concept of availability of variables, as stated above. And the rule $(\forall R)_T$ is replaced by the new rule:

$$\frac{\cdots [\Gamma \Rightarrow \Delta, A[z/x] \mid \cdots}{\cdots [\Gamma \Rightarrow \Delta, \forall x A \mid \cdots} (\forall R)_{VC}.$$

$\boxed{l}$ Those rules with subscript T are replaced. $(\forall R)_T$ is replaced by

$$\frac{\mathcal{T}_1 \quad \mathcal{T}_2 \quad \cdots \quad \mathcal{T}_k}{\cdots \mid \Gamma_1 \overset{\alpha_1}{\Rightarrow} \Delta_1, \forall x A \mid \Gamma_2 \overset{\alpha_2}{\Rightarrow} \Delta_2 \mid \cdots \mid \Gamma_k \overset{\alpha_k}{\Rightarrow} \Delta_k} (\forall R)_K$$

where

$$\begin{aligned}
\mathcal{T}_1 \equiv \quad & \cdots \mid \Gamma_1 \overset{\alpha_1}{\Rightarrow} \Delta_1 \mid \quad \overset{\{z\}}{\Rightarrow} A[z/x] \mid \Gamma_2 \overset{\alpha_2}{\Rightarrow} \Delta_2 \mid \cdots \mid \Gamma_k \overset{\alpha_k}{\Rightarrow} \Delta_k, \\
\mathcal{T}_2 \equiv \quad & \cdots \mid \Gamma_1 \overset{\alpha_1}{\Rightarrow} \Delta_1 \mid \Gamma_2 \overset{\alpha_2}{\Rightarrow} \Delta_2 \mid \quad \overset{\{z\}}{\Rightarrow} A[z/x] \mid \cdots \mid \Gamma_k \overset{\alpha_k}{\Rightarrow} \Delta_k, \\
\cdots, & \\
\mathcal{T}_k \equiv \quad & \cdots \mid \Gamma_1 \overset{\alpha_1}{\Rightarrow} \Delta_1 \mid \Gamma_2 \overset{\alpha_2}{\Rightarrow} \Delta_2 \mid \cdots \mid \Gamma_k \overset{\alpha_k}{\Rightarrow} \Delta_k \mid \quad \overset{\{z\}}{\Rightarrow} A[z/x].
\end{aligned}$$

The subscript K is for *sKip*. Note that the eigenvariable z has no free occurrences in the lower TS, because of the shape of $\mathcal{T}_k$ and the definition of a TS. $(\to\mathrm{R})_\mathrm{T}$ and $(\neg\mathrm{R})_\mathrm{T}$ are also replaced by $(\to\mathrm{R})_\mathrm{K}$ and $(\neg\mathrm{R})_\mathrm{K}$ respectively, which are of the same form as $(\forall\mathrm{R})_\mathrm{K}$. For example in $(\to\mathrm{R})_\mathrm{K}$,

$$\mathcal{T}_i \equiv \quad \cdots \mid \Gamma_1 \overset{\alpha_1}{\Rightarrow} \Delta_1 \mid \cdots \mid \Gamma_i \overset{\alpha_i}{\Rightarrow} \Delta_i \mid A \overset{\emptyset}{\Rightarrow} B \mid \cdots \mid \Gamma_k \overset{\alpha_k}{\Rightarrow} \Delta_k.$$

In $\mathbf{TN}^3_4\mathsf{dl}$, the rule $(\forall\mathrm{R})_\mathrm{VC}$ remains the same as that of $\mathbf{TN}^3_4\mathsf{d}$.

3.2 Proof of Kripke completeness

The proofs of Kripke completeness of $\mathbf{GN}^3_4[\mathsf{d}][\mathsf{l}]$ are by a simple application of the tree-sequent method. The sketch is as follows. First we prove Kripke completeness of the corresponding TS system; given an unprovable TS $\mathcal{T}$ (i.e. $\mathbf{TN}^3_4[\mathsf{d}][\mathsf{l}] \nvdash \mathcal{T}$), we can extend $\mathcal{T}$ into a $\mathbf{TN}^3_4[\mathsf{d}][\mathsf{l}]$-*saturated* (possibly infinite) tree-sequent $\mathcal{T}_\omega$, which induces a *counter* $\mathsf{N}^3_4[\mathsf{d}][\mathsf{l}]$-*model* for $\mathcal{T}$. Then we define the *formulaic translation* of a TS $\mathcal{T}$, which we denote by $\mathcal{T}^f$. With the lemma that $\mathbf{TN}^3_4[\mathsf{d}][\mathsf{l}] \vdash \mathcal{T}$ implies $\mathbf{GN}^3_4[\mathsf{d}][\mathsf{l}] \vdash \mathcal{T}^f$ and Kripke completeness of $\mathbf{TN}^3_4[\mathsf{d}][\mathsf{l}]$, we can conclude that if $\mathbf{GN}^3_4[\mathsf{d}][\mathsf{l}] \nvdash A$ then the TS $[\overset{\alpha}{\Rightarrow} A]$ has a counter $\mathsf{N}^3_4[\mathsf{d}][\mathsf{l}]$-model, hence Kripke completeness of $\mathbf{GN}^3_4[\mathsf{d}][\mathsf{l}]$.

Definition 3.7 (Counter model for tree-sequent)
Let $\mathcal{M} = (M, \leq, U, I^+, I^-)$ be an $\mathsf{N}^3_4[\mathsf{d}][\mathsf{l}]$-model, $U : M \to \mathcal{P}W$, and $\mathcal{T}$ a TS of $\mathbf{TN}^3_4[\mathsf{d}][\mathsf{l}]$. $\mathcal{M}$ is said to be a *counter model for* $\mathcal{T}$ if:

1. the tree-structure of $\mathcal{T}$ can be embedded in $(M, \leq)$;
2. the set of all variables can be embedded in W;
3. for each node $(a : \Gamma_a \overset{\alpha_a}{\Rightarrow} \Delta_a)$ of $\mathcal{T}$, $a \models^+ A[\underline{\vec{x}}/\vec{x}]$ for each $A \in \Gamma_a$, and $a \not\models^+ B[\underline{\vec{y}}/\vec{y}]$ for each $B \in \Delta_a$, where $\vec{x}$ or $\vec{y}$ is an enumeration of the free variables in A or B.

If $\mathcal{T}$ has no counter models, then $\mathcal{T}$ is said to be *unrefutable*.

It is easily seen that every TS derived in $\mathbf{TN}^3_4[\mathsf{d}][\mathsf{l}]$ is unrefutable, which justifies such unfamiliar rules as (D), $(\forall\mathrm{R})_\mathrm{T}$ and $(\forall\mathrm{R})_\mathrm{K}$.

An *infinite tree-sequent* is a TS whose tree-structure, associating sequents, and associating sets of variables are all possibly infinite, satisfying the condition on availability of variables.

$\mathbf{TN}^3_4[\mathsf{d}][\mathsf{l}]$-*saturatedness* is introduced as a natural extension of **LK**-saturatedness. Here **LK** is meant to be a sequent system used to formalize classical logic.[9] The con-

[9] To be precise, **LK** is meant to be the system **G1c** in [26], with the definition of a sequent modified to be just as in this paper and $\neg$ taken primitive instead of $\perp$.

cept of *saturated sequent* with respect to a certain sequent system is often introduced for completeness proofs; see e.g. [12].

Definition 3.8 ($\mathbf{TN}^3_4$[d][l]-saturatedness)
An infinite tree-sequent $\mathcal{T}$ of $\mathbf{TN}^3_4$[l] is $\mathbf{TN}^3_4$[l]-*saturated* if it satisfies the following conditions:

1. If $(b : \Gamma_b \overset{\alpha_b}{\Rightarrow} \Delta_b)$ is a successor of $(a : \Gamma_a \overset{\alpha_a}{\Rightarrow} \Delta_a)$ in $\mathcal{T}$, then $\Gamma_a \subseteq \Gamma_b$;
2. For each node $(a : \Gamma_a \overset{\alpha_a}{\Rightarrow} \Delta_a)$ of $\mathcal{T}$,
   (a) [($\wedge$L)-saturated] If $A \wedge B \in \Gamma_a$, then $A \in \Gamma_a$ and $B \in \Gamma_a$;
   (b) [($\wedge$R)-saturated] If $A \wedge B \in \Delta_a$, then $A \in \Delta_a$ or $B \in \Delta_a$;
   (c) [($\to$L)-saturated] If $A \to B \in \Gamma_a$, then $A \in \Delta_a$ or $B \in \Gamma_a$;
   (d) [$(\to\mathrm{R})_{\mathrm{T}}$-saturated] If $A \to B \in \Delta_a$, then there exists a successor $(b : \Gamma_b \overset{\alpha_b}{\Rightarrow} \Delta_b)$ of a such that $A \in \Gamma_b$ and $B \in \Delta_b$;
   (e) [($\neg$L)-saturated] If $\neg A \in \Gamma_a$, then $A \in \Delta_a$;
   (f) [$(\neg\mathrm{R})_{\mathrm{T}}$-saturated] If $\neg A \in \Delta_a$, then there exists a successor $(b : \Gamma_b \overset{\alpha_b}{\Rightarrow} \Delta_b)$ of a such that $A \in \Gamma_b$;
   (g) [($\forall$L)-saturated] If $\forall x A \in \Gamma_a$, then $A[y/x] \in \Gamma_a$ for every y available at a ;
   (h) [$(\forall\mathrm{R})_{\mathrm{T}}$-saturated] If $\forall x A \in \Delta_a$, then there exist a successor $(b : \Gamma_b \overset{\alpha_b}{\Rightarrow} \Delta_b)$ of a and a variable y such that $A[y/x] \in \Delta_b$;
   (i) [($\sim\wedge$L)-saturated] If $\sim(A \wedge B) \in \Gamma_a$, then $\sim A \in \Gamma_a$ or $\sim B \in \Gamma_a$;
   (j) [($\sim\wedge$R)-saturated] If $\sim(A \wedge B) \in \Delta_a$, then $\sim A \in \Delta_a$ and $\sim B \in \Delta_a$;
   (k) [($\sim\to$L)-saturated] If $\sim(A \to B) \in \Gamma_a$, then $A \in \Gamma_a$ and $\sim B \in \Gamma_a$;
   (l) [($\sim\to$R)-saturated] If $\sim(A \to B) \in \Delta_a$, then $A \in \Delta_a$ or $\sim B \in \Delta_a$;
   (m) [($\sim\neg$L)-saturated] If $\sim\neg A \in \Gamma_a$, then $A \in \Gamma_a$;
   (n) [($\sim\neg$R)-saturated] If $\sim\neg A \in \Delta_a$, then $A \in \Delta_a$;
   (o) [($\sim\sim$L)-saturated] If $\sim\sim A \in \Gamma_a$, then $A \in \Gamma_a$;
   (p) [($\sim\sim$R)-saturated] If $\sim\sim A \in \Delta_a$, then $A \in \Delta_a$;
   (q) [($\sim\forall$L)-saturated] If $\sim\forall x A \in \Gamma_a$, then $\sim A[y/x] \in \Gamma_a$ for some variable y ;
   (r) [($\sim\forall$R)-saturated] If $\sim\forall x A \in \Delta_a$, then $\sim A[y/x] \in \Delta_a$ for every y available at a.

An infinite tree-sequent $\mathcal{T}$ of $\mathbf{TN}^3_4$d[l] is $\mathbf{TN}^3_4$d[l]-*saturated* if it satisfies the conditions above, replacing "for every y available at a" by "for every variable y".

Lemma 3.9 (Kripke completeness of $\mathbf{TN}^3_4$[d][l])
Let $\mathbf{TN}^3_4$[d][l] $\nvdash \mathcal{T}$ and at least one variable is available at the root of $\mathcal{T}$. Then $\mathcal{T}$ has a counter model.

Proof. First we extend $\mathcal{T}$ into a $\mathbf{TN}^3_4$[d][l]-saturated (possibly infinite) tree-sequent $\mathcal{T}_\omega$. We construct an infinite sequence of (finite) TS's $\mathcal{T}_0, \mathcal{T}_1, \ldots$ and the union of them (now it possibly becomes infinite) is what we would like to obtain.

Let $B_1, B_2, \ldots$ be an enumeration of all the formulas of N3. We rearrange the sequence and obtain another sequence

$$B_1 \mid B_1, B_2 \mid B_1, B_2, B_3 \mid \ldots,$$

which we denote by $A_1, A_2, \ldots$. In $A_1, A_2, \ldots$ every formula of N3 appears infinitely many times. Since every term of N3 is nothing but a variable we can enumerate them as $x_1, x_2, \ldots$.

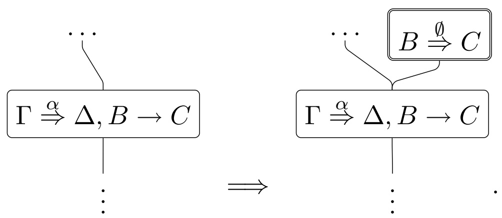
First we consider the case of **TN3**. The modifications of the proof for its variants we shall describe later.

Let $\mathcal{T}_0 :\equiv \mathcal{T}$. The i-th step, which is the step of extension from $\mathcal{T}_{i-1}$ to $\mathcal{T}_i$, consists of *inheritance* and *reduction* of the formula A_i, using a finite number of variables $x_1, x_2, \ldots, x_i$ (hence the step involves only a finite number of operations). Note that in each operation unprovability of the TS is preserved. The operations executed in the i-th step are as follows:

1. [inheritance] For each node $(a : \Gamma_a \overset{\alpha_a}{\Rightarrow} \Delta_a)$ such that $A_i \in \Gamma_a$, add A_i to the antecedents of all the successors of a. Unprovability is preserved because of the rule (D) of **TN3**.
2. [reduction] According to the shape of A_i, one of the following operations is executed for each node $(a : \Gamma_a \overset{\alpha_a}{\Rightarrow} \Delta_a)$:

(a) $[A_i \equiv B \wedge C]$ If $A_i \in \Gamma_a$, then add B and C to Γ_a. Unprovability is preserved because of the rule ($\wedge$L) of **TN3**. If $A_i \in \Delta_a$, then add either B or C to Δ_a, so that unprovability is preserved. This is possible because of the rule ($\wedge$R); if neither choice preserves unprovability, then by the rule the original TS must be provable, which is a contradiction. Each operation below also preserves unprovability because of the corresponding rule of **TN3**, although we do not put explicitly.

(b) $[A_i \equiv B \to C]$ If $A_i \in \Gamma_a$, then add B to Δ_a or C to Γ_a, so that unprovability is preserved. If $A_i \in \Delta_a$, then make a new leaf $(b : B \overset{\emptyset}{\Rightarrow} C)$ as an immediate successor of a:



(c) $[A_i \equiv \neg B]$ If $A_i \in \Gamma_a$, then add B to Δ_a. If $A_i \in \Delta_a$, then make a new leaf $(b : B \overset{\emptyset}{\Rightarrow}\quad)$ as an immediate successor of a.

(d) $[A_i \equiv \forall x B]$ If $A_i \in \Gamma_a$, then add $B[x_j/x]$ to Γ_a, for each $x_j \in \{x_1, \ldots, x_i\}$ which is available at a. If $A_i \in \Delta_a$, then take a fresh variable x_m and make a new leaf $(b : \overset{\{x_m\}}{\Rightarrow} B[x_m/x])$ as an immediate successor of a; we can take fresh x_m since the original TS is finite.

(e) $[A_i \equiv {\sim}(B \wedge C)]$ If $A_i \in \Gamma_a$, add ${\sim}B$ or ${\sim}C$ to Γ_a so that unprovability is preserved. If $A_i \in \Delta_a$, add ${\sim}B$ and ${\sim}C$ to Δ_a.

(f) $[A_i \equiv {\sim}(B \to C)]$ If $A_i \in \Gamma_a$, add B and ${\sim}C$ to Γ_a. If $A_i \in \Delta_a$, add B or ${\sim}C$ to Δ_a so that unprovability is preserved.

(g) $[A_i \equiv {\sim}\neg B$ or $A_i \equiv {\sim}{\sim}B]$ If $A_i \in \Gamma_a$ (or Δ_a), add B to Γ_a (or Δ_a, respectively).

(h) $[A_i \equiv {\sim}\forall x B]$ If $A_i \in \Gamma_a$, then take a fresh variable x_m, add ${\sim}B[x_m/x]$ to Γ_a, and also add x_m to α_a. If $A_i \in \Delta_a$, add ${\sim}B[x_j/x]$ to Δ_a, for each $x_j \in \{x_1, \ldots, x_i\}$ which is available at a.

For variants of **N3**, the following modifications are to be made:

$\boxed{4}$ Nothing.

$\boxed{\mathsf{d}}$ Omit conditions which involves availability of variables, and the operation (2d) is replaced by

(d) $[A_i \equiv \forall x B]$ If $A_i \in \Gamma_a$, then add $B[x_j/x]$ to Γ_a, for each x_j in $\{x_1, \ldots, x_i\}$. If $A_i \in \Delta_a$, then take a fresh variable x_m and add $B[x_m/x]$ to Δ_a; unprovability is preserved because of the rule $(\forall\mathrm{R})_{\mathrm{VC}}$.

$\boxed{\mathsf{l}}$ The operations (2b), (2c) and (2d) are replaced: (2d) is replaced by

(d) $[A_i \equiv \forall x B]$ If $A_i \in \Gamma_a$, then add $B[x_j/x]$ to Γ_a, for each $x_j \in \{x_1, \ldots, x_i\}$ which is available at a. Assume $A_i \in \Delta_a$. Let the TS to which the operation is about to be applied denoted by

$$\cdots \mid \Gamma_1 \overset{\alpha_1}{\Rightarrow} \Delta_1 (\equiv \Gamma_a \overset{\alpha_a}{\Rightarrow} \Delta_a) \mid \Gamma_2 \overset{\alpha_2}{\Rightarrow} \Delta_2 \mid \cdots \mid \Gamma_k \overset{\alpha_k}{\Rightarrow} \Delta_k.$$

Now take a fresh variable x_m and let the TS's $\mathcal{S}_1, \ldots, \mathcal{S}_k$ be

$$\mathcal{S}_l :\equiv \cdots \mid \Gamma_1 \overset{\alpha_1}{\Rightarrow} \Delta_1 \mid \cdots \mid \Gamma_l \overset{\alpha_l}{\Rightarrow} \Delta_l \mid \overset{\{x_m\}}{\Rightarrow} B[x_m/x] \mid \cdots \mid \Gamma_k \overset{\alpha_k}{\Rightarrow} \Delta_k. \quad (l \in [1, k])$$

Take an unprovable $\mathcal{S}_l$ as a new TS; this is possible because of the rule $(\forall\mathrm{R})_{\mathrm{K}}$.

The modified (2b) and (2c) are just the same as above, inserting a new node in such a position that unprovability is preserved.

Let $\mathcal{T}_\omega$ be the union of $\mathcal{T}_0, \mathcal{T}_1, \ldots$, i.e. the tree-structure, associating sequents and associating sets of variables of $\mathcal{T}_\omega$ are the unions of those of $\mathcal{T}_0, \mathcal{T}_1, \ldots$. It is easily verified that $\mathcal{T}_\omega$ is $\mathbf{T}\mathsf{N}^3_4[\mathsf{d}][\mathsf{l}]$-saturated.

We construct an $\mathsf{N3}$-model $\mathcal{M} = (M, \leq, U, I^+, I^-)$ using syntactical objects, namely the $\mathbf{T}\mathsf{N3}$-saturated $\mathcal{T}_\omega$ obtained above. Let $(M, \leq)$ be the tree-structure of $\mathcal{T}_\omega$, $U(a)$ be the set of available variables at a. For each node $(a : \Gamma_a \overset{\alpha_a}{\Rightarrow} \Delta_a)$ of $\mathcal{T}_\omega$ and each predicate symbol p (which is m-ary), the verum / falsum interpretation of p at a is defined by:

$$p^{I^+(a)} := \{(y_1, \ldots, y_m) \mid p(y_1, \ldots, y_m) \in \Gamma_a\},$$
$$p^{I^-(a)} := \{(y_1, \ldots, y_m) \mid {\sim} p(y_1, \ldots, y_m) \in \Gamma_a\}.$$

It is easily verified that $\mathcal{M}$ satisfies the conditions for $\mathsf{N3}$-model. Indeed, the condition on $\mathcal{T}$ that some variables are available at its root yields that $U(a) \neq \emptyset$ for every $a \in M$. If $a \leq b$ and $(y_1, \ldots, y_m) \in p^{I^+(a)}$, then $p(y_1, \ldots, y_m) \in \Gamma_a$ and by the operation [inheritance] $p(y_1, \ldots, y_m) \in \Gamma_b$, hence $(y_1, \ldots, y_m) \in p^{I^+(b)}$. And the fact that every finite sub-tree-sequent of $\mathcal{T}_\omega$ is unprovable in $\mathbf{T}\mathsf{N3}$ and the initial TS (Fal) yield that $p^{I^+(a)} \cap p^{I^-(a)} = \emptyset$.

For $\mathsf{N}^3_4[\mathsf{l}]$ $\mathcal{M}$ is constructed in the same way. For those with d (i.e. $\mathsf{N}^3_4\mathsf{d}[\mathsf{l}]$), $U(a)$ is defined to be just the set of all variables for every $a \in M$.

Since $\mathcal{T}$ is a sub-tree-sequent of $\mathcal{T}_\omega$, we can take the identity map as an embedding of the tree-structure of $\mathcal{T}$ in $(M, \leq)$, and also can take the set of all variables as W, where $U : M \to \mathcal{P}W$. Then it is again easily verified by induction on the construction of formulas that $\mathcal{M}$ is a counter model for $\mathcal{T}$. ∎

DEFINITION 3.10 (Formulaic translation of TS)
Let $\mathcal{T}$ be a pre-tree-sequent of $\mathsf{N}_4^3[\mathsf{l}]$. The *formulaic translation* of $\mathcal{T}$, denoted by $\mathcal{T}^f$, is defined inductively on the height of $\mathcal{T}$:

$$[\Gamma \overset{\alpha}{\Rightarrow} \Delta \mid \mathcal{T}_1 \ldots \mathcal{T}_m]^f :\equiv \forall \overrightarrow{\alpha}\,(\,(\bigwedge \Gamma) \to (\bigvee \Delta) \vee \mathcal{T}_1^f \vee \cdots \vee \mathcal{T}_m^f\,).$$

Let $\mathcal{T}$ be a TS of $\mathsf{N}_4^3\mathsf{d}[\mathsf{l}]$. The *formulaic translation* of $\mathcal{T}$, again denoted by $\mathcal{T}^f$, is a universal closure of $\mathcal{T}^p$, which in turn is defined inductively on the height of $\mathcal{T}$:

$$[\Gamma \Rightarrow \Delta \mid \mathcal{T}_1 \ldots \mathcal{T}_m]^p :\equiv (\bigwedge \Gamma) \to (\bigvee \Delta) \vee \mathcal{T}_1^p \vee \cdots \vee \mathcal{T}_m^p$$

LEMMA 3.11
If $\mathbf{T}\mathsf{N}_4^3[\mathsf{d}][\mathsf{l}] \vdash \mathcal{T}$, then $\mathbf{G}\mathsf{N}_4^3[\mathsf{d}][\mathsf{l}] \vdash \mathcal{T}^f$.

To prove this lemma we prove a couple of sublemmas, whose proofs are easy by induction on the height of a in $\mathcal{T}$.

SUBLEMMA 3.12
Let $\mathcal{T}$ be a TS of $\mathbf{T}\mathsf{N}_4^3[\mathsf{l}]$, a a node of $\mathcal{T}$, $\mathcal{T}'$ be a pTS which consists of a and all of its successors, $\mathcal{T}_1', \ldots, \mathcal{T}_k'$ pre-tree-sequents of $\mathbf{T}\mathsf{N}_4^3[\mathsf{l}]$, and $\mathcal{T}_1, \ldots, \mathcal{T}_k$ be a tree-sequent obtained by replacing $\mathcal{T}'$ by $\mathcal{T}_1', \ldots, \mathcal{T}_k'$, respectively (Fig. 3).

FIG. 3. $\mathcal{T}, \mathcal{T}_1, \ldots, \mathcal{T}_k$

If $\mathbf{G}\mathsf{N}_4^3[\mathsf{l}] \vdash \mathcal{T}_1'^f, \ldots, \mathcal{T}_k'^f \Rightarrow \mathcal{T}'^f$, then $\mathbf{G}\mathsf{N}_4^3[\mathsf{l}] \vdash \mathcal{T}_1^f, \ldots, \mathcal{T}_k^f \Rightarrow \mathcal{T}^f$.

Let $\mathcal{T}$ be a TS of $\mathbf{T}\mathsf{N}_4^3\mathsf{d}[\mathsf{l}]$, $\mathcal{T}', \mathcal{T}_1', \ldots, \mathcal{T}_k', \mathcal{T}_1, \ldots, \mathcal{T}_k$ be just as above. If $\mathbf{G}\mathsf{N}_4^3\mathsf{d}[\mathsf{l}] \vdash \mathcal{T}_1'^p, \ldots, \mathcal{T}_k'^p \Rightarrow \mathcal{T}'^p$, then $\mathbf{G}\mathsf{N}_4^3\mathsf{d}[\mathsf{l}] \vdash \mathcal{T}_1^p, \ldots, \mathcal{T}_k^p \Rightarrow \mathcal{T}^p$.

SUBLEMMA 3.13
Let $\mathcal{T}$ and $\mathcal{T}'$ be just as in the sublemma above. If $\mathbf{G}\mathsf{N}_4^3[\mathsf{d}][\mathsf{l}] \vdash \mathcal{T}'^f$, then $\mathbf{G}\mathsf{N}_4^3[\mathsf{d}][\mathsf{l}] \vdash \mathcal{T}^f$.

PROOF. (of Lemma 3.11) The case for N_4^3 is easy. The proof is by induction on the derivation of $\mathcal{T}$ in $\mathbf{T}\mathsf{N}_4^3$. When $\mathcal{T}$ is an initial TS, use the second sublemma. For the step cases where $\mathcal{T}$ is derived by a derivation rule with a hypothesis (or hypotheses), use the first sublemma.

For those logics with l the proofs are just the same; however, the step case where $\mathcal{T}$ is derived by the rule $(\to\mathrm{R})_\mathrm{K}$, $(\neg\mathrm{R})_\mathrm{K}$ or $(\forall\mathrm{R})_\mathrm{K}$ is rather complicated. Here we present the case for $(\to\mathrm{R})_\mathrm{K}$. It suffices to prove that

$$\begin{aligned}&\mathsf{GN4l}\text{ (hence }\mathbf{G}\mathsf{N}_4^3[\mathsf{d}]\mathsf{l}) \vdash E \to F \vee \forall \vec{x}(C \to D \vee (A \to B)),\\ &\quad E \to F \vee (A \to B \vee \forall \vec{x}(C \to D)) \Rightarrow E \to F \vee (A \to B) \vee \forall \vec{x}(C \to D) \qquad (\clubsuit)\end{aligned}$$

where $\vec{x}$ have their free occurrences only in C or D. Using this fact repeatedly, we can finish the proof.

$$(A \to B) \to \forall\vec{x}(C \to D), \forall\vec{x}(C \to D \lor (A \to B)) \Rightarrow \forall\vec{x}(C \to D) \quad (\diamondsuit)$$
$$\forall\vec{x}(C \to D) \to (A \to B), A \to B \lor \forall\vec{x}(C \to D) \Rightarrow A \to B \quad (\heartsuit)$$

The above two sequents are derivable in $\mathbf{G}\mathsf{N}_4^3[\mathsf{d}]\mathsf{l}$, as is easily verified.

$$\dfrac{\dfrac{}{((A \to B) \to \forall\vec{x}(C \to D)) \lor (\forall\vec{x}(C \to D) \to (A \to B))}\,(\mathrm{Li}) \quad \dfrac{(\diamondsuit) \quad (\heartsuit)}{\cdots}\,(\lor\mathrm{L})}{\dfrac{\forall\vec{x}(C \to D \lor (A \to B)), A \to B \lor \forall\vec{x}(C \to D) \Rightarrow (A \to B) \lor \forall\vec{x}(C \to D)}{(\clubsuit)}}\,(\mathrm{C})$$

For those logics with d, the proof is again almost the same; however, we cannot apply the first sublemma when $\mathcal{T}$ is derived by the rule $(\forall\mathrm{R})_{\mathrm{VC}}$. Instead we use the character of logics with D, that is:

$$\forall x(B \to C(x)) \cong_{\mathsf{N}_4^3\mathsf{d}[\mathsf{l}]} B \to \forall x C(x), \quad \forall x(B \lor C(x)) \cong_{\mathsf{N}_4^3\mathsf{d}[\mathsf{l}]} B \lor \forall x C(x),$$

where x has no free occurrences in B and $\cong$ denotes logical equivalence. Suppose $\mathbf{G}\mathsf{N}_4^3\mathsf{d}[\mathsf{l}] \vdash (\cdots[\Gamma \Rightarrow \Delta, A[z/x] \mid \cdots)^f$ as an induction hypothesis. Then $(\cdots[\Gamma \Rightarrow \Delta, A[z/x] \mid \cdots)^f$ is in the form

$$\forall\vec{x}\forall z(B_1 \to C_1 \lor (B_2 \to C_2 \lor (\cdots \lor (B_n \to C_n \lor A[z/x])\cdots))),$$

where z has no free occurrence in any B_i or C_i. We repeatedly apply the logical equivalences above and obtain

$$\begin{aligned}(\cdots[\Gamma \Rightarrow \Delta, A[z/x] \mid \cdots)^f &\equiv \forall\vec{x}\forall z(B_1 \to C_1 \lor (\cdots \lor (B_n \to C_n \lor A[z/x])\cdots)) \\ &\cong_{\mathsf{N}_4^3\mathsf{d}[\mathsf{l}]} \forall\vec{x}(B_1 \to C_1 \lor (\cdots \lor (B_n \to C_n \lor \forall x A)\cdots)) \\ &\equiv (\cdots[\Gamma \Rightarrow \Delta, \forall x A \mid \cdots)^f.\end{aligned}$$

Then the induction hypothesis yields $\mathbf{G}\mathsf{N}_4^3\mathsf{d}[\mathsf{l}] \vdash (\cdots[\Gamma \Rightarrow \Delta, \forall x A \mid \cdots)^f$. ∎

Using Kripke completeness of $\mathbf{T}\mathsf{N}_4^3[\mathsf{d}][\mathsf{l}]$ and the results on translations obtained above, we can conclude Kripke completeness of $\mathbf{G}\mathsf{N}_4^3[\mathsf{d}][\mathsf{l}]$.

THEOREM 3.14 (Kripke completeness of $\mathbf{G}\mathsf{N}_4^3[\mathsf{d}][\mathsf{l}]$)
If $\mathsf{N}_4^3[\mathsf{d}][\mathsf{l}] \models A$, then $\mathbf{G}\mathsf{N}_4^3[\mathsf{d}][\mathsf{l}] \vdash A$.

PROOF. Let $\mathbf{G}\mathsf{N}_4^3[\mathsf{d}][\mathsf{l}] \not\vdash A$, and $\mathcal{T}$ be a TS which consists of only its root; for $\mathsf{N}_4^3[\mathsf{l}]$ $\mathcal{T} :\equiv [\ \overset{\alpha}{\Rightarrow} A]$ where α is a nonempty finite set of variables which contains every free variable in A, and for $\mathsf{N}_4^3\mathsf{d}[\mathsf{l}]$ $\mathcal{T} :\equiv [\ \Rightarrow A]$. Then $\mathbf{T}\mathsf{N}_4^3[\mathsf{d}][\mathsf{l}] \not\vdash \mathcal{T}$ by Lemma 3.11, and by Lemma 3.9 there exists a counter model $\mathcal{M}$ for $\mathcal{T}$, which is also a counter model for A, i.e. $\mathcal{M} \not\models A$. ∎

4 Kripke completeness of GN3[d][l]o

4.1 Proof by an embedding of CI *– for* GN3[l]o

As stated in the introduction, Kripke completeness of **G**N3[d][l]o can hardly be shown by a simple application of the tree-sequent method. For **G**N3[d][l]o we present two different proofs in this section; given an unprovable formula A, we construct a counter $\mathsf{N3[d][l]o}$-model for it. Again as stated above, a counter model $\mathcal{M} = (M, \leq, U, I^+, I^-)$ we shall construct is such that: each $a \in M$ which is not omniscient has an immediate successor a_g which is maximal in $(M, \leq)$ and is omniscient. Since $\mathbf{G}\mathsf{N3[d][l]o}$ is stronger than $\mathbf{G}\mathsf{N3[d][l]}$, A is also unprovable in $\mathbf{G}\mathsf{N3[d][l]}$; hence by completeness in the previous section there exists a counter $\mathbf{G}\mathsf{N3[d][l]}$-model $\mathcal{M}'$. The problem is how to obtain an omniscient possible world a_g which will be associated to each node a of $\mathcal{M}'$. The first proof, presented in this subsection, utilizes an embedding of CI in $\mathbf{G}\mathsf{N3[l]o}$ for this purpose. As we point out later, this method is not applicable to $\mathbf{G}\mathsf{N3d[l]o}$.

First we present some derivations possible in $\mathbf{G}\mathsf{N}^3_4\mathsf{[d][l]o}$:

Lemma 4.1
In $\mathbf{G}\mathsf{N4o}$ (hence also in $\mathbf{G}\mathsf{N}^3_4\mathsf{[d][l]o}$), we can make the following derivations:

$$\frac{}{\neg{\sim}A, \neg A \Rightarrow}\ (\mathrm{Om1}) \qquad \frac{{\sim}A, \Gamma \Rightarrow}{\neg A, \Gamma \Rightarrow}\ (\mathrm{Om2}) \qquad \frac{}{A \to B \Rightarrow \neg\neg({\sim}A \vee B)}\ (\mathrm{Om3})$$

Proof. (Om1)

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(Om)}
\UnaryInfC{$\Rightarrow \neg\neg(A \vee {\sim}A)$}
\AxiomC{}
\RightLabel{(Id)}
\UnaryInfC{$A \Rightarrow A$}
\doubleLine
\RightLabel{(W)($\neg$L)}
\UnaryInfC{$A, \neg{\sim}A, \neg A \Rightarrow$}
\AxiomC{}
\RightLabel{(Id)}
\UnaryInfC{${\sim}A \Rightarrow {\sim}A$}
\doubleLine
\UnaryInfC{${\sim}A, \neg{\sim}A, \neg A \Rightarrow$}
\RightLabel{($\vee$L)}
\BinaryInfC{$A \vee {\sim}A, \neg{\sim}A, \neg A \Rightarrow$}
\doubleLine
\RightLabel{($\neg$R)($\neg$L)}
\UnaryInfC{$\neg\neg(A \vee {\sim}A), \neg{\sim}A, \neg A \Rightarrow$}
\RightLabel{(C)}
\BinaryInfC{$\neg{\sim}A, \neg A \Rightarrow$}
\end{prooftree}
$$

(Om2)

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(Om1)}
\UnaryInfC{$\neg{\sim}A, \neg A \Rightarrow$}
\RightLabel{($\neg$R)}
\UnaryInfC{$\neg A \Rightarrow \neg\neg{\sim}A$}
\AxiomC{}
\RightLabel{hyp.}
\UnaryInfC{${\sim}A, \Gamma \Rightarrow$}
\doubleLine
\RightLabel{($\neg$R, L)}
\UnaryInfC{$\neg\neg{\sim}A, \Gamma \Rightarrow$}
\RightLabel{(C)}
\BinaryInfC{$\neg A, \Gamma \Rightarrow$}
\end{prooftree}
$$

(Om3)

$$
\begin{prooftree}
\AxiomC{(Om)}
\AxiomC{}
\RightLabel{(Id)}
\UnaryInfC{$A \Rightarrow A$}
\AxiomC{}
\RightLabel{(Id)}
\UnaryInfC{$B \Rightarrow B$}
\doubleLine
\RightLabel{($\to$L)}
\BinaryInfC{$A \to B, A \Rightarrow {\sim}A \vee B$}
\RightLabel{($\neg$L)}
\UnaryInfC{$A \to B, A, \neg({\sim}A \vee B) \Rightarrow$}
\AxiomC{}
\RightLabel{(Id)}
\UnaryInfC{${\sim}A \Rightarrow {\sim}A$}
\doubleLine
\UnaryInfC{$A \to B, {\sim}A \Rightarrow {\sim}A \vee B$}
\RightLabel{($\neg$L)}
\UnaryInfC{$A \to B, {\sim}A, \neg({\sim}A \vee B) \Rightarrow$}
\RightLabel{($\vee$L)}
\BinaryInfC{$A \to B, A \vee {\sim}A, \neg({\sim}A \vee B) \Rightarrow$}
\doubleLine
\RightLabel{($\neg$R)($\neg$L)}
\UnaryInfC{$A \to B, \neg\neg(A \vee {\sim}A), \neg({\sim}A \vee B) \Rightarrow$}
\RightLabel{($\neg$R)}
\UnaryInfC{$A \to B, \neg\neg(A \vee {\sim}A) \Rightarrow \neg\neg({\sim}A \vee B)$}
\RightLabel{(C)}
\BinaryInfC{$A \to B \Rightarrow \neg\neg({\sim}A \vee B)$}
\end{prooftree}
$$

∎

In fact (Om) is equivalent to (Om3): take $B \equiv A$ in (Om3), and use (C) with $\Rightarrow A \to A$. Moreover, where (Fal) is available (i.e. in $\mathsf{N3[d][l]o}$), (Om1), (Om2) and (Om) are all equivalent. Since the proof above shows that (Om)$\Rightarrow$(Om1)$\Rightarrow$(Om2), it suffices to show (Om2)⇒(Om):

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\sim A, \sim\sim A \Rightarrow}\,(\mathrm{Fal})}{\sim(A \vee \sim A) \Rightarrow}\,(\sim\vee\mathrm{L})}{\neg(A \vee \sim A) \Rightarrow}\,(\mathrm{Om2})}{\Rightarrow \neg\neg(A \vee \sim A)}\,(\neg\mathrm{R})$$

Now we make some remarks. All of $\vee$, $\to$ and $\exists$ can be introduced as defined symbols in Cl in terms of $\wedge$, $\neg$ and $\forall$; however in this paper we assume the logical connectives of Cl include $\to$, to make correspondence with $\mathsf{N3[d][l]o}$. In what follows a *structure* in Tarski-type semantics for Cl is often called a Cl-*model*. Cl-model can be regarded as an Int-model consisting of only one possible world. We denote the domain of a Cl-model $\mathcal{A}$ by $|\mathcal{A}|$, and the interpretation of a predicate symbol p by $p_{\mathcal{A}}$. We will often denote a formula of Cl by $A_{\neg}$ in order to make clear that A contains no $\sim$'s. $A_{\sim\neg}$ is a formula of $\mathsf{N3}$ which is obtained by replacing some (possibly all or no) $\neg$'s in A by $\sim$. The subscript $_{\sim\neg}$ is also often used to make clear that it is a formula of $\mathsf{N3}$. Then $A_{\neg}$ is a formula obtained from $A_{\sim\neg}$ by replacing every $\sim$ therein by $\neg$.

Lemma 4.2
Let $\mathcal{A}$ be a Cl-model, and $\mathcal{M}_{\mathcal{A}} = (M, \leq, U, I^+, I^-)$ an $\mathsf{N3[d][l]o}$-model defined by $M := \{0\}$, $U(0) := |\mathcal{A}|$, $p^{I^+(0)} := p_{\mathcal{A}}$ and $p^{I^-(0)} := U(0)^m \setminus p_{\mathcal{A}}$. Then for an arbitrary closed formula $A_{\neg}$ of Cl, $\mathcal{A} \models A_{\neg}$ iff $0 \models^+ A_{\sim\neg}$ (or equivalently $\mathcal{M} \models A_{\sim\neg}$). Moreover, $\mathcal{A} \not\models A_{\neg}$ iff $0 \models^- A_{\sim\neg}$.

Proof. It is easily verified by induction that, for an omniscient possible world a (i.e. $p^{I^+(a)} \cup p^{I^-(a)} = U(a)^m$) which is maximal in $(M, \leq)$ and a closed formula A of $\mathsf{N3}$, exactly one of $a \models^+ A$ or $a \models^- A$ holds; hence the first *iff* instantly yields the second one.

The first *iff* is shown by induction on the construction of A; we present just one step case. Assume $A_{\neg} \equiv \neg B_{\neg}$. Then $A_{\sim\neg} \equiv \neg B_{\sim\neg}$ or $\sim B_{\sim\neg}$. $\mathcal{A} \models A_{\neg}$ iff $\mathcal{A} \not\models B_{\neg}$, which is equivalent to $0 \not\models^+ B_{\sim\neg}$ by the induction hypothesis. This is equivalent to $0 \models^+ \neg B_{\sim\neg}$ since 0 has no successors, and is also equivalent to $0 \models^+ \sim B_{\sim\neg}$ by the above argument. ∎

Using the lemmas above, we can introduce the main lemma in this subsection:

Lemma 4.3 (Embedding of **LK** in **G**$\mathsf{N3[d][l]o}$)
The following are all equivalent:

1. $\mathbf{LK} \vdash \Gamma_{\neg} \Rightarrow \Delta_{\neg}$;
2. $\mathbf{G}\mathsf{N3[d][l]o} \vdash \Gamma_{\sim\neg}, \sim\Delta_{\sim\neg} \Rightarrow$;
3. $\mathbf{G}\mathsf{N3[d][l]o} \vdash \Gamma_{\sim\neg}, \neg\Delta_{\sim\neg} \Rightarrow$.

Proof. [2 ⇔ 3] is obvious by (Om2) of Lemma 4.1 and $\mathbf{G}\mathsf{N3[d][l]o} \vdash \sim A \Rightarrow \neg A$.

[2 ⇒ 1] is shown semantically. Let $\mathcal{A}$ an arbitrary Cl-model. Kripke soundness of $\mathbf{G}\mathsf{N3[d][l]o}$ yields $\mathcal{M}_{\mathcal{A}} \models \Gamma_{\sim\neg}, \sim\Delta_{\sim\neg} \Rightarrow$, and by Lemma 4.2 we have $\mathcal{A} \models \Gamma_{\neg}, \neg\Delta_{\neg} \Rightarrow$, hence $\mathcal{A} \models \Gamma_{\neg} \Rightarrow \Delta_{\neg}$. Then by completeness of **LK**, $\mathbf{LK} \vdash \Gamma_{\neg} \Rightarrow \Delta_{\neg}$.

[1 ⇒ 2] is by the induction on derivation in **LK**. Note that we can restrict initial sequents (Id) of **LK** to the form $p(\vec{x}) \Rightarrow p(\vec{x})$, an atomic formula in each side. In the following $A_{\sim\neg}$ is denoted by A' to avoid proofs being too wide. The derivation rule of **LK** applied at last is presented in the left, and the corresponding proof figure in $\mathbf{GN3}[\mathsf{d}][\mathsf{l}]\mathsf{o}$ is in the right:

$$\frac{}{p(\vec{x}) \Rightarrow p(\vec{x})}\ (\mathrm{Id}) \qquad \frac{}{p(\vec{x}), {\sim}p(\vec{x}) \Rightarrow}\ (\mathrm{Fal})$$

The case of (W) or a rule introducing $\wedge$ or $\forall$ is easy.

$$\frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta}\ (\neg\mathrm{L}) \qquad \dfrac{\dfrac{\text{ind. hyp.}}{\Gamma', {\sim}\Delta', {\sim}A' \Rightarrow}}{\neg A', \Gamma', {\sim}\Delta' \Rightarrow}\ (\mathrm{Om2}) \quad \text{and} \quad \dfrac{\text{ind. hyp.}}{{\sim}A', \Gamma', {\sim}\Delta' \Rightarrow}$$

$$\frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A}\ (\neg\mathrm{R}) \qquad \dfrac{\dfrac{\text{ind. hyp.}}{\Gamma', {\sim}\Delta', A' \Rightarrow}}{\Gamma', {\sim}\Delta', {\sim}\neg A' \ \text{(or } {\sim}{\sim}A') \Rightarrow}\ ({\sim}\neg\mathrm{L}) \text{ or } ({\sim}{\sim}\mathrm{L})$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \to B, \Gamma \Rightarrow \Delta}\ (\to\mathrm{L}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\text{ind. hyp.}}{{\sim}A', \Gamma', {\sim}\Delta' \Rightarrow} \quad \dfrac{\text{ind. hyp.}}{B', \Gamma', {\sim}\Delta' \Rightarrow}}{{\sim}A' \vee B', \Gamma', {\sim}\Delta' \Rightarrow}\ (\vee\mathrm{L})}{\neg\neg({\sim}A' \vee B'), \Gamma', {\sim}\Delta' \Rightarrow}\ (\neg\mathrm{R})(\neg\mathrm{L})}{A' \to B', \Gamma', {\sim}\Delta' \Rightarrow}\ (\mathrm{Om3})\ (\mathrm{C})$$

$$\frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B}\ (\to\mathrm{R}) \qquad \dfrac{\dfrac{\text{ind. hyp.}}{\Gamma', {\sim}\Delta', A', {\sim}B' \Rightarrow}}{\Gamma', {\sim}\Delta', {\sim}(A' \to B') \Rightarrow}\ ({\sim}{\to}\mathrm{L})$$

∎

Corollary 4.4
If $\mathbf{GN3}[\mathsf{d}][\mathsf{l}]\mathsf{o} \nvdash \Gamma_{\sim\neg} \Rightarrow \Delta_{\sim\neg}$, then $\mathbf{LK} \nvdash \Gamma_{\neg} \Rightarrow$.

We prove another several lemmas needed later.

Lemma 4.5
Let $\Gamma \Rightarrow \Delta$ be an infinite sequent of **LK** which is *consistent*, that is, for every finite subsets $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ we have $\mathbf{LK} \nvdash \Gamma' \Rightarrow \Delta'$. Then there exists a Cl-model $\mathcal{A}$ such that:

1. the set of all variables can be embedded in $|\mathcal{A}|$;
2. $\mathcal{A} \models A[\underline{\vec{x}}/\vec{x}]$ for every $A \in \Gamma$;
3. $\mathcal{A} \not\models B[\underline{\vec{x}}/\vec{x}]$ for every $B \in \Delta$.

Proof. First increase variables twofold, adding x'_i for each original variable x_i. Then an infinite number of variables $x'_1, x'_2, \ldots$ have no occurrence in $\Gamma \Rightarrow \Delta$, which is an (infinite) sequent of the original formal language. Now with the condition that $\Gamma \Rightarrow \Delta$ is consistent we can extend it to a **LK**-saturated infinite sequent $\widetilde{\Gamma} \Rightarrow \widetilde{\Delta}$, which induces a Cl-model $\mathcal{A}$ where $|\mathcal{A}| = \{x_1, x'_1, x_2, x'_2, \ldots\}$ and the conditions 2. and 3. are satisfied. ∎

Definition 4.6 (Tree-sequent system **TN3**[l]o)
A *tree-sequent* of **TN3**[l]o is the same as that of **TN3**[l], respectively, and initial sequents and derivation rules of **TN3**[l]o are those of **TN3**[l] plus

$$\frac{}{\cdots[\overset{\alpha}{\Rightarrow} \neg\neg(A \vee {\sim}A) \mid \cdots}\ (\mathrm{Om}), \qquad \frac{\cdots[\Gamma \overset{\alpha}{\Rightarrow} \Delta, A \mid \cdots \quad \cdots[A, \Sigma \overset{\alpha}{\Rightarrow} \Pi \mid \cdots}{\cdots[\Gamma, \Sigma \overset{\alpha}{\Rightarrow} \Delta, \Pi \mid \cdots}\ (\mathrm{C}).$$

The cut rule (C) is necessary in the proof of the following lemma. If it is the case (we do not know whether or not) that (C) can be eliminated in **GN3**[l]o, then so is (C) of **TN3**[l]o.

Lemma 4.7
Let $\mathcal{T}$ be a tree-sequent of **TN3**[l]o. If $\mathcal{T}$ has a node $(a : \Gamma_a \overset{\alpha}{\Rightarrow} \Delta_a)$ such that **GN3**[l]o $\vdash \Gamma_a \Rightarrow \Delta_a$, then **TN3**[l]o $\vdash \mathcal{T}$.

Proof. By induction on derivation in **GN3**[l]o. For initial sequents (Li) of **GN3**lo,

$$\frac{\mathcal{T}_1 \quad \cdots \quad \dfrac{\mathcal{T}_{i,1} \quad \cdots \quad \mathcal{T}_{i,k+1}}{\mathcal{T}_i}\,(\to\mathrm{R})_{\mathrm{K}} \quad \cdots \quad \mathcal{T}_k}{\cdots \mid \overset{\alpha_1}{\Rightarrow} (A \to B) \vee (B \to A) \mid \Gamma_2 \overset{\alpha_2}{\Rightarrow} \Delta_2 \mid \cdots \mid \Gamma_k \overset{\alpha_k}{\Rightarrow} \Delta_k}\ (\to\mathrm{R})_{\mathrm{K}}, (\vee\mathrm{R})$$

where

$$\mathcal{T}_i \equiv \cdots \mid \overset{\alpha_1}{\Rightarrow} B \to A \mid \cdots \mid \Gamma_i \overset{\alpha_i}{\Rightarrow} \Delta_i \mid A \overset{\emptyset}{\Rightarrow} B \mid \cdots \mid \Gamma_k \overset{\alpha_k}{\Rightarrow} \Delta_k, \qquad i \in [1,k]$$
$$\mathcal{T}_{i,j} \equiv \cdots \mid \overset{\alpha_1}{\Rightarrow} \ \mid \cdots \mid A \overset{\emptyset}{\Rightarrow} B \mid \cdots \mid B \overset{\emptyset}{\Rightarrow} A \mid \cdots \mid \Gamma_k \overset{\alpha_k}{\Rightarrow} \Delta_k. \qquad j \in [1,k+1]$$

Every $\mathcal{T}_{i,j}$ is provable in **TN3**lo by (Id), (W) and (D).

It is easy to check the cases for the other initial sequents or the rules which admit side formulas in its conclusion, i.e. other than $(\ \ )_{\mathrm{S}}$. For the rule (C) of **GN3**[l]o we need (C) of **TN3**[l]o.

We present only the case for $(\to\mathrm{R})_{\mathrm{S}}$; for $(\neg\mathrm{R})_{\mathrm{S}}$ and $(\forall\mathrm{R})_{\mathrm{S,\ VC}}$ proofs are just the same.

For $\dfrac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \to B}\ (\to\mathrm{R})_{\mathrm{S}}$, we have $\dfrac{\dfrac{\dfrac{\text{ind. hyp.}}{\cdots[\ \overset{\alpha}{\Rightarrow}\ \mid \cdots[A, \Gamma \overset{\emptyset}{\Rightarrow} B] \cdots] \cdots}}{\cdots[\Gamma \overset{\alpha}{\Rightarrow}\ \mid \cdots[A \overset{\emptyset}{\Rightarrow} B] \cdots] \cdots}\,(\mathrm{D})}{\cdots[\Gamma \overset{\alpha}{\Rightarrow} A \to B \mid \cdots}\ (\to\mathrm{R})_{\mathrm{T}}$ in **GN3**o.

In **GN3**lo, the proof is similar; first drop Γ, then apply $(\to\mathrm{R})_{\mathrm{K}}$. ∎

We define the *formulaic translation* of a TS $\mathcal{T}$ of **TN3**[l]o, denoted by $\mathcal{T}^f$, as that of **TN3**[l]: $[\Gamma \overset{\alpha}{\Rightarrow} \Delta \mid \mathcal{T}_1 \ldots \mathcal{T}_m]^f :\equiv \forall\overrightarrow{\alpha}\big((\bigwedge\Gamma) \to (\bigvee\Delta) \vee \mathcal{T}_1^f \vee \cdots \vee \mathcal{T}_m^f\big)$.

Lemma 4.8
If **TN3**[l]o $\vdash \mathcal{T}$, then **GN3**[l]o $\vdash \mathcal{T}^f$.

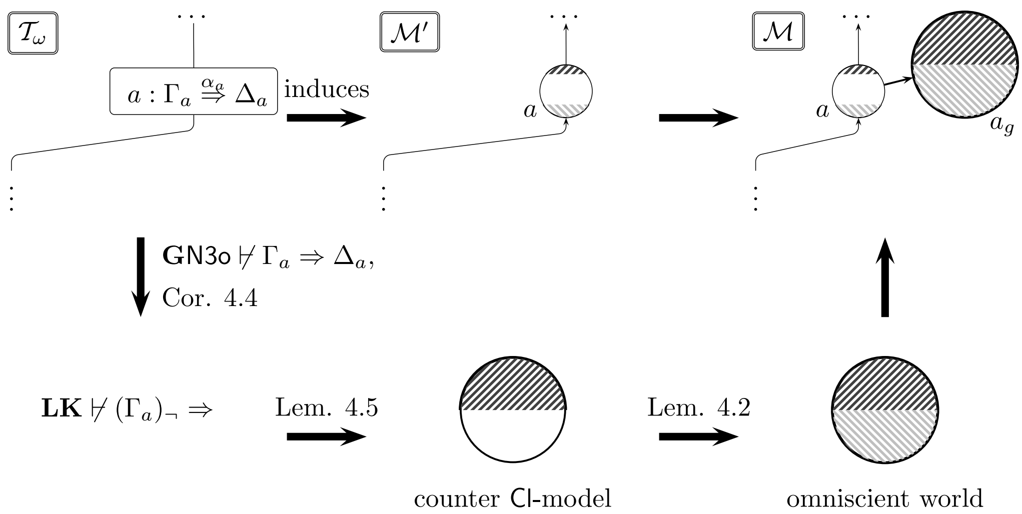
Proof. Similar to that of Lemma 3.11. ∎

Theorem 4.9 (Kripke completeness of **GN3**[l]o)
If N3[l]o $\models A$, then **GN3**[l]o $\vdash A$.

Proof. Let **G**N3[l]o $\not\vdash A$. We shall construct a counter N3[l]o-model for A in the following way. The case of **G**N3o is presented ahead of **G**N3lo.

For **G**N3o, first we construct a counter N3-model $\mathcal{M}'$ by extending the TS $\mathcal{T} :\equiv [\overset{\alpha}{\Rightarrow} A]$ into a **T**N3-saturated infinite TS $\mathcal{T}_\omega$, as in the proof of Kripke completeness of **T**N3. By Lemma 4.8 $\mathcal{T}$ is unprovable in **T**N3o, and obviously the procedures of extension preserve unprovability of the TS; hence every finite sub-tree-sequent of $\mathcal{T}_\omega$ is unprovable in **T**N3o. Now Lemma 4.7 yields that for each node $(a : \Gamma_a \overset{\alpha_a}{\Rightarrow} \Delta_a)$ of $\mathcal{T}_\omega$, an infinite sequent $\Gamma_a \Rightarrow \Delta_a$ is consistent in **G**N3o, i.e. every finite subsequent of $\Gamma_a \Rightarrow \Delta_a$ is unprovable in **G**N3o.

Next we associate an omniscient possible world a_g to each node a of N3-model $\mathcal{M}'$ and obtain a counter N3o-model $\mathcal{M}$. Such omniscient worlds are obtained as follows: the fact stated above that every $\Gamma_a \Rightarrow \Delta_a$ (which is a node of $\mathcal{T}_\omega$) is consistent in **G**N3o and Corollary 4.4 yield that an infinite sequent $(\Gamma_a)_\neg \Rightarrow$ is consistent in **LK**. Now by Lemma 4.5 we obtain a counter Cl-model for $(\Gamma_a)_\neg \Rightarrow$ with a sufficiently large domain, which in turn induces an omniscient possible world a_g of N3o-model by Lemma 4.2. For each node a of $\mathcal{M}'$, we refer to the node $(a : \Gamma_a \overset{\alpha_a}{\Rightarrow} \Delta_a)$ of $\mathcal{T}_\omega$ which induces it, then add an omniscient world a_g obtained as above (i.e. by the fact that $(\Gamma_a)_\neg \Rightarrow$ is consistent in **LK**) as an immediate successor of a (see Fig. 4.1).



For **G**N3lo, the proof is just the same. First extend the TS $\mathcal{T} :\equiv [\overset{\alpha}{\Rightarrow} A]$ into a **T**N3l-saturated infinite TS $\mathcal{T}_\omega$, just as in the proof of completeness of **T**N3l. Let $\mathcal{M}'$ be a counter N3l-model for A induced by $\mathcal{T}_\omega$. Let us denote $\mathcal{T}_\omega$ by $\Gamma_1 \overset{\alpha_1}{\Rightarrow} \Delta_1 \mid \Gamma_2 \overset{\alpha_2}{\Rightarrow} \Delta_2 \mid \cdots$ and $\Gamma_\omega := \bigcup_i \Gamma_i$, $\Delta_\omega := \bigcup_i \Delta_i$. Then by Lemma 4.7 each $\Gamma_i \Rightarrow \Delta_i$ is consistent in **G**N3lo, hence $\Gamma_\omega \Rightarrow \Delta_\omega$ is also. Corollary 4.4 yields that $(\Gamma_\omega)_\neg \Rightarrow$ is consistent in **LK**, and induces an omniscient world g in the same way as above. By adding g to $\mathcal{M}'$ as the maximum, we obtain a counter N3lo-model $\mathcal{M}$ for A.

It remains to be proved that $\mathcal{M}$ is certainly an N3[l]o-model, and $\mathcal{M} \not\models A$. In the following only the case for **G**N3o is in consideration; for **G**N3lo the proof is just the same.

Since $U(a)$ is the set of the available variables and $U(a_g)$ is $\{x_1, x_1', x_2, x_2', \dots\}$ by Lemma 4.5, $U(a) \subseteq U(a_g)$. If $\vec{x} \in p^{I^+(a)}$ (or $\vec{x} \in p^{I^-(a)}$), then by the definition of $\mathcal{M}'$, $p(\vec{x}) \in \Gamma_a$ (or $\sim p(\vec{x}) \in \Gamma_a$) where $(a : \Gamma_a \overset{\alpha_a}{\Rightarrow} \Delta_a)$ is the corresponding node of $\mathcal{T}_\omega$; hence $p(\vec{x})$ is in $(\Gamma_a)_\neg$ (or not in $(\Gamma_a)_\neg$, respectively). By the construction of a_g, i.e. by Lemma 4.2, $\vec{x} \in p^{I^+(a_g)}$ (or $\vec{x} \in p^{I^-(a_g)}$). It is easy to verify that $\mathcal{M}$ satisfies the other conditions of $\mathsf{N3o}$-models.

In order to show $\mathcal{M} \not\models A$, it suffices to show that: for each node $(a : \Gamma_a \overset{\alpha_a}{\Rightarrow} \Delta_a)$ of $\mathcal{T}_\omega$, $B \in \Gamma_a$ (or $B \in \Delta_a$) implies that $a \models^+ B[\underline{\vec{x}}/\vec{x}]$ (or $a \not\models^+ B[\underline{\vec{x}}/\vec{x}]$, respectively) holds also in $\mathcal{M}$ (it holds in $\mathcal{M}'$ by definition). The proof is easy by induction; the point is that the addition of a_g does not affect such conditions as that for $a \not\models^+ \neg B$, that is, "there exists *at least one* world $b \geq a$ such that $b \models^+ B$". ∎

Remark 4.10

The method above is not applicable to $\mathbf{G}\mathsf{N3d[l]o}$. The problem lies in the proof of Lemma 4.5; we cannot construct a counter CI-model without infinitely many additional variables.

4.2 *Proof by TSg method – for* $\mathbf{G}\mathsf{N3[d][l]o}$

The method in the previous subsection is not applicable to $\mathbf{G}\mathsf{N3d[l]o}$; that is because in the method we must construct omniscient possible worlds *after we have finished* extending an unprovable formula into an *infinite* TS. To avoid this problem, it seems reasonable to consider a method in which we extend both the TS and the seeds of omniscient possible worlds *simultaneously*; this is the idea of the proofs in this subsection.

In the proofs here we make usage of a *tree-sequent with guardians*, TSg in short. Roughly speaking, a TSg is a TS each node of which has an extra sequent and a finite set of variables; we denote a node a which is associated with $\Gamma_a \overset{\alpha_a}{\Rightarrow} \Delta_a$ and an extra $\Sigma_a \overset{\beta_a}{\Rightarrow} \Pi_a$ by $(a : \Gamma_a \overset{\alpha_a}{\Rightarrow} \Delta_a \uparrow \Sigma_a \overset{\beta_a}{\Rightarrow} \Pi_a)$. An extra sequent $\Sigma_a \overset{\beta_a}{\Rightarrow} \Pi_a$ (presented in the right) is said to be the *guardian* of a, and is the seed of an omniscient world a_g which will be assigned to a. The TS-translation of a TSg, which we shall define later, will make clear the idea of TSg.

Using TSg's, we can prove Kripke completeness in the same way as that of $\mathbf{G}\mathsf{N}^3_4\mathsf{[d][l]}$. Completeness of the corresponding TSg system is shown by extending an unprovable TSg into a saturated one, which induces a counter model. Then using the formulaic translation of a TSg, we can conclude completeness of the Gentzen-style sequent system.

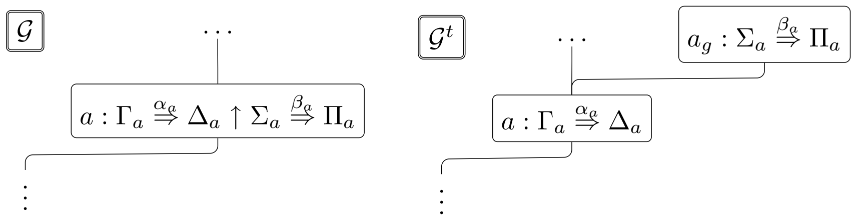
Now we proceed to precise arguments, defining TSg systems $\mathbf{Tg}\mathsf{N3[d][l]o}$.

Definition 4.11 (TSg of $\mathbf{Tg}\mathsf{N3[d][l]o}$)

A *tree-sequent with guardians* $\mathcal{G}$ of $\mathbf{Tg}\mathsf{N3o}$ is a finite tree, each node a of which is associated with two sequents of $\mathbf{G}\mathsf{N3o}$ and finite sets of variables, denoted by $(a : \Gamma_a \overset{\alpha_a}{\Rightarrow} \Delta_a \uparrow \Sigma_a \overset{\beta_a}{\Rightarrow} \Pi_a)$, and satisfies the following condition:

Let $\mathcal{G}^t$ be a labelled tree obtained by, for each node a, omitting the guardian

$\Sigma_a \overset{\beta_a}{\Rightarrow} \Pi_a$ and adding a new immediate successor $(a_g : \Sigma_a \overset{\beta_a}{\Rightarrow} \Pi_a)$ to a:



Then $\mathcal{G}^t$ is a tree-sequent of **T**N3; that is, $\mathcal{G}^t$ satisfies the conditions as to availability of variables.

We shall call $\Gamma_a \overset{\alpha_a}{\Rightarrow} \Delta_a$ the *sequent* (or *left sequent*) of a, and $\Sigma_a \overset{\beta_a}{\Rightarrow} \Pi_a$ the *guardian* (or *guardian sequent*). The TS $\mathcal{G}^t$ is called the *TS-translation* of a TSg $\mathcal{G}$. A variable x is said to be *l-available at* a node a of $\mathcal{G}$ if it is available at a in $\mathcal{G}^t$. x is *g-available at* a if it is available at a_g in $\mathcal{G}^t$.

For variants, the definition is modified as follows:

$\boxed{\mathsf{d}}$ A TSg is just a labelled tree each node of which is associated with two sequents.

$\boxed{\mathsf{l}}$ A TSg $\mathcal{G}$ and its TS-translation $\mathcal{G}^t$ are simpler:

$$\mathcal{G} \equiv \Gamma_1 \overset{\alpha_1}{\Rightarrow} \Delta_1 \mid \Gamma_2 \overset{\alpha_2}{\Rightarrow} \Delta_2 \mid \cdots \mid \Gamma_k \overset{\alpha_k}{\Rightarrow} \Delta_k \uparrow \Sigma \overset{\beta}{\Rightarrow} \Pi,$$
$$\mathcal{G}^t \equiv \Gamma_1 \overset{\alpha_1}{\Rightarrow} \Delta_1 \mid \Gamma_2 \overset{\alpha_2}{\Rightarrow} \Delta_2 \mid \cdots \mid \Gamma_k \overset{\alpha_k}{\Rightarrow} \Delta_k \mid \Sigma \overset{\beta}{\Rightarrow} \Pi,$$

for **Tg**N3lo. For **Tg**N3dlo, omit the sets of variables. Here $\Sigma \overset{\beta}{\Rightarrow} \Pi$ is the only guardian of $\mathcal{G}$, and denoted by g.

For example, it follows immediately from the definition above that $\alpha_a \cap \beta_a = \emptyset$ for each node $(a : \Gamma_a \overset{\alpha_a}{\Rightarrow} \Delta_a \uparrow \Sigma_a \overset{\beta_a}{\Rightarrow} \Pi_a)$ of a TSg.

Omitting the condition which involves availability of variables, we obtain the definition of *pre-TSg* of **Tg**N3[l]o, pTSg in short. Again pTSg's emerge as subtrees of TSg's.

In what follows we denote a TSg in the same way as we do a TS, using [,] and |.

Definition 4.12 (TSg system **Tg**N3[d][l]o)
Initial TSg's and derivation rules of **Tg**N3o are presented below, ahead of those for its variants, followed by some remarks:

$$\frac{}{\cdots [A \overset{\alpha}{\Rightarrow} A \uparrow \Sigma \overset{\beta}{\Rightarrow} \Pi \mid \cdots}\ (\mathrm{Id}) \qquad \frac{}{\cdots [A, {\sim}A \overset{\alpha}{\Rightarrow} \quad \uparrow \Sigma \overset{\beta}{\Rightarrow} \Pi \mid \cdots}\ (\mathrm{Fal})$$

$$\frac{}{\cdots [\Gamma \overset{\alpha}{\Rightarrow} \Delta \uparrow A \overset{\alpha}{\Rightarrow} A \mid \cdots}\ (\mathrm{gId}) \qquad \frac{}{\cdots [\Gamma \overset{\alpha}{\Rightarrow} \Delta \uparrow A, {\sim}A \overset{\alpha}{\Rightarrow} \quad \mid \cdots}\ (\mathrm{gFal})$$

$$\frac{}{\cdots [\Gamma \overset{\alpha}{\Rightarrow} \Delta \uparrow \overset{\beta}{\Rightarrow} A \vee {\sim}A \mid \cdots}\ (\mathrm{gOm})$$

TgN3o has all the derivation rules of **T**N3, which are applied only to left sequents:

(W), (D), ($\wedge$L), ($\wedge$R), ($\to$L), $(\to\mathrm{R})_{\mathrm{T}}$, ($\neg$L), $(\neg\mathrm{R})_{\mathrm{T}}$, ($\forall$L), $(\forall\mathrm{R})_{\mathrm{T}}$, ($\sim\wedge$L), ($\sim\wedge$R), ($\sim\to$L), ($\sim\to$R), ($\sim\neg$L), ($\sim\neg$R), ($\sim\sim$L), ($\sim\sim$R), $(\sim\forall\mathrm{L})_{\mathrm{VC}}$ and ($\sim\forall$R).

For example, (W) of **Tg**N3o is: $$\frac{\cdots[\Gamma \overset{\alpha}{\Rightarrow} \Delta \uparrow \Sigma \overset{\beta}{\Rightarrow} \Pi \mid \cdots}{\cdots[\Gamma', \Gamma \overset{\alpha}{\Rightarrow} \Delta, \Delta' \uparrow \Sigma \overset{\beta}{\Rightarrow} \Pi \mid \cdots} \ (\mathrm{W})$$

(S) is a structural rule unique to TSg's; considering the TS-translations, (S) can be regarded as a special case of (D).

$$\frac{\cdots[\Gamma \overset{\alpha}{\Rightarrow} \Delta \uparrow A, \Sigma \overset{\beta}{\Rightarrow} \Pi \mid \cdots}{\cdots[A, \Gamma \overset{\alpha}{\Rightarrow} \Delta \uparrow \Sigma \overset{\beta}{\Rightarrow} \Pi \mid \cdots} \ (\text{Slide, S})$$

The rules (g$\cdots$) involve only guardians but no left sequents; for them we indicate only guardians.

$$\frac{\cdots \uparrow \Sigma \overset{\beta}{\Rightarrow} \Pi \mid \cdots}{\cdots \uparrow \Sigma', \Sigma \overset{\beta}{\Rightarrow} \Pi, \Pi' \mid \cdots} \ (\mathrm{gW}) \qquad \frac{\cdots \uparrow \Sigma \overset{\beta}{\Rightarrow} \Pi, A \mid \cdots \quad \cdots \uparrow A, \Sigma \overset{\beta}{\Rightarrow} \Pi \mid \cdots}{\cdots \uparrow \Sigma \overset{\beta}{\Rightarrow} \Pi \mid \cdots} \ (\mathrm{gC})$$

$$\frac{\cdots \uparrow A, B, \Sigma \overset{\beta}{\Rightarrow} \Pi \mid \cdots}{\cdots \uparrow A \wedge B, \Sigma \overset{\beta}{\Rightarrow} \Pi \mid \cdots} \ (\mathrm{g}\wedge\mathrm{L}) \qquad \frac{\cdots \uparrow \Sigma \overset{\beta}{\Rightarrow} \Pi, A \mid \cdots \quad \cdots \uparrow \Sigma \overset{\beta}{\Rightarrow} \Pi, B \mid \cdots}{\cdots \uparrow \Sigma \overset{\beta}{\Rightarrow} \Pi, A \wedge B \mid \cdots} \ (\mathrm{g}\wedge\mathrm{R})$$

$$\frac{\cdots \uparrow \Sigma \overset{\beta}{\Rightarrow} \Pi, A \mid \cdots \quad \cdots \uparrow B, \Pi \overset{\beta}{\Rightarrow} \Sigma \mid \cdots}{\cdots \uparrow A \to B, \Sigma \overset{\beta}{\Rightarrow} \Pi \mid \cdots} \ (\mathrm{g}{\to}\mathrm{L}) \qquad \frac{\cdots \uparrow A, \Sigma \overset{\beta}{\Rightarrow} \Pi, B \mid \cdots}{\cdots \uparrow \Sigma \overset{\beta}{\Rightarrow} \Pi, A \to B \mid \cdots} \ (\mathrm{g}{\to}\mathrm{R})$$

$$\frac{\cdots \uparrow \Sigma \overset{\beta}{\Rightarrow} \Pi, A \mid \cdots}{\cdots \uparrow \neg A, \Sigma \overset{\beta}{\Rightarrow} \Pi \mid \cdots} \ (\mathrm{g}\neg\mathrm{L}) \qquad \frac{\cdots \uparrow A, \Sigma \overset{\beta}{\Rightarrow} \Pi \mid \cdots}{\cdots \uparrow \Sigma \overset{\beta}{\Rightarrow} \Pi, \neg A \mid \cdots} \ (\mathrm{g}\neg\mathrm{R})$$

$$\frac{\cdots \uparrow A[y/x], \Sigma \overset{\beta}{\Rightarrow} \Pi \mid \cdots}{\cdots \uparrow \forall x A, \Sigma \overset{\beta}{\Rightarrow} \Pi \mid \cdots} \ (\mathrm{g}\forall\mathrm{L}) \qquad \frac{\cdots \uparrow \Sigma \overset{\beta\cup\{z\}}{\Rightarrow} \Pi, A[z/x] \mid \cdots}{\cdots \uparrow \Sigma \overset{\beta}{\Rightarrow} \Pi, \forall x A \mid \cdots} \ (\mathrm{g}\forall\mathrm{R})_{\mathrm{VC}}$$

$$\frac{\cdots \uparrow {\sim}A, \Sigma \overset{\beta}{\Rightarrow} \Pi \mid \cdots \quad \cdots \uparrow {\sim}B, \Sigma \overset{\beta}{\Rightarrow} \Pi \mid \cdots}{\cdots \uparrow {\sim}(A \wedge B), \Sigma \overset{\beta}{\Rightarrow} \Pi \mid \cdots} \ (\mathrm{g}{\sim}\wedge\mathrm{L})$$

$$\frac{\cdots \uparrow \Sigma \overset{\beta}{\Rightarrow} \Pi, {\sim}A, {\sim}B \mid \cdots}{\cdots \uparrow \Sigma \overset{\beta}{\Rightarrow} \Pi, {\sim}(A \wedge B) \mid \cdots} \ (\mathrm{g}{\sim}\wedge\mathrm{R})$$

$$\frac{\cdots \uparrow A, {\sim}B, \Sigma \overset{\beta}{\Rightarrow} \Pi \mid \cdots}{\cdots \uparrow {\sim}(A \to B), \Sigma \overset{\beta}{\Rightarrow} \Pi \mid \cdots} \ (\mathrm{g}{\sim}{\to}\mathrm{L})$$

$$\frac{\cdots \uparrow \Sigma \overset{\beta}{\Rightarrow} \Pi, A \mid \cdots \quad \cdots \uparrow \Sigma \overset{\beta}{\Rightarrow} \Pi, {\sim}B \mid \cdots}{\cdots \uparrow \Sigma \overset{\beta}{\Rightarrow} \Pi, {\sim}(A \to B) \mid \cdots} \ (\mathrm{g}{\sim}{\to}\mathrm{R})$$

$$\frac{\cdots \uparrow A, \Sigma \overset{\beta}{\Rightarrow} \Pi \mid \cdots}{\cdots \uparrow {\sim}\neg A, \Sigma \overset{\beta}{\Rightarrow} \Pi \mid \cdots} \ (\mathrm{g}{\sim}\neg\mathrm{L}) \qquad \frac{\cdots \uparrow \Sigma \overset{\beta}{\Rightarrow} \Pi, A \mid \cdots}{\cdots \uparrow \Sigma \overset{\beta}{\Rightarrow} \Pi, {\sim}\neg A \mid \cdots} \ (\mathrm{g}{\sim}\neg\mathrm{R})$$

$$\frac{\cdots \uparrow A, \Sigma \overset{\beta}{\Rightarrow} \Pi \mid \cdots}{\cdots \uparrow {\sim}{\sim}A, \Sigma \overset{\beta}{\Rightarrow} \Pi \mid \cdots}\ (\mathrm{g}{\sim}{\sim}\mathrm{L}) \qquad \frac{\cdots \uparrow \Sigma \overset{\beta}{\Rightarrow} \Pi, A \mid \cdots}{\cdots \uparrow \Sigma \overset{\beta}{\Rightarrow} \Pi, {\sim}{\sim}A \mid \cdots}\ (\mathrm{g}{\sim}{\sim}\mathrm{R})$$

$$\frac{\cdots \uparrow {\sim}A[z/x], \Sigma \overset{\beta\cup\{z\}}{\Rightarrow} \Pi \mid \cdots}{\cdots \uparrow {\sim}\forall x A, \Sigma \overset{\beta}{\Rightarrow} \Pi \mid \cdots}\ (\mathrm{g}{\sim}\forall\mathrm{L})_{\mathrm{VC}} \qquad \frac{\cdots \uparrow \Sigma \overset{\beta}{\Rightarrow} \Pi, {\sim}A[y/x] \mid \cdots}{\cdots \uparrow \Sigma \overset{\beta}{\Rightarrow} \Pi, {\sim}\forall x A \mid \cdots}\ (\mathrm{g}{\sim}\forall\mathrm{R})$$

It may be seen that the logical rules applied to guardians are just as those of **LK**; it is reasonable in view of the fact that a guardian sequent is a seed of a guardian possible world, which is omniscient and maximal in the tree-structure.

For variants, the following modifications are made:

$\boxed{\mathsf{d}}$ Replace the rule $(\forall\mathrm{R})_{\mathrm{T}}$ by $(\forall\mathrm{R})_{\mathrm{VC}}$.

$\boxed{\mathsf{l}}$ $(\to\mathrm{R})_{\mathrm{T}}$, $(\neg\mathrm{R})_{\mathrm{T}}$ and $(\forall\mathrm{R})_{\mathrm{T}}$ are replaced by $(\to\mathrm{R})_{\mathrm{K}}$, $(\neg\mathrm{R})_{\mathrm{K}}$ and $(\forall\mathrm{R})_{\mathrm{K}}$, respectively. And (S) is replaced by

$$\frac{\cdots \mid \Gamma \overset{\alpha}{\Rightarrow} \Delta \mid \cdots \uparrow A, \Sigma \overset{\beta}{\Rightarrow} \Pi}{\cdots \mid A, \Gamma \overset{\alpha}{\Rightarrow} \Delta \mid \cdots \uparrow \Sigma \overset{\beta}{\Rightarrow} \Pi}\ (\mathrm{S})$$

First we prove Kripke completeness of the TSg systems.

Definition 4.13 (Counter model for TSg)
An $\mathsf{N3[d][l]o}$-model $\mathcal{M}$ is a *counter model for* a TSg $\mathcal{G}$ of $\mathbf{Tg}\mathsf{N3[d][l]o}$ if $\mathcal{M}$ is a counter model for the TS-translation $\mathcal{G}^t$ in the sense of Definition 3.7.

Definition 4.14 ($\mathbf{Tg}\mathsf{N3[d][l]o}$-saturatedness)
An infinite TSg $\mathcal{G}$ is $\mathbf{Tg}\mathsf{N3[d][l]o}$-*saturated* if it satisfies the following conditions:

1. the TS-translation, $\mathcal{G}^t$, is $\mathbf{T}\mathsf{N3[d][l]}$-saturated;
2. for each guardian $\Sigma \overset{\beta}{\Rightarrow} \Pi$ and every atomic formula $p(\vec{x})$, if $\vec{x}$ are available there, then either $p(\vec{x})$ or ${\sim}p(\vec{x})$ is in Σ.

The second condition is necessary for a guardian to induce an omniscient possible world.

Lemma 4.15 (Kripke completeness of $\mathbf{Tg}\mathsf{N3[d][l]o}$)
Let $\mathbf{Tg}\mathsf{N3[d][l]o} \nvdash \mathcal{G}$ and at least one variable is l-available at the root of $\mathcal{G}$. Then $\mathcal{G}$ has a counter model.

Proof. We extend $\mathcal{G}$ into a $\mathbf{Tg}\mathsf{N3[d][l]o}$-saturated infinite TSg step by step, just as the proof of Lemma 3.9. Let $A_1, A_2, \ldots$ the same sequence, $x_1, x_2, \ldots$ also, and $\mathcal{G}_0 :\equiv \mathcal{G}$. The operation done in the i-th step, the step from $\mathcal{G}_{i-1}$ to $\mathcal{G}_i$ as to the formula A_i, is as follows. Again note that unprovability is preserved in each operation.

1. Apply the same operations [inheritance] and [reduction] to the left-sequents of $\mathcal{G}$. For example, if $(a : \Gamma_a \overset{\alpha_a}{\Rightarrow} \Delta_a \uparrow \Sigma_a \overset{\beta_a}{\Rightarrow} \Pi_a)$ and $A_i \in \Gamma_a$, then add A_i to the antecedent of the left sequent of each successor of a, not involving guardians;
2. [slide] For each node $(a : \Gamma_a \overset{\alpha_a}{\Rightarrow} \Delta_a \uparrow \Sigma_a \overset{\beta_a}{\Rightarrow} \Pi_a)$, if $A_i \in \Gamma_a$ then add A_i to Σ_a. This operation preserves unprovability because of the rule (S), and can again be

regarded as a special case of [inheritance], considering $\mathcal{G}^t$. For those logics with l, the operation is as follows: if $\cdots \mid \Gamma \overset{\alpha}{\Rightarrow} \Delta \mid \cdots \uparrow \Sigma \overset{\beta}{\Rightarrow} \Pi$ and $A_i \in \Gamma$, then add A_i to Σ;

3. [g-reduction] Reduction of A_i which appears in guardians. According to the shape of A_i, one of the following operations is executed to each node ($a : \Gamma_a \overset{\alpha_a}{\Rightarrow} \Delta_a \uparrow \Sigma_a \overset{\beta_a}{\Rightarrow} \Pi_a$), or $\uparrow \Sigma \overset{\beta}{\Rightarrow} \Pi$ for those logics with l:
   (a) [$A_i \equiv p(\vec{x})$] If $\vec{x}$ are g-available at a, then add $p(\vec{x})$ or $\sim p(\vec{x})$ to Σ_a, so that unprovability is preserved. The choice is possible: if not, we can derive a contradiction as follows.

$$\dfrac{\text{(gOm)} \qquad \dfrac{\cdots \uparrow p(\vec{x}), \Sigma_a \overset{\beta_a}{\Rightarrow} \Pi_a \mid \cdots \qquad \cdots \uparrow \sim p(\vec{x}), \Sigma_a \overset{\beta_a}{\Rightarrow} \Pi_a \mid \cdots}{\cdots \uparrow p(\vec{x}) \vee \sim p(\vec{x}), \Sigma_a \overset{\beta_a}{\Rightarrow} \Pi_a \mid \cdots}\,(\mathrm{g}\vee\mathrm{L})}{\cdots \uparrow \Sigma_a \overset{\beta_a}{\Rightarrow} \Pi_a \mid \cdots}\,(\mathrm{gC})$$

   (b) [$A_i \equiv B \wedge C$] If $A_i \in \Sigma_a$, then add both B and C to Σ_a. Unprovability is preserved by the rule (g$\wedge$L). If $A_i \in \Pi_a$, then add B or C to Π_a, so that unprovability is preserved. This is possible by (g$\wedge$R) ;
   (c) [$A_i \equiv B \to C$] If $A_i \in \Sigma_a$, then add B to Π_a or C to Σ_a, so that unprovability is preserved. If $A_i \in \Pi_a$, then add B to Σ_a and C to Π_a ;
   (d) [$A_i \equiv \neg B$] If $A_i \in \Sigma_a$ (or Π_a), then add B to Π_a (or Σ_a, respectively) ;
   (e) [$A_i \equiv \forall x B$] If $A_i \in \Sigma_a$, then add $B[y/x]$ to Σ_a, for every y which is g-available at a and is in $\{x_1, \ldots, x_i\}$. If $A_i \in \Pi_a$, then take a fresh variable x_m, add $A[x_m/x]$ to Π_a and also add x_m to β_a ;
   (f) [$A_i \equiv \sim(B \wedge C)$] If $A_i \in \Sigma_a$, add $\sim B$ or $\sim C$ to Σ_a, so that unprovability is preserved. If $A_i \in \Pi_a$, add $\sim B$ and $\sim C$ to Π_a ;
   (g) [$A_i \equiv \sim(B \to C)$] If $A_i \in \Sigma_a$, add B and $\sim C$ to Σ_a. If $A_i \in \Pi_a$, add B or $\sim C$ to Π_a so that unprovability is preserved;
   (h) [$A_i \equiv \sim\neg B$ or $A_i \equiv \sim\sim B$] If $A_i \in \Sigma_a$ (or Π_a), add B to Σ_a (or Π_a, respectively) ;
   (i) [$A_i \equiv \sim\forall x B$] If $A_i \in \Sigma_a$, then take a fresh x_m, add $\sim B[x_m/x]$ to Σ_a, and also add x_m to β_a. If $A_i \in \Pi_a$, add $\sim B[y/x]$ to Π_a for every y which is g-available at a and in $\{x_1, \ldots, x_i\}$.

For **Tg**N3d[l]o, omit conditions which involve availability of variables.

Let $\mathcal{G}_\omega$ be the union of $\mathcal{G}_0, \mathcal{G}_1, \ldots$. Then $\mathcal{G}_\omega$ is **Tg**N3[d][l]o-saturated: condition 1. is easily verified, and 2. is by the operation (3a).

Since an infinite TS ${\mathcal{G}_\omega}^t$ is **T**N3[d][l]-saturated, it induces an N3[d][l]-model $\mathcal{M}$ in the same way as the proof of Lemma 3.9. Moreover, condition 2. of **Tg**N3[d][l]o-saturatedness yields that $\mathcal{M}$ is actually an N3[d][l]o-model, where the omniscient world a_g for each world a (or the only omniscient world g for those logics with l) is induced by a_g of ${\mathcal{G}_\omega}^t$ (or g of it, respectively). By its construction $\mathcal{M}$ is a counter model for $\mathcal{G}$, which completes the proof. ∎

Now we introduce the formulaic translation of a TSg, and with the lemma above conclude Kripke completeness of **G**N3[d][l]o.

Definition 4.16 (Formulaic translation of TSg)
The *formulaic translation* of a pre-TSg $\mathcal{G}$ of **Tg**N3o, denoted by $\mathcal{G}^f$, is defined inductively on the height of $\mathcal{G}$:

$$[\Gamma \overset{\alpha}{\Rightarrow} \Delta \uparrow \Sigma \overset{\beta}{\Rightarrow} \Pi \mid \mathcal{G}_1 \dots \mathcal{G}_m]^f$$
$$:\equiv \forall\vec{\alpha}\Big((\bigwedge \Gamma) \to (\bigvee \Delta) \ \vee\ \forall\vec{\beta}\neg\neg\Big((\bigwedge \Sigma) \to (\bigvee \Pi) \Big) \ \vee \mathcal{G}_1^f \vee \cdots \vee \mathcal{G}_m^f \Big).$$

The *formulaic translation* of a TSg $\mathcal{G}$ of **Tg**N3do, again denoted by $\mathcal{G}^f$, is a universal closure of $\mathcal{G}^p$, which in turn is defined inductively on the height of $\mathcal{G}$:

$$[\Gamma \Rightarrow \Delta \uparrow \Sigma \Rightarrow \Pi \mid \mathcal{G}_1 \dots \mathcal{G}_m]^p$$
$$:\equiv (\bigwedge \Gamma) \to (\bigvee \Delta) \ \vee\ \neg\neg\Big((\bigwedge \Sigma) \to (\bigvee \Pi) \Big) \ \vee \mathcal{G}_1^p \vee \cdots \vee \mathcal{G}_m^p.$$

For a pre-TSg of **Tg**N3lo, its formulaic translation is defined inductively by:

$$(\Gamma \overset{\alpha}{\Rightarrow} \Delta \uparrow \Sigma \overset{\beta}{\Rightarrow} \Delta)^f :\equiv \forall\vec{\alpha}\Big((\bigwedge \Gamma) \to (\bigvee \Delta) \vee \forall\vec{\beta}\neg\neg((\bigwedge \Sigma) \to (\bigvee \Pi)) \Big),$$
$$(\Gamma \overset{\alpha}{\Rightarrow} \Delta \mid \mathcal{G})^f :\equiv \forall\vec{\alpha}((\bigwedge \Gamma) \to (\bigvee \Delta) \vee \mathcal{G}^f).$$

For a pre-TSg of **Tg**N3dlo, $\mathcal{G}^p$ is defined as follows, and $\mathcal{G}^f$ is its universal closure:

$$(\Gamma \Rightarrow \Delta \uparrow \Sigma \Rightarrow \Pi)^p :\equiv (\bigwedge \Gamma) \to (\bigvee \Delta) \vee \neg\neg((\bigwedge \Sigma) \to (\bigvee \Pi)),$$
$$(\Gamma \Rightarrow \Delta \mid \mathcal{G})^f :\equiv (\bigwedge \Gamma) \to (\bigvee \Delta) \vee \mathcal{G}^f.$$

Lemma 4.17
If **Tg**N3[d][l]o $\vdash \mathcal{G}$, then **G**N3[d][l]o $\vdash \mathcal{G}^f$.

Proof. Since the counterparts of lemma 3.12 and 3.13 are easily verified, we can assume that the node to which a derivation rule is applied (or the node which is in the form indicated in an initial TSg) is nothing but the root. Now we prove the lemma by the induction on the derivation of $\mathcal{G}$ in **Tg**N3[d][l]o.

The cases for (Id), (Fal), (gId) and (gFal) are easy.

For the (gOm), by (Om) of **G**N3[d][l]o.

For the rules which are common in **T**N3[d] and **Tg**N3[d]o such as (D) or ($\sim\forall$R), the proof is just as that of lemma 3.11.

For the remaining rules involving guardians, we present only the proofs for complicated cases here. First we prove the following fact needed later:

$$\mathsf{Int}\,(\text{hence } \mathbf{GN}^3_4[\mathsf{d}][\mathsf{l}][\mathsf{o}]) \vdash \neg\neg(A \vee \neg A) \qquad (\star)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A \Rightarrow A \vee \neg A}\,(\vee\mathrm{R})}{A, \neg(A \vee \neg A) \Rightarrow}\,(\neg\mathrm{L})}{\neg(A \vee \neg A) \Rightarrow \neg A}\,(\neg\mathrm{R})}{\neg(A \vee \neg A) \Rightarrow A \vee \neg A}\,(\vee\mathrm{R})}{\neg(A \vee \neg A), \neg(A \vee \neg A) \Rightarrow}\,(\neg\mathrm{L})}{\Rightarrow \neg\neg(A \vee \neg A)}\,(\neg\mathrm{R})$$

For (g¬R), it suffices to show that $\mathbf{G}\mathsf{N3[d][l]o} \vdash \neg\neg(C \land A \to D) \Rightarrow \neg\neg(C \to D \lor \neg A)$.

$$
\dfrac{(\star) \qquad \cfrac{\cfrac{\dfrac{\dfrac{\vdots}{C, A, C \land A \to D \Rightarrow D \lor \neg A} \quad \dfrac{\vdots}{C, \neg A, C \land A \to D \Rightarrow D \lor \neg A}}{C, A \lor \neg A, C \land A \to D \Rightarrow D \lor \neg A}\ (\lor\mathrm{L})}{A \lor \neg A, C \land A \to D \Rightarrow C \to D \lor \neg A}\ (\to\mathrm{R})}{\neg\neg(A \lor \neg A), \neg\neg(C \land A \to D) \Rightarrow \neg\neg(C \to D \lor \neg A)}\ (\neg\mathrm{L})(\neg\mathrm{R})}{\neg\neg(C \land A \to D) \Rightarrow \neg\neg(C \to D \lor \neg A)}\ (\mathrm{C})
$$

For (g→R), the proof is similar to above, using (⋆).
For (g∀R), it suffices to show that

$$\mathbf{G}\mathsf{N3[d][l]o} \vdash \forall z\neg\neg(C \to D \lor A[z/x]) \Rightarrow \neg\neg(C \to D \lor \forall x A)$$

where z is free in neither C nor D.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{C \Rightarrow C}\ (\mathrm{Id}) \quad \dfrac{\dfrac{}{D, {\sim}D \Rightarrow}\ (\mathrm{Fal}) \quad \dfrac{}{A[z/x], {\sim}A[z/x] \Rightarrow}\ (\mathrm{Fal})}{D \lor A[z/x], {\sim}D, {\sim}A[z/x] \Rightarrow}\ (\lor\mathrm{L})}{C \to D \lor A[z/x], C, {\sim}D, {\sim}A[z/x] \Rightarrow}\ (\to\mathrm{L})}{\forall z\neg\neg(C \to D \lor A[z/x]), C, {\sim}D, {\sim}A[z/x] \Rightarrow}\ (\neg\mathrm{R})(\neg\mathrm{L})(\forall\mathrm{L})}{\forall z\neg\neg(C \to D \lor A[z/x]), C, {\sim}D, {\sim}\forall x A \Rightarrow}\ ({\sim}\forall\mathrm{L})_{\mathrm{VC}}}{\forall z\neg\neg(C \to D \lor A[z/x]), {\sim}(C \to D \lor \forall x A) \Rightarrow}\ ({\sim}\lor\mathrm{L})({\sim}{\to}\mathrm{L})}{\forall z\neg\neg(C \to D \lor A[z/x]), \neg(C \to D \lor \forall x A) \Rightarrow}\ (\mathrm{Om2})}{\forall z\neg\neg(C \to D \lor A[z/x]) \Rightarrow \neg\neg(C \to D \lor \forall x A)}\ (\neg\mathrm{R})
$$

For (S), it suffices to show that

$$\mathbf{G}\mathsf{N3[l]o} \vdash \forall\vec{x}\big(C \to D \lor \forall\vec{y}\neg\neg(A \land E \to F)\big) \Rightarrow \forall\vec{x}\big(A \land C \to D \lor \forall\vec{y}\neg\neg(E \to F)\big)$$

where $\vec{y}$ have no free occurrences in C, D or A. This is easy. ■

THEOREM 4.18 (Kripke completeness of $\mathbf{G}\mathsf{N3[d][l]o}$)
If $\mathsf{N3[d][l]o} \models A$, then $\mathbf{G}\mathsf{N3[d][l]o} \vdash A$.

PROOF. Similar to that of Theorem 3.14. Take $\mathcal{G} :\equiv [\ \overset{\alpha}{\Rightarrow} A \uparrow \overset{\emptyset}{\Rightarrow}\ ]$, and use Lemma 4.15 and 4.17. ■

5 Remarks on logics with N4[d][l]o

As stated in the introduction, Kripke completeness of logics N4[d][l]o is remain unproved in this paper. Here we are to show that their proofs cannot be done using our methods.

The key is an axiom $\forall x\neg\neg A \to \neg\neg\forall x A$, called the *double negation shift*, *DNS* in short. This is a theorem of those logics $\mathsf{N3[d][l]o}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A[z/x], {\sim}A[z/x] \Rightarrow}\ (\mathrm{Fal})}{\forall x\neg\neg A, {\sim}A[z/x] \Rightarrow}\ (\neg\mathrm{R}), (\neg\mathrm{L}), (\forall\mathrm{L})}{\forall x\neg\neg A, {\sim}\forall xA \Rightarrow}\ ({\sim}\forall\mathrm{L})}{\forall x\neg\neg A, \neg\forall xA \Rightarrow}\ (\mathrm{Om2})}{\Rightarrow \forall x\neg\neg A \to \neg\neg\forall xA}\ (\neg\mathrm{R}), (\to\mathrm{R})$$

On the other hand, DNS is not provable in even the strongest system among $\mathsf{GN4[d][l]o}$, i.e. $\mathsf{GN4dlo}$, since it has the following counter $\mathsf{N4dlo}$-model $\mathcal{M} = (M, \leq, U, I^+, I^-)$: $(M, \leq) = (\omega, \leq)$, $U(n) = \omega$ for every $n \in \omega$, p a unary predicate symbol, $p^{I^+(n)} = \{1, 2, \ldots, n\}$ and $p^{I^-(n)} = \omega$. Then $\mathcal{M} \not\models \forall x\neg\neg p(x) \to \neg\neg\forall xp(x)$.

The intermediate logic MH, which is Int plus DNS, is characterized by the class of Int-models such that: for each possible world a, there exists $b \geq a$ which is maximal $(*)$ [5]. Counter models which the two methods in this paper construct all have the property $(*)$; omniscient worlds are always maximal. However, since the axiom DNS does not involve strong negation, DNS is valid in every $\mathsf{N4}$-model (hence also in every $\mathsf{N}^3_4\mathsf{[d][l][o]}$-model) which has the property $(*)$. Hence we cannot construct a counter $\mathsf{N4o}$-model for DNS by the methods.

Acknowledgements

The authors are grateful to the referee for his comments and detailed historical remarks. The first author would like to thank Norihiro KAMIDE, Izumi TAKEUTI, Takeshi YAMAGUCHI and Tatsuya SHIMURA for their invaluable comments and encouragements.

References

[1] Seiki Akama. Tableaux for logic programming with strong negation. In Didier Galmiche, editor, *Automated Reasoning with Analytic Tableaux and Related Methods, International Conference, TABLEAUX'97*, volume 1227 of *Lecture Notes in Computer Science*, pages 31–42. Springer-Verlag, May 1997.

[2] Ahmad Almukdad and David Nelson. Constructible falsity and inexact predicates. *The Journal of Symbolic Logic*, 49(1):231–233, 1984.

[3] Giovanna Corsi. Completeness theorem for Dummett's LC quantified and some of its extensions. *Studia Logica*, 51:317–335, 1992.

[4] Giovanna Corsi and Silvio Ghilardi. Directed frames. *Archive for Mathematical Logic*, 29:53–67, 1989.

[5] Dov M. Gabbay. *Semantical Investigations in Heyting's Intuitionistic Logic*, volume 148 of *Synthese Library*. D. Reidel, 1981.

[6] Valentine Goranko. The craig interpolation theorem for propositional logics with strong negation. *Studia Logica*, 44:291–317, 1985.

[7] Sabine Görnemann. A logic stronger than intuitionism. *The Journal of Symbolic Logic*, 36:249–261, 1971.

[8] Andrzej Grzegorczyk. A philosophically plausible formal interpretation of intuitionistic logic. *Indagationes Mathematicae*, 26:596–601, 1964.

[9] Yuri Gurevich. Intuitionistic logic with strong negation. *Studia Logica*, 36:49–59, 1977.

[10] Heinrich Herre and David Pearce. Disjunctive logic programming, constructivity and strong negation. In *Logic in AI. Proceedings of the European Workshop JELIA 92*, volume 633 of *Lecture Notes in Artificial Intelligence*. Springer-Verlag, 1992.

[11] Arend Heyting. *Intuitionism. An Introduction*. North-Holland, 1956.

[12] Ryo Kashima. Sequent calculi of non-classical logics – Proofs of completeness theorems by sequent calculi (in Japanese). In *Proceedings of Mathematical Society of Japan Annual Colloquium of Foundations of Mathematics*, pages 49–67, 1999.

[13] Marcus Kracht. On extensions of intermediate logics by strong negation. *Journal of Philosophical Logic*, 27:49–73, 1998.

[14] Saul A. Kripke. Semantical analysis of intuitionistic logic I. In J. Crossley and M. Dummett, editors, *Formal Systems and Recursive Functions*, pages 92–129. North-Holland, 1965.

[15] A. A. Markov. Konstruktivnaja logika. *Uspehi Matematičeskih Nauk*, 5:187–188, 1950.

[16] David Nelson. Constructible falsity. *The Journal of Symbolic Logic*, 14:16–26, 1949.

[17] Sergei P. Odintsov. Algebraic semantics for paraconsistent Nelson's logic. *Journal of Logic and Computation*, 13(4):453–468, 2003.

[18] Sergei P. Odintsov and Heinrich Wansing. Inconsistency-tolerant description logic, Motivation and basic systems. In Vincent F. Hendricks and Jacek Malinowski, editors, *Trends in Logic, 50 Years of Studia Logica*, volume 21 of *Trends in Logic*, pages 301–335. Kluwer Academic Publishers, Dordrecht, 2003.

[19] David Pearce and Gerd Wagner. Reasoning with negative information, I: Strong negation in logic programs. In *Language, Knowledge and Intentionality (Acta Philosophica Fennica 49)*, pages 405–439, 1990.

[20] David Pearce and Gerd Wagner. Logic programming with strong negation. In Peter Schroeder-Heister, editor, *Extensions of Logic Programming*, volume 475 of *Lecture Notes in Artificial Intelligence*, pages 311–326. Springer-Verlag, 1991.

[21] Graham Priest and Richard Routley. Introduction: Paraconsistent logics. *Studia Logica*, 43:3–16, 1984.

[22] Andrzej Sendlewski. Some investigations of varieties of N-lattices. *Studia Logica*, 43:257–280, 1984.

[23] Andrzej Sendlewski. Nelson algebras through Heyting ones. *Studia Logica*, 49:106–126, 1990.

[24] Andrzej Sendlewski. Axiomatic extensions of the constructive logic with strong negation and the disjunction property. *Studia Logica*, 55:377–388, 1995.

[25] Richmond H. Thomason. A semantical study of constructive falsity. *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, 15:247–257, 1969.

[26] Anne S. Troelstra and Helmut Schwichtenberg. *Basic Proof Theory.* Number 43 in Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, 1996.

[27] Anne S. Troelstra and Dirk van Dalen. *Constructivism in Mathematics, Vol. I.* North-Holland, 1988.

[28] Dirk van Dalen. Intuitionistic logic. In D. M. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic III*, volume 166 of *Synthese Library*, pages 225–340. D. Reidel, 1986.

[29] Franz von Kutschera. Ein verallgemeinerter Widerlegungsbegriff für Gentzenkalküle. *Archiv für Mathematische Logik und Grundlagenforschung*, 12:104–118, 1969.

[30] Gerd Wagner. *Vivid Logic. Knowledge-Based Reasoning with Two Kinds of Negation*, volume 764 of *Lecture Notes in Artificial Intelligence*. Springer-Verlag, 1994.

[31] Heinrich Wansing. *The Logic of Information Structures*, volume 681 of *Lecture Notes in Artificial Intelligence*. Springer-Verlag, 1993.

[32] Heinrich Wansing. Higher-arity Gentzen systems for Nelson's logics. In Julian Nida-Rümelin, editor, *Rationality, Realism, Revision, Proceedings of the 3rd international congress of the Society for Analytical Philosophy*, volume 23 of *Perspectives in Analytical Philosophy*, pages 105–109. Walter de Gruyter, September 1999.

Received March 31, 2003