Pseudo Functorial Semantics

Ichiro Hasuo [a,b,*]

[a] *Research Institute for Mathematical Sciences, Kyoto University, Japan*

[b] *PRESTO Research Promotion Program, Japan Science and Technology Agency*

Abstract

Categories with algebraic structure—the most prominent example being monoidal categories—satisfy equational axioms only up-to coherent isomorphisms. Therefore they are *pseudo algebras*. We extend Lawvere's *functorial semantics* to such pseudo structure: in contrast to standard *strict* algebras which are identified with product-preserving functors, pseudo algebras are product-preserving *pseudo* functors. This identification paves a way to a uniform theory of pseudo algebras. To demonstrate its use we prove a lifting result of pseudo algebraic structure to a category of coalgebras, a result that is crucial in our coalgebraic study of software components with the *microcosm principle*.

1 Introduction

In this note we describe the details of our approach to categories with algebraic structure. They are fundamentally different from "algebras" in the usual sense: they satisfy equational axioms only up-to coherent isomorphisms. Therefore they are *pseudo algebras*, following the categorical tradition of naming *pseudo* notions.

To start with, our representation of algebraic structure is by a *Lawvere theory* [12,10], that is, a category $\mathbb{L}$ with finite products with its objects freely generated by the terminal category $\mathbf{1}$. It is standard that a *set* with $\mathbb{L}$-structure is identified with a functor

$$\mathbb{L} \longrightarrow \mathbf{Sets}$$

* Corresponding author. Research Institute for Mathematical Sciences, Kyoto University, 606-8502 Japan. Tel: +81 75 753 7251, fax: +81 75 753 7266.
Email address: `ichiro@kurims.kyoto-u.ac.jp` (Ichiro Hasuo).
URL: `http://www.kurims.kyoto-u.ac.jp/~ichiro` (Ichiro Hasuo).

which preserves finite products; thus semantics of $\mathbb{L}$ is given by such functors.

For a *category* with $\mathbb{L}$-structure, replacing **Sets** by **CAT** is not enough: equational axioms are represented as commutative diagrams in $\mathbb{L}$; and they are carried to diagrams in **CAT** that commute on-the-nose. Therefore a functor $\mathbb{L} \to$ **CAT** necessarily satisfy equational axioms up-to identity. We have to relax this requirement.

In this note we find the relaxed requirement to be a *finite product preserving pseudo functor* $\mathbb{L} \to$ **CAT**. *Pseudo functors* (see e.g. [3]) are almost functors but they preserve identities and composition only up-to coherent isomorphisms; hence they carry a commutative diagram in $\mathbb{L}$ to a diagram in **CAT** filled in with an isomorphism.

The idea is quite straightforward, and for "monoidal" theories (instead of Lawvere theories that are cartesian) the same idea has been already employed in Segal's categorical approach to conformal field theory [16]. However for our application we want to be able to handle Lawvere theories—algebraic structure with the cartesian context—and preservation of finite products is a delicate issue in pursuing the idea of using pseudo functors.

We provide a rigorous definition when a pseudo functor is finite product preserving, i.e., a definition of pseudo $\mathbb{L}$-algebra, which we would prefer to call $\mathbb{L}$-*category*. Justifying the definition is the main goal of §2: to this goal, we first define pseudo algebras in a conventional way, for algebraic structure specified by *algebraic specification* (Σ, E) that is simply a pair of operations and equational axioms. Pseudo (Σ, E)-algebras are called (Σ, E)-*categories*; standard examples such as monoidal categories are obviously instances of (Σ, E)-categories.

Then we prove that there is a equivalence between the 2-category of (Σ, E)-categories and that of $\mathbb{L}_{(\Sigma,E)}$-categories, where $\mathbb{L}_{(\Sigma,E)}$ is the Lawvere theory induced by (Σ, E). Note that the relationship (equivalence) is stronger than *biequivalence*, a standard "being the same" relation between 2-categories. This result will justify identifying pseudo algebras with pseudo functors.

In its course almost any single notion is accompanied by a *coherence condition* that needs to be taken care of. We give them a rigorous account. For example we formulate the coherence condition for (Σ, E)-categories in terms of equational logic; this for example yields as its instance the coherence condition for monoidal categories. (To be precise, what is yielded is the consequence "every diagram commutes" of the *coherence theorem* e.g. in [14, §VII.2])

In §3 we use our formalization of pseudo algebra to prove a result that, when an endofunctor $F : \mathbb{C} \to \mathbb{C}$ is "lax-compatible" with the $\mathbb{L}$-structure of $\mathbb{C}$, then the $\mathbb{L}$-structure lifts to the category $\mathbf{Coalg}(F)$ of F-coalgebras. This is a crucial result in our coalgebraic study [6] of software components in the setting of the *microcosm principle*.

1.1 Related work

The definition of pseudo algebra in [5] looks equivalent to ours. His definition is for algebraic structure specified by a Lawvere theory (as is ours) but a Lawvere theory is thought of rather as a *clone*, that is a generalized form of *operad* (see e.g. [13,4]). Consequently the definition's pseudo functorial aspect is (at least) not emphasized.

In the same reference [5] the relationship of his definition of pseudo algebra (for a Lawvere theory) to pseudo algebras for a 2-monad is elaborated. For the latter the standard reference is [2]. For our application of the component calculus, it is important that we can accommodate the *microcosm principle* (see [1,7]): the same algebraic structure is carried once by a category $\mathbb{C}$, and another time by an object $X \in \mathbb{C}$. To this end Lawvere theories are more useful than (2-)monads as representation of algebraic structure.

1.2 Notations

A signature Σ as a presheaf $\Sigma : \mathbb{N} \to \mathbf{Sets}$. Here $\mathbb{N}$ denotes the set of natural numbers, considered as a discrete category. The set of Σ-*terms*, with varying numbers of variables, then forms a presheaf $T_\Sigma : \mathbf{Nat} \to \mathbf{Sets}$, where $\mathbf{Nat}$ is the category of natural numbers and functions between them (a full subcategory of $\mathbf{Sets}$). Note that the domain $\mathbf{Nat}$ of T_Σ is extended from that of Σ, namely $\mathbb{N}$. For example, if Σ has a binary operation $\sigma \in \Sigma(2)$, then the term $\bigl(x_0, x_1, x_2 \vdash \sigma(x_0, \sigma(x_1, x_2))\bigr)$ resides in $T_\Sigma(3)$. Along the arrow $! : 3 \to 1$ in $\mathbf{Nat}$, it is mapped to

$$\bigl(x_0 \vdash \sigma(x_0, \sigma(x_0, x_0))\bigr) \;\in\; T_\Sigma(1) \enspace .$$

We emphasize that operations, terms and formal equality between terms are all under certain variable environments $x_0, \ldots, x_{n-1} \vdash \cdots$, although we often omit an environment when there is no risk of confusion.

2 Pseudo Algebra via Pseudo Functor

We concentrate on the one-sorted setting; extension to the many-sorted setting is straightforward.

2.1 (Σ, E)-category

First we present the definition of (Σ, E)-category. The definition is fairly straightforward—although writing down the coherence condition requires some work. This is the notion that we would like to describe later alternatively in more categorical terms, namely by pseudo functors.

A (Σ, E)-category is a category equipped with interpretation of operations in Σ:

$$[\![\sigma]\!]_{\mathbb{C}} \ : \ \mathbb{C}^n \longrightarrow \mathbb{C}$$

for each $n \in \mathbb{N}$ and $\sigma \in \Sigma(n)$. Here $\mathbb{C}^n$ is a shorthand for the n-fold product $(\cdots((\mathbb{C} \times \mathbb{C}) \times \mathbb{C}) \times \cdots \times \mathbb{C})$. We will often denote $[\![\sigma]\!]_{\mathbb{C}}$ by $[\![\sigma]\!]$ when the base category $\mathbb{C}$ is evident.

Moreover these interpreted operations must satisfy the equations in E up-to coherent isomorphisms. To put it rigorous, we need to fix some notations.

Definition 2.1 *The interpretation $[\![\_]\!]_{\mathbb{C}}$ of operations has an obvious extension to an arbitrary Σ-term $t \in T_\Sigma(n)$. Concretely, for each $i \in [0, n-1]$ and $\sigma \in \Sigma(m)$,*

$$\begin{aligned} [\![x_0, \ldots, x_{n-1} \vdash x_i]\!]_{\mathbb{C}} &:= \mathbb{C}^n \xrightarrow{\pi_i} \mathbb{C} \ , \\ [\![x_0, \ldots, x_{n-1} \vdash \sigma(t_0, \ldots, t_{m-1})]\!]_{\mathbb{C}} &:= \mathbb{C}^n \xrightarrow{\langle [\![t_0]\!]_{\mathbb{C}}, \ldots, [\![t_{m-1}]\!]_{\mathbb{C}} \rangle} \mathbb{C}^m \xrightarrow{[\![\sigma]\!]_{\mathbb{C}}} \mathbb{C} \ . \end{aligned} \tag{1}$$

Here the tupling notation $\langle f_0, \ldots, f_{m-1} \rangle$ is again a shorthand for $\langle \ldots \langle \langle f_0, f_1 \rangle, f_2 \rangle \ldots, f_{m-1} \rangle$.

The interpreted operations $[\![\sigma]\!]_{\mathbb{C}}$ must satisfy equations up-to isomorphisms, that is, for each equation $(x_0, \ldots, x_{n-1} \ \vdash \ s \ = \ t) \in E$ we have a natural isomorphism

$$\begin{tikzcd} \mathbb{C}^n \arrow[r, bend left, "{[\![s]\!]_{\mathbb{C}}}"] \arrow[r, bend right, "{[\![t]\!]_{\mathbb{C}}}"'] \arrow[r, phantom, "\cong \Downarrow \gamma_s^t"] & \mathbb{C} \end{tikzcd} \ . \tag{2}$$

Let us describe the coherence condition to which the above mediating isomorphisms are subject. We denote by $\sim$ the formal equality relation between terms, derived from the given set E of equations. That is, $\sim$ is derived using the following axioms. The axiomatization is a common one for equational logic.

$$\frac{(s=t)\in E}{s\sim t}\ (\textsc{Ax})\qquad \frac{s'\sim t' \quad s_0\sim t_0 \quad \cdots \quad s_{n-1}\sim t_{n-1}}{s'[s_0/x_0,\dots,s_{m-1}/x_{m-1}]\sim t'[t_0/x_0,\dots,t_{m-1}/x_{m-1}]}\ (\textsc{Cong})$$

$$\frac{}{s\sim s}\ (\textsc{Refl})\quad \frac{s\sim t \quad t\sim u}{s\sim u}\ (\textsc{Trans})\quad \frac{t\sim s}{s\sim t}\ (\textsc{Symm})$$

Here $s'[s_0/x_0,\dots,s_{m-1}/x_{m-1}]$ in (Cong) denotes (simultaneous) substitution for an m-ary term $x_0,\dots,x_{m-1}\vdash s'$.

We assign, to each derivation tree $\mathcal{D}$ composed by these rules, a natural isomorphism $\gamma_{\mathcal{D}}$ inductively as follows. The left column shows the last rule applied in $\mathcal{D}$.

$$\frac{(s=t)\in E}{s\sim t}\ (\textsc{Ax})\qquad \Longrightarrow \gamma_{\mathcal{D}} := \gamma_s^t \text{ from (2)}\ ;$$

$$\frac{\begin{array}{cccc}\mathcal{D}' & \mathcal{D}_0 & & \mathcal{D}_{n-1}\\ s'\sim t' & s_0\sim t_0 & \cdots & s_{n-1}\sim t_{n-1}\end{array}}{s'[s_0/x_0,\dots,s_{m-1}/x_{m-1}]\sim t'[t_0/x_0,\dots,t_{m-1}/x_{m-1}]}\ (\textsc{Cong})$$

$$\Longrightarrow \gamma_{\mathcal{D}} := \begin{tikzcd} \mathbb{C}^n \arrow[r, bend left, "\langle [\![s_0]\!],\dots,[\![s_{m-1}]\!]\rangle"] \arrow[r, bend right, "\langle [\![t_0]\!],\dots,[\![t_{m-1}]\!]\rangle"'] \arrow[r, phantom, "\Downarrow\langle\gamma_{\mathcal{D}'},\dots,\gamma_{\mathcal{D}_{m-1}}\rangle"] \arrow[rr, bend left=60, "{[\![s'[s_0/x_0,\dots,s_{m-1}/x_{m-1}]]\!]}"] \arrow[rr, bend right=60, "{[\![t'[t_0/t_0,\dots,t_{m-1}/t_{m-1}]]\!]}"'] & \mathbb{C}^m \arrow[r, bend left, "[\![s']\!]"] \arrow[r, bend right, "[\![t']\!]"'] \arrow[r, phantom, "\Downarrow\gamma_{\mathcal{D}'}"] \arrow[u, phantom, "\parallel"] \arrow[d, phantom, "\parallel"] & \mathbb{C} \end{tikzcd}\ ;$$

$$\frac{}{s\sim s}\ (\textsc{Refl})\qquad \Longrightarrow \gamma_{\mathcal{D}} := \mathrm{id}\ ;$$

$$\frac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2\\ s\sim t & t\sim u\end{array}}{s\sim u}\ (\textsc{Trans})\qquad \Longrightarrow \gamma_{\mathcal{D}} := \gamma_{\mathcal{D}_2}\bullet\gamma_{\mathcal{D}_1}\ ;$$

$$\frac{\begin{array}{c}\mathcal{D}_1\\ t\sim s\end{array}}{s\sim t}\ (\textsc{Symm})\qquad \Longrightarrow \gamma_{\mathcal{D}} := \gamma_{\mathcal{D}}^{-1}\ .$$

Here $\bullet$ denotes vertical composition of natural transformations. It is obvious that, when $\mathcal{D}$ is a derivation of $s\sim t$, then the assigned natural isomorphism $\gamma_{\mathcal{D}}$ is of the type $[\![s]\!]_{\mathbb{C}} \Rightarrow [\![t]\!]_{\mathbb{C}}$. For the rule (Cong) we used the following fact.

Lemma 2.2 *Let $x_0,\dots,x_{m-1}\vdash t'$ be a Σ-term with m variables, and $x_0,\dots,x_{n-1}\vdash t_i$ be ones with n variables, for each $i\in[0,m-1]$. For the interpretation of terms defined in Def. 2.1, we have*

$$[\![\, t'[t_0/x_0,\dots,t_{m-1}/x_{m-1}] \,]\!]_{\mathbb{C}} = \mathbb{C}^n \xrightarrow{\langle [\![t_0]\!],\dots,[\![t_{m-1}]\!]\rangle} \mathbb{C}^m \xrightarrow{\quad [\![t']\!] \quad} \mathbb{C}\ .$$

PROOF. By induction of the construction of the term t'. $\square$

Now we are ready to state the coherence condition.

Condition 2.3 *If $\mathcal{D}$ and $\mathcal{D}'$ are two derivation trees of the same equality $s \sim t$, then $\gamma_{\mathcal{D}} = \gamma_{\mathcal{D}'}$.*

Assuming this coherence condition, the notation γ_s^t is valid not only for an equational axiom $(s = t) \in E$ but also for any formal equality $s \sim t$ that is derived from axioms in E. In particular, well-definedness of γ_s^t is guaranteed.

We have thus characterized the class of "every diagram" that should commute, by means of equational reasoning. To summarize:

Definition 2.4 ((Σ, E)-category) *A* (Σ, E)-category *is a triple*

$$\Big(\mathbb{C},\ \{[\![\sigma]\!]_{\mathbb{C}}\}_{\sigma \in \Sigma},\ \{\gamma_s^t\}_{(s=t)\in E} \Big)$$

which we will often denote simply by $(\mathbb{C}, [\![\_]\!]_{\mathbb{C}}, \gamma)$. Here:

- *$\mathbb{C}$ is a locally small category;*
- *$[\![\sigma]\!]_{\mathbb{C}} : \mathbb{C}^n \to \mathbb{C}$ is a functor for each $n \in \mathbb{N}$ and each operation $\sigma \in \Sigma(n)$; and*
- *$\gamma_s^t : [\![s]\!]_{\mathbb{C}} \Rightarrow [\![t]\!]_{\mathbb{C}} : \mathbb{C}^n \to \mathbb{C}$ is a natural isomorphism for each equation $(s = t) \in E$. These must satisfy the coherence condition (Cond. 2.3); hence giving rise to a natural isomorphism γ_s^t for each derived formal equality $s \sim t$.*

It follows immediately from the coherence condition (Cond. 2.3) that:

$$\gamma_s^s = \mathrm{id}\ ; \qquad \gamma_t^u \bullet \gamma_s^t = \gamma_s^u\ ; \qquad \text{and} \quad (\gamma_s^t)^{-1} = \gamma_t^s\ . \tag{3}$$

These equations correspond to the rules (Refl), (Trans), and (Symm), respectively.

We proceed to introduce suitable notions of *morphism* of (Σ, E)-categories, and *transformations* between them. They altogether form a 2-category (Σ, E)-**CAT**.

Definition 2.5 (Morphism and transformation for (Σ, E)-categories) *A* morphism $(\mathbb{C}, [\![\_]\!]_{\mathbb{C}}, \gamma) \to (\mathbb{D}, [\![\_]\!]_{\mathbb{D}}, \delta)$ *of (Σ, E)-categories is a functor $F : \mathbb{C} \to \mathbb{D}$ equipped with*

- *a natural isomorphism F_σ, for each operation $\sigma \in \Sigma(n)$, of the following type.*

$$\begin{tikzcd}
\mathbb{C}^n \arrow[r, "F^{\times n}"] \arrow[d, "{[\![\sigma]\!]_{\mathbb{C}}}"'] \arrow[dr, phantom, "\cong\ \swArrow F_\sigma"] & \mathbb{D}^n \arrow[d, "{[\![\sigma]\!]_{\mathbb{D}}}"] \\
\mathbb{C} \arrow[r, "F"'] & \mathbb{D}
\end{tikzcd}$$

Here $F^{\times n}$ is a shorthand for the n-fold product $(\cdots((F \times F) \times F) \times \cdots \times F)$. [1] *These F_σ's have an obvious extension F_t to an arbitrary Σ-term t, defined by induction on construction of t. Namely (see also Def. 2.1),*

$$F_{x_i} := \mathrm{id} \ ,$$

$$F_{\sigma(t_0,\dots,t_{m-1})} := \begin{tikzcd} \mathbb{C}^n \arrow[r, "F^{\times n}"] \arrow[d, "{\langle \llbracket t_0 \rrbracket_{\mathbb{C}},\dots,\llbracket t_{m-1} \rrbracket_{\mathbb{C}} \rangle}"'] \arrow[dr, phantom, "{\Downarrow \langle F_{t_0},\dots,F_{t_{m-1}} \rangle}"] & \mathbb{D}^n \arrow[d, "{\langle \llbracket t_0 \rrbracket_{\mathbb{D}},\dots,\llbracket t_{m-1} \rrbracket_{\mathbb{D}} \rangle}"] \\ \mathbb{C}^m \arrow[r, "F^{\times m}"] \arrow[d, "\llbracket \sigma \rrbracket_{\mathbb{C}}"'] \arrow[dr, phantom, "\Downarrow F_\sigma"] & \mathbb{D}^m \arrow[d, "\llbracket \sigma \rrbracket_{\mathbb{D}}"] \\ \mathbb{C} \arrow[r, "F"'] & \mathbb{D} \end{tikzcd}$$

Due to these natural isomorphisms $\{F_\sigma\}_{\sigma \in \Sigma}$, the functor F "preserves algebraic structure up-to isomorphisms." They are subject to the following coherence condition:

- *for each equation $(s = t) \in E$, we have the following 2-cells equal.*

$$\begin{tikzcd} \mathbb{C}^n \arrow[r, "F^{\times n}"] \arrow[d, bend right=50, "\llbracket t \rrbracket_{\mathbb{C}}"'] \arrow[d, bend left=20, "\llbracket s \rrbracket_{\mathbb{C}}"] \arrow[d, phantom, "\overset{\gamma^t_s}{\Leftarrow}"] \arrow[dr, phantom, "\overset{\cong}{\swarrow} F_s"] & \mathbb{D}^n \arrow[d, bend left, "\llbracket s \rrbracket_{\mathbb{D}}"] \\ \mathbb{C} \arrow[r, "F"'] & \mathbb{D} \end{tikzcd} = \begin{tikzcd} \mathbb{C}^n \arrow[r, "F^{\times n}"] \arrow[d, bend right, "\llbracket t \rrbracket_{\mathbb{C}}"'] \arrow[dr, phantom, "\overset{\cong}{\swarrow} F_t"] & \mathbb{D}^n \arrow[d, bend right=20, "\llbracket t \rrbracket_{\mathbb{D}}"'] \arrow[d, bend left=50, "\llbracket s \rrbracket_{\mathbb{D}}"] \arrow[d, phantom, "\overset{\delta^t_s}{\Leftarrow}"] \\ \mathbb{C} \arrow[r, "F"'] & \mathbb{D} \end{tikzcd}$$

Here γ^t_s and δ^t_s are natural isomorphisms up-to which $\mathbb{C}$ and $\mathbb{D}$ satisfy the equation $s = t$ (Def. 2.4).

It is easy to see that, once this coherence holds for each $(s = t) \in E$, it also holds for any formal equality $s \sim t$ derived from E.

A transformation *between such morphisms is a natural transformation $\varphi : F \Rightarrow G$, subject to the coherence condition:*

- *for each operation $\sigma \in \Sigma(n)$, the following 2-cells are equal.*

$$\begin{tikzcd} \mathbb{C}^n \arrow[r, bend left, "F^{\times n}"] \arrow[r, bend right, "G^{\times n}"'] \arrow[r, phantom, "\Downarrow \varphi^{\times n}"] \arrow[d, "\llbracket \sigma \rrbracket_{\mathbb{C}}"'] \arrow[dr, phantom, "\swarrow G_\sigma"] & \mathbb{D}^n \arrow[d, "\llbracket \sigma \rrbracket_{\mathbb{D}}"] \\ \mathbb{C} \arrow[r, bend right, "G"'] & \mathbb{D} \end{tikzcd} = \begin{tikzcd} \mathbb{C}^n \arrow[r, bend left, "F^{\times n}"] \arrow[d, "\llbracket \sigma \rrbracket_{\mathbb{C}}"'] \arrow[dr, phantom, "\swarrow F_\sigma"] & \mathbb{D}^n \arrow[d, "\llbracket \sigma \rrbracket_{\mathbb{D}}"] \\ \mathbb{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\Downarrow \varphi"] & \mathbb{D} \end{tikzcd}$$

Example 2.6 *Let us take the usual specification for monoids as (Σ, E) with two operations (unit and multiplication) and equations (associativity and unit law). Then the 2-category (Σ, E)-**CAT** is that of monoidal categories, strong (as opposed to strict or lax/oplax) monoidal functors, and monoidal transformations.*

[1] Unlike the case of $\mathbb{C}^n$, the notation F^n is confusing since it can also be taken as the n-fold composition $F \circ F \circ \cdots \circ F$. Hence the notation $F^{\times n}$.

In particular (Σ, E)-categories in this case are the same thing as monoidal categories. When we look at their definitions, the latter's "coherence condition" requires commutativity of a smaller class of diagrams (the pentagon and the other); however the coherence theorem for monoidal categories (see e.g. [14, § VII.2]) ensures that "every diagram commutes"—every diagram in a larger family which we specified in terms of equational reasoning (Cond. 2.3).

Remark 2.7 *A symmetric monoidal category is not a (Σ, E)-category for the specification (Σ, E) for commutative monoids! The coherence condition requires that $\mathsf{s}_{X,X} = \mathrm{id} : X \otimes X \xrightarrow{\cong} X \otimes X$, which is stronger than the requirement for symmetric monoidal categories. This is what is referred to as "symmetry is not commutativity."*

2.2 $\mathbb{L}$-category

The notion of (Σ, E)-category was the formalization of a "pseudo algebra" carried by a category, in a rather conventional manner. Now we introduce its "Segalic" formalization [16,9] that uses pseudo functors. Later in §2.3 we will show that these two formalizations are indeed equivalent.

Let $\mathbb{L}$ be a Lawvere theory.

Definition 2.8 ($\mathbb{L}$-category) *An* $\mathbb{L}$-category *is a pseudo functor*

$$C \;:\; \mathbb{L} \longrightarrow \mathbf{CAT}$$

which is "finite product preserving" in the following sense:

(1) for each $n \in \mathbb{L}$,

$$Cn = (C1)^n = (\cdots((C1 \times C1) \times C1) \times \cdots \times C1) \;\; ;$$

(2) for each $n \in \mathbb{L}$ and $i \in [0, n-1]$,

$$C(n \xrightarrow{\pi_i} 1) \quad = \quad (C1)^n \xrightarrow{\pi_i} (C1) \;\; ,$$

where π_i on the right is the i-th projection in $\mathbf{CAT}$. More precisely, $\pi_i : (\cdots(C1 \times C1) \times \cdots \times C1) \longrightarrow C1$ is given by

$$\pi_i = \begin{cases} \pi_r \circ (\pi_l)^{n-i} & (2 \leq i) \\ \pi_l^{n-1} & (i = 1) \end{cases}$$

where π_l and π_r are left- and right-projections from a binary product;

(3) *Let us denote by $C_{b,a}$ the mediating iso 2-cell up-to which $C : \mathbb{L} \to \mathbf{CAT}$ preserves composition $b \circ a$.*

$$\underline{\textit{in } \mathbb{L}} \quad \begin{tikzcd} m \arrow[d, "a"] \\ n \arrow[d, "b"] \\ k \end{tikzcd} \qquad \underline{\textit{in } \mathbf{CAT}} \quad \begin{tikzcd} & Cm \arrow[dl, "Ca"'] \arrow[dd, bend left, "C(b\circ a)"] \\ Cn \arrow[dr, "Cb"'] \arrow[r, phantom, "\overset{C_{b,a}}{\underset{\cong}{\Longrightarrow}}"] & {} \\ & Ck \end{tikzcd} \tag{4}$$

We require that, when b is a projection $\pi_i : n \to 1$ (with $i \in [0, n-1]$), the mediating 2-cell $C_{\pi_i,a}$ is the identity:

$$C_{\pi_i,a} = \mathrm{id} \quad : \quad C\pi_i \circ Ca \Longrightarrow C(\pi_i \circ a) \ .$$

Note that this is not only asserting the equality $C\pi_i \circ Ca = C(\pi_i \circ a)$; but also the chosen mediating isomorphism $C_{\pi_i,a}$ is the identity.

We also require that when $b = \mathrm{id}$, the mediating 2-cell $C_{\mathrm{id},a}$ is the identity.

Remark 2.9 *In the definition, the codomain of a pseudo functor C need not be* **CAT***; it can in fact be any 2-category with chosen cartesian structure. We will not need such generality.*

Formalizing a suitable notion of pseudo functor preserving finite products might seem straightforward, but the actual conditions above are in fact quite delicate. One can even find them somewhat arbitrary; we now look at their equivalent conditions and consequences.

Lemma 2.10 *Assuming a pseudo functor $C : \mathbb{L} \to$* **Sets** *satisfies the conditions (1–2) in Def. 2.8, (3) is equivalent to the following one.*

(3') *Let p be a* basic morphism *in $\mathbb{L}$, that is, one in the image of the embedding $\mathbf{Nat}^{\mathrm{op}} \to \mathbb{L}$. In yet other words, it is a morphism made up using only projections and diagonals.*

Then the pseudo functor C preserves composition of the form $p \circ a$ up-to identity, that is,

$$C_{p,a} = \mathrm{id} \quad : \quad Cp \circ Ca \Longrightarrow C(p \circ a) \ .$$

PROOF. Since a projection π_i is a basic morphism, obviously (3') implies (3). We shall prove the converse.

A basic morphism $p : n \to k$ in $\mathbb{L}$ is identified with an arrow $n \to k$ in $\mathbf{Nat}^{\mathrm{op}}$, and further with an arrow $k \to n$ in **Nat**; this is a function. We shall denote all these three by the same symbol p. Then obviously we have

$$p = \langle \pi_{p(0)}, \pi_{p(1)}, \ldots, \pi_{p(k-1)} \rangle \quad : \quad n \to k \qquad \text{in } \mathbb{L},$$

where p's in the subscripts are for the function from $[0, k-1]$ to $[0, m-1]$.

Now let $a : m \to n$ be an arrow in $\mathbb{L}$ and $i \in [0, k-1]$. We have

$$
\begin{tikzcd}
Cn \arrow[rr, "Cp"] \arrow[dr, phantom, "\Downarrow C_{p,a}"] & & Ck \arrow[d, "\pi_i"] \\
Cm \arrow[u, "Ca"] \arrow[urr, "C(p\circ a)"'] & & C1
\end{tikzcd}
=
\begin{tikzcd}
Cn \arrow[rr, "Cp"] \arrow[r, phantom, "\Downarrow C_{p,a}"] & & Ck \arrow[d, "C\pi_i"] \\
Cm \arrow[u, "Ca"] \arrow[urr, "C(p\circ a)" description] \arrow[rr, "C(\pi_{p(i)}\circ a)"'] & \arrow[ur, phantom, "\Downarrow C_{\pi_i, p\circ a}"] & C1
\end{tikzcd}
\quad \text{by (3)}
$$

$$
=
\begin{tikzcd}
Cn \arrow[rr, "Cp"] \arrow[drr, "C(\pi_{p(i)})" description] \arrow[d, phantom, "\Downarrow C_{\pi_{p(i)},a}"] & \arrow[r, phantom, "\Downarrow C_{\pi_i,p}"] & Ck \arrow[d, "C\pi_i"] \\
Cm \arrow[u, "Ca"] \arrow[rr, "C(\pi_{p(i)}\circ a)"'] & & C1
\end{tikzcd}
\quad \text{by coh. cond.}
$$

$$
= \mathrm{id} \quad \text{by (3).}
$$

Since this holds for each $i \in [0, k-1]$, the 2-cell $C_{p,a}$ is the tuple $\langle \mathrm{id}, \ldots, \mathrm{id} \rangle = \mathrm{id}$. $\square$

A pseudo functor in general preserves identities only up-to isomorphisms. However, additional conditions on $\mathbb{L}$-categories force strict preservation.

Lemma 2.11 *Given an $\mathbb{L}$-category $C : \mathbb{L} \to \mathbf{CAT}$, as a pseudo functor it is equipped with an iso 2-cell*

$$
C_n \quad : \quad \mathrm{id}_{Cn} \overset{\cong}{\Longrightarrow} C(\mathrm{id}_n)
$$

up-to which it preserves the identity. This C_n is in fact the identity; in particular $C(id_n) = \mathrm{id}_{Cn}$.

PROOF. When $n = 0$, $C0$ is a (chosen) terminal object, so that there is exactly one arrow $C0 \to C0$.

When $n = 1$, the coherence condition on the mediating 2-cells of C requires the following 2-cells equal.

$$
\begin{tikzcd}
C1 \arrow[d, bend right=60, "\mathrm{id}_{C1}"'{name=L}] \arrow[d, bend left=60, "C\mathrm{id}"{name=R}] \arrow[from=L, to=R, phantom, "\overset{C_1}{\Rightarrow}"] \arrow[dd, bend left=90, "C\mathrm{id}"{name=O}] \\
C1 \arrow[d, "C\mathrm{id}"'] \arrow[r, phantom, "\overset{C_{\mathrm{id}_1,\mathrm{id}_1}}{\Rightarrow}"] & {} \\
C1
\end{tikzcd}
=
\begin{tikzcd}
C1 \arrow[d, bend right=60, "C\mathrm{id}"'{name=A}] \arrow[d, bend left=60, "C\mathrm{id}"{name=B}] \arrow[from=A, to=B, phantom, "\overset{\mathrm{id}}{\Rightarrow}"] \\
C1
\end{tikzcd}
$$

By the condition (3) in Def. 2.8 we have $C_{\mathrm{id}_1,\mathrm{id}_1} = \mathrm{id}$; and we have $C(\mathrm{id}_1) = \mathrm{id}_{C1}$ due to the condition (2) in Def. 2.8. This proves that the 2-cell C_1 in the diagram above is also the identity. The case when $n \geq 2$ is similar. $\square$

It was part of the definition that we have $C_{\mathrm{id},a} = \mathrm{id}$, that is, postcomposition of an identity is preserved on-the-nose. So is precomposition.

Lemma 2.12 *For an* $\mathbb{L}$*-category* $C : \mathbb{L} \to \mathbf{CAT}$*, the mediating iso 2-cell* $C_{b,\mathrm{id}}$ *(see (4)) is the identity.*

PROOF. By the coherence condition we have

$$
\begin{tikzcd}
Cn \arrow[d, bend right=60, "\mathrm{id}_{Cn}"'] \arrow[d, bend left=60, "C\mathrm{id}"] \arrow[d, phantom, "\overset{C_n}{\Rightarrow}"] \arrow[dd, bend left=90, "Cb"] \\
Cn \arrow[d, "Cb"'] \arrow[d, phantom, shift left=6ex, "\overset{C_{b,\mathrm{id}_n}}{\Rightarrow}"] \\
Cm
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
Cn \arrow[d, bend right=60, "Cb"'] \arrow[d, bend left=60, "Cb"] \arrow[d, phantom, "\overset{\mathrm{id}}{\Rightarrow}"] \\
Cm
\end{tikzcd}
.
$$

By Lem. 2.11 $C_n = \mathrm{id}$; hence we have $C_{b,\mathrm{id}_n} = \mathrm{id}$. □

We now introduce suitable 1-cells and 2-cells for the 2-category of $\mathbb{L}$-categories.

Definition 2.13 (Morphism of $\mathbb{L}$**-categories)** *A* morphism *from an* $\mathbb{L}$*-category* $C : \mathbb{L} \to \mathbf{CAT}$ *to* D *is a pseudo (or "strong") natural transformation* F *from* C *to* D*:*

$$
\begin{tikzcd}
\mathbb{L} \arrow[r, bend left, "C"] \arrow[r, bend right, "D"'] \arrow[r, phantom, "\Downarrow F"] & \mathbf{CAT}
\end{tikzcd}
$$

that is "finite product preserving" in the following sense:

- *For each* $a : n \to m$ *in* $\mathbb{L}$*, let us denote by* F_a *the natural isomorphism that fills in the pseudo naturality diagram of* F*. That is,*

$$
\underline{\text{in } \mathbb{L}} \quad
\begin{tikzcd}
n \arrow[d, "a"] \\
m
\end{tikzcd}
\qquad\qquad
\underline{\text{in } \mathbf{CAT}} \quad
\begin{tikzcd}
Cn \arrow[r, "F_n"] \arrow[d, "Ca"'] \arrow[dr, phantom, "\cong \;\; \swarrow F_a"] & Dn \arrow[d, "Da"] \\
Cm \arrow[r, "F_m"'] & Dm
\end{tikzcd}
\tag{5}
$$

  We require that, for a projection $a = \pi_i : n \to 1$*, the mediating 2-cell* F_{π_i} *is the identity.*

The "finite product preservation" condition has the following consequence. Therefore a morphism of $\mathbb{L}$-categories is really a functor $F : C1 \to D1$ that preserves $\mathbb{L}$-structure up-to coherent isomorphism.

Lemma 2.14 *Components of a morphism* $F : C \to D$ *of* $\mathbb{L}$*-categories are completely determined solely by its* 1*-component. Specifically, let us denote* F*'s* 1*-component* $F_1 : C1 \to D1$ *(which is a functor) also by* F*. Then a component* $F_n : Cn \to Dn$ *is the n-fold product* $F^{\times n}$ *of* F*.*

$$
\begin{tikzcd}
Cn \arrow[r, "F_n"] \arrow[d, equal] & Dn \arrow[d, equal] \\
(C1)^n \arrow[r, "F^{\times n} = (\cdots(F \times F) \times \cdots \times F)"'] & (D1)^n
\end{tikzcd}
$$

PROOF. Obvious from Def. 2.13 and the condition (2) in Def. 2.8. $\square$

Definition 2.15 (Transformation for $\mathbb{L}$-categories) *A* transformation $\varphi : F \Rightarrow G : C \to D$ *between morphisms of* $\mathbb{L}$*-categories is a modification between strong natural transformations*

$$\begin{tikzcd}[column sep=huge] \mathbb{L} \arrow[r, bend left, "C"] \arrow[r, bend right, "D"'] \arrow[r, phantom, "F\Downarrow \;\overset{\varphi}{\Rrightarrow}\; \Downarrow G"] & \mathbf{CAT} \end{tikzcd} .$$

The coherence condition for modifications, as well as the product preservation property of F and G, again forces components of φ to be determined by its 1*-component. That is, since we have*

$$\underline{\textit{in } \mathbb{L}} \quad \begin{tikzcd} n \arrow[d, "\pi_i"] \\ 1 \end{tikzcd} \qquad \underline{\textit{in } \mathbf{CAT}} \quad \begin{tikzcd} Cn \arrow[r, bend left, "F^{\times n}"] \arrow[d, "\pi_i"'] \arrow[dr, phantom, "/\!/"] & Dn \arrow[d, "\pi_i"] \\ C1 \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\varphi_1"] & D1 \end{tikzcd} = \begin{tikzcd} Cn \arrow[r, bend left, "F^{\times n}"] \arrow[r, bend right, "G^{\times n}"'] \arrow[r, phantom, "\Downarrow\varphi_n"] \arrow[d, "\pi_i"'] \arrow[dr, phantom, "/\!/"] & Dn \arrow[d, "\pi_i"] \\ C1 \arrow[r, bend right, "G"'] & D1 \end{tikzcd} ;$$

this forces φ_n to be equal to $(\varphi_1)^{\times n}$, the n-fold product of φ_1.

Given a Lawvere theory $\mathbb{L}$, $\mathbb{L}$-categories, morphisms and transformations form a 2-category. We denote it by $\mathbb{L}$-$\mathbf{CAT}$.

2.3 Equivalence between (Σ, E)-categories and $\mathbb{L}_{(\Sigma,E)}$-categories

Now we shall show that the two formalizations of "pseudo algebra" carried by a category—rather conventionally as (Σ, E)-category and using pseudo functors as $\mathbb{L}$-category—are in fact equivalent.

To put it precisely, let (Σ, E) be a (one-sorted) algebraic specification, and $\mathbb{L}_{(\Sigma,E)}$ be the Lawvere theory induced by (Σ, E). An arrow $a : n \to 1$ in $\mathbb{L}_{(\Sigma,E)}$ is identified with $[t]$, an n-ary Σ-term

$$x_0, \ldots, x_{n-1} \vdash t$$

modulo the formal equality $\sim$ derived from the axiom set E. An arrow $n \to m$ in $\mathbb{L}_{(\Sigma,E)}$ is a m-fold tuple of such. The following will be the main statement in this section; its precise formulation (with a minor side condition) is later in Thm. 2.29.

The 2-categories (Σ, E)-$\mathbf{CAT}$ and $\mathbb{L}_{(\Sigma,E)}$-$\mathbf{CAT}$ are equivalent.

The statement itself calls for some explanation. In category theory, when one goes to one higher dimension (from sets to categories, or from categories to

2-categories) the weakest possible notion of being "the same" is weakened. In a *set* two elements are "the same" when they are identical. In a *category*, *isomorphisms* between objects give a weaker (and usually more appropriate) notion of being "the same." Further, in a *2-category* the weakest notion is *equivalence*; and *biequivalence* (see e.g. [15]) in a *3-category*, etc. Categories themselves are organized in a 2-category, so two categories are "the same" when they are equivalent; 2-categories are "the same" when they are biequivalent.

As to Thm. 2.29, since (Σ, E)-$\mathbf{CAT}$ and $\mathbb{L}_{(\Sigma,E)}$-$\mathbf{CAT}$ are 2-categories, the weakest possible form of the result that they are "the same" asserts that they are biequivalent. This in fact might follow easily from well-known coherence results. In Thm. 2.29, however, we establish a stronger relation between them, namely equivalence. Unfortunately it does not go as far as establishing an isomorphism; the reason is explained later in Rem. 2.30.

2.3.1 A 2-functor $\Phi : \mathbb{L}_{(\Sigma,E)}$-$\mathbf{CAT} \to (\Sigma, E)$-$\mathbf{CAT}$

We start with describing a 2-functor $\Phi : \mathbb{L}_{(\Sigma,E)}$-$\mathbf{CAT} \to (\Sigma, E)$-$\mathbf{CAT}$. Later it is shown to be faithful, full and surjective on objects, hence establishing an equivalence of 2-categories.

On objects, given an $\mathbb{L}_{(\Sigma,E)}$-category $C : \mathbb{L}_{(\Sigma,E)} \to \mathbf{CAT}$, we define ΦC to be

- its underlying category is given by $C1 \in \mathbf{CAT}$;
- its interpretation $[\![\sigma]\!]_{\Phi C}$ of an operation $\sigma \in \Sigma(n)$ is given by $(C1)^n \xrightarrow{C[\sigma]} C1$, C's action on an arrow $[\sigma] : n \to 1$ in $\mathbb{L}_{(\Sigma,E)}$. Recall that $[\_\,]$ denotes taking a quotient modulo $\sim$, the formal equality derived from axioms in E.

To describe γ_s^t up-to which an equation $(s = t) \in E$ holds, we need some preparation.

We have defined $[\![\sigma]\!]_{\Phi C} := C\sigma$, for each operation $\sigma \in \Sigma(n)$. As in Def. 2.1, the interpretation $[\![\_\,]\!]_{\Phi C}$ is extended to interpretation $[\![t]\!]_{\Phi C}$ of any Σ-term t in an inductive manner.

Now we shall introduce, for each Σ-term $x_0, \ldots, x_{n-1} \vdash t$, a natural isomorphism γ_t.

$$\begin{tikzcd} (C1)^n \arrow[r, bend left, "{[\![t]\!]_{\Phi C}}"] \arrow[r, bend right, "{C[t]}"'] \arrow[r, phantom, "\cong\Downarrow\gamma_t"] & C1 \end{tikzcd} \tag{6}$$

Here both of $[\![t]\!]_{\Phi C}$ and $C[t]$ are "interpretation of a term t" induced by C; the former being built up inductively from the interpretation of operations $[\![\sigma]\!]_{\Phi C} = C\sigma$, and the latter simply being C's action on arrows. When C is a strict functor they are *exactly* the same; now because C is only a pseudo functor they are only *essentially* the same, that is, the same up-to the coherent isomorphism γ_t.

Definition 2.16 (Natural isomorphism γ_t) *Given an $\mathbb{L}_{(\Sigma,E)}$-functor C, a natural isomorphism γ_t (see (6)) is defined inductively on the construction of a term t.*

- *If t is a variable $x_0, \ldots, x_{n-1} \vdash x_i$, the term t is a projection $\pi_i : n \to 1$ as an arrow in $\mathbb{L}_{(\Sigma,E)}$. Hence*

$$
\begin{aligned}
&C[t] = C\pi_i = \pi_i && \text{by Def. 2.8(2),}\\
&[\![t]\!]_{\Phi C} = \pi_i && \text{by Def. 2.1.}
\end{aligned}
$$

  We set $\gamma_{x_i} := \mathrm{id}$ in this case.
- *If t is a composed term $x_0, \ldots, x_{n-1} \vdash \sigma(t_0, \ldots, t_{m-1})$ with $\sigma \in \Sigma(m)$, we set γ_t to be the following composed 2-cell.*

$$
\gamma_{\sigma(t_0,\ldots,t_{m-1})} \quad := \quad
\begin{tikzcd}
C1^n \arrow[r, bend left, "{\langle[\![t_0]\!],\ldots,[\![t_{m-1}]\!]\rangle}"] \arrow[r, bend right, "{\langle C[t_0],\ldots,C[t_{m-1}]\rangle = C\langle[t_0],\ldots,[t_{m-1}]\rangle}"'] \arrow[r, phantom, "\Downarrow\langle\gamma_{t_0},\ldots,\gamma_{t_{m-1}}\rangle"] \arrow[rr, bend left=60, "{[\![\sigma(t_0,\ldots,t_{m-1})]\!]}"] \arrow[rr, bend right=60, "{C([\sigma(t_0,\ldots,t_{m-1})])}"'] & C1^m \arrow[r, "{[\![\sigma]\!] = C[\sigma]}"] \arrow[u, phantom, "\parallel"] \arrow[d, phantom, "\Downarrow C_{[\sigma],\langle[t_0],\ldots,[t_{m-1}]\rangle}"] & C1
\end{tikzcd}
\quad ;
$$

  here the upper square commutes due to Def. 2.1; γ_{t_i} is already defined by the induction hypothesis; and the natural isomorphism $C_{[\sigma],\langle[t_0],\ldots,[t_{m-1}]\rangle}$ is the one up-to which the pseudo functor C preserves composition.

The natural isomorphism $\gamma_t : [\![t]\!]_{\Phi C} \Rightarrow C[t]$, thus defined for each Σ-term t, is compatible with substitution (cf. Lem. 2.2).

Lemma 2.17 *Let $x_0, \ldots, x_{m-1} \vdash t'$ be a Σ-term with m variables, and $x_0, \ldots, x_{n-1} \vdash t_i$ be ones with n variables, for each $i \in [0, m-1]$. Then we have*

$$
\gamma_{t'[t_0/x_0,\ldots,t_{m-1}/x_{m-1}]} \quad = \quad
\begin{tikzcd}
C1^n \arrow[r, bend left, "{\langle[\![t_0]\!],\ldots,[\![t_{m-1}]\!]\rangle}"] \arrow[r, bend right, "{\langle C[t_0],\ldots,C[t_{m-1}]\rangle = C\langle[t_0],\ldots,[t_{m-1}]\rangle}"'] \arrow[r, phantom, "\Downarrow\langle\gamma_{t_0},\ldots,\gamma_{t_{m-1}}\rangle"] \arrow[rr, bend left=60, "{[\![t'[t_0/x_0,\ldots,t_{m-1}/x_{m-1}]]\!]}"] \arrow[rr, bend right=60, "{C([t'[t_0/x_0,\ldots,t_{m-1}/x_{m-1}]]) = C([t']\circ\langle[t_0],\ldots,[t_{m-1}]\rangle)}"'] & C1^m \arrow[r, bend left, "{[\![t']\!]}"] \arrow[r, bend right, "{C[t']}"'] \arrow[r, phantom, "\Downarrow\gamma_{t'}"] \arrow[u, phantom, "\parallel"] \arrow[d, phantom, "\Downarrow C_{[t'],\langle[t_0],\ldots,[t_{m-1}]\rangle}"] & C1
\end{tikzcd}
\quad .
$$

PROOF. The upper square commutes due to Lem. 2.2. We prove the lemma by induction on the construction of the term t'.

When $t' = x_i$, a variable, we have

$$
\begin{array}{lll}
\gamma_{t'[t_0/x_0,\ldots,t_{m-1}/x_{m-1}]} = \gamma_{t_i} & & \\
\gamma_{t'} = \gamma_{x_i} = \mathrm{id} & & \text{by Def. 2.16} \\
C_{[t'],\langle[t_0],\ldots,[t_{m-1}]\rangle} = C_{\pi_i,\langle[t_0],\ldots,[t_{m-1}]\rangle} = \mathrm{id} & & \text{by Def. 2.8(3)}
\end{array}
$$

therefore both the sides of the equality reduce to γ_{t_i}.

Let t' be a composed term $\sigma(s_0,\ldots,s_{k-1})$ with $\sigma \in \Sigma(k)$ and $x_0,\ldots,x_{m-1} \vdash s_j$. First we observe that the substitution distributes to subterms:

$$
t'[t_0/x_0,\ldots,t_{m-1}/x_{m-1}] = \sigma\Big(s_o[\vec{t}/\vec{x}],\ldots,s_{k-1}[\vec{t}/\vec{x}]\Big)\ . \tag{7}
$$

We use this in the following calculation, as well as the coherence condition on the mediating 2-cells $C_{b,a}$ of a pseudo functor C.

$$
\gamma_{t'[t_0/x_0,\ldots,t_{m-1}/x_{m-1}]} =
\begin{tikzcd}
& (C1)^m \arrow[dr, "{\langle[\![s_0]\!],\ldots,[\![s_{k-1}]\!]\rangle}"] & & \\
(C1)^n \arrow[ur, "{\langle[\![t_0]\!],\ldots,[\![t_{m-1}]\!]\rangle}"] \arrow[rr, "{C\langle[s_0[\vec{t}/\vec{x}]],\ldots,[s_{k-1}[\vec{t}/\vec{x}]]\rangle}"'] \arrow[r, phantom, "{\Downarrow\langle\gamma_{s_0[\vec{t}/\vec{x}]},\ldots,\gamma_{s_{k-1}[\vec{t}/\vec{x}]}\rangle}"] \arrow[rrr, bend right=50] & & (C1)^k \arrow[r, "{[\![\sigma]\!]=C[\sigma]}"] \arrow[d, phantom, "{\Downarrow C_{[\sigma],\langle[s_0[\vec{t}/\vec{x}]],\ldots,[s_{k-1}[\vec{t}/\vec{x}]]\rangle}}"] & C1 \\
& & {} &
\end{tikzcd}
\quad \text{by Def. 2.16}
$$

$$
=
\begin{tikzcd}
& (C1)^m \arrow[dr, bend left] \arrow[dr, bend right] \arrow[dr, phantom, "\Downarrow\langle\overrightarrow{\gamma_s}\rangle"] & & \\
(C1)^n \arrow[ur, bend left] \arrow[ur, bend right] \arrow[ur, phantom, "\Downarrow\langle\overrightarrow{\gamma_t}\rangle"] \arrow[rr] \arrow[r, phantom, "{\Downarrow C_{\langle[\vec{s}]\rangle,\langle[\vec{t}]\rangle}}"] \arrow[rrr, bend right=50] & & (C1)^k \arrow[r, "{[\![\sigma]\!]=C[\sigma]}"] \arrow[d, phantom, "{\Downarrow C_{[\sigma],\langle[s_0[\vec{t}/\vec{x}]],\ldots,[s_{k-1}[\vec{t}/\vec{x}]]\rangle}}"] & C1 \\
& & {} &
\end{tikzcd}
\quad \text{by ind. hyp.}
$$

$$
=
\begin{tikzcd}
& (C1)^m \arrow[dr, bend left] \arrow[dr, bend right] \arrow[dr, phantom, "\Downarrow\langle\overrightarrow{\gamma_s}\rangle"] \arrow[rr, bend right=60] & & \\
(C1)^n \arrow[ur, bend left] \arrow[ur, bend right] \arrow[ur, phantom, "\Downarrow\langle\overrightarrow{\gamma_t}\rangle"] \arrow[rrr, bend right=50] & & (C1)^k \arrow[r, "{[\![\sigma]\!]=C[\sigma]}"] \arrow[d, phantom, "{\Downarrow C_{[\sigma],\langle[\vec{s}]\rangle}}"] & C1 \\
& & {\Downarrow C_{[\sigma]\circ\langle[\vec{s}]\rangle,\langle[\vec{t}]\rangle}} &
\end{tikzcd}
\quad \text{by coh. cond.}
$$

$$
=
\begin{tikzcd}
& (C1)^m \arrow[dr] \arrow[rr, bend right=60] & & \\
(C1)^n \arrow[ur, bend left] \arrow[ur, bend right] \arrow[ur, phantom, "\Downarrow\langle\overrightarrow{\gamma_t}\rangle"] \arrow[rrr, bend right=50] & & (C1)^k \arrow[r, "{[\![\sigma]\!]=C[\sigma]}"] \arrow[d, phantom, "\Downarrow\gamma_{t'}"] & C1 \\
& & {\Downarrow C_{[\sigma]\circ\langle[\vec{s}]\rangle,\langle[\vec{t}]\rangle}} &
\end{tikzcd}
\quad \text{by Def. 2.16.}
$$

This concludes the proof. $\square$

We use this natural isomorphism $\gamma_t : [\![t]\!]_{\Phi C} \Rightarrow C[t]$ to define a natural isomorphism $\gamma_s^t : [\![s]\!]_{\Phi C} \Rightarrow [\![t]\!]_{\Phi C}$ up-to which an equation $(s = t) \in E$ is satisfied in the (Σ, E)-category ΦC.

Definition 2.18 (Natural isomorphism γ_s^t) *Given an $\mathbb{L}_{(\Sigma,E)}$-category $C : \mathbb{L}_{(\Sigma,E)} \to$ **CAT** and an equation $(s = t) \in E$, we define a natural isomorphism*

γ_s^t *to be the following composite.*

$$\gamma_s^t \quad := \quad \begin{tikzcd}[column sep=huge] (C1)^n \arrow[r, bend left=40, "{[\![s]\!]_{\Phi C}}"] \arrow[r, "{C[s]=C[t]}" description] \arrow[r, bend right=40, "{[\![t]\!]_{\Phi C}}"'] \arrow[r, phantom, bend left=20, "\Downarrow\ \gamma_s"] \arrow[r, phantom, bend right=20, "\Downarrow\ \gamma_t^{-1}"] & C1 \end{tikzcd}$$

Here γ_s and γ_t are the ones introduced in Def. 2.16. We have $C[s] = C[t]$: since $s = t$ is in E as an axiom, we have $[s] = [t]$ as arrows of $\mathbb{L}_{(\Sigma,E)}$.

Finally, to summarize:

Definition 2.19 (Φ on 0-cells) *Given an $\mathbb{L}_{(\Sigma,E)}$-category $C : \mathbb{L}_{(\Sigma,E)} \to$ **CAT**, we define a (Σ, E)-category ΦC as follows (recall Def. 2.4).*

- *Its underlying category is given by $C1 \in$ **CAT**.*
- *Its interpretation $[\![\sigma]\!]_{\Phi C}$ of an operation $\sigma \in \Sigma(n)$ is given by $(C1)^n \xrightarrow{C[\sigma]} C1$, C's action on an arrow $[\sigma] : n \to 1$ in $\mathbb{L}_{(\Sigma,E)}$.*
- *The isomorphism γ_s^t—up-to which each equation $(s = t) \in E$ is satisfied—is as introduced in Def. 2.18.*

Lemma 2.20 *The isomorphisms $\{\gamma_s^t\}_{(s=t)\in E}$ indeed satisfy the coherence condition (Cond. 2.3).*

PROOF. We prove the following statement which yields the lemma immediately:

For each derivation $\mathcal{D}$ of $s \sim t$, we have

$$\gamma_{\mathcal{D}} \quad = \quad \begin{tikzcd}[column sep=huge] (C1)^n \arrow[r, bend left=40, "{[\![s]\!]_{\Phi C}}"] \arrow[r, "{C[s]=C[t]}" description] \arrow[r, bend right=40, "{[\![t]\!]_{\Phi C}}"'] \arrow[r, phantom, bend left=20, "\Downarrow\ \gamma_s"] \arrow[r, phantom, bend right=20, "\Downarrow\ \gamma_t^{-1}"] & C1 \end{tikzcd} \quad . \tag{8}$$

The composite $\gamma_t^{-1} \bullet \gamma_s$ was the definition of the 2-cell γ_s^t for equational axioms; now we shall prove that the same composite determines $\gamma_{\mathcal{D}}$ for any derivation.

We prove (8) by induction on derivation.

- When the last rule applied in $\mathcal{D}$ is the (Ax) rule, then

$$\begin{aligned} \gamma_{\mathcal{D}} &= \gamma_s^t && \text{by def. of } \gamma_{\mathcal{D}} \\ &= \gamma_t^{-1} \bullet \gamma_s && \text{by Def. 2.18.} \end{aligned}$$

- When $\mathcal{D}$ is of the form

$$\frac{\overset{\mathcal{D}'}{s' \sim t'} \quad \overset{\mathcal{D}_0}{s_0 \sim t_0} \quad \cdots \quad \overset{\mathcal{D}_{n-1}}{s_{n-1} \sim t_{n-1}}}{s'[s_0/x_0, \dots, s_{m-1}/x_{m-1}] \sim t'[t_0/x_0, \dots, t_{m-1}/x_{m-1}]} \ (\textsc{Cong})$$

  we have

$$\begin{aligned}
\gamma_{\mathcal{D}} &= \begin{tikzcd}[column sep=huge] (C1)^n \arrow[r, bend left, "\langle[\![s_0]\!],\dots,[\![s_{m-1}]\!]\rangle"] \arrow[r, bend right, "\langle[\![t_0]\!],\dots,[\![t_{m-1}]\!]\rangle"'] \arrow[r, phantom, "\Downarrow\langle\gamma_{\mathcal{D}'},\dots,\gamma_{\mathcal{D}_{m-1}}\rangle"] & (C1)^m \arrow[r, bend left, "[\![s']\!]"] \arrow[r, bend right, "[\![t']\!]"'] \arrow[r, phantom, "\Downarrow\gamma_{\mathcal{D}'}"] & (C1) \end{tikzcd} && \text{by def. of } \gamma_{\mathcal{D}} \\
&= \begin{tikzcd}[column sep=huge] (C1)^n \arrow[r, bend left, ""{name=A}] \arrow[r, bend right, ""'{name=B}] \arrow[r, "\Downarrow\langle\gamma_{s_0},\dots,\gamma_{s_{m-1}}\rangle", "\Downarrow\langle\gamma_{t_0}^{-1},\dots,\gamma_{t_{m-1}}^{-1}\rangle"'] & (C1)^m \arrow[r, bend left] \arrow[r, bend right] \arrow[r, "\Downarrow\gamma_{s'}", "\Downarrow\gamma_{t'}^{-1}"'] & (C1) \end{tikzcd} && \text{by ind. hyp.} \\
&= \begin{tikzcd}[column sep=large] & (C1)^m \arrow[dr, "\Downarrow\gamma_{s'}"] \arrow[dr, bend left] & \\ (C1)^n \arrow[ur, "\Downarrow\langle\gamma_{s_0},\dots,\gamma_{s_{m-1}}\rangle"] \arrow[ur, bend left] \arrow[dr, "\Downarrow\langle\gamma_{t_0}^{-1},\dots,\gamma_{t_{m-1}}^{-1}\rangle"'] \arrow[dr, bend right] \arrow[rr, "{C[s'[\vec{s}/\vec{x}]]=C[t'[\vec{t}/\vec{x}]]}" description] \arrow[rr, phantom, shift left=3, "\Downarrow C_{[s'],\langle[s_0],\dots,[s_{m-1}]\rangle}"] \arrow[rr, phantom, shift right=3, "\Downarrow C^{-1}_{[t'],\langle[t_0],\dots,[t_{m-1}]\rangle}"'] & & (C1) \\ & (C1)^m \arrow[ur, "\Downarrow\gamma_{t'}^{-1}"'] \arrow[ur, bend right] & \end{tikzcd} && (\dagger) \\
&= \gamma^{-1}_{t'[t_0/x_0,\dots,t_{m-1}/x_{m-1}]} \bullet \gamma_{s'[s_0/x_0,\dots,s_{m-1}/x_{m-1}]} && \text{by Lem. 2.17.}
\end{aligned}$$

  Here $(\dagger)$ is because, since $s' \sim t'$ and $s_i \sim t_i$ we have $C_{[s'],\langle[s_0],\dots,[s_{m-1}]\rangle} = C_{[t'],\langle[t_0],\dots,[t_{m-1}]\rangle}$.
- The proof is obvious when the last rule in $\mathcal{D}$ is either (Refl), (Trans) or (Symm). $\square$

We have described the action on objects of $\Phi : \mathbb{L}_{(\Sigma,E)}$-**CAT** $\to (\Sigma, E)$-**CAT**. We proceed to its action on 1- and 2-cells.

Definition 2.21 (Φ on 1-cells) *Given a morphism $F : C \to D$ of $\mathbb{L}_{(\Sigma,E)}$-categories, we define a morphism*

$$\Phi F : \Phi C \longrightarrow \Phi D$$

of (Σ, E)-categories (cf. Def. 2.5) as follows.

- *ΦF's underlying functor is F's 1-component $F_1 : C1 \to D1$.*
- *For each operation $\sigma \in \Sigma(n)$, we need a natural isomorphism up-to which ΦF preserves (interpretation of) σ. For this we simply take the pseudo naturality 2-cell $F_{[\sigma]}$ of F:*

$$\underline{\text{in } \mathbb{L}} \quad \begin{tikzcd} n \arrow[d, "\sigma"] \\ 1 \end{tikzcd} \qquad \underline{\text{in } \mathbf{CAT}} \quad \begin{tikzcd}[column sep=huge] Cn \arrow[r, "F_n = F^{\times n}"] \arrow[d, "{C[\sigma]=[\![\sigma]\!]_{\Phi C}}"'] \arrow[dr, phantom, "\overset{\cong}{\swarrow} F_{[\sigma]}"] & Dn \arrow[d, "{D[\sigma]=[\![\sigma]\!]_{\Phi D}}"] \\ C1 \arrow[r, "F"'] & D1 \end{tikzcd}$$

That is,

$$\Phi F \;=\; \Big(\, F_1 : C1 \to D1, \; \{F_{[\sigma]}\}_{\sigma\in\Sigma} \, \Big) \;, \qquad (\Phi F)_\sigma \;=\; F_{[\sigma]} \;\; .$$

As described in Def. 2.5, the mediating isomorphism $(\Phi F)_\sigma$ for each operation σ is extended to one $(\Phi F)_t$ for each term t, in an inductive manner. Note that $(\Phi F)_t$—defined inductively on construction of a term t—is not exactly the same as $F_{[t]}$ which is part of the definition of a pseudo natural transformation $F : C \to D$. They are, however, related via γ_t from Def. 2.16. This relationship is used (Cor. 2.23) in proving that ΦF in Def. 2.21 indeed satisfy the coherence condition (Def. 2.5).

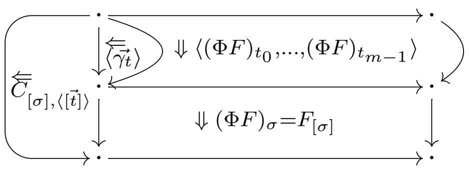
Lemma 2.22 *For each Σ-term t, we have the following natural isomorphisms equal.*

$$\begin{tikzcd} Cn \arrow[r, "F^{\times n}"] \arrow[d, bend right, "{C[t]}"'] \arrow[d, bend left, "{[\![t]\!]_{\Phi C}}"] \arrow[d, phantom, "\overset{\gamma_t}{\Leftarrow}"] \arrow[dr, phantom, "\cong\;(\Phi F)_t"] & Dn \arrow[d, bend left, "{[\![t]\!]_{\mathbb{D}}}"] \\ C1 \arrow[r, "F"'] & D1 \end{tikzcd} \;=\; \begin{tikzcd} Cn \arrow[r, "F^{\times n}"] \arrow[d, bend right, "{C[t]}"'] \arrow[dr, phantom, "\cong\;F_{[t]}"] & Dn \arrow[d, bend right, "{D[t]}"'] \arrow[d, bend left, "{[\![t]\!]_{\Phi D}}"] \arrow[d, phantom, "\overset{\delta_t}{\Leftarrow}"] \\ C1 \arrow[r, "F"'] & D1 \end{tikzcd}$$

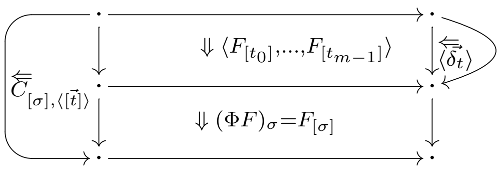
Here δ_t is to D what γ_t is to C; see Def. 2.16.

PROOF. By induction on construction of a term t. When t is a variable x_i, then $[t] : n \to 1$ as an arrow in $\mathbb{L}_{(\Sigma,E)}$ is a projection π_i. Therefore, due to the finite product preservation in Def. 2.13, we have $F_{[t]} = \mathrm{id}$. By definition the other 2-cells in the equation are all identities too; hence both sides are the identity.

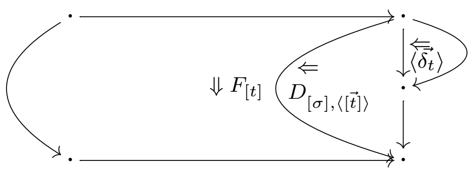
When t is a composed term $\sigma(t_0, \ldots, t_{m-1})$, we proceed as follows.

(LHS) = 

by def. of γ_t and $(\Phi F)_t$

= 

by ind. hyp.

= 

by coh. cond. on F

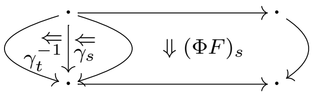
= (RHS) .

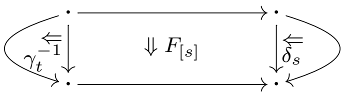
This concludes the proof.

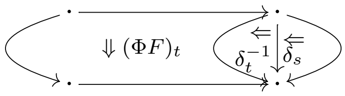
Corollary 2.23 *The data $\Phi F = (F_1, \{F_{[\sigma]}\}_{\sigma\in\Sigma})$ in Def. 2.21 indeed satisfy the coherence condition in Def. 2.5. That is, for each equation $(s = t) \in E$, we have the follows 2-cells equal.*

$$
\begin{tikzcd}
Cn \arrow[r, "F^{\times n}"] \arrow[d, bend right=60, "{[\![t]\!]_{\Phi C}}"'] \arrow[d, bend right=20, "{[\![s]\!]_{\Phi C}}"] \arrow[d, phantom, bend right=40, "\overset{\gamma_s^t}{\Leftarrow}"] \arrow[dr, phantom, "\overset{\cong}{\swarrow}(\Phi F)_s"] & Dn \arrow[d, bend left=20, "{[\![s]\!]_{\Phi D}}"] \\
C1 \arrow[r, "F"'] & D1
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
Cn \arrow[r, "F^{\times n}"] \arrow[d, bend right=20, "{[\![t]\!]_{\Phi C}}"'] \arrow[dr, phantom, "\overset{\cong}{\swarrow}(\Phi F)_t"] & Dn \arrow[d, bend right=20, "{[\![t]\!]_{\Phi D}}"'] \arrow[d, bend left=60, "{[\![s]\!]_{\Phi D}}"] \arrow[d, phantom, bend left=40, "\overset{\delta_s^t}{\Leftarrow}"] \\
C1 \arrow[r, "F"'] & D1
\end{tikzcd}
$$

PROOF.

(LHS) =  by def. of γ_s^t

=  by Lem. 2.22

=  by Lem. 2.22

= (RHS) . □

Definition 2.24 (Φ on 2-cells) *Given a transformation $\varphi : F \Rightarrow G : C \to D$ for $\mathbb{L}_{(\Sigma,E)}$-categories, we define a transformation for (Σ, E)-categories (cf. Def. 2.5)*

$$\Phi\varphi \;:\; \Phi F \Longrightarrow \Phi G \;:\; \Phi C \longrightarrow \Phi D$$

to be φ's 1-component

$$\varphi_1 \;:\; F_1 \Longrightarrow G_1 \;\; .$$

Lemma 2.25 *The transformation $\Phi\varphi = \varphi_1$ in the definition indeed satisfies the coherence condition. That is, for each operation $\sigma \in \Sigma_n$ we have the following 2-cells equal.*

$$
\underline{\textit{in } \mathbf{CAT}}\quad
\begin{tikzcd}
Cn \arrow[r, bend left, "F^{\times n}"] \arrow[d, "{[\![\sigma]\!]_{\Phi C}}"'] \arrow[dr, phantom, "\Downarrow (\Phi F)_\sigma"] & Dn \arrow[d, "{[\![\sigma]\!]_{\Phi D}}"] \\
C1 \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\varphi_1"] & D1
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
Cn \arrow[r, bend left, "F^{\times n}"] \arrow[r, bend right, "G^{\times n}"'] \arrow[r, phantom, "\Downarrow(\varphi_1)^{\times n}"] \arrow[d, "{[\![\sigma]\!]_{\Phi C}}"'] \arrow[dr, phantom, "\Downarrow (\Phi G)_\sigma"] & Dn \arrow[d, "{[\![\sigma]\!]_{\Phi D}}"] \\
C1 \arrow[r, bend right, "G"'] & D1
\end{tikzcd}
$$

PROOF. Immediate from the coherence condition for $\varphi : F \Rightarrow G$ as a 2-cell in $\mathbb{L}_{(\Sigma,E)}$-**CAT**, and that $\varphi_n = (\varphi_1)^{\times n}$ (see a remark in Def. 2.15). □

Obviously Φ's action on 0-, 1- and 2-cells that we have introduced preserve identities and composition. Therefore we have defined a 2-functor $\Phi : \mathbb{L}_{(\Sigma,E)}\text{-}\mathbf{CAT} \to (\Sigma, E)\text{-}\mathbf{CAT}$.

2.3.2 $\Phi : \mathbb{L}_{(\Sigma,E)}\text{-}\mathbf{CAT} \to (\Sigma, E)\text{-}\mathbf{CAT}$ *is an equivalence*

To prove our main goal (Thm. 2.29), we shall now establish that the 2-functor Φ is an equivalence (Lem. 2.26 and 2.28).

Lemma 2.26 *Φ is full and faithful as a 2-functor. That is, for any C, D in $\mathbb{L}_{(\Sigma,E)}\text{-}\mathbf{CAT}$, Φ's action*

$$\Phi_{C,D} \;:\; \mathbb{L}_{(\Sigma,E)}\text{-}\mathbf{CAT}(C, D) \longrightarrow (\Sigma, E)\text{-}\mathbf{CAT}(\Phi C, \Phi D)$$

is an isomorphism of categories.

PROOF. For $\Phi_{C,D}$, we construct its (local) inverse

$$\Psi_{C,D} \;:\; (\Sigma, E)\text{-}\mathbf{CAT}(\Phi C, \Phi D) \longrightarrow \mathbb{L}_{(\Sigma,E)}\text{-}\mathbf{CAT}(C, D) \;. \tag{9}$$

In the sequel we denote $\Psi_{C,D}$ simply by Ψ. Note that we are not aiming at a global inverse $\Psi : \mathbb{L}_{(\Sigma,E)}\text{-}\mathbf{CAT} \to (\Sigma, E)\text{-}\mathbf{CAT}$; this is purely for notational convenience. Given a 1-cell $F : \Phi C \to \Phi D$ in $(\Sigma, E)\text{-}\mathbf{CAT}$, we define $\Psi F : C \to D$ to be as follows.

- Its components are

$$(\Psi F)_n \;:=\; F^{\times n} \quad : \quad Cn \longrightarrow Dn \;;$$

- for each arrow $a : n \to 1$ in $\mathbb{L}_{(\Sigma,E)}$, the mediating isomorphism $(\Psi F)_a$ (in the diagram (5)) is defined to be the following composite.

$$(\Psi F)_a \quad := \quad \begin{tikzcd} Cn \arrow[r, "F^{\times n}"] \arrow[d, bend right=60, "Ca=C[t]"'] \arrow[d, "{[\![t]\!]_{\Phi C}}"] \arrow[d, phantom, shift right=3, "\overset{\gamma_t}{\Leftarrow}"] \arrow[dr, phantom, "\overset{\cong}{\Leftarrow}_{F_t}"] & Dn \arrow[d, "{[\![t]\!]_{\Phi D}}"'] \arrow[d, bend left=60, "Da=D[t]"] \arrow[d, phantom, shift left=3, "\overset{\delta_t^{-1}}{\Leftarrow}"] \\ C1 \arrow[r, "F"'] & D1 \end{tikzcd} \tag{10}$$

Here t is a Σ-term such that $[t] = a$; F_t is F's mediating isomorphism extended to a term t (Def. 2.5); and γ_t and δ_t are induced by $C, D : \mathbb{L}_{(\Sigma,E)} \to \mathbf{CAT}$ as in Def. 2.16.

We have to check that $(\Phi F)_a$ is well-defined; the above definition apparently depends on the choice of the term t. Let $s \sim t$, hence $a = [s] = [t]$. We have

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, bend right=60] \arrow[d] \arrow[d, phantom, shift right=3, "\underset{\gamma_t}{\Leftarrow}"] \arrow[dr, phantom, "\Downarrow F_t"] & \cdot \arrow[d] \arrow[d, bend left=60] \arrow[d, phantom, shift left=3, "\underset{\delta_t^{-1}}{\Leftarrow}"] \\ \cdot \arrow[r] & \cdot \end{tikzcd} \quad \overset{(\dagger)}{=} \quad \begin{tikzcd} \cdot \arrow[r] \arrow[d, bend right=60] \arrow[d] \arrow[d, bend left=60] \arrow[d, phantom, shift right=3, "\underset{\gamma_t}{\Leftarrow}"] \arrow[d, phantom, shift left=3, "\underset{\gamma_s^t}{\Leftarrow}"] \arrow[dr, phantom, "\Downarrow F_s"] & \cdot \arrow[d, bend right=60] \arrow[d] \arrow[d, bend left=60] \arrow[d, phantom, shift right=3, "\underset{\delta_t^s}{\Leftarrow}"] \arrow[d, phantom, shift left=3, "\underset{\delta_t^{-1}}{\Leftarrow}"] \\ \cdot \arrow[r] & \cdot \end{tikzcd} \quad \overset{(\ddagger)}{=} \quad \begin{tikzcd} \cdot \arrow[r] \arrow[d, bend right=60] \arrow[d] \arrow[d, phantom, shift right=3, "\underset{\gamma_s}{\Leftarrow}"] \arrow[dr, phantom, "\Downarrow F_s"] & \cdot \arrow[d] \arrow[d, bend left=60] \arrow[d, phantom, shift left=3, "\underset{\delta_s^{-1}}{\Leftarrow}"] \\ \cdot \arrow[r] & \cdot \end{tikzcd} \;.$$

Here (†) holds due to the coherence condition in Def. 2.5; (‡) holds due to the definition of γ^t_s for ΦC (Def. 2.18), namely $\gamma^t_s = \gamma_t^{-1} \bullet \gamma_s$.

As to the coherence condition for a pseudo natural transformation (compatibility with identities and composition in $\mathbb{L}_{(\Sigma,E)}$), showing that $\{(\Psi F)_a\}_a$ satisfies it is easy. The proof uses the definition of γ_t (Def. 2.16). Thus we have shown that $\Psi F : C \to D$ is indeed a 1-cell in $\mathbb{L}_{(\Sigma,E)}$-**CAT**.

Now we describe $\Psi = \Psi_{C,D}$'s (see (9)) action on arrows, that is, on 2-cells in (Σ, E)-**CAT**. Given

$$\varphi \;:\; F \Longrightarrow G \;:\; \Phi C \longrightarrow \Phi D \qquad \text{in } (\Sigma, E)\text{-}\mathbf{CAT},$$

$\Psi\varphi : \Psi F \Rightarrow \Psi G$ is given by the components

$$(\Psi\varphi)_n \;:=\; \varphi^{\times n} \quad .$$

We need to check its coherence condition (as a modification). Let $a : n \to 1$ be an arrow in $\mathbb{L}_{(\Sigma,E)}$, and t be a Σ-term such that $a = [t]$. Then

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow (\Psi F)_a"] & \cdot \arrow[d] \\ \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\varphi"] & \cdot \end{tikzcd}
\overset{(10)}{=}
\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[d, bend right=80] \arrow[d, phantom, swap, "\overset{\Leftarrow}{\gamma_t}"] \arrow[dr, phantom, "\Downarrow F_t"] & \cdot \arrow[d] \arrow[d, bend left=80] \arrow[d, phantom, "\overset{\Leftarrow}{\delta_t^{-1}}"] \\ \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\varphi"] & \cdot \end{tikzcd}
\overset{(\dagger)}{=}
\begin{tikzcd} \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\varphi^{\times n}"] \arrow[d] \arrow[d, bend right=80] \arrow[d, phantom, swap, "\overset{\Leftarrow}{\gamma_t}"] \arrow[dr, phantom, "\Downarrow G_t"] & \cdot \arrow[d] \arrow[d, bend left=80] \arrow[d, phantom, "\overset{\Leftarrow}{\delta_t^{-1}}"] \\ \cdot \arrow[r] & \cdot \end{tikzcd}
\overset{(10)}{=}
\begin{tikzcd} \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\varphi^{\times n}"] \arrow[d] \arrow[dr, phantom, "\Downarrow (\Psi G)_a"] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd} \quad .$$

Here (†) holds due to the coherence condition for φ as a 2-cell in (Σ, E)-**CAT** (Def. 2.5). Functoriality of $\Psi_{C,D}$ is obvious; thus we have described the functor $\Psi_{C,D}$ in (9).

It remains to be shown that $\Psi_{C,D}$ is the inverse of $\Phi_{C,D}$. To show that $\Phi_{C,D} \circ \Psi_{C,D} = \mathrm{id}$ we have to prove, among others, the equality $(\Phi_{C,D}\Psi_{C,D}F)_\sigma = F_\sigma$ of mediating isomorphisms. We have

$$(\Phi_{C,D}\Psi_{C,D}F)_\sigma \overset{\text{Def. 2.21}}{=} (\Psi_{C,D}F)_{[\sigma]} \overset{(10)}{=}
\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[d, bend right=80] \arrow[d, phantom, swap, "\overset{\Leftarrow}{\gamma_\sigma}"] \arrow[dr, phantom, "\Downarrow F_\sigma"] & \cdot \arrow[d] \arrow[d, bend left=80] \arrow[d, phantom, "\overset{\Leftarrow}{\delta_\sigma^{-1}}"] \\ \cdot \arrow[r] & \cdot \end{tikzcd}
\overset{(\dagger)}{=} F_\sigma \quad ,$$

where (†) holds because γ_σ and δ_σ are identities for an operation σ, due to Def. 2.16.

To show the other way $\Psi_{C,D} \circ \Phi_{C,D} = \mathrm{id}$, again the only nontrivial point is the equality $(\Phi_{C,D}\Psi_{C,D}F)_\sigma = F_\sigma$ of mediating isomorphisms. We have to show, for each arrow $a : n \to 1$ in $\mathbb{L}_{(\Sigma,E)}$ and a term t such that $a = [t]$,

$$F_a \quad = \quad \begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[d, bend right=80] \arrow[d, phantom, swap, "\overset{\Leftarrow}{\gamma_t}"] \arrow[dr, phantom, "\Downarrow (\Phi F)_t"] & \cdot \arrow[d] \arrow[d, bend left=80] \arrow[d, phantom, "\overset{\Leftarrow}{\delta_t^{-1}}"] \\ \cdot \arrow[r] & \cdot \end{tikzcd} \quad .$$

This is proved by induction on construction of a term t, using coherence condition for F as a pseudo natural transformation and the definition of γ_t (Def. 2.16). This concludes the proof of Lem. 2.26. $\square$

We shall now show that the 2-functor Φ is surjective on objects; this together with Lem. 2.26 will establish that Φ is an equivalence of 2-categories. For this, however, we need the following (rather minor) side condition.

Condition 2.27 *The set E of equational axioms does not derive equality of two operations in Σ. That is, for each pair of distinct operations $\sigma, \tau \in \Sigma(n)$, we never derive*

$$x_0, \ldots, x_{n-1} \vdash \sigma \sim \tau$$

from the axioms in E. In yet other words: the following composition of functions is injective:

$$\Sigma(n) \hookrightarrow T_\Sigma(n) \twoheadrightarrow T_\Sigma(n)/E \quad .$$

Lemma 2.28 *Assume that an algebraic specification (Σ, E) satisfies Cond. 2.27. Then the 2-functor $\Phi : \mathbb{L}_{(\Sigma,E)}$-**CAT** $\to (\Sigma, E)$-**CAT** is surjective on objects.*

PROOF. Given a (Σ, E)-category

$$\Big(\mathbb{C},\ \{[\![\sigma]\!]_\mathbb{C}\}_{\sigma\in\Sigma},\ \{\gamma_s^t\}_{(s=t)\in E} \Big) \ ,$$

we construct an $\mathbb{L}_{(\Sigma,E)}$-category

$$\Psi\mathbb{C} \ : \ \mathbb{L}_{(\Sigma,E)} \longrightarrow \mathbf{CAT}$$

in the following way. Later $\Psi\mathbb{C}$ is shown to be carried to $\mathbb{C}$ by Φ. The construction involves the axiom of choice.

The action on objects of $\Psi\mathbb{C}$ is as expected: $n \mapsto (\mathbb{C})^n$. On arrows: given an arrow $a : n \to 1$ in $\mathbb{L}_{(\Sigma,E)}$,

- first we choose a *representative* term $t \in T_\Sigma(n)$ such that $a = [t]$. This is under the following restriction:

  when there exists an operation $\sigma \in \Sigma(n)$ such that $a = [\sigma]$, then we choose this σ to be the representative of a. (11)

  By Cond. 2.27 such an operation σ is unique if it exists;
- then we let

$$(\Psi\mathbb{C})(a) \ := \ [\![t]\!]_\mathbb{C} \quad : \quad \mathbb{C}^n \longrightarrow \mathbb{C} \ .$$

We are left to specify the coherent isomorphisms $(\Psi\mathbb{C})_{b,a}$ up-to which composition is preserved. Given two successive arrows

$$n \xrightarrow{a} m \xrightarrow{b} 1 \qquad \text{in } \mathbb{L}_{(\Sigma,E)},$$

let $a = \langle [s_0], \ldots, [s_{m-1}] \rangle$, $b = [t]$ and $b \circ a = [u]$ be our choices of representatives; hence

$$(\Psi\mathbb{C}) \quad : \quad a \mapsto \langle [\![s_0]\!]_{\mathbb{C}}, \ldots, [\![s_{m-1}]\!]_{\mathbb{C}} \rangle \ , \quad b \mapsto [\![t]\!]_{\mathbb{C}} \ , \quad b \circ a \mapsto [\![u]\!]_{\mathbb{C}} \ .$$

Now we have

$$\begin{aligned} b \circ a &= [t] \circ \langle [s_0], \ldots, [s_{m-1}] \rangle \\ &= [t \circ \langle s_0, \ldots, s_{m-1} \rangle] \qquad\qquad \text{by the rule (Cong) of deriving } \sim. \end{aligned}$$

Therefore $[t \circ \langle s_0, \ldots, s_{m-1} \rangle] = [u]$; that is

$$t \circ \langle s_0, \ldots, s_{m-1} \rangle \sim u \ .$$

This makes the natural isomorphism $\gamma^u_{t \circ \langle s_0, \ldots, s_{m-1} \rangle}$ (induced by a (Σ, E)-category $\mathbb{C}$, see Def. 2.4) available. We set this to be the mediating isomorphism $(\Psi\mathbb{C})_{b,a}$. That is,

$$\underline{\text{in } \mathbb{L}} \quad \begin{array}{c} n \\ \downarrow a \\ m \\ \downarrow b \\ 1 \end{array} \quad \overset{\Psi\mathbb{C}}{\longmapsto} \quad \underline{\text{in } \mathbf{CAT}} \quad \begin{tikzcd} & \mathbb{C}^n \arrow[dl, "(\Psi\mathbb{C})(a) = \langle [\![s_0]\!], \ldots, [\![s_{m-1}]\!] \rangle"'] \arrow[dd, bend left=60, "(\Psi\mathbb{C})(b \circ a) = [\![u]\!]"] \arrow[dd, phantom, "\overset{\gamma^u_{t \circ \langle s_0, \ldots, s_{m-1} \rangle}}{\underset{\cong}{\Longrightarrow}}"] \\ \mathbb{C}^m \arrow[dr, "(\Psi\mathbb{C})(b) = [\![t]\!]"'] & \\ & \mathbb{C} \end{tikzcd}$$

The isomorphisms $(\Psi\mathbb{C})_{b,a}$ thus defined satisfy the coherence condition (which is part of the definition of pseudo functor) due to the coherence condition on γ (Cond. 2.3). Therefore $\Psi\mathbb{C}$ is indeed an $\mathbb{L}_{(\Sigma,E)}$-category.

It remains to be shown that $\Phi(\Psi\mathbb{C}) = \mathbb{C}$. Their base categories are obviously the same. Their interpretation of operations is the same too, because

$$[\![\sigma]\!]_{\Phi\Psi\mathbb{C}} = (\Psi\mathbb{C})(\sigma) = [\![t]\!]_\sigma$$

where t is the representative for $[\sigma]$, which is forced to be σ by the restriction (11). Finally, we have to show the equality between their mediating isomorphisms for equations $(s = t) \in E$. The mediating isomorphism for $(s = t) \in E$ of $\Phi(\Psi\mathbb{C})$ is given by the composite $\gamma_t^{-1} \bullet \gamma_s$, where γ_s and γ_t are the natural isomorphisms induced by $\Psi\mathbb{C}$, see Def. 2.16.

$$[\![s]\!]_{\mathbb{C}} \overset{\gamma_s}{\Longrightarrow} (\Psi\mathbb{C})([s]) = (\Psi\mathbb{C})([t]) \overset{\gamma_t^{-1}}{\Longrightarrow} [\![t]\!]_{\mathbb{C}}$$

Here let u be our choice of the representative term for $[s] = [t]$, in defining $\Psi\mathbb{C}$. Then we have $(\Psi\mathbb{C})([s]) = (\Psi\mathbb{C})([t]) = [\![u]\!]_{\mathbb{C}}$. Moreover, it is easy to see (by induction on construction of terms) that the natural isomorphism

$$[\![s]\!]_{\mathbb{C}} \overset{\gamma_s}{\Longrightarrow} [\![u]\!]_{\mathbb{C}} = (\Psi\mathbb{C})([s])$$

coincides with γ^u_s, the natural isomorphism accompanying a (Σ, E)-category $\mathbb{C}$ (Def. 2.4). Therefore we have, using the equations (3),

$$\gamma_t^{-1} \bullet \gamma_s = (\gamma^u_t)^{-1} \bullet \gamma^u_s = \gamma^t_u \bullet \gamma^u_s = \gamma^t_s \ .$$

This concludes the proof. □

Theorem 2.29 *Assume that an algebraic specification (Σ, E) satisfies Cond. 2.27. Then the 2-categories (Σ, E)-$\mathbf{CAT}$ and $\mathbb{L}_{(\Sigma,E)}$-$\mathbf{CAT}$ are equivalent.*

PROOF. By Lem. 2.26 and 2.28. □

Remark 2.30 *The 2-functor Φ fails to be an isomorphism essentially due to the choice that we have to make in defining $\Psi\mathbb{C}$ in the proof of Lem. 2.28. For example, take the usual specification for monoids (Expl. 2.6) as (Σ, E); in defining its corresponding $\mathbb{L}_{(\Sigma,E)}$-category $\Psi\mathbb{C}$, we have to fix what*

$$(\Psi\mathbb{C})\Big(3 \xrightarrow{[x_0\cdot(x_1\cdot x_2)] = [(x_0\cdot x_1)\cdot x_2]} 1\Big) \quad : \quad \mathbb{C}^3 \longrightarrow \mathbb{C}$$

should be. And there is no canonical choice between the following two candidates (they are indeed different):

$$(X_0, X_1, X_2) \quad \longmapsto \quad X_0 \otimes (X_1 \otimes X_2) \quad \textit{or} \quad (X_0 \otimes X_1) \otimes X_2 \ .$$

Instead of the 2-category of monoidal categories, what is isomorphic to $\mathbb{L}_{(\Sigma,E)}$-$\mathbf{CAT}$ (for the current (Σ, E)) is that of unbiased monoidal categories*, see [13].*

3 Lifting $\mathbb{L}$-structure to a category of coalgebras

In this section we obtain some elementary results on $\mathbb{L}$-categories, one of which is about lifting the $\mathbb{L}$-structure on $\mathbb{C}$ to the one on the category $\mathbf{Coalg}(F)$ of F-coalgebras, for $F : \mathbb{C} \to \mathbb{C}$.

Notation 3.1 *In the previous section $\mathbb{L}$-categories have been denoted by $C, D, \ldots$ to distinguish them from (Σ, E)-categories. From now on we work exclusively with $\mathbb{L}$-categories since their formulation (more categorical) makes them easier to reason about. Consequently we shall denote $\mathbb{L}$-categories by $\mathbb{C}, \mathbb{D}, \ldots$ to emphasize that they are a category with pseudo $\mathbb{L}$-structure. Accordingly morphisms/transformations of $\mathbb{L}$-categories (Def. 2.13, 2.15) shall be referred to as* $\mathbb{L}$-functors *and* $\mathbb{L}$-natural transformations*, respectively. They are functors/natural transformations that respect $\mathbb{L}$-structure.*

Definition 3.2 (Lax/oplax $\mathbb{L}$-functor) *Let $\mathbb{C}, \mathbb{D}$ be $\mathbb{L}$-categories. A* lax $\mathbb{L}$-functor *$F : \mathbb{C} \to \mathbb{D}$ is a lax natural transformation*

$$\begin{tikzcd} \mathbb{L} \arrow[r, bend left, "\mathbb{C}"] \arrow[r, bend right, "\mathbb{D}"'] \arrow[r, phantom, "\Downarrow F"] & \mathbf{CAT} \end{tikzcd} .$$

That is, it is a morphism of $\mathbb{L}$-categories (Def. 2.13) for which a naturality diagram is filled in with a 2-cell that is not necessarily invertible.

$$\underline{\textit{in } \mathbb{L}} \quad \begin{tikzcd} n \arrow[d, "a"] \\ m \end{tikzcd} \qquad \underline{\textit{in } \mathbf{CAT}} \quad \begin{tikzcd} Cn \arrow[r, "F_n"] \arrow[d, "Ca"'] \arrow[dr, phantom, "\swarrow F_a"] & Dn \arrow[d, "Da"] \\ Cm \arrow[r, "F_m"'] & Dm \end{tikzcd}$$

An oplax $\mathbb{L}$-functor *is defined in a similar way, as an oplax natural transformation. A* strict $\mathbb{L}$-functor *is a (strict) natural transformation of the same type.*

Theorem 3.3 *Let $\mathbb{C}$ be an $\mathbb{L}$-category and $F : \mathbb{C} \to \mathbb{C}$ be a lax $\mathbb{L}$-functor. Then the category $\mathbf{Coalg}(F)$ of coalgebras is an $\mathbb{L}$-category in a canonical way. Moreover, the forgetful functor $U : \mathbf{Coalg}(F) \to \mathbb{L}$ is a strict $\mathbb{L}$-functor.*

The theorem is in fact a special case of Thm. 3.5 later, due to the following well-known characterization. The notion of *inserter* comes from [11,17]; see also [8].

Lemma 3.4 *Given a category $\mathbb{C}$ and a functor $F : \mathbb{C} \to \mathbb{C}$, the category $\mathbf{Coalg}(F)$ is an inserter $\mathrm{Ins}(\mathrm{id}, F)$ of the following diagram in $\mathbf{CAT}$.*

$$\begin{tikzcd} \mathbb{C} \arrow[r, shift left, "\mathrm{id}"] \arrow[r, shift right, "F"'] & \mathbb{C} \end{tikzcd} \qquad \square$$

Theorem 3.5 *Let $\mathbb{C}, \mathbb{D}$ be $\mathbb{L}$-categories, $G : \mathbb{C} \to \mathbb{D}$ be an oplax $\mathbb{L}$-functor and $F : \mathbb{C} \to \mathbb{D}$ be a lax $\mathbb{L}$-functor. Then the inserter $\mathrm{Ins}(G, F)$ is canonically an $\mathbb{L}$-category. Moreover, the forgetful functor $U : \mathrm{Ins}(G, F) \to \mathbb{C}$ is a strict $\mathbb{L}$-functor.*

PROOF. Let us denote the inserter by U and ρ as follows.

$$\begin{tikzcd} Ins(G,F) \arrow[r, bend left, "U"] \arrow[r, bend right, "U"'] & \mathbb{C} \arrow[r, bend left, "G"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\swarrow \rho"] & \mathbb{D} \end{tikzcd}$$

We describe a $\mathbb{L}$-category $\mathrm{Ins}(G, F)$, that is a pseudo functor

$$\mathrm{Ins}(G, F) \; : \; \mathbb{L} \longrightarrow \mathbf{CAT} \; .$$

On objects it carries n to $(\mathrm{Ins}(G,F))^n$. For its action on an arrow $a : n \to 1$ in $\mathbb{L}$, consider the following 2-cell.

$$\begin{tikzcd}
 & \mathbb{C}^n \arrow[r, "\mathbb{C}a"] \arrow[dr, "G^{\times n}"'] \arrow[dd, phantom, "\swarrow\rho^{\times n}"] & \mathbb{C} \arrow[dr, "G"] \arrow[d, phantom, "\swarrow G_a"] & \\
Ins(G,F)^n \arrow[ur, "U^{\times n}"] \arrow[dr, "U^{\times n}"'] & & \mathbb{D}^n \arrow[r, "\mathbb{D}a"] \arrow[d, phantom, "\searrow F_a"] & \mathbb{D} \\
 & \mathbb{C}^n \arrow[r, "\mathbb{C}a"'] \arrow[ur, "F^{\times n}"] & \mathbb{C} \arrow[ur, "F"'] &
\end{tikzcd} \tag{12}$$

Universality of the inserter $\mathrm{Ins}(G,F)$ then induces a 1-cell $\mathrm{Ins}(G,F)^n \to \mathrm{Ins}(G,F)$ which we define to be the pseudo functor's action $\mathrm{Ins}(G,F)(a)$ on a.

$$\begin{tikzcd}
\mathrm{Ins}(G,F)^n \arrow[r, dashed, "\mathrm{Ins}(G,F)(a)"] & Ins(G,F) \arrow[r, bend left, "U"] \arrow[r, bend right, "U"'] & \mathbb{C} \arrow[r, bend left, "G"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\swarrow\rho"] & \mathbb{D}
\end{tikzcd} \tag{13}$$

By the definition of inserter, it is the unique 1-cell such that:

- the two arrows $\mathrm{Ins}(G,F) \to \mathbb{C}$ on the upper/lower edges of (12,13) are equal, that is,
$$\mathbb{C}a \circ U^{\times n} = U \circ \mathrm{Ins}(G,F)(a) \ ; \tag{14}$$
- the composed 2-cell in (12) is equal to the one in (13).

Finite product preservation forces the action on $a : n \to m$ to be the tuple of action on each component of a:

$$\mathrm{Ins}(G,F)(a) \quad = \quad \langle\ \mathrm{Ins}(G,F)(\pi_0 \circ a),\ \ldots,\ \mathrm{Ins}(G,F)(\pi_{m-1} \circ a)\ \rangle \ .$$

Now let $n \xrightarrow{a} m \xrightarrow{b} 1$ be successive arrows in $\mathbb{L}$; we describe the mediating 2-cell $\mathrm{Ins}(G,F)_{b,a}$ up-to which their composition is preserved. The action $\mathrm{Ins}(G,F)(b \circ a)$ is induced by the following 2-cell:

$$\begin{tikzcd}
 & \mathbb{C}^n \arrow[rr, "\mathbb{C}(b\circ a)"] \arrow[dr, "G^{\times n}"'] \arrow[dd, phantom, "\swarrow\rho^{\times n}"] & & \mathbb{C} \arrow[dr, "G"] & \\
Ins(G,F)^n \arrow[ur, "U^{\times n}"] \arrow[dr, "U^{\times n}"'] & & \mathbb{D}^n \arrow[rr, "\mathbb{D}(b\circ a)"] \arrow[ur, phantom, "\swarrow G_{b\circ a}"] \arrow[dr, phantom, "\searrow F_{b\circ a}"] & & \mathbb{D} \\
 & \mathbb{C}^n \arrow[rr, "\mathbb{C}(b\circ a)"'] \arrow[ur, "F^{\times n}"] & & \mathbb{C} \arrow[ur, "F"'] &
\end{tikzcd} \tag{15}$$

whereas $\mathrm{Ins}(G,F)(b) \circ \mathrm{Ins}(G,F)(a)$ is by

$$\begin{tikzcd}
 & \mathbb{C}^n \arrow[r, "\mathbb{C}a"] \arrow[dr, "G^{\times n}"'] \arrow[dd, phantom, "\swarrow\rho^{\times n}"] & \mathbb{C}^m \arrow[r, "\mathbb{C}b"] \arrow[dr, "G^{\times m}"'] \arrow[d, phantom, "\swarrow G_a"] & \mathbb{C} \arrow[dr, "G"] \arrow[d, phantom, "\swarrow G_b"] & \\
Ins(G,F)^n \arrow[ur, "U^{\times n}"] \arrow[dr, "U^{\times n}"'] & & \mathbb{D}^n \arrow[r, "\mathbb{D}a"] \arrow[d, phantom, "\searrow F_a"] & \mathbb{D}^m \arrow[r, "\mathbb{D}b"] \arrow[d, phantom, "\searrow F_b"] & \mathbb{D} \\
 & \mathbb{C}^n \arrow[r, "\mathbb{C}a"'] \arrow[ur, "F^{\times n}"] & \mathbb{C}^m \arrow[r, "\mathbb{C}b"'] \arrow[ur, "F^{\times m}"] & \mathbb{C} \arrow[ur, "F"'] &
\end{tikzcd} \ . \tag{16}$$

The relationship between the two is

$$\begin{tikzcd}
& \cdot \arrow[r] \arrow[rr, bend right, "\cong\Downarrow\mathbb{C}_{b,a}"] & \cdot \arrow[r] & \cdot \arrow[dr] & \\
\cdot \arrow[ur] \arrow[dr] & & \Downarrow(15) & & \cdot \\
& \cdot \arrow[r] \arrow[rr, bend left, "\cong\Downarrow\mathbb{C}_{b,a}"'] & \cdot \arrow[r] & \cdot \arrow[ur] &
\end{tikzcd} = (16)$$

which follows from the coherence condition on F and G as (op)lax natural transformations. This modification $\mathbb{C}_{b,a}$ between the two cones (15,16) gives rise to a 2-cell

$$\mathrm{Ins}(G,F)_{b,a} \quad : \quad \mathrm{Ins}(G,F)(b) \circ \mathrm{Ins}(G,F)(a) \overset{\cong}{\Longrightarrow} \mathrm{Ins}(G,F)(b \circ a) \ .$$

by the 2-universality of the inserter $\mathrm{Ins}(G,F)$. They satisfy the coherence condition due to the coherence of $\mathbb{C}_{b,a}$; thus we have obtained an $\mathbb{L}$-functor $\mathrm{Ins}(G,F)$.

To show that the forgetful functor $U : \mathrm{Ins}(G,F) \to \mathbb{C}$ is a strict $\mathbb{L}$-functor, we need to show that for each arrow $a : n \to 1$ in $\mathbb{L}$ the following diagram commutes (up-to identity).

$$\begin{tikzcd}
\mathrm{Ins}(G,F)^n \arrow[r, "U^{\times n}"] \arrow[d, "\mathrm{Ins}(G,F)(a)"'] & \mathbb{C}^n \arrow[d, "\mathbb{C}a"] \\
\mathrm{Ins}(G,F) \arrow[r, "U"'] & \mathbb{C}
\end{tikzcd}$$

We have already derived this in (14) from the definition of $\mathrm{Ins}(G,F)(a)$. This concludes the proof. □

4 Future work

In this note we have stuck to a Lawvere theory $\mathbb{L}$ which is a (1-)category. For defining $\mathbb{L}$-category, however, there is no reason to do so. The definition works all the same if $\mathbb{L}$ is a 2-category, and this extended expressive power allows us to have a symmetric monoidal category as an example. (Note that symmetric monoidal categories are *not* $\mathbb{L}$-categories for the theory $\mathbb{L}$ of commutative monoids; this is the well-known "symmetry vs. commutativity" issue) The corresponding conventional notion of pseudo algebra—which is defined for operations and equations—then becomes far more complicated; it has to account for 1- and 2-operations and equations. The details are yet to be elaborated.

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References

[1] J.C. Baez and J. Dolan. Higher dimensional algebra III: n-categories and the algebra of opetopes. *Adv. Math*, 135:145–206, 1998.

[2] R. Blackwell, G. Kelly and A. Power. Two-dimensional monad theory. *Journ. of Pure & Appl. Algebra*, 59:1–41, 1989.

[3] F. Borceux. *Handbook of Categorical Algebra*, vol. 50, 51 and 52 of *Encyclopedia of Mathematics*. Cambridge Univ. Press, 1994.

[4] P.L. Curien. Operads, clones, and distributive laws, 2008. Preprint, `www.pps.jussieu.fr/~curien`.

[5] T.M. Fiore. Pseudo limits, biadjoints, and pseudo algebras: Categorical foundations of conformal field theory. *Memoirs of the AMS*, 182, 2006.

[6] I. Hasuo, C. Heunen, B. Jacobs and A. Sokolova. Coalgebraic components in a many-sorted microcosm. In *Conference on Algebra and Coalgebra in Computer Science (CALCO 2009)*, Lect. Notes Comp. Sci. Springer-Verlag, 2009. To appear.

[7] I. Hasuo, B. Jacobs and A. Sokolova. The microcosm principle and concurrency in coalgebra. In *Foundations of Software Science and Computation Structures*, vol. 4962 of *Lect. Notes Comp. Sci.*, pp. 246–260. Springer-Verlag, 2008.

[8] C. Hermida and B. Jacobs. Structural induction and coinduction in a fibrational setting. *Inf. & Comp.*, 145:107–152, 1998.

[9] C. Hermida. A roadmap to the unification of weak categorical structures: transformations and equivalences among the various notions of pseudo-algebra, 2004.
`maggie.cs.queensu.ca/chermida/papers/roadmap.pdf`.

[10] M. Hyland and A.J. Power. Discrete Lawvere theories and computational effects. *Theor. Comp. Sci.*, 366(1–2):144–162, 2006.

[11] G.M. Kelly. Elementary observations on 2-categorical limits. *Bull. Austr. Math. Soc.*, 39:301–317, 1989.

[12] F.W. Lawvere. *Functorial Semantics of Algebraic Theories and Some Algebraic Problems in the Context of Functorial Semantics of Algebraic Theories*. PhD thesis, Columbia University, 1963. Reprints in Theory and Applications of Categories, 5 (2004) 1–121.

[13] T. Leinster. *Higher Operads, Higher Categories*, vol. 298 of *London Math. Soc. Lect. Note Series*. Cambridge Univ. Press, 2004. Available online at `http://arxiv.org/abs/math.CT/0305049`.

[14] S. Mac Lane. *Categories for the Working Mathematician*. Springer, Berlin, 2nd edn., 1998.

[15] A.J. Power. Why tricategories? *Inf. & Comp.*, 120(2):251–262, 1995.

[16] G. Segal. The definition of conformal field theory. In U. Tillmann, editor, *Topology, Geometry and Quantum Field Theory*, vol. 308 of *London Math. Soc. Lect. Note Series*, pp. 423–577. Cambridge Univ. Press, 2004.

[17] R. Street. Fibrations and Yoneda's lemma in a 2-category. In G.M. Kelly, editor, *Proc. Sydney Category Theory Seminar 1972/1973*, no. 420 in Lect. Notes Math., pp. 104–133. Springer, Berlin, 1974.