Codensity Games for Bisimilarity

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Introduction and Motivations

Bisimulation & Bisimilarity, the First Example

Definition (Kripke structure)

A **Kripke structure** is a set $\boldsymbol{X}$ equipped with functions

$$c_1 \colon X \longrightarrow 2^{\mathsf{AP}} \quad \text{and} \quad c_2 \colon X \longrightarrow \mathcal{P}X, \quad \text{that is}$$
$$c := \langle c_1, c_2 \rangle \colon X \longrightarrow 2^{\mathsf{AP}} \times \mathcal{P}X,$$

where **AP** is a set of *atomic propositions*, and $\mathcal{P}X = \{U \subseteq X\}$.

Bisimulation & Bisimilarity, the First Example

Since Milner & Park, bisimulation is a standard equivalence notion between systems with potential branching.

Definition (bisimulation)

$R \subseteq X$ is a **bisimulation** over $c \colon X \to 2^{\mathsf{AP}} \times \mathcal{P}X$ if, for each $(x, y) \in R$,

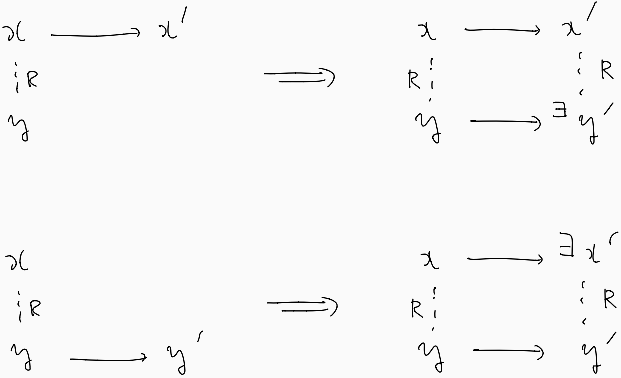
- **(On atomic propositions)** $c_1(x) = c_1(y) \ \in 2^{\mathsf{AP}}$
- **(Mimicking I)**
  $x' \in c_2(x) \implies \exists y'. \big(y' \in c_2(y) \wedge (x', y') \in R \big)$
- **(Mimicking II)**
  $y' \in c_2(y) \implies \exists x'. \big(x' \in c_2(x) \wedge (x', y') \in R \big)$

Definition (bisimilarity)

The **bisimilarity** over $c \colon X \to 2^{\mathsf{AP}} \times \mathcal{P}X$ is the greatest bisimulation.

Bisimulation & Bisimilarity, the First Example

The mimicking conditions:



Bisimulation & Bisimilarity, via Fixed Points

Definition (relation lifting Φ)

Let $\Phi \colon 2^{X \times X} \longrightarrow 2^{(2^{\mathrm{AP}} \times \mathcal{P}X) \times (2^{\mathrm{AP}} \times \mathcal{P}X)}$ be defined by

$$\Phi(R) \;:=\; \left\{ \begin{pmatrix} (\alpha, S), \\ (\beta, T) \end{pmatrix} \;\middle|\; \begin{array}{l} \alpha = \beta, \\ \forall x' \in S.\, \exists y' \in T.\, (x', y') \in R, \\ \forall y' \in T.\, \exists x' \in S.\, (x', y') \in R \end{array} \right\}.$$

Φ *lifts* a relation from X to $2^{\mathrm{AP}} \times \mathcal{P}X$.

Definition (pullback c^*)

$c \colon X \longrightarrow 2^{\mathrm{AP}} \times \mathcal{P}X$ induces a function

$$c^* \colon 2^{(2^{\mathrm{AP}} \times \mathcal{P}X) \times (2^{\mathrm{AP}} \times \mathcal{P}X)} \longrightarrow 2^{X \times X} \quad \text{by}$$

$$c^*(T) := \left\{ (x, y) \;\middle|\; \bigl(c(x), c(y)\bigr) \in T \right\}.$$

Bisimulation & Bisimilarity, via Fixed Points

We have obtained (obviously monotone) functions

$$2^{X \times X} \xrightarrow[\text{lifting}]{\Phi} 2^{(2^{\mathsf{AP}} \times \mathcal{P}X) \times (2^{\mathsf{AP}} \times \mathcal{P}X)} \xrightarrow[\text{pull back along } c]{c^*} 2^{X \times X}.$$

Proposition

$R \subseteq X$ is a bisimulation iff $R \subseteq c^*(\Phi(R))$.

Let's recall some fixed-point theory.

Let L be a complete lattice, $f \colon L \to L$ be monotone.

- **[Knaster–Tarski]**:
  the greatest *post-fixed* point is the greatest *fixed* point νf.
  (Bisimilarity is a fixed point)
- **[Cousot–Cousot]**: the (transfinite) sequence
  $\top \sqsupseteq f(\top) \sqsupseteq f^2(\top) \sqsupseteq \cdots$ stabilizes to νf.
  (Foundation of the partition-refinement algorithm)

The Second Example: Bisimulation Metric

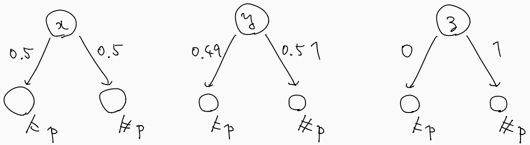
Definition (Markov chain)

A **Markov chain** is a set X equipped with functions

$$c_1 \colon X \longrightarrow 2^{\mathsf{AP}} \quad \text{and} \quad c_2 \colon X \longrightarrow \mathcal{D}X, \quad \text{that is}$$
$$c := \langle c_1, c_2 \rangle \colon X \longrightarrow 2^{\mathsf{AP}} \times \mathcal{D}X,$$

where $\mathcal{D}X = \{\, d \colon X \to [0,1] \mid \sum_{x \in X} d(x) = 1 \,\}$ is the set of probability distributions over X.

Bisimulation and bisimilarity is known for MCs [Larsen & Skou '91].
However they are too strict... y should be "closer" to x than z



The Second Example: Bisimulation Metric

Let $(\mathbf{PMet_1})_X$ be the set of (1-b'dd) *pseudometrics* over X.

Definition (pseudometric lifting Φ)

Let $\Phi\colon (\mathbf{PMet_1})_X \longrightarrow (\mathbf{PMet_1})_{2^{\mathsf{AP}} \times \mathcal{D}X}$ be defined by

$$(\Phi m)\big(\,(\alpha, d), (\beta, e)\,\big) \;:=\; \begin{cases} 1 & \text{if } \alpha \neq \beta \\ (\mathcal{K}m)(d, e) & \text{if } \alpha = \beta \end{cases}$$

where $\mathcal{K}m$ is the *Kantorovich metric* over $\mathcal{D}X$ by m:

$$(\mathcal{K}m)(d, e) \;:=\; \sup \Big\{ \,\big|\, \textstyle\sum_x f(x)d(x) - \sum_X f(x)e(z) \,\big| \;\Big|\; f\colon (X, m) \to ([0,1], \text{Eucl. met.}) \text{ is non-expansive} \Big\}$$

Φ *lifts* a pseudometric.

Note the role of f: we "observe" its expectations such as $\sum_x f(x)d(x)$.

The Second Example: Bisimulation Metric

Definition (pullback c^*)

$c \colon X \longrightarrow 2^{\mathsf{AP}} \times \mathcal{D}X$ induces a function

$$c^* \colon (\mathbf{PMet}_1)_{2^{\mathsf{AP}} \times \mathcal{D}X} \longrightarrow (\mathbf{PMet}_1)_X$$
$$c^*(n)(x, y) := n\big(\, c(x),\ c(y) \,\big)$$

Thus we have obtained functions

$$(\mathbf{PMet}_1)_X \xrightarrow[\text{lifting}]{\Phi} (\mathbf{PMet}_1)_{2^{\mathsf{AP}} \times \mathcal{D}X} \xrightarrow[\text{pull back along } c]{c^*} (\mathbf{PMet}_1)_X$$

that are "monotonic":

- Let $m \sqsubseteq m'$ in $(\mathbf{PMet}_1)_X$ be defined by

  $$\forall x, y \in X. \quad m(x, y) \geq m'(x, y)$$

  ("the indistinguishability order")
- Then $m \sqsubseteq m'$ implies $c^*(\Phi(m)) \sqsubseteq c^*(\Phi(m'))$

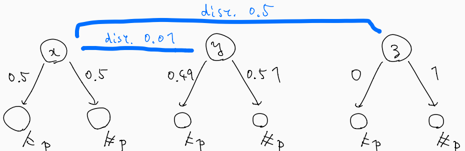
The Second Example: Bisimulation Metric

$$(\mathrm{PMet}_1)_X \xrightarrow[\text{lifting}]{\Phi} (\mathrm{PMet}_1)_{2^{\mathsf{AP}} \times \mathcal{D}X} \xrightarrow[\text{pull back along } c]{c^*} (\mathrm{PMet}_1)_X$$

Definition (bisimulation metric [Desharnais, Gupta, Jagadeesan, Panangaden '04]**)**

The **bisimulation metric** $m_{(X,c)}$ is the greatest fixed point of $c^* \circ \Phi$.

- **Observation-respecting:**
  distinguishes x, y such that $\exists p. (x \models p \land y \not\models p)$
- **Transition-invariant:**
  makes $c\colon X \to 2^{\mathsf{AP}} \times \mathcal{D}X$ *non-expansive*
- The **least distinguishing** among the above



A Scenario in Common

$$2^{X\times X} \xrightarrow[\text{lifting}]{\Phi} 2^{(2^{AP}\times\mathcal{P}X)\times(2^{AP}\times\mathcal{P}X)} \xrightarrow[\text{pull back along } c]{c^*} 2^{X\times X}$$

$$(\mathbf{PMet}_1)_X \xrightarrow[\text{lifting}]{\Phi} (\mathbf{PMet}_1)_{2^{AP}\times\mathcal{D}X} \xrightarrow[\text{pull back along } c]{c^*} (\mathbf{PMet}_1)_X$$

1. Identify an **indistinguishability structure (IS)** $\mathbb{E}$
   - Binary relations, pseudometrics, ...
2. **Lift** a functor F to $\Phi\colon \mathbb{E}_X \to \mathbb{E}_{FX}$
   - Given F, not always unique, although often there's a canonical one
3. **Pull back** along the dynamics $c\colon X \to FX$
4. **Bisimulation** is a post-fixed point $P \sqsubseteq c^*(\Phi(P))$
   Bisimilarity is the greatest fixed point
   - wrt. the *indistinguishability order* $\sqsubseteq$

$\Longrightarrow$ we strive for categorical formalization!

Fibration for **Indistinguishability Structures** and **Decent Maps**

The Categorical Setting

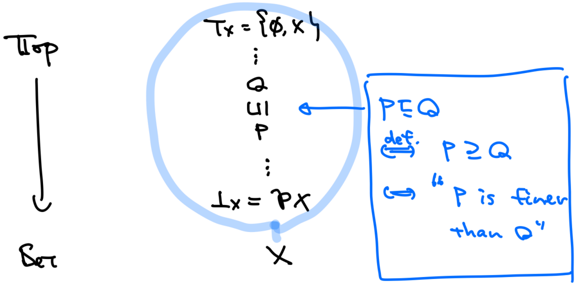
$$\begin{array}{c} \mathbb{E} \\ \downarrow p \\ \mathbb{C} \circlearrowleft F \end{array} \qquad\qquad X \xrightarrow{\;c\;} FX$$

- The usual **coalgebra** business [Rutten, Jacobs, ...]
  - $F\colon \mathbb{C} \to \mathbb{C}$, a **behavior type** functor
  - $X \in \mathbb{C}$, a state space
  - $c\colon X \to FX$, a coalgebra (describing dynamics)
- A **$\mathbf{CLat}_{\sqcap}$-fibration** $\begin{array}{c}\mathbb{E}\\ \downarrow p\\ \mathbb{C}\end{array}$
  - $P \in \mathbb{E}_X$, an **indistinguishability predicate** over X
    binary relation, pseudo-metric, preorder, σ-field, topology, ...
  - $\mathbb{E}_X$ is a **complete lattice**, wrt. the **indistinguishability order** $\sqsubseteq$
  - $f\colon X \to Y$ (in $\mathbb{C}$) induces a **"pullback" map $c^*\colon \mathbb{E}_Y \to \mathbb{E}_X$**
    that is **$\sqcap$-preserving**

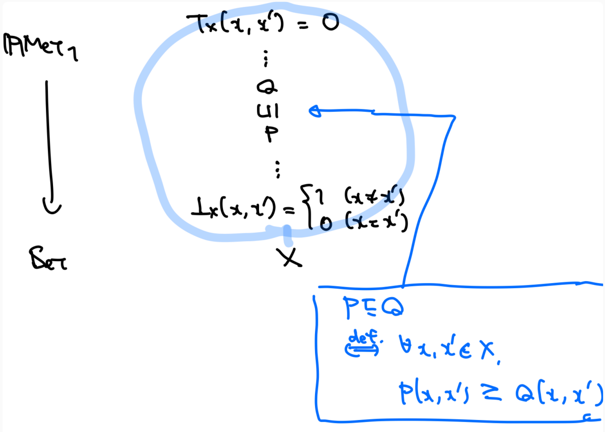
Examples: $\mathbf{CLat}_{\sqcap}$-Fibration

	indist. pred. $P \in \mathbb{E}_X$	$P \sqsubseteq Q$	$\bigsqcap P_i$
Top $\downarrow$ **Set**	$\mathcal{O} \subseteq \mathcal{P}X$, topology	$P \supseteq Q$	gen. from $\bigcup P_i$
Meas $\downarrow$ **Set**	$\mathcal{B} \subseteq \mathcal{P}X$, σ-field	$P \supseteq Q$	gen. from $\bigcup P_i$
$\mathbf{PMet}_1$ $\downarrow$ **Set**	$m\colon X \times X \to [0,1]$ pseudometric	$P(x,y) \geq Q(x,y)$	$\sup_i P_i(x,y)$
ERel $\downarrow$ **Set**	$R \subseteq X \times X$ endorelation	$P \subseteq Q$	$\bigcap P_i$
Pre $\downarrow$ **Set**	$\preceq \subseteq X \times X$ preorder	$P \subseteq Q$	$\bigcap P_i$
EqRel $\downarrow$ **Set**	$R \subseteq X \times X$ equiv. rel.	$P \subseteq Q$	$\bigcap P_i$

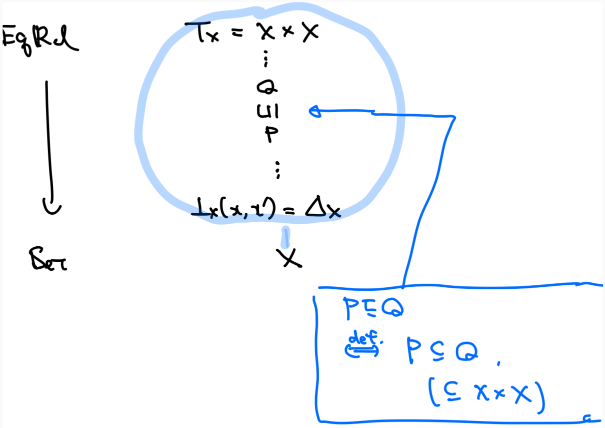
Examples: $\mathbf{CLat}_{\sqcap}$-Fibration



Examples: $\mathbf{CLat}_{\sqcap}$-Fibration



Examples: $\mathbf{CLat}_{\sqcap}$-Fibration



$\mathbf{CLat}_{\sqcap}$-Fibration and Decent Maps

$\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{C}\end{smallmatrix}$: a $\mathbf{CLat}_{\sqcap}$-fibration

- $X \in \mathbb{C}$ comes with $\mathbb{E}_X$, the set of indistinguishability predicates
- What is the category $\mathbb{E}$?

  $\Longrightarrow$ a "patch up" of $(\mathbb{E}_X)_{X \in \mathbb{C}}$ (the Grothendieck construction)
  - **object:** (X, P) where $X \in \mathbb{C}$ and $P \in \mathbb{E}_X$
  - **arrow:**

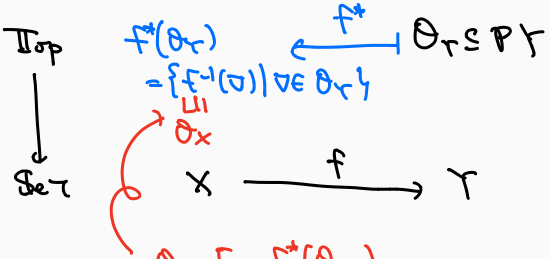
$$\frac{f\colon (X,P) \longrightarrow (Y,Q) \text{ in } \mathbb{E}}{f\colon X \longrightarrow Y \text{ in } \mathbb{C}, \text{ s.t. } P \sqsubseteq f^*Q}, \text{ as in } \begin{array}{c}\mathbb{E}\\ \Big\downarrow p\\ \mathbb{C}\end{array} \qquad \begin{array}{ccc} f^*Q & \xrightarrow{\overline{f}(Q)} & Q \\ \uparrow & & \\ P & & \\ & & \\ X & \xrightarrow{f} & Y \end{array}$$

Definition (decent map)

$f\colon X \to Y$ is **decent** wrt. $P \in \mathbb{E}_X, Q \in \mathbb{E}_Y$ if $P \sqsubseteq f^*Q$.

Instances: continuity, measurability, non-expansiveness, relation-preservation, monotonicity, ...

Examples: decent maps



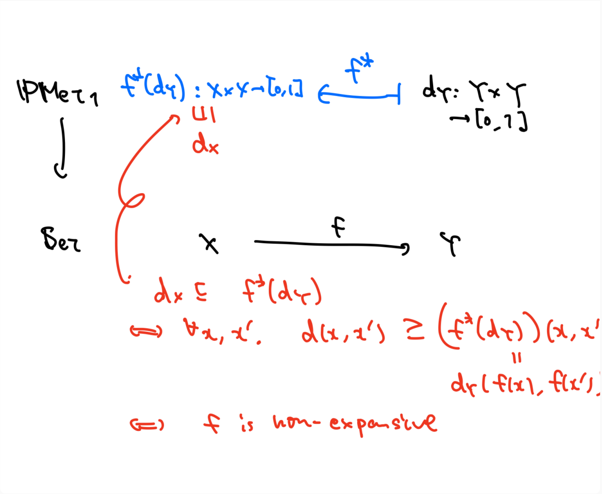
$$\Theta_X \sqsubseteq f^*(\Theta_Y)$$

$$\iff \Theta_X \supseteq f^*(\Theta_Y)$$

$$\iff \forall U \in \Theta_Y.\ f^{-1}(U) \in \Theta_X$$

$\iff$ f is cont'd.

Examples: decent maps



Summary: Indistinguishability Structures and Decent Maps

- In a $\mathbf{CLat}_{\sqcap}$-fibration $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{C}\end{smallmatrix}$,
  - the fiber $\mathbb{E}_X$ consists of **indistinguishability predicates** $P, Q, \ldots$
  - the **indistinguishability order**: $P \sqsubseteq Q$ iff "P is more discriminative than Q."
- An arrow $f \colon X \longrightarrow Y$ in $\mathbb{C}$ is a **decent map**

$$f \colon (X, P) \stackrel{\cdot}{\longrightarrow} (Y, Q) \qquad \text{in } \mathbb{E}$$

iff $P \sqsubseteq f^*(Q)$.

  - "f maps *similar* elements (wrt. P) to *similar* elements (wrt. Q)"

Codensity Lifting and Codensity Bisimilarity

[Katsumata, Sato, Sprunger, Dubut, Komorida, H., ...]

Codensity Lifting

The common scenario (recap)

1. Identify an **indistinguishability structure (IS)** $\mathbb{E}$ ✓
2. **Lift** a functor F to $\Phi\colon \mathbb{E}_X \to \mathbb{E}_{FX}$ **??**
3. **Pull back** along the dynamics $c\colon X \to FX$ ✓
4. **Bisimulation** is a post-fixed point $P \sqsubseteq c^*(\Phi(P))$
   Bisimilarity is the greatest fixed point ✓

We want a lifting Φ, as in $\begin{array}{ccc} \mathbb{E} & \xrightarrow{\Phi} & \mathbb{E} \\ \downarrow p & & \downarrow p \\ \mathbb{C} & \xrightarrow[F]{} & \mathbb{C} \end{array}$, that is

- **generic** (uniformly defined)
- **comprehensive** (covering known bisimilarity-like notions)
- **intuitive** ((in)distinguishability, observability, ...)

Codensity lifting [Katsumata & Sato, CALCO'15] is an answer.

It derives from $\top\top$-lifting [Lindley, Stark, Abadi, ...].

Codensity Lifting

Definition (codensity lifting [Katsumata & Sato, CALCO'15])

- $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{C}\end{smallmatrix}$ be a $\mathbf{CLat}_{\sqcap}$-fibration,
- $F : \mathbb{C} \longrightarrow \mathbb{C}$,
- (**modality**) $\tau : F\Omega \to \Omega$ be an algebra in $\mathbb{C}$, and
- (**observation indistinguishability**) $\underline{\mathbf{\Omega}} \in \mathbb{E}_{\Omega}$.

The *codensity lifting* $F^{\underline{\mathbf{\Omega}},\tau} : \mathbb{E} \to \mathbb{E}$ is defined by

$$F^{\underline{\mathbf{\Omega}},\tau} P = \bigsqcap_{k \in \mathbb{E}(P,\underline{\mathbf{\Omega}})} \Big(\tau \circ F\big(p(k)\big)\Big)^{*} \underline{\mathbf{\Omega}}.$$

$$\begin{array}{ccc} \mathbb{E} & \xrightarrow{F^{\underline{\mathbf{\Omega}},\tau}} & \mathbb{E} \\ \downarrow p & & \downarrow p \\ \mathbb{C} & \xrightarrow[F]{} & \mathbb{C} \end{array}$$

Codensity Lifting

Definition (codensity lifting [Katsumata & Sato, CALCO'15])

- $\begin{array}{c}\mathbb{E}\\ \downarrow p\\ \mathbb{C}\end{array}$ be a $\mathbf{CLat}_{\sqcap}$-fibration,
- $F : \mathbb{C} \longrightarrow \mathbb{C}$,
- (**modality**) $\tau : F\Omega \to \Omega$ be an algebra in $\mathbb{C}$, and
- (**observation indistinguishability**) $\underline{\Omega} \in \mathbb{E}_\Omega$.

The *codensity lifting* $F^{\underline{\Omega},\tau} : \mathbb{E} \to \mathbb{E}$ is defined by

$$F^{\underline{\Omega},\tau} P = \prod_{k \in \mathbb{E}(P,\underline{\Omega})} \Big(\tau \circ F\big(p(k)\big)\Big)^* \underline{\Omega}.$$

- **Least distinguishability** s.t. every **1-step observation** is decent
  - "1-step observation": satisfaction of **modal formulas of depth 1**
  - here modality is $\tau : F\Omega \to \Omega$, and
    a prop. fml. is $k : (X, P) \stackrel{\cdot}{\longrightarrow} (\Omega, \underline{\Omega})$, decent

Codensity Bisimulation

The common scenario (recap)

1. Identify an **indistinguishability structure (IS)** $\mathbb{E}$ ✓
2. **Lift** a functor F to $\Phi\colon \mathbb{E}_X \to \mathbb{E}_{FX}$ ✓
3. **Pull back** along the dynamics $c\colon X \to FX$ ✓
4. **Bisimulation** is a post-fixed point $P \sqsubseteq c^*(\Phi(P))$
   Bisimilarity is the greatest fixed point ✓

Definition (codensity bisimulation)

Let $c : X \to FX$ be a F-coalgebra.

- A **codensity bisimulation** is $P \in \mathbb{E}_X$ s.t. $P \sqsubseteq c^*(F^{\underline{\Omega},\tau}(P))$.
- A **codensity bisimilarity** is the greatest fixed point of $c^* \circ F^{\underline{\Omega},\tau}$.

Codensity Bisimulation

Definition (codensity bisimulation)

Let $\boldsymbol{c} : \boldsymbol{X} \to \boldsymbol{FX}$ be a $\boldsymbol{F}$-coalgebra.

- A **codensity bisimulation** is $\boldsymbol{P} \in \mathbb{E}_{\boldsymbol{X}}$ s.t. $\boldsymbol{P} \sqsubseteq \boldsymbol{c}^*(\boldsymbol{F}^{\underline{\boldsymbol{\Omega}},\boldsymbol{\tau}}(\boldsymbol{P}))$.
- A **codensity bisimilarity** is the greatest fixed point $\boldsymbol{\nu}\left(\boldsymbol{c}^* \circ \boldsymbol{F}^{\underline{\boldsymbol{\Omega}},\boldsymbol{\tau}}\right)$.

Examples: **(relational) bisimulation** for Kripke structures,
bisimulation metric for Markov chains,
those for a wide class of coalgebras, **bisimulation topology**, etc.

$\Longrightarrow$
Signifies the roles of **observations**, **predicates** and **distinguishability** in bisimulation-like notions

Codensity Games

(Untrimmed) Codensity Games

A uniform game notion that characterizes codensity bisimulation (and hence many known bisimulation-like notions)

position	pl.	possible moves
$P \in \mathbb{E}_X$	S	$k \in \mathbb{C}(X, \Omega)$ s.t. $\tau \circ F(p(k)) \circ c \notin \mathbb{E}(P, \underline{\Omega})$
$k \in \mathbb{C}(X, \Omega)$	D	$P' \in \mathbb{E}_X$ s.t. $k \notin \mathbb{E}(P', \underline{\Omega})$

- Played by **Spoiler** and **Duplicator**
- **Safety game**, in that an infinite play is won by Duplicator

Theorem

The following are equivalent

- Duplicator wins at $P \in \mathbb{E}_X$
- P is below bisimilarity, that is, $P \sqsubseteq \nu\big(c^* \circ F^{\underline{\Omega}, \tau}\big)$

Trimming the Arena

position	pl.	possible moves
$P \in \mathbb{E}_X$	S	$k \in \mathbb{C}(X, \Omega)$ s.t. $\tau \circ F(p(k)) \circ c \notin \mathbb{E}(P, \underline{\Omega})$
$k \in \mathbb{C}(X, \Omega)$	D	$P' \in \mathbb{E}_X$ s.t. $k \notin \mathbb{E}(P', \underline{\Omega})$

This **untrimmed** game tends to have *big* arenas...

... we can restrict to **generators** in a fiber

Definition (generator)

$\mathcal{G} \subseteq |\mathbb{E}_X|$ is a *generator* of $\mathbb{E}_X$ if, for any $P \in \mathbb{E}_X$, there is $\mathcal{P} \subseteq \mathcal{G}$ s.t. $\bigsqcup_{Q \in \mathcal{P}} Q = P$.

Definition (trimmed codensity game)

position	pl.	possible moves
$P \in \mathcal{G}$	S	$k \in \mathbb{C}(X, \Omega)$ s.t. $\tau \circ F(p(k)) \circ c \notin \mathbb{E}(P, \underline{\Omega})$
$k \in \mathbb{C}(X, \Omega)$	D	$P' \in \mathbb{E}_X$ s.t. $k \notin \mathbb{E}(P', \underline{\Omega})$

Trimming the Arena

Definition (generator)

$\mathcal{G} \subseteq |\mathbb{E}_{\boldsymbol{X}}|$ is a *generator* of $\mathbb{E}_{\boldsymbol{X}}$ if, for any $\boldsymbol{P} \in \mathbb{E}_{\boldsymbol{X}}$, there is $\mathcal{P} \subseteq \mathcal{G}$ s.t. $\bigsqcup_{\boldsymbol{Q} \in \mathcal{P}} \boldsymbol{Q} = \boldsymbol{P}$.

Definition (trimmed codensity game)

position	pl.	possible moves
$\boldsymbol{P} \in \mathcal{G}$	S	$\boldsymbol{k} \in \mathbb{C}(\boldsymbol{X}, \boldsymbol{\Omega})$ s.t. $\boldsymbol{\tau} \circ \boldsymbol{F}(\boldsymbol{p}(\boldsymbol{k})) \circ \boldsymbol{c} \notin \mathbb{E}(\boldsymbol{P}, \underline{\boldsymbol{\Omega}})$
$\boldsymbol{k} \in \mathbb{C}(\boldsymbol{X}, \boldsymbol{\Omega})$	D	$\boldsymbol{P}' \in \mathbb{E}_{\boldsymbol{X}}$ s.t. $\boldsymbol{k} \notin \mathbb{E}(\boldsymbol{P}', \underline{\boldsymbol{\Omega}})$

A recipe for generators:

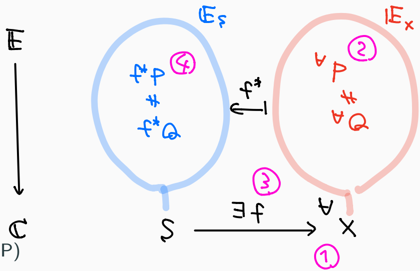
- $\boldsymbol{S} \in \mathbb{C}$ is a *fibered separator* if, for any $\boldsymbol{X} \in \mathbb{C}$ and $\boldsymbol{P}, \boldsymbol{Q} \in \mathbb{E}_{\boldsymbol{X}}$, $\left(\forall \boldsymbol{f} \in \mathbb{C}(\boldsymbol{S}, \boldsymbol{X}).\ \boldsymbol{f}^*\boldsymbol{P} = \boldsymbol{f}^*\boldsymbol{Q}\right) \implies \boldsymbol{P} = \boldsymbol{Q}$.
- Let $\mathcal{G}_{\boldsymbol{S}}$ be a generator of $\mathbb{E}_{\boldsymbol{S}}$. Then, for each $\boldsymbol{X} \in \mathbb{C}$, $\{\boldsymbol{f}_*\boldsymbol{P} \mid \boldsymbol{P} \in \mathcal{G}_{\boldsymbol{S}}, \boldsymbol{f} \in \mathbb{C}(\boldsymbol{S}, \boldsymbol{X})\}$ is a generator of $\mathbb{E}_{\boldsymbol{X}}$.

Trimming the Arena

A recipe for generators:

- $S \in \mathbb{C}$ is a *fibered separator* if, for any $X \in \mathbb{C}$ and $P, Q \in \mathbb{E}_X$,
  $(\forall f \in \mathbb{C}(S, X).\ f^*P = f^*Q) \implies P = Q$.
- Let $\mathcal{G}_S$ be a generator of $\mathbb{E}_S$. Then, for each $X \in \mathbb{C}$,
  $\{f_*P \mid P \in \mathcal{G}_S, f \in \mathbb{C}(S, X)\}$ is a generator of $\mathbb{E}_X$.

Fibered separator S



Trimming the Arena

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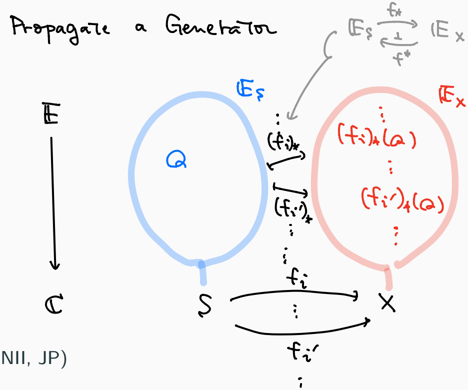
  $(\forall \boldsymbol{f} \in \mathbb{C}(\boldsymbol{S}, \boldsymbol{X}).\ \boldsymbol{f}^* \boldsymbol{P} = \boldsymbol{f}^* \boldsymbol{Q}) \implies \boldsymbol{P} = \boldsymbol{Q}$.
- Let $\mathcal{G}_{\boldsymbol{S}}$ be a generator of $\mathbb{E}_{\boldsymbol{S}}$. Then, for each $\boldsymbol{X} \in \mathbb{C}$,
  $\{\boldsymbol{f}_* \boldsymbol{P} \mid \boldsymbol{P} \in \mathcal{G}_{\boldsymbol{S}}, \boldsymbol{f} \in \mathbb{C}(\boldsymbol{S}, \boldsymbol{X})\}$ is a generator of $\mathbb{E}_{\boldsymbol{X}}$.



Examples

Table 1: Codensity Bisimilarity Game for Conventional Bisimilarity

position	pl.	possible moves
$(x, y) \in X \times X$	S	$k \in \mathbf{Set}(X, 2)$ s.t. $\exists x' \in c(x).\ k(x') = \top \Leftrightarrow \exists y' \in c(y).\ k(y') = \top$ not satisfied
$k \in \mathbf{Set}(X, 2)$	D	(x'', y'') s.t. $k(x'') \neq k(y'')$

Examples

Table 2: Codensity Bisimilarity Game for Deterministic Automata and Their Language Equivalence

position	pl.	possible moves
$(x, y) \in X \times X$	S	If $\pi_1(x) \neq \pi_1(y)$ then S wins If $\pi_1(x) = \pi_1(y)$ then $a \in \Sigma$ and $k \in \mathbf{Set}(X, 2)$ s.t. $k(\pi_2(x)(a)) \neq k(\pi_2(y)(a))$
$a \in \Sigma$ and $k \in \mathbf{Set}(X, 2)$	D	$(x'', y'') \in X \times X$ s.t. $k(x'') \neq k(y'')$

Examples

Table 3: Codensity Bisimilarity Game for Deterministic Automata and Bisimulation Topology

position	pl.	possible moves
$\mathcal{O} \in \mathbf{Top}_{\boldsymbol{X}}$	S	$\boldsymbol{a} \in \{\epsilon\} \cup \boldsymbol{\Sigma}$ and $\boldsymbol{k} \in \mathbf{Set}(\boldsymbol{X}, 2)$ s.t. $\tau_{\boldsymbol{A}} \circ (\boldsymbol{A}_{\boldsymbol{\Sigma}}\boldsymbol{k}) \circ \boldsymbol{c} \notin \mathbf{Top}(\mathcal{O}, \underline{\boldsymbol{\Omega}}(\boldsymbol{a}))$
$\boldsymbol{a} \in \{\epsilon\} \cup \boldsymbol{\Sigma}$ and $\boldsymbol{k} \in \mathbf{Set}(\boldsymbol{X}, 2)$	D	$\mathcal{O}' \in \mathbf{Top}_{\boldsymbol{X}}$ s.t. $\boldsymbol{k} \notin \mathbf{Top}(\mathcal{O}', \underline{\boldsymbol{\Omega}}(\boldsymbol{a}))$