Kripke Completeness of First-Order Constructive Logics with Strong Negation

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Overview

- Introducing **Strong Negation** and Nelson's Constructive Logic N
- Introducing variants of N, esp. by the **Axiom of Potential Omniscience**
- Completeness proofs are given by the **Tree-Sequent** method

Notations

- Logical Symbols:
  $\wedge$, $\rightarrow$, $\forall$,
  $\neg$ (Heyting's negation), $\sim$ (strong negation)
- $\vee$ and $\exists$ can be defined:
  $A \vee B \equiv {\sim}({\sim}A \wedge {\sim}B)$, $\quad \exists x A \equiv {\sim}\forall x {\sim} A$
- $\Gamma \Rightarrow \Delta$: A sequent.
  Γ and Δ are finite **sets** of formulas
- $A \leftrightarrow B$ is for $(A \rightarrow B) \wedge (B \rightarrow A)$
- GL: Gentzen-style sequent system for logic L.
  TL: Tree-sequent system for L.

Strong Negation $\sim A$

Introduced by Nelson and Markov, axiomatized by:

$$A \to (\sim A \to B),$$

$$\sim(A \wedge B) \leftrightarrow \sim A \vee \sim B, \quad \sim(A \vee B) \leftrightarrow \sim A \wedge \sim B,$$

$$\sim(A \to B) \leftrightarrow A \wedge \sim B, \quad \sim\sim A \leftrightarrow A \quad \sim\neg A \leftrightarrow A,$$

$$\sim\forall x A \leftrightarrow \exists x \sim A, \quad \sim\exists x A \leftrightarrow \forall x \sim A.$$

Nelson's constructive logic N, is Int plus $\sim$.

Strong Negation vs. Heyting's One

Heyting's Negation

$\neg A \leftrightarrow (A \to \bot)$,

Strong Negation vs. Heyting's One

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"the compound sentences are not a product of experiment, they arise from reasoning. This concerns also negation: we see that the lemon is yellow, we do not see that it is not blue" (Grzegorczyk)

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$\neg A \leftrightarrow (A \rightarrow \perp)$,

Negative information is reduced to positive one.

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Strong Negation

Strong Negation vs. Heyting's One

Heyting's Negation

$\neg A \leftrightarrow (A \rightarrow \bot)$,
Negative information is reduced to positive one.

Strong Negation

"we can not only **verify** a simple proposition such as **This door is locked.** by direct inspection, but also **falsify** it" (Kracht)

Strong Negation vs. Heyting's One

Heyting's Negation

$\neg A \leftrightarrow (A \rightarrow \bot)$,

Negative information is reduced to positive one.

Strong Negation

Negative/Positive informations are equally primitive!

Kripke Semantics for N

Int-model:

$$(M, \leq, U, I^+)$$

Kripke Semantics for N

N-model:

$$(M, \leq, U, I^+, I^-)$$

Kripke Semantics for N

N-model:

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I^+: **verum** interpretation

I^-: **falsum** interpretation

Kripke Semantics for N

N-model:

$$(M, \leq, U, I^+, I^-)$$

I^+: **verum** interpretation
extended to $a \models^+ A$ (a **verifies** A).

I^-: **falsum** interpretation
extended to $a \models^- A$ (a **falsifies** A).

Kripke Semantics for N

$$a \models^+ p(\underline{u_1}, \ldots, \underline{u_m}) \iff (u_1, \ldots, u_m) \in p^{I^+(a)}\ ;$$

$$a \models^+ A \wedge B \iff a \models^+ A \quad \text{and} \quad a \models^+ B\ ;$$

$$a \models^+ A \to B \iff \text{for every } b \geq a, \quad b \models^+ A \quad \text{implies} \quad b \models^+ B\ ;$$

$$a \models^+ \neg A \iff \text{for every } b \geq a, \quad b \not\models^+ A\ ;$$

$$a \models^+ {\sim} A \iff a \models^- A\ ;$$

$$a \models^+ \forall x A \iff \text{for every } b \geq a \text{ and every } u \in U(b), \quad b \models^+ A[\underline{u}/x]\ ;$$

Kripke Semantics for N

$a \models^{-} p(\underline{u_1}, \ldots, \underline{u_m}) \iff (u_1, \ldots, u_m) \in p^{I^{-}(a)}$;

$a \models^{-} A \wedge B \iff a \models^{-} A$ or $a \models^{-} B$;

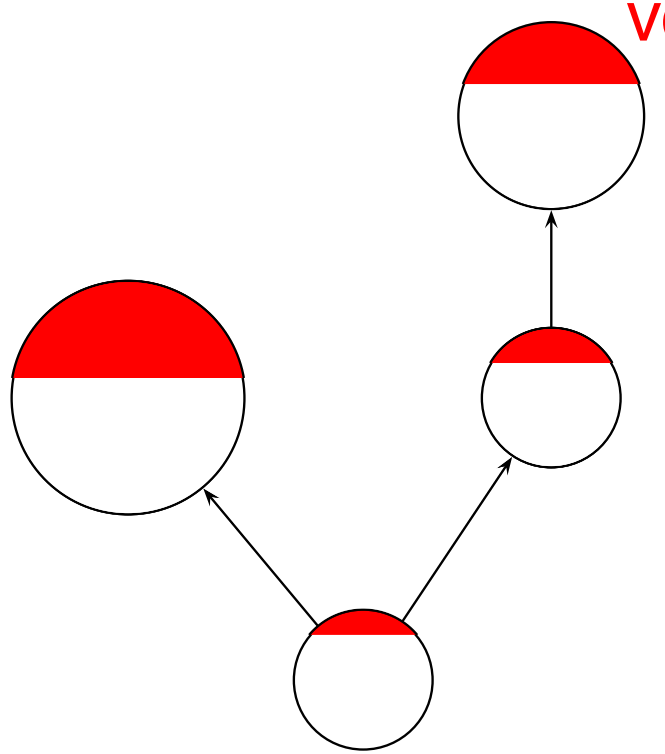
$a \models^{-} A \to B \iff a \models^{+} A$ and $a \models^{-} B$;

$a \models^{-} \neg A \iff a \models^{+} A$;

$a \models^{-} {\sim} A \iff a \models^{+} A$;

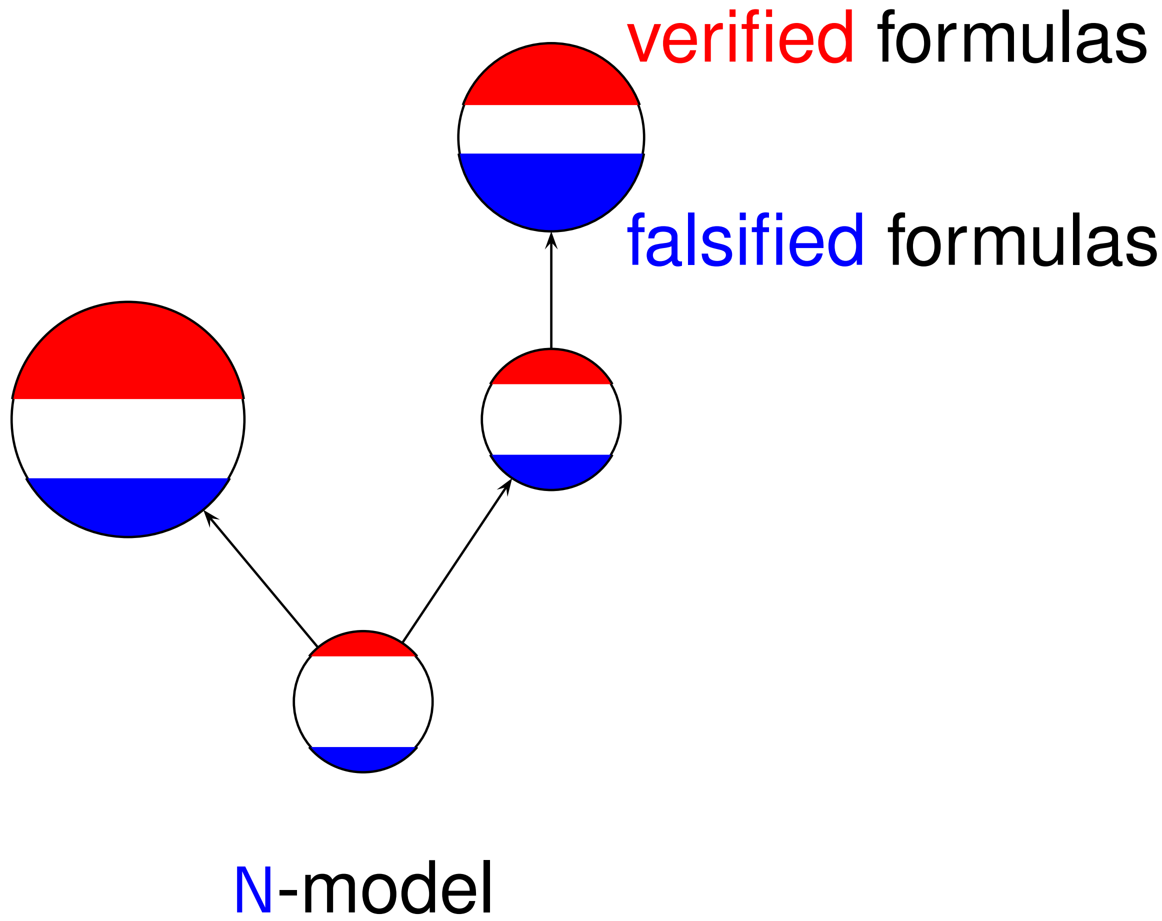
$a \models^{-} \forall x A \iff$ for some $u \in U(a)$, $a \models^{-} A[\underline{u}/x]$.

Kripke Semantics for N



Int-model

Kripke Semantics for N



Sequent System GN $\sim$***-free part***

$$\frac{}{A \Rightarrow A}\ \text{(Identity)}$$

$$\frac{\Gamma \Rightarrow \Delta}{\Sigma, \Gamma \Rightarrow \Delta, \Pi}\ \text{(Weakening)} \qquad \frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \text{(Cut)}$$

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ (\wedge\mathsf{L}) \qquad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\ (\wedge\mathsf{R})$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \rightarrow B, \Gamma \Rightarrow \Delta}\ (\rightarrow\mathsf{L}) \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B}\ (\rightarrow\mathsf{R})_{\mathsf{S}}$$

$$\frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta}\ (\neg\mathsf{L}) \qquad \frac{A, \Gamma \Rightarrow}{\Gamma \Rightarrow \neg A}\ (\neg\mathsf{R})_{\mathsf{S}}$$

$$\frac{A[y/x], \Gamma \Rightarrow \Delta}{\forall x A, \Gamma \Rightarrow \Delta}\ (\forall\mathsf{L}) \qquad \frac{\Gamma \Rightarrow A[z/x]}{\Gamma \Rightarrow \forall x A}\ (\forall\mathsf{R})_{\mathsf{S,\ VC}}$$

(Equal to Maehara's $\mathbf{LJ'}$)

Sequent System GN *Introducing* $\sim$

$$\frac{}{A, {\sim}A \Rightarrow} \text{ (Ex Falso)}$$

$$\frac{{\sim}A, \Gamma \Rightarrow \Delta \quad {\sim}B, \Gamma \Rightarrow \Delta}{{\sim}(A \wedge B), \Gamma \Rightarrow \Delta} \ ({\sim}{\wedge}\mathrm{L}) \qquad \frac{\Gamma \Rightarrow \Delta, {\sim}A, {\sim}B}{\Gamma \Rightarrow \Delta, {\sim}(A \wedge B)} \ ({\sim}{\wedge}\mathrm{R})$$

$$\frac{A, {\sim}B, \Gamma \Rightarrow \Delta}{{\sim}(A \rightarrow B), \Gamma \Rightarrow \Delta} \ ({\sim}{\rightarrow}\mathrm{L}) \qquad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, {\sim}B}{\Gamma \Rightarrow \Delta, {\sim}(A \rightarrow B)} \ ({\sim}{\rightarrow}\mathrm{R})$$

$$\frac{A, \Gamma \Rightarrow \Delta}{{\sim}\neg A, \Gamma \Rightarrow \Delta} \ ({\sim}{\neg}\mathrm{L}) \qquad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, {\sim}\neg A} \ ({\sim}{\neg}\mathrm{R})$$

$$\frac{A, \Gamma \Rightarrow \Delta}{{\sim}{\sim}A, \Gamma \Rightarrow \Delta} \ ({\sim}{\sim}\mathrm{L}) \qquad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, {\sim}{\sim}A} \ ({\sim}{\sim}\mathrm{R})$$

$$\frac{{\sim}A[z/x], \Gamma \Rightarrow \Delta}{{\sim}\forall x A, \Gamma \Rightarrow \Delta} \ ({\sim}{\forall}\mathrm{L})_{\mathrm{VC}} \qquad \frac{\Gamma \Rightarrow \Delta, {\sim}A[y/x]}{\Gamma \Rightarrow \Delta, {\sim}\forall x A} \ ({\sim}{\forall}\mathrm{R})$$

The Axiom of Potential Omniscience

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The axiom of potential omniscience

$$\neg\neg(A \vee {\sim}A)$$

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The axiom of potential omniscience

$$\neg\neg(A \vee \sim A)$$

Introduced by Hasuo, interpreted as:

> We can eventually either verify or falsify a statement, with proper additional information.

Variants of N

N: Int plus $\sim$.

D: Add $\forall x(A(x) \vee B) \rightarrow \forall x A(x) \vee B$, the axiom of **constant domain**

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L: $(A \to B) \vee (B \to A)$

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L: $(A \rightarrow B) \vee (B \rightarrow A)$ $\boxed{\Rightarrow \text{linearly ordered models}}$

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- D: Add $\forall x(A(x) \lor B) \to \forall x A(x) \lor B$, the axiom of **constant domain** $\boxed{\Rightarrow \text{contant domain models}}$
- L: $(A \to B) \lor (B \to A)$ $\boxed{\Rightarrow \text{linearly ordered models}}$
- O: Add $\neg\neg(A \lor {\sim}A)$, the axiom of **potential omniscience**

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for $a \in M$ and A (closed formula),
$\exists b \geq a$ s.t. $b \models^{+} A$ or $b \models^{-} A$.

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P: Omit the axiom $A \rightarrow ({\sim}A \rightarrow B)$.

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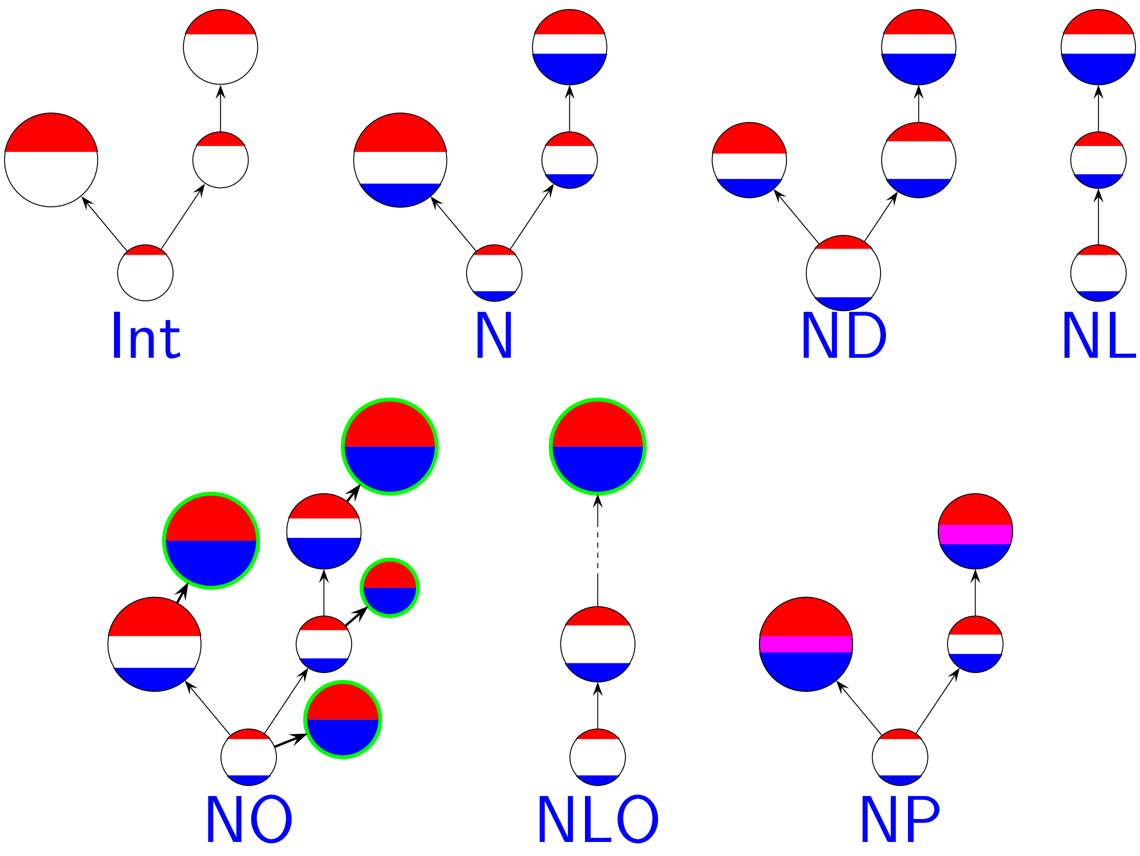
> for $a \in M$ and A (closed formula),
> $\exists b \geq a$ s.t. $b \models^{+} A$ or $b \models^{-} A$.

P: Omit the axiom $A \rightarrow (\sim A \rightarrow B)$.

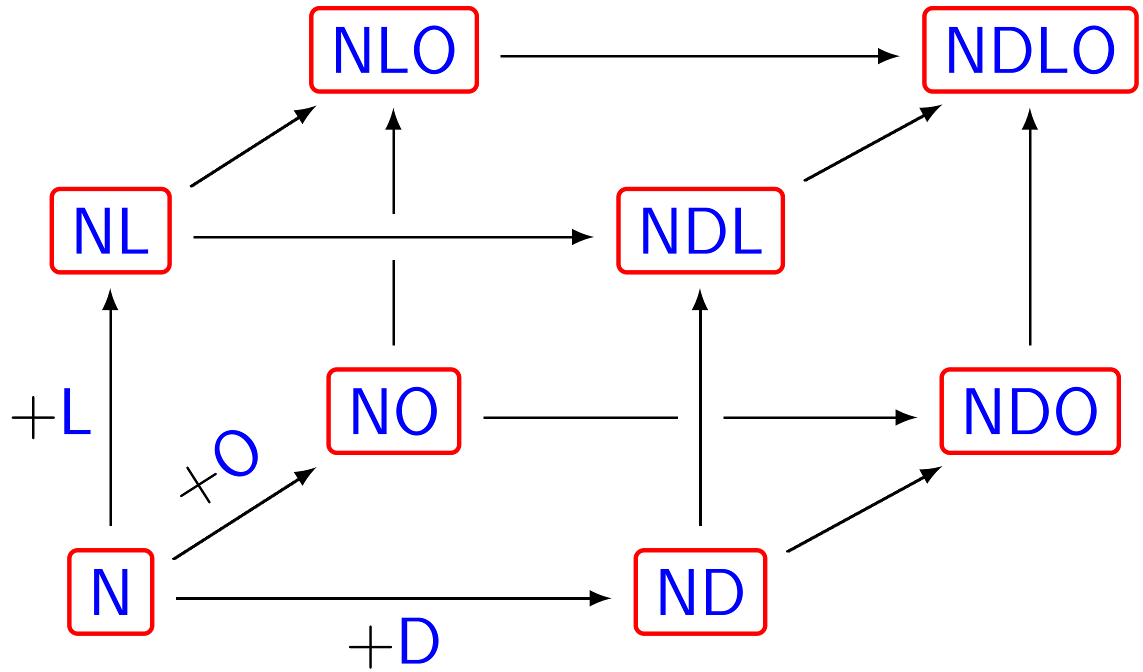
> I^{+} and I^{-} are not disjoint $\Rightarrow$ paraconsistency!

Corresponding Kripke Models



Logics in Consideration – The N-family

Those without P:



For the enclosed, completeness is shown.
N: van Dalen (1986), ND: Thomason (1969)
The others: Hasuo

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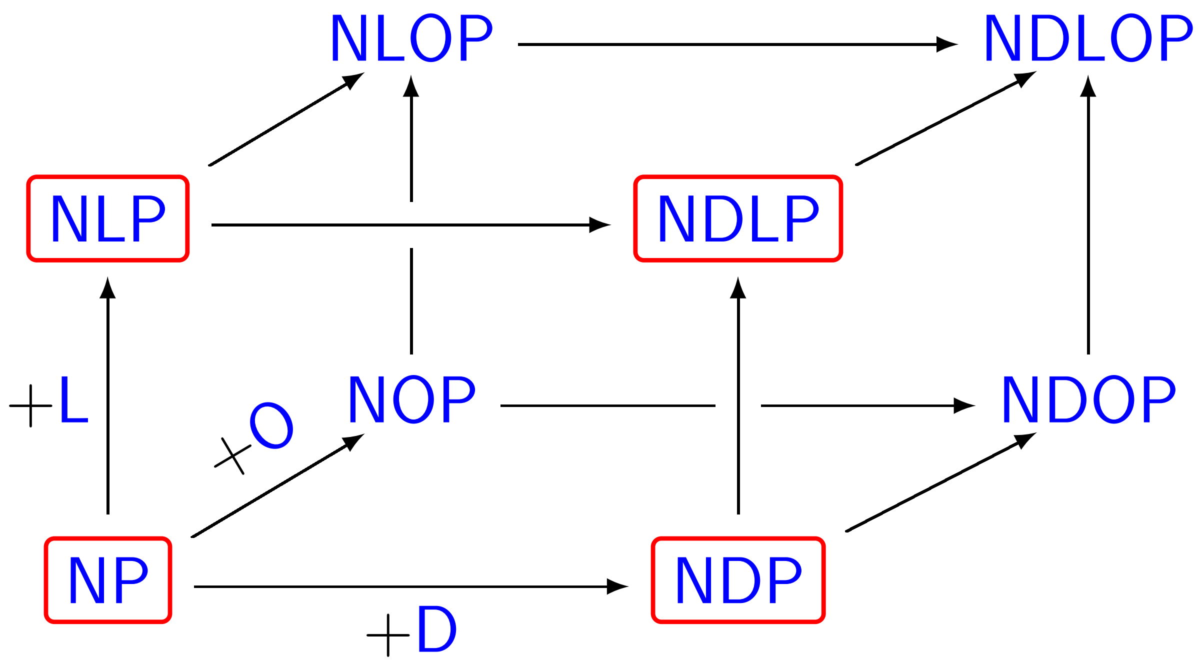

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Completeness Proofs for Logics without O

The **Tree-Sequent** method (Kashima) gives unified proofs for N[D][L][P]
(also for Int and intermediate logics e.g. CD, LC).

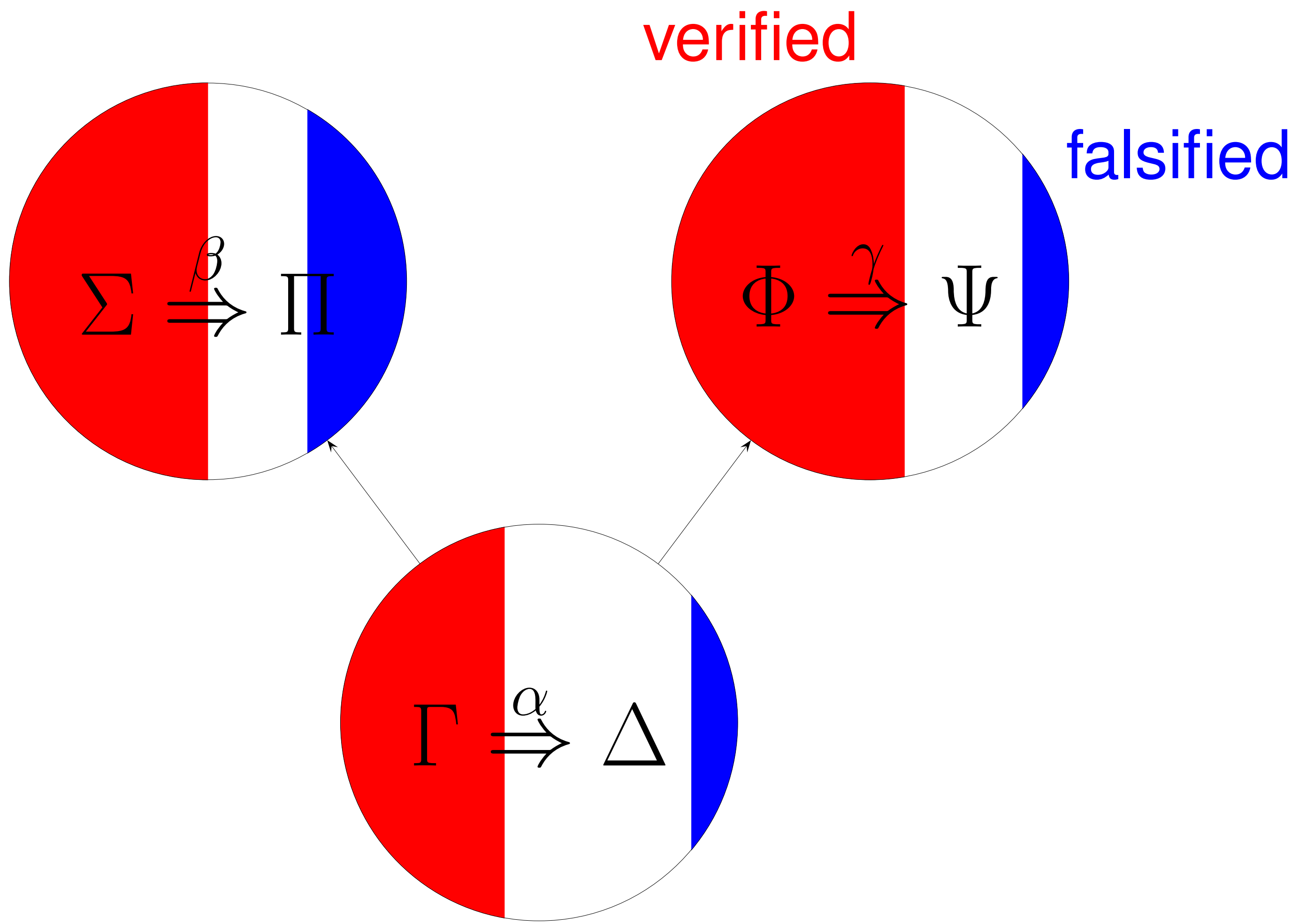
Sloppily, the tree-sequent (TS) method is

a kind of semantic tableaux.

Tree-Sequent and its Counter Model

$$
\begin{tikzcd}
\Sigma \overset{\beta}{\Rightarrow} \Pi & & \Phi \overset{\gamma}{\Rightarrow} \Psi \\
& \Gamma \overset{\alpha}{\Rightarrow} \Delta \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

Tree-Sequent and its Counter Model



Formulas in LHS are **verified**,
those in RHS are **not verified**.

Outline of the Tree-Sequent method

For logic L,

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For logic L,

1. Define the TS system $\mathbb{T}\mathsf{L}$.

A TS of $\mathbb{T}\mathsf{N}$ is a tree of sequents, since N-models are trees.

Accordingly a TS of $\mathbb{T}\mathsf{NL}$ is a sequence of sequents.

Outline of the Tree-Sequent method

For logic L,

1. Define the TS system $\mathbb{T}$L.

2. Completeness of the TS system, i.e.

$$\boxed{\mathbb{T}\mathsf{L} \nvdash \mathcal{T} \quad \Rightarrow \quad \mathcal{T} \text{ has a counter model}}$$

is easy.

Extend $\mathcal{T}$ into a **saturated** TS, which induces a counter model.

Outline of the Tree-Sequent method

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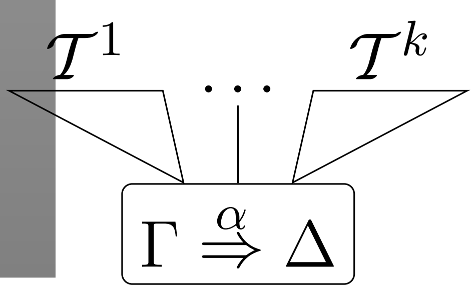
is easy.

3. Define formulaic translation $\mathcal{T}^f$ of a TS,

and prove $\boxed{\mathbb{T}\mathsf{L} \vdash \mathcal{T} \quad \Rightarrow \quad \mathbb{G}\mathsf{L} \vdash \mathcal{T}^f}$

e.g. in $\mathbb{T}$N,



$$\overset{f}{\mapsto} \quad \forall \overrightarrow{\alpha}\Big(\big(\bigwedge \Gamma\big) \to \big(\bigvee \Delta\big) \vee \mathcal{T}_1^f \vee \cdots \vee \mathcal{T}_k^f \Big)$$

Outline of the Tree-Sequent method

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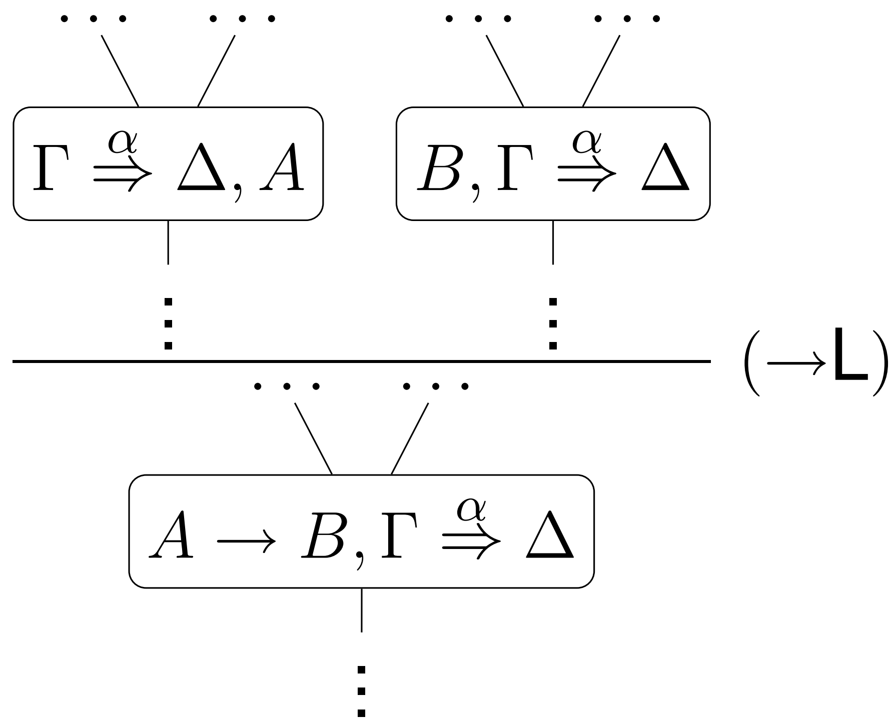
4. Let $\mathrm{G}\mathsf{L} \nvdash A$.

Then $\mathrm{T}\mathsf{L} \nvdash \boxed{\overset{\alpha}{\Rightarrow} A}$ by 3,
hence A has a counter model by 2.

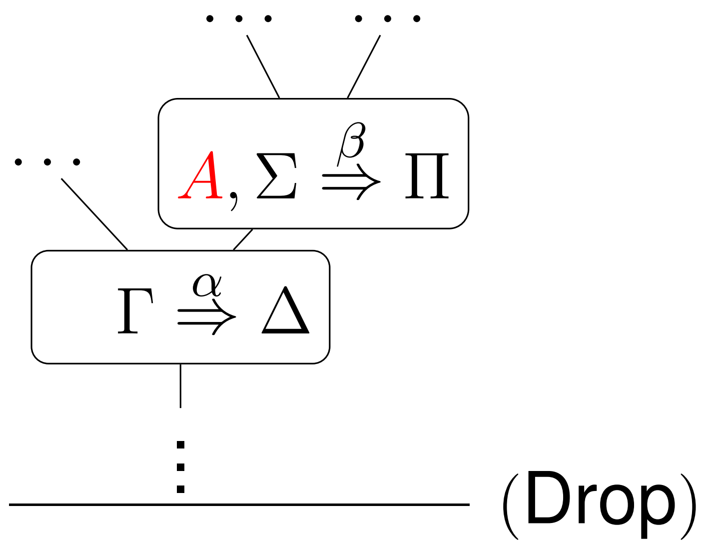
$\mathbb{T}$N – *TS system for logic* N

$$\dfrac{\begin{array}{c}\cdots \qquad \cdots\\ \boxed{\Gamma \overset{\alpha}{\Rightarrow} \Delta, A}\\ \vdots\end{array} \qquad \begin{array}{c}\cdots \qquad \cdots\\ \boxed{B, \Gamma \overset{\alpha}{\Rightarrow} \Delta}\\ \vdots\end{array}}{}\ (\rightarrow\mathsf{L})$$

$\mathbb{T}$N – *TS system for logic* N



TN – *TS system for logic* N



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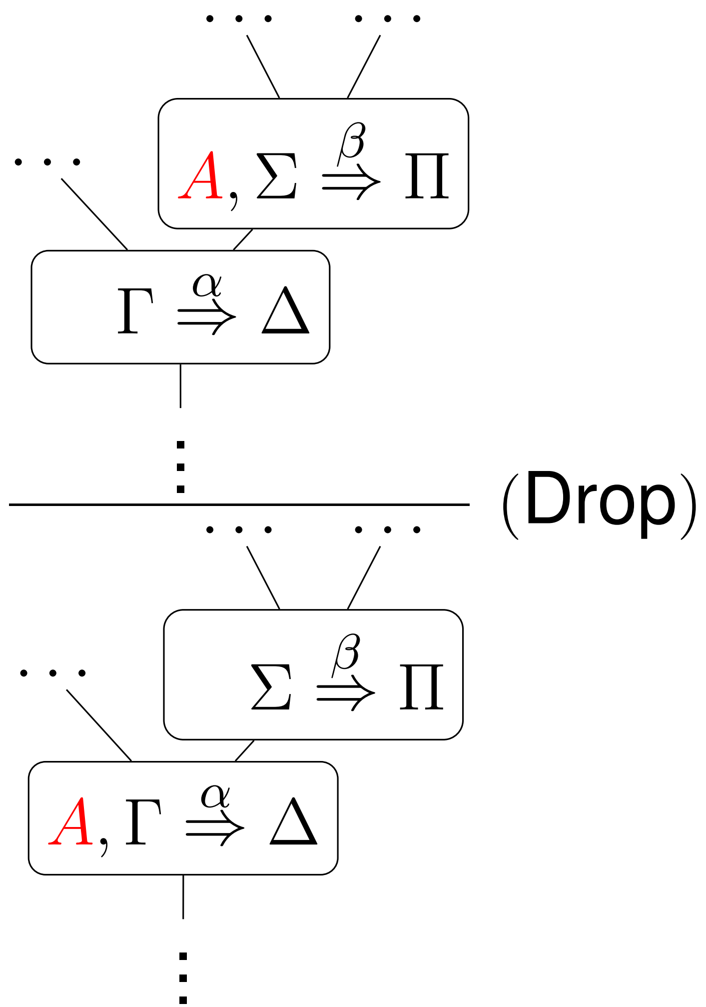
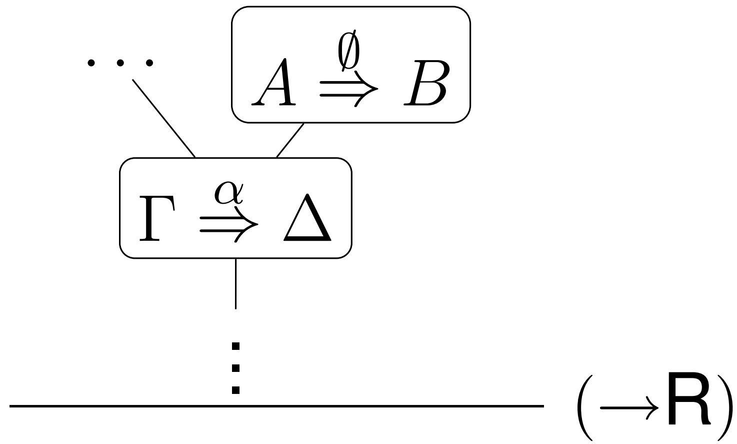


$\mathbb{T}$N – *TS system for logic* N



TN – *TS system for logic* N

$$\cdots \qquad \boxed{A \overset{\emptyset}{\Rightarrow} B}$$

$$\boxed{\Gamma \overset{\alpha}{\Rightarrow} \Delta}$$

$$\vdots$$

$$\frac{\qquad\qquad\qquad}{\cdots \qquad \text{(trimmed!)}} \;(\to\mathrm{R})$$

$$\boxed{\Gamma \overset{\alpha}{\Rightarrow} \Delta, A \to B}$$

$$\vdots$$

Completeness Proofs for Logics with O *(1)*

(Again) O is for **the axiom of potential omniscience**.

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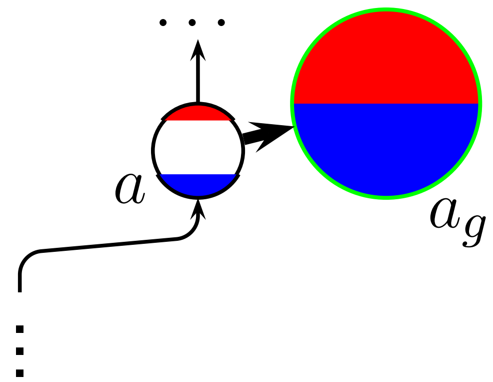
How can we obtain **omniscient worlds**
(where every closed formula is either verified or falsified)?



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One of our proofs
– Utilizes an embedding of LK

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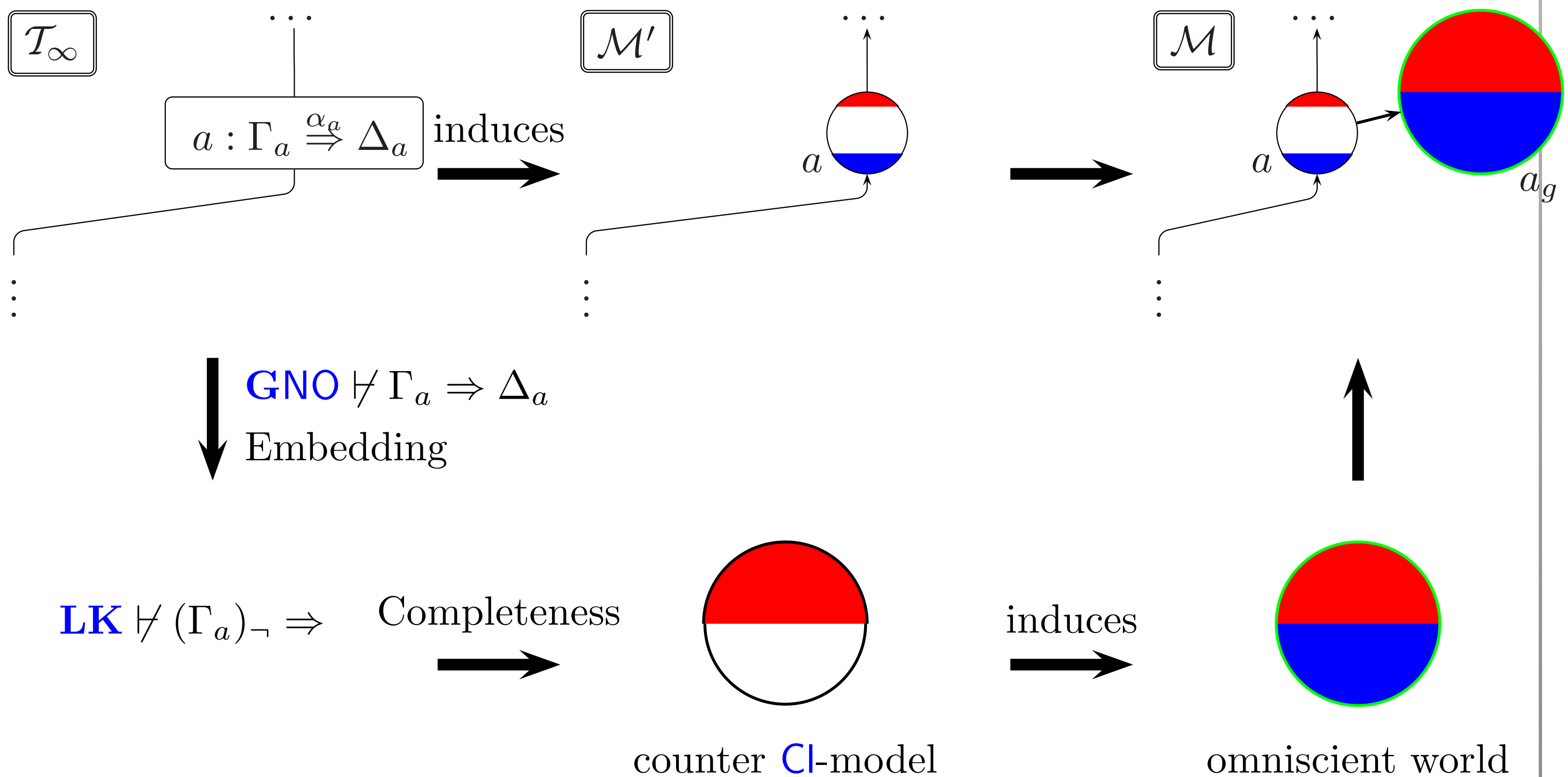
One of our proofs
– Utilizes an embedding of $\mathbf{LK}$

Replacing an arbitrary number of $\neg$ by $\sim$,
we obtain $A_{\sim\neg}$ from A.

[Lemma] (Embedding)
$\mathbf{LK} \vdash \Gamma \Rightarrow \Delta$ *iff* $\mathrm{GN[D][L]O} \vdash \Gamma_{\sim\neg}, {\sim}\Delta_{\sim\neg} \Rightarrow$

Completeness Proofs for Logics with O *(1)*



Completeness Proofs for Logics with O *(2)*

The other proof is By **Tree-Sequents with Guardians**.

$$\cdots$$

$$\boxed{a : \Gamma_a \overset{\alpha_a}{\Rightarrow} \Delta_a \uparrow \Sigma_a \overset{\beta_a}{\Rightarrow} \Pi_a}$$

$$\vdots$$

The second sequent is **guardian**, a seed of an omniscient world.

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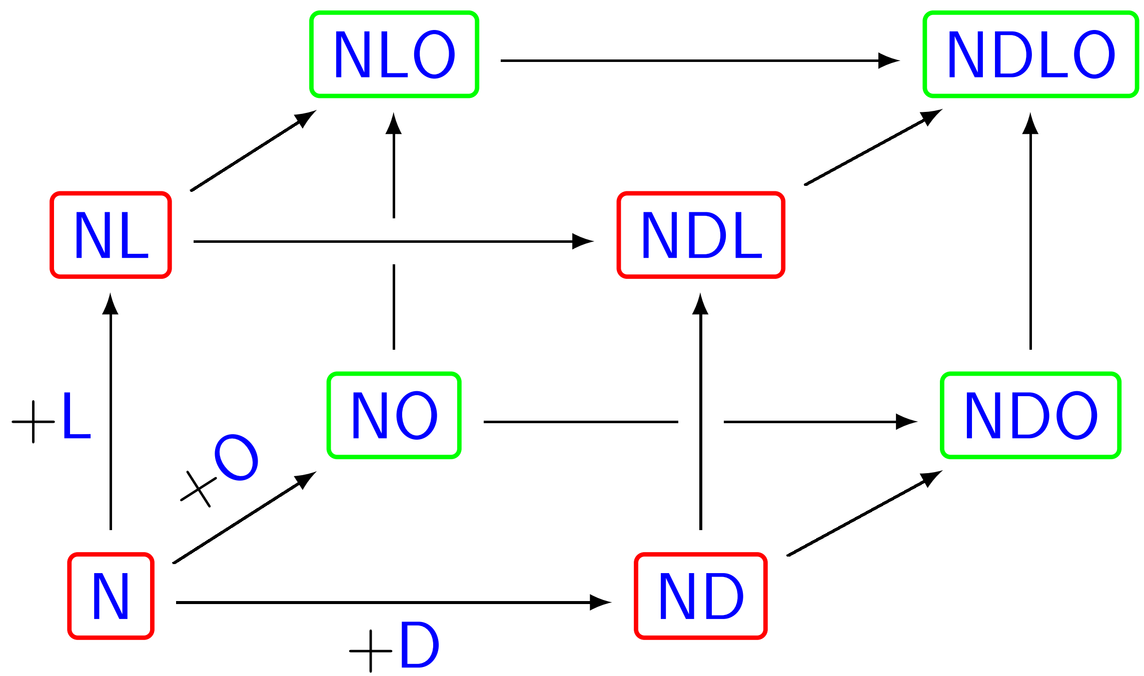
$$\vdots$$

Proofs are just like in the TS method.

Problem (1):
Logics with Both O and P

Those without P:



logic: completeness by TS

logic: completeness by embedding or TSg

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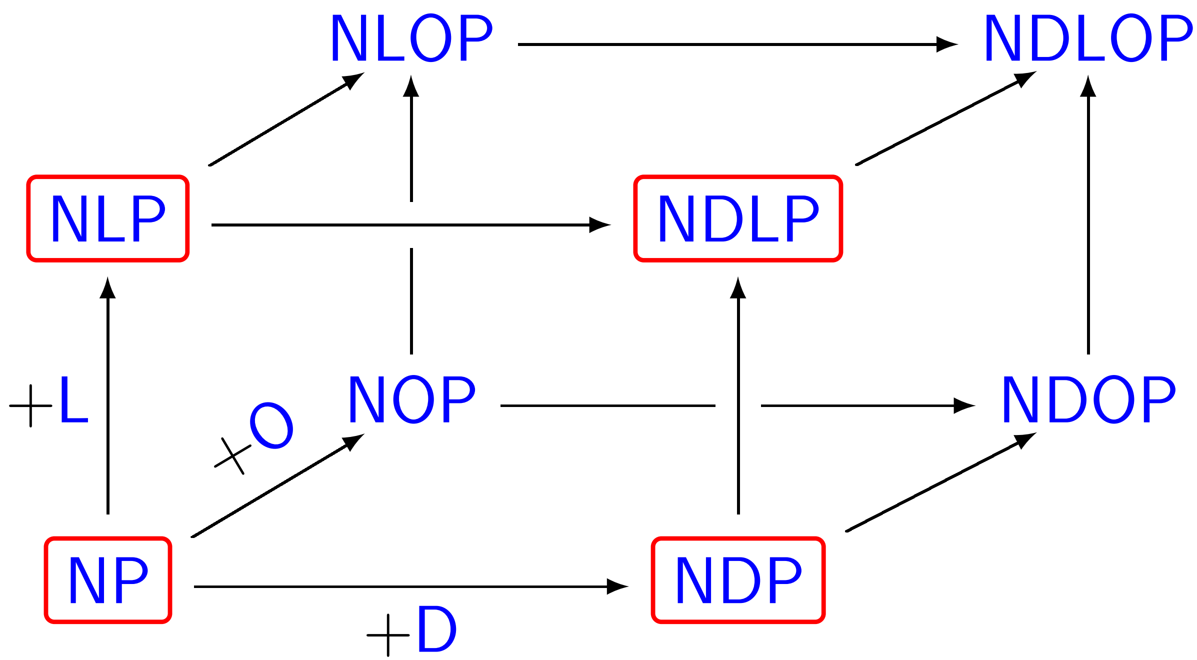

logic: completeness by TS

Problem (1):
Logics with Both O *and* P

Two methods here,
embedding of LK,
tree-sequent with guardians
cannot be applied.

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NOP $\nvdash$ (DNS), since we have a counter model.

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[Gabbay, 1981] MH (= Int plus (DNS)) is characterized by the frames s.t.

for $\forall a \in M$, $\exists b \geq a$ s.t. b is maximal.

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[Gabbay, 1981] MH (= Int plus (DNS)) is characterized by the frames s.t.

for $\forall a \in M$, $\exists b \geq a$ s.t. b is maximal.

Counter models by our methods satisfy the above property! (Omniscient worlds are maximal)

Problem (2): Completeness of (NO plus $\neg A \vee \neg\neg A$)

prop-Int plus $\neg A \vee \neg\neg A$
(**The weak law of excluded middle**) is characterized by frames with their maximums.

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[Question] (Quantified)
Is (NO plus $\neg A \vee \neg\neg A$) characterized by N-models with its maximum omniscient?

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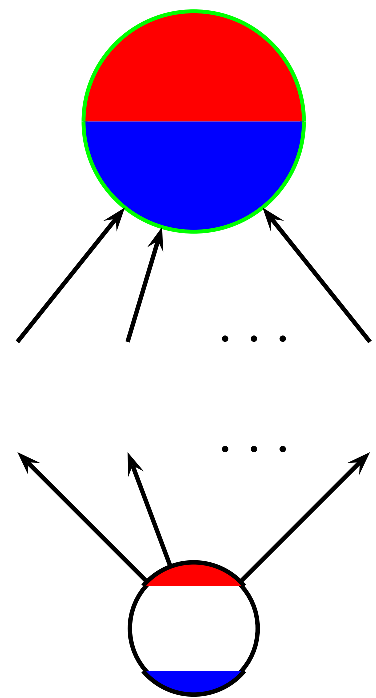
[Question] (Quantified)
Is (NO plus $\neg A \vee \neg\neg A$) characterized by N-models with its maximum omniscient?



Problem (2): Completeness of (NO plus $\neg A \vee \neg\neg A$)

[Corsi and Ghilardi, 1989]

KC (= Int plus $\neg A \vee \neg\neg A$) is characterized by directed frames,

i.e.

$$\boxed{a \leq b,\ a \leq c \Rightarrow \exists d \quad \text{s.t.} \quad b \leq d,\ c \leq d.}$$

(Existence of the maximum is too strong!)

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[Fact] NO $\vdash$ (DNS).

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[Ono, 1987]

Int + $\neg A \vee \neg\neg A$ + $\forall x \neg\neg A \rightarrow \neg\neg\forall x A$ (DNS) + (the axiom of constant domain)

is characterized by constant domain frames with **the maximum**.

Thank You for Your Attention