Coalgebraic Trace Semantics for Probabilistic Systems

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Overview

- **Initial algebra in $\mathbf{Sets}$ yields final coalgebra in $\mathbf{Rel}$**
  - Application: context-free grammars/languages
  - $\mathbf{Rel} \cong \mathbf{Sets}_{\mathcal{P}}$, Kleisli category
  - In Proc. CALCO'05.
- Generalization:
  Same for subdistribution monad $\mathcal{D}$, instead of $\mathcal{P}$
  - Application: trace distribution of probabilistic automata
  - In Proc. CALCO-jnr.

Both results are presented in CALCO'05,
Monday 5 September, 15.40–, so now is the time for...

Sneak Preview

Coalgebraic Trace Semantics

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Coalgebras

Consider a functor

$$FX = 1 + \Sigma \times X \quad , \quad \text{where} \quad 1 = \{\checkmark\} \; .$$

An F-coalgebra

$$\begin{tikzcd} 1 + \Sigma \times X \\ X \arrow[u, "c"] \end{tikzcd}$$

is a system in which,

each state $x \in X$ either

$$\begin{cases} \text{terminates,} & \text{i.e.} \quad c(x) = \checkmark, \qquad \text{or} \\ \text{outputs } a \in \Sigma \text{ and moves to } y \in X, & \text{i.e.} \quad c(x) = (a, y). \end{cases}$$

Final coalgebra semantics

Behavior of F-coalgebra is

the finite/infinite sequence of outputs,
i.e. an element of $\Sigma^\infty = \Sigma^* + \Sigma^\omega$.

- Σ^∞ carries a **final** F-coalgebra.

$$\begin{tikzcd}
1 + \Sigma \times \Sigma^\infty & \checkmark & (a, \sigma) \\
\Sigma^\infty \arrow[u, "\cong", "\zeta"'] & \langle\rangle \arrow[u, mapsto] & a \cdot \sigma \arrow[u, mapsto]
\end{tikzcd}$$

Final coalgebra semantics

- Unique coalgebra morphism yields the **behavior map**.

$$\begin{tikzcd}
1 + \Sigma \times X \arrow[r, dashed, "1 + \Sigma \times \mathsf{beh}_c"] & 1 + \Sigma \times \Sigma^\infty \\
X \arrow[u, "c"] \arrow[r, dashed, "\mathsf{beh}_c"'] & \Sigma^\infty \arrow[u, "\cong"', "\zeta"]
\end{tikzcd}$$

- This amounts to solving the (co)recursive equation:

$$\mathsf{beh}_c(x) = \begin{cases} \langle\rangle & \text{if } c(x) = \checkmark \\ a \cdot \mathsf{beh}_c(y) & \text{if } c(x) = (a, y) \end{cases}$$

So far so standard...

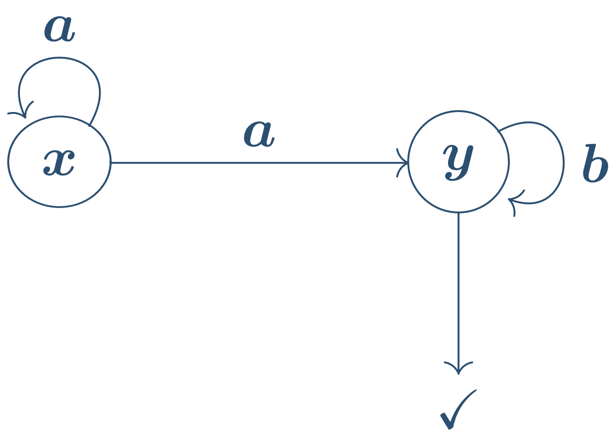
Non-deterministic systems

Consider a coalgebra

$$\begin{tikzcd}\mathcal{P}(1 + \Sigma \times X) \\ X \arrow[u, "c"]\end{tikzcd}$$

.

Example



$$c(x) = \{(a, x),\ (a, y)\}$$

$$c(y) = \{\checkmark,\ (b, y)\}$$

This is **non-deterministic**, or **branching-time**.

Trace semantics

[Lambek]

Structure map of a final coalgebra is an isomorphism

Final coalgebra semantics doesn't work since an isomorphism

$$\begin{tikzcd} \mathcal{P}FZ \\ Z \arrow[u, "\cong"'] \end{tikzcd}$$

is impossible.

Trace semantics

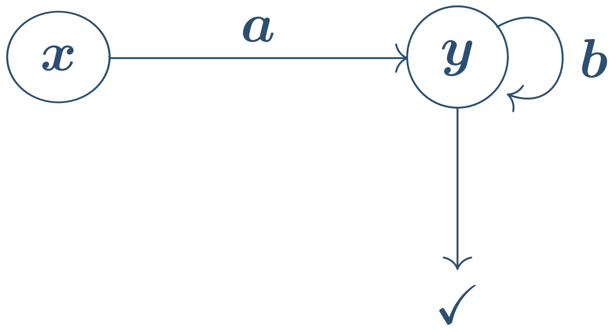
Some candidates for “**semantics**” of non-deterministic systems...

- bisimulation equivalence (finest)
- **trace equivalence**, which we are interested in

trace = set of linear-time, sequential behavior which can possibly happen

Trace semantics

Example



$$\mathbf{trace}(x) = \{a,\ ab,\ abb,\ \ldots\}$$

$$\mathbf{trace}(y) = \{\langle\rangle,\ b,\ bb,\ \ldots\}$$

What is “trace”, from coalgebraic perspective?

Trace semantics

[IH & BJ, CALCO'05]

- We move our base category from $\mathbf{Sets}$ to $\mathbf{Rel}$.

- Then the map $\mathbf{trace} : X \longrightarrow \mathcal{P}(\Sigma^*)$ is actually obtained via **final coalgebra** again!

- Notice that: Σ^* is the initial $(1 + \Sigma \times -)$-algebra.

Main technical result (CALCO)

Let F be a "shapely" functor in $\mathbf{Sets}$.
(Lifts to $\mathbf{Rel}$)

Let
$$\begin{tikzcd} FA \arrow[d, "\alpha", "\cong"'] \\ A \end{tikzcd}$$
be the initial F-algebra in $\mathbf{Sets}$.

Theorem

Coalgebra $A \xrightarrow{\mathsf{graph}(\alpha^{-1})} FA$ is final in $\mathbf{Rel}$.

Main technical result (CALCO)

Note: $\mathbf{Rel} \cong \mathbf{Sets}_{\mathcal{P}}$, Kleisli category.

$$\mathbf{Sets}_{\mathcal{P}} \begin{cases} \text{object} & \text{that of } \mathbf{Sets} \\[2ex] \text{arrow} & \dfrac{X \to Y \qquad \text{in } \mathbf{Sets}_{\mathcal{P}}}{X \to \mathcal{P}Y \qquad \text{in } \mathbf{Sets}} \end{cases}$$

Let $$\begin{tikzcd} FA \arrow[d, "\alpha", "\cong"'] \\ A \end{tikzcd}$$ be the initial F-algebra in $\mathbf{Sets}$.

Theorem

Coalgebra $A \xrightarrow{\alpha^{-1}} FA \xrightarrow{\{-\}} \mathcal{P}FA$ is final in $\mathbf{Sets}_{\mathcal{P}}$.

Can we generalize? Other monads?

Main technical result (CALCO-jnr.)

The same holds for **subdistribution monad** $\mathcal{D}$:

$$\mathcal{D}X = \{d : X \to [0,1] \mid \sum_{x \in X} d(x) \leq 1\}$$

Let

$$\begin{tikzcd} FA \arrow[d, "\alpha", "\cong"'] \\ A \end{tikzcd}$$

be the initial F-algebra in $\mathbf{Sets}$.

Theorem

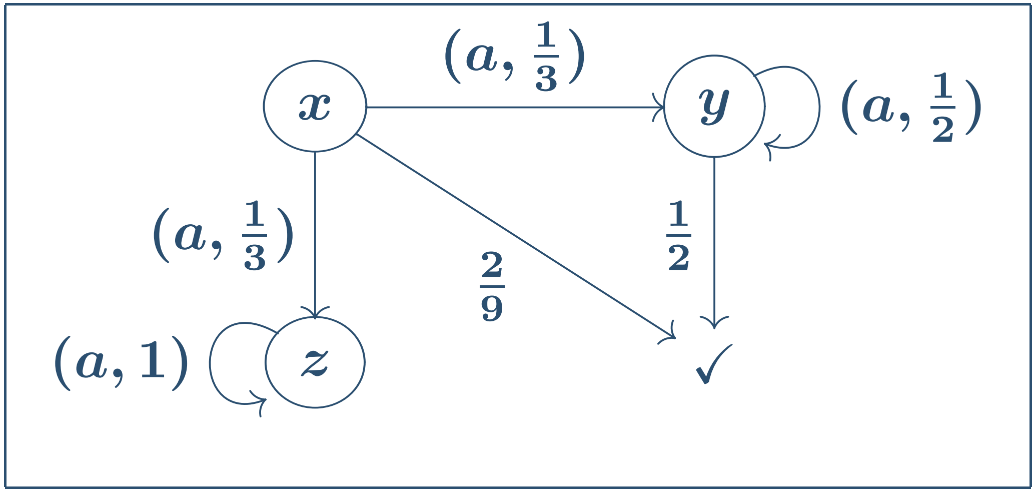
Coalgebra $A \xrightarrow{\alpha^{-1}} FA \xrightarrow{\eta^{\mathcal{D}}} \mathcal{D}FA$ is final in $\mathbf{Sets}_{\mathcal{D}}$.

This gives trace semantics for **probabilistic systems**.

Example: probabilistic systems

$$\begin{tikzcd}
\mathcal{D}(1+\Sigma\times X) \\
X \arrow[u, "c"']
\end{tikzcd}$$

Example



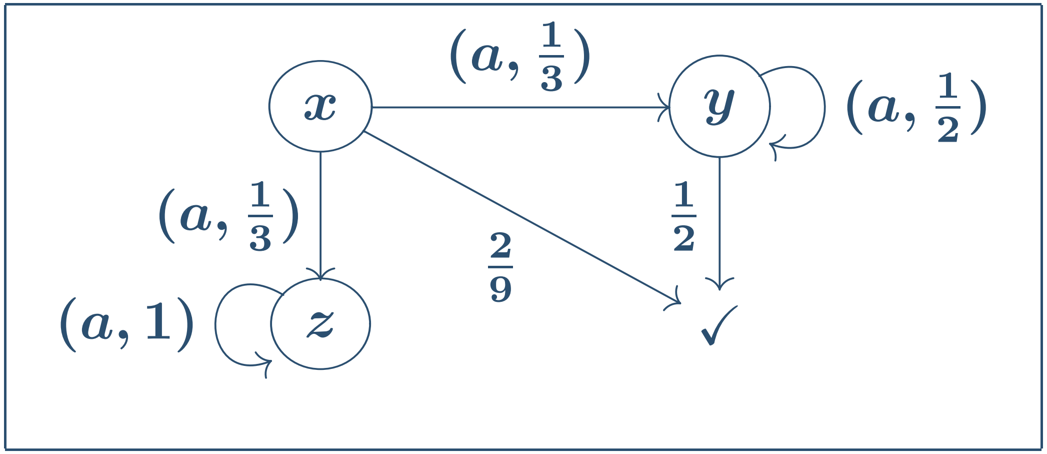
$$c(x) = \begin{bmatrix} \checkmark & \mapsto & 2/9 \\ (a, y) & \mapsto & 1/3 \\ (a, z) & \mapsto & 1/3 \end{bmatrix}$$

$$c(y) = \cdots$$

Remaining $1/9$ of $c(x)$: probability of **deadlock**

Example: probabilistic systems

Example



By our finality result we obtain $\mathbf{trace} : X \to \mathcal{D}(\Sigma^*)$.

$$\mathbf{trace}(x) : \langle\rangle \mapsto \frac{2}{9} \qquad a \mapsto \frac{1}{3} \cdot \frac{1}{2} \qquad a^2 \mapsto \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{2} \qquad \cdots$$

Remaining of $\mathbf{trace}(x)$ is understood:

$1/3$ for **livelock**, $1/9$ for deadlock

Conclusion

We have previewed:

- Initial algebra in $\mathbf{Sets}$ yields final coalgebra in $\mathbf{Rel} = \mathbf{Sets}_{\mathcal{P}}$.
  - Trace semantics for **non-deterministic** systems.
- Initial algebra in $\mathbf{Sets}$ yields final coalgebra in $\mathbf{Sets}_{\mathcal{D}}$.
  - Trace semantics for **probabilistic** systems.

Proofs and more examples
(e.g. context-free grammars/languages) will be in CALCO.

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- Initial algebra in $\mathbf{Sets}$ yields final coalgebra in $\mathbf{Sets}_{\mathcal{D}}$.
  - Trace semantics for **probabilistic** systems.

Proofs and more examples
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See you on Monday!

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