Coalgebraic Trace Semantics
for Probabilistic Systems

Ichiro Hasuo       Bart Jacobs
Radboud University Nijmegen, the Netherlands
Overview

- **Initial algebra in** \( \text{Sets} \) **yields final coalgebra in** \( \text{Rel} \)
  - Application: context-free grammars/languages
  - \( \text{Rel} \cong \text{Sets}_P \), Kleisli category
  - In Proc. CALCO’05.

- **Generalization:**
  - **Same for subdistribution monad** \( \mathcal{D} \), instead of \( P \)
  - Application: trace distribution of probabilistic automata
  - In Proc. CALCO-jnr.

Both results are presented in CALCO’05, Monday 5 September, 15.40–, so now is the time for...
Sneak Preview

Coalgebraic Trace Semantics

Ichiro Hasuo       Bart Jacobs
Radboud University Nijmegen, the Netherlands
Coalgebras

Consider a functor

$$FX = 1 + \Sigma \times X$$

where $$1 = \{\checkmark\}$$.

An $$F$$-coalgebra is a system in which,

$$\begin{array}{c}
1 + \Sigma \times X \\
\downarrow c \\
X
\end{array}$$

each state $$x \in X$$ either

$$\begin{cases}
\text{terminates,} & \text{i.e. } c(x) = \checkmark, \\
\text{outputs } a \in \Sigma \text{ and moves to } y \in X, & \text{i.e. } c(x) = (a, y).
\end{cases}$$
Final coalgebra semantics

Behavior of $F$-coalgebra is
the finite/infinite sequence of outputs,
i.e. an element of $\Sigma^\infty = \Sigma^* + \Sigma^\omega$.

- $\Sigma^\infty$ carries a final $F$-coalgebra.

\[
\begin{array}{c}
1 + \Sigma \times \Sigma^\infty \\
\Downarrow \Sigma^\infty \\
\Downarrow \zeta \\
\Downarrow (a, \sigma) \\
\Downarrow a \cdot \sigma
\end{array}
\]
Final coalgebra semantics

- Unique coalgebra morphism yields the behavior map.

\[
1 + \Sigma \times X \xrightarrow{c} 1 + \Sigma \times \text{beh}_c \xrightarrow{\zeta} 1 + \Sigma \times \Sigma^\infty
\]

This amounts to solving the (co)recursive equation:

\[
\text{beh}_c(x) = \begin{cases} 
\langle \rangle & \text{if } c(x) = \checkmark \\
 a \cdot \text{beh}_c(y) & \text{if } c(x) = (a, y)
\end{cases}
\]

So far so standard...
Non-deterministic systems

Consider a coalgebra

\[ P(1 + \Sigma \times X) \]

\[ \xrightarrow{c} X \]

Example

This is non-deterministic, or branching-time.
Trace semantics

[Trace semantics]

Structure map of a final coalgebra is an isomorphism

Final coalgebra semantics doesn’t work since an isomorphism is impossible.
Trace semantics

Some candidates for "semantics" of non-deterministic systems...

- bisimulation equivalence (finest)
- trace equivalence, which we are interested in

\[
\text{trace} = \text{set of linear-time, sequential behavior which can possibly happen}
\]
Trace semantics

Example

\[
\begin{align*}
\text{trace}(x) &= \{a, ab, abb, \ldots\} \\
\text{trace}(y) &= \{\langle\rangle, b, bb, \ldots\}
\end{align*}
\]

What is “trace”, from coalgebraic perspective?
Trace semantics

[IH & BJ, CALCO’05]

- We move our base category from \( \text{Sets} \) to \( \text{Rel} \).

- Then the map \( \text{trace} : X \rightarrow \mathcal{P}(\Sigma^*) \) is actually obtained via **final coalgebra** again!

- Notice that: \( \Sigma^* \) is the initial \( (1 + \Sigma \times -) \)-algebra.
Main technical result (CALCO)

Let $F$ be a “shapely” functor in $\text{Sets}$.

(Lifts to $\text{Rel}$)

\[
\begin{array}{c}
FA \\
\alpha \downarrow \\
A
\end{array}
\]

Let $\cong \alpha$ be the initial $F$-algebra in $\text{Sets}$.

**Theorem**

$\text{Coalgebra } A \xrightarrow{\text{graph}(\alpha^{-1})} FA$ is final in $\text{Rel}$. 
Main technical result (CALCO)

Note: \( \text{Rel} \cong \text{Sets}_\mathcal{P} \), Kleisli category.

\[
\begin{align*}
\text{Sets}_\mathcal{P} & \left\{ \begin{array}{l}
\text{object that of \text{Sets}} \\
\text{arrow} \end{array} \right\} \\
\begin{array}{c}
X 
\rightarrow Y \\
X 
\rightarrow \mathcal{P}Y
\end{array}
\end{align*}
\]

Let \( \alpha \) be the initial \( F \)-algebra in \( \text{Sets} \).

Theorem

Coalgebra \( A \xrightarrow{\alpha^{-1}} FA \xrightarrow{\{-\}} \mathcal{P}FA \) is final in \( \text{Sets}_\mathcal{P} \).

Can we generalize? Other monads?
Main technical result (CALCO-jnr.)

The same holds for subdistribution monad $\mathcal{D}$:

$$\mathcal{D}X = \{d : X \to [0, 1] \mid \sum_{x \in X} d(x) \leq 1\}$$

Let $\alpha : FA \cong A$ be the initial $F$-algebra in $\text{Sets}$.

Theorem

Coalgebra $A \xrightarrow{\alpha^{-1}} FA \xrightarrow{\eta^\mathcal{D}} \mathcal{D}FA$ is final in $\text{Sets}_{\mathcal{D}}$.

This gives trace semantics for probabilistic systems.
Example: probabilistic systems

\[ \mathcal{D}(1 + \Sigma \times X) \]

\[ c \]

\[ X \]

Example

\[
\begin{array}{c}
\begin{array}{c}
(x, \frac{1}{3}) \\
\downarrow \\
(y, \frac{1}{2})
\end{array} \\
\begin{array}{c}
(a, \frac{1}{3}) \\
\downarrow \\
(z)
\end{array}
\end{array}
\]

\[
c(x) = \left[ \begin{array}{c}
\checkmark \rightarrow 2/9 \\
(a, y) \rightarrow 1/3 \\
(a, z) \rightarrow 1/3
\end{array} \right]
\]

\[
c(y) = \cdots
\]

Remaining 1/9 of \(c(x)\): probability of deadlock
Example: probabilistic systems

By our finality result we obtain

\[ \text{trace} : X \rightarrow \mathcal{D}(\Sigma^*) \]

\[
\begin{align*}
\text{trace}(x) : & \quad \langle \rangle \mapsto \frac{2}{9} \\
& \quad a \mapsto \frac{1}{3} \cdot \frac{1}{2} \\
& \quad a^2 \mapsto \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{2} \\
& \quad \ldots
\end{align*}
\]

Remaining of \( \text{trace}(x) \) is understood:

- 1/3 for **livelock**
- 1/9 for **deadlock**
Conclusion

We have previewed:

- Initial algebra in $\mathbf{Sets}$ yields final coalgebra in $\mathbf{Rel} = \mathbf{Sets}_P$.
  - Trace semantics for **non-deterministic** systems.

- Initial algebra in $\mathbf{Sets}$ yields final coalgebra in $\mathbf{Sets}_D$.
  - Trace semantics for **probabilistic** systems.

Proofs and more examples
(e.g. context-free grammars/languages) will be in CALCO.
Conclusion

We have previewed:

- Initial algebra in \( \text{Sets} \) yields final coalgebra in \( \text{Rel} = \text{Sets}^{\mathcal{P}} \).
  - Trace semantics for \textbf{non-deterministic} systems.

- Initial algebra in \( \text{Sets} \) yields final coalgebra in \( \text{Sets}^{\mathcal{D}} \).
  - Trace semantics for \textbf{probabilistic} systems.

Proofs and more examples (e.g. context-free grammars/languages) will be in CALCO.

See you on Monday!
Contact: Ichiro Hasuo  www.cs.ru.nl/~ichiro  ichiro@cs.ru.nl