Generic Forward & Backward Simulation II
Probabilistic Simulation

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For Purely Probabilistic Systems...

Jonsson-Larsen simulation for prob. sys.

[Jonsson-Larsen, LICS'91]

via *weight function*

${}_u\backslash{}^v$	$\perp$	$\cdots\ y'\ \cdots$
$\perp$	$\lesseqgtr \mathbf{0}$	$\geq \mathbf{0}$
$\vdots$ x' $\vdots$	$\mathbf{0}$	$\begin{cases} \geq \mathbf{0} & (x' R y') \\ \mathbf{0} & (\text{o.w.}) \end{cases}$

For Purely Probabilistic Systems...

coalgebraic

Kleisli simulation
[H., CONCUR'06]

fwd.
$$\begin{array}{ccc} FX & \xleftarrow{Ff} & FY \\ c\uparrow & \sqsubseteq & \uparrow d \\ X & \xleftarrow[f]{} & Y \end{array}$$

bwd.
$$\begin{array}{ccc} FX & \xrightarrow{Fb} & FY \\ c\uparrow & \sqsubseteq & \uparrow d \\ X & \xrightarrow[b]{} & Y \end{array}$$

Hughes-Jacobs simulation
[Hughes-Jacobs, TCS'04]

$$\begin{array}{ccccc} BX & \xleftarrow{B\pi_1} & BR & \xrightarrow{B\pi_2} & BY \\ c\uparrow & \sqsubseteq & r\uparrow & \sqsubseteq & \uparrow d \\ X & \xleftarrow[\pi_1]{} & R & \xrightarrow[\pi_2]{} & Y \end{array}$$

Jonsson-Larsen simulation for prob. sys.
[Jonsson-Larsen, LICS'91]

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specializes

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[Hughes-Jacobs, TCS'04]

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soundness theorem

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Hasuo (Kyoto, JP)

Generic *Kleisli* Simulation

- I. Hasuo. *Generic forward and backward simulations*. CONCUR'06.

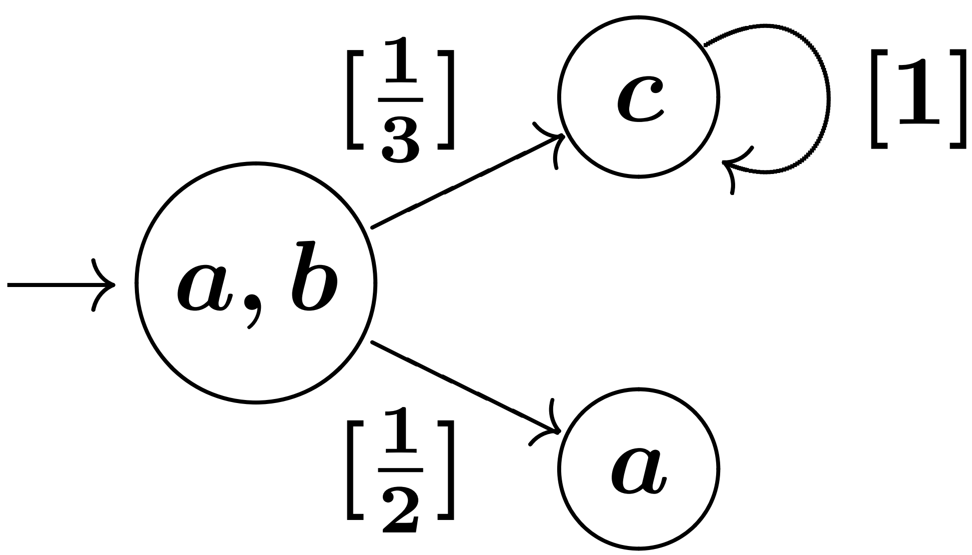
- Coalgebraic generalization of
  - N. Lynch and F. Vaandrager. *Forward and backward simulations I. Untimed systems*. Inf.&Comp.'95.

I

Background:

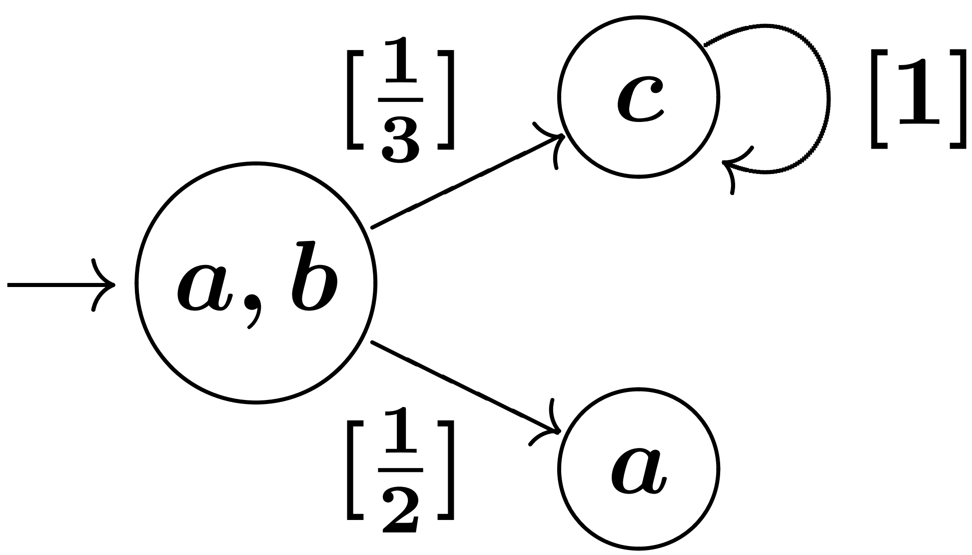
Jonsson-Larsen Simulation for Prob. Systems

Probabilistic System



- Randomized algorithms, complexity
- Pervasive in distributed/concurrent applications
  - Network protocols flipping coins

Probabilistic System



- Randomized algorithms, complexity
- Pervasive in distributed/concurrent applications
  - Network protocols flipping coins
- N.B. Only *purely probabilistic* systems in the current work
  - Unlike Segala's *probabilistic automata*

Hasuo (Kyoto, JP)

Theory of Probabilistic Systems

- Bisimulation
  - When are two systems *equivalent?*
  - Definitions based on *weight functions* or *equivalence classes* (these are equivalent)

Theory of Probabilistic Systems

- Bisimulation
  - When are two systems *equivalent?*
  - Definitions based on *weight functions* or *equivalence classes* (these are equivalent)
- Simulation: two different views/uses
  - As def. of "refinement relation" [Jonsson-Larsen, LICS'91] [Baier-Katoen-Hermanns-Wolf, Inf.&Comp.'05]
  - As a proof method for trace inclusion [Lynch-Vaandrager, Inf.&Comp.'95] [H., CONCUR'06]

Simulation-Based Verification: a Scenario

specification system

$\mathcal{S}$

- small enough, obviously satisfies a safety property P

implementation system

$\mathcal{I}$

- bigger
- Goal Prove P

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<u>Thm.</u> (Soundness)
∃ simulation ⇒ trace incl.

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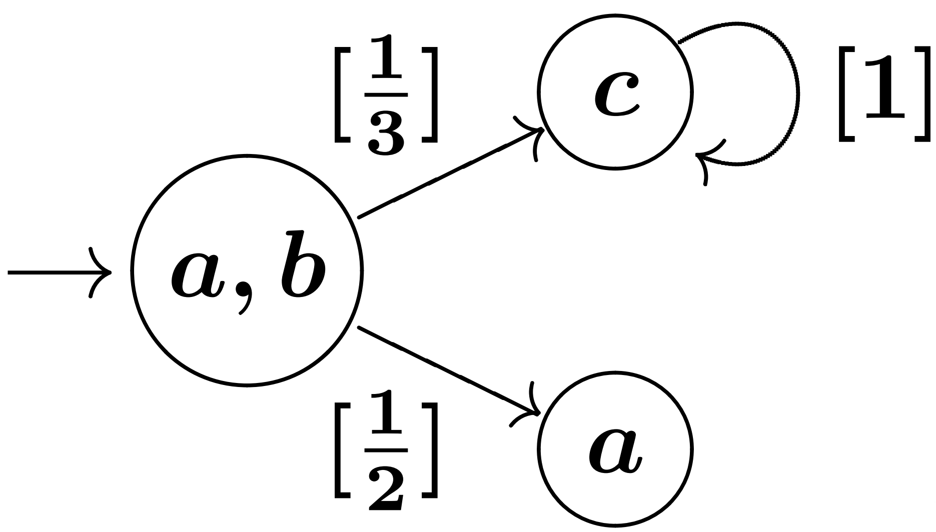
- Trace incl.: arbitrary many steps
- Simulation: stepwise

DTMC



Definition. A *discrete-time Markov chain (DTMC)* is

$$(X, x_0, l, p)$$

where

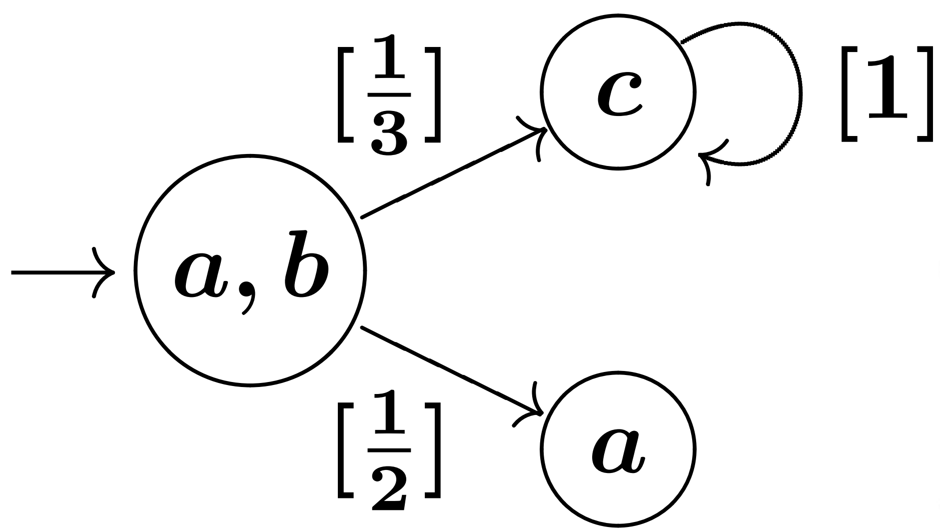
- X is a *state space*;
- $x_0 \in X$ is an *initial state*;
- $l : X \to \mathcal{P}(\mathsf{AP})$ is a *labeling function*,

$$l(x) = \{\text{atomic propositions true at } x\}$$

- $p : X \to \mathcal{D}X$ is a *transition function*, where

$$\begin{aligned} \mathcal{D}X &= \{\text{prob. subdistr. over } X\} \\ &= \{d : X \to [0,1] \mid \sum_x d(x) \leq 1\} \ . \end{aligned}$$

DTMC



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DTMC:
"Probabilistic Kripke model"

Jonsson-Larsen Simulation

[Jonsson-Larsen, LICS'91]

Definition. **(JL-simulation)** Let $\mathcal{X} = (X, x_0, l, p)$ and $\mathcal{Y} = (Y, y_0, m, q)$ be DTMCs. A *JL-simulation* from $\mathcal{X}$ to $\mathcal{Y}$ is a relation $R \subseteq X \times Y$ which satisfies the following.

1. $x_0 R y_0$.
2. xRy implies $l(x) = m(y)$.
3. For each $x \in X$ and $y \in Y$ such that xRy, there exists a *weight function*

   $$\Delta_{x,y} \;:\; (\{\bot\} + X) \times (\{\bot\} + Y) \longrightarrow [0,1]$$

   such that

   (a) $\Delta_{x,y}(u,v) > 0$ implies either
   - $u = \bot$, or
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   (b) $\Delta_{x,y}(\bot,\bot) + \sum_{y' \in Y} \Delta_{x,y}(\bot, y') = 1 - \sum_{x' \in X} p(x)(x')$;

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$$\downarrow$$

$$q(y)(y')$$

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- Why this definition?
  - Relation to non-det. version

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(b) $\Delta_{x,y}(\bot,\bot) + \sum_{y' \in Y} \Delta_{x,y}(\bot, y') = 1 - \sum_{x' \in X} p(x)(x')$;

(c) for each $x' \in X$:
$\Delta_{x,y}(x', \bot) + \sum_{y' \in Y} \Delta_{x,y}(x', y') = p(x)(x')$;

(d) $\Delta_{x,y}(\bot,\bot) + \sum_{x' \in X} \Delta_{x,y}(x', \bot) = 1 - \sum_{y' \in Y} q(y)(y')$;

(e) for each $y' \in Y$:
$\Delta_{x,y}(\bot, y') + \sum_{x' \in X} \Delta_{x,y}(x', y') = q(y)(y')$.

- Why this definition?
  - Relation to non-det. version
- Asymmetry?

Jonsson-Larsen Simulation

[Jonsson-Larsen, LICS'91]

Definition. (JL-simulation) Let $\mathcal{X} = (X, x_0, l, p)$ and $\mathcal{Y} = (Y, y_0, m, q)$ be DTMCs. A *JL-simulation* from $\mathcal{X}$ to $\mathcal{Y}$ is a relation $R \subseteq X \times Y$ which satisfies the following.

1. $x_0 R y_0$.
2. xRy implies $l(x) = m(y)$.
3. For each $x \in X$ and $y \in Y$ such that xRy, there exists a *weight function*

$$\Delta_{x,y} \;:\; (\{\perp\} + X) \times (\{\perp\} + Y) \longrightarrow [0,1]$$

such that

(a) $\Delta_{x,y}(u,v) > 0$ implies either
- $u = \perp$, or
- $u = x' \in X$, $v = y' \in Y$ and $x' R y'$;

(b) $\Delta_{x,y}(\perp,\perp) + \sum_{y' \in Y} \Delta_{x,y}(\perp, y') = 1 - \sum_{x' \in X} p(x)(x')$;

(c) for each $x' \in X$:
$\Delta_{x,y}(x',\perp) + \sum_{y' \in Y} \Delta_{x,y}(x', y') = p(x)(x')$;

(d) $\Delta_{x,y}(\perp,\perp) + \sum_{x' \in X} \Delta_{x,y}(x', \perp) = 1 - \sum_{y' \in Y} q(y)(y')$;

(e) for each $y' \in Y$:
$\Delta_{x,y}(\perp, y') + \sum_{x' \in X} \Delta_{x,y}(x', y') = q(y)(y')$.

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- Asymmetry?
- Adaptation to other types of systems?

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- Why this definition?
  - Relation to non-det. version
- Asymmetry?
- Adaptation to other types of systems?
- Soundness?

For Purely Probabilistic Systems...

coalgebraic

Kleisli simulation

[H., CONCUR'06]

<u>fwd.</u>

$$\begin{tikzcd} FX & FY \arrow[l, "Ff"', "\shortmid" marking] \\ X \arrow[u, "c"] \arrow[ur, phantom, "\sqsubseteq"] & Y \arrow[l, "f", "\shortmid" marking] \arrow[u, "d"'] \end{tikzcd}$$

<u>bwd.</u>

$$\begin{tikzcd} FX \arrow[r, "Fb", "\shortmid" marking] & FY \\ X \arrow[u, "c"] \arrow[r, "b"', "\shortmid" marking] \arrow[ur, phantom, "\sqsubseteq"] & Y \arrow[u, "d"'] \end{tikzcd}$$

specializes

Hughes-Jacobs simulation

[Hughes-Jacobs, TCS'04]

$$\begin{tikzcd} BX & BR \arrow[l, "B\pi_1"'] \arrow[r, "B\pi_2"] & BY \\ X \arrow[u, "c"] \arrow[ur, phantom, "\sqsubseteq"] & R \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] \arrow[u, "r"] \arrow[ur, phantom, "\sqsubseteq"] & Y \arrow[u, "d"'] \end{tikzcd}$$

specializes

Jonsson-Larsen simulation for prob. sys.

[Jonsson-Larsen, LICS'91]

via *weight function*

$u \backslash v$	$\bot$	$\cdots\ y'\ \cdots$
$\bot$	$\lesseqgtr 0$	≥ 0
$\vdots$ x' $\vdots$	0	$\begin{cases} \geq 0 & (x' R y') \\ 0 & (\text{o.w.}) \end{cases}$

Hasuo (Kyoto, JP)

For Purely Probabilistic Systems...

coalgebraic

Kleisli simulation

[H., CONCUR'06]

fwd.
$$\begin{tikzcd} FX & FY \arrow[l, "Ff"'] \arrow[dl, phantom, "\sqsubseteq"] \\ X \arrow[u, "c"] & Y \arrow[l, "f"] \arrow[u, "d"'] \end{tikzcd}$$

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$$\begin{tikzcd} FX \arrow[r, "Fb"] \arrow[dr, phantom, "\sqsubseteq"] & FY \\ X \arrow[u, "c"] \arrow[r, "b"'] & Y \arrow[u, "d"'] \end{tikzcd}$$

specializes

Hughes-Jacobs simulation

[Hughes-Jacobs, TCS'04]

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Another understanding provides better understanding

alizes

Jonsson-Larsen simulation for prob. sys.

[Jonsson-Larsen, LICS'91]

via *weight function*

$u \backslash v$	$\perp$	$\cdots\ y'\ \cdots$
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II

Generic Kleisli Simulation

[H., CONCUR'06]

Kleisli Simulation

- Uniform definition for a variety of systems
  - Parameters:

T	branching type	non-determinism, probability, weighted, ...
F	transition/action type	LTS, Kripke models, CFG, ...

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- Generic soundness theorem:
  $\exists$ simulation $\Rightarrow$ trace inclusion

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- Forward, backward, hybrid (fwd.-bwd, bwd.-fwd)

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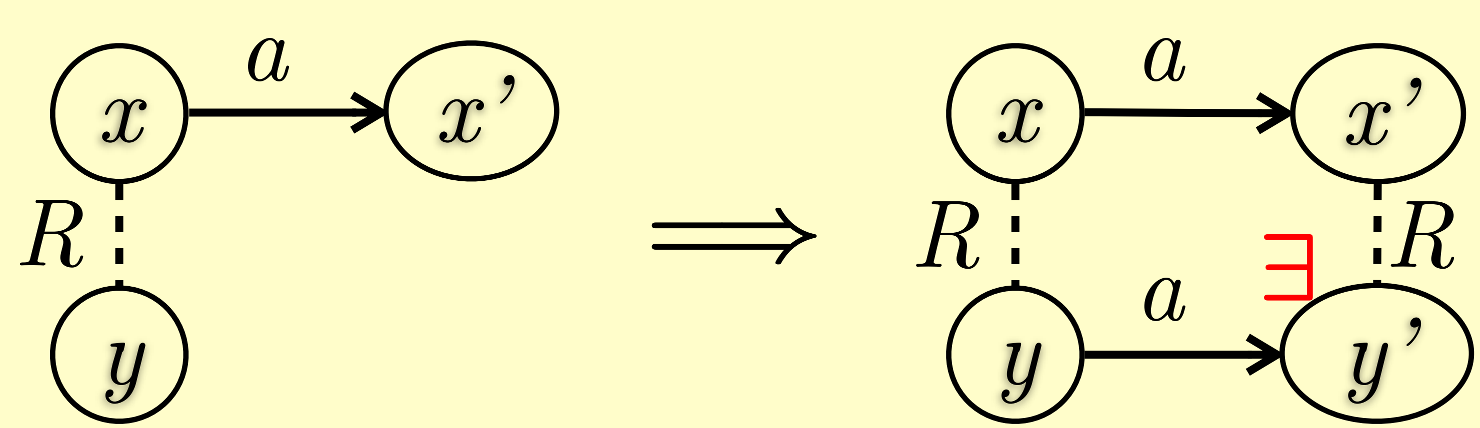
- Forward, backward, hybrid (fwd.-bwd, bwd.-fwd)
- Let's start with instances...

Kleisli Simulation for LTS

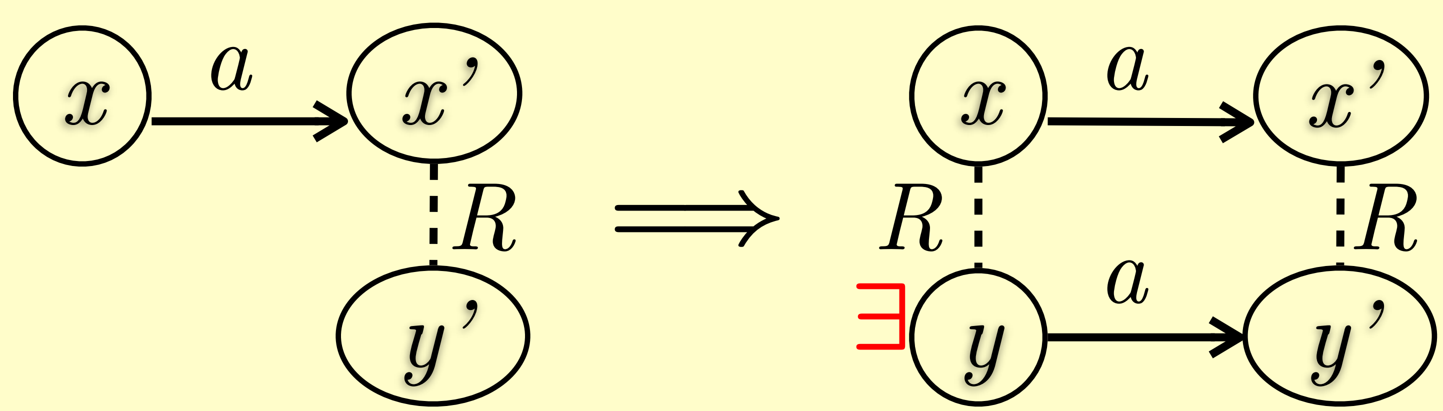
- Coincides with the standard definition
  e.g. in [Lynch-Vaandrager, Inf.&Comp.'95]

Forward simulation

A relation R between states of two systems, s.t.



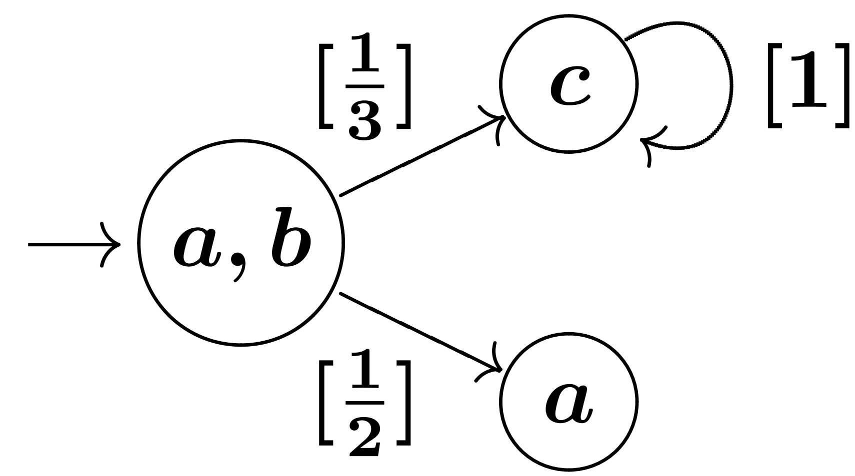
Backward simulation



Kleisli Simulation for Probabilistic LTS

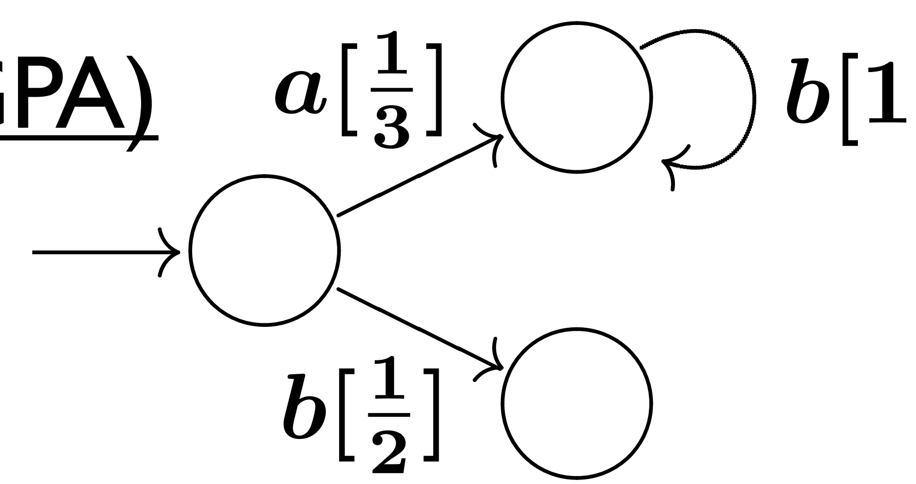
<u>Discrete Time Markov Chain (DTMC)</u>

- "Probabilistic Kripke model"
- Labeled *states*



<u>Generative Probabilistic Automaton (GPA)</u>

- "Probabilistic LTS"
- Labeled *transitions*



Kleisli Simulation for Probabilistic LTS

Definition.
A *forward simulation* from (X, x_0, c) to (Y, y_0, d) is a function

$$f \;:\; Y \longrightarrow \mathcal{D}X$$

such that

$$f(y_0)(x_0) = 1 \qquad \text{(INIT)}$$

$$\sum_{x \in X} f(y)(x) \cdot c(x)(a, x') \leq \sum_{y' \in Y} d(y)(a, y') \cdot f(y')(x')$$

for each $y \in Y$, $a \in \mathsf{Ac}$ and $x' \in X$
(ACT)

$\mathcal{D}X$ Subdistribution opr.

$$= \left\{ d : X \to [0,1] \;\middle|\; \sum_{x} d(x) \leq 1 \right\}$$

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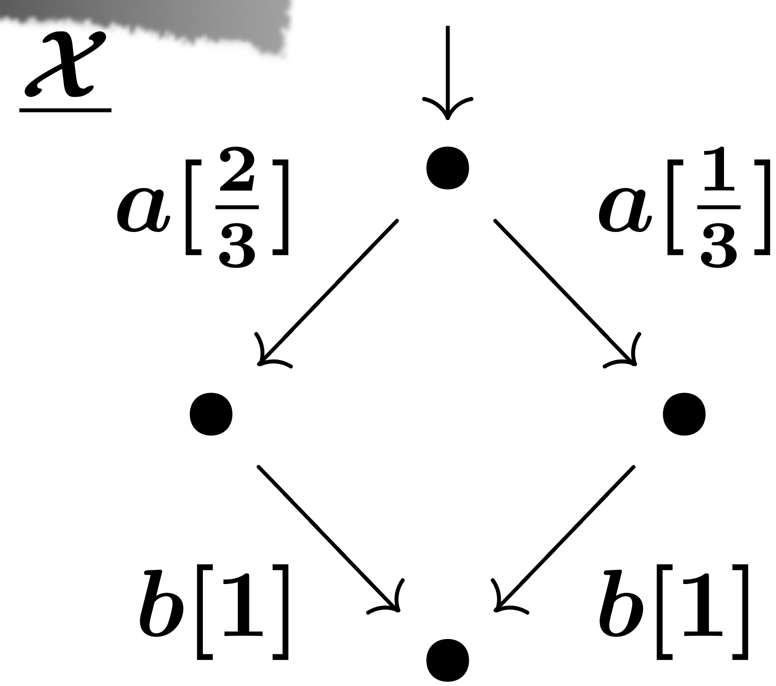

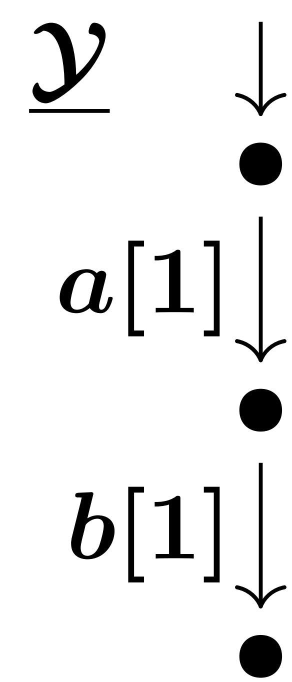

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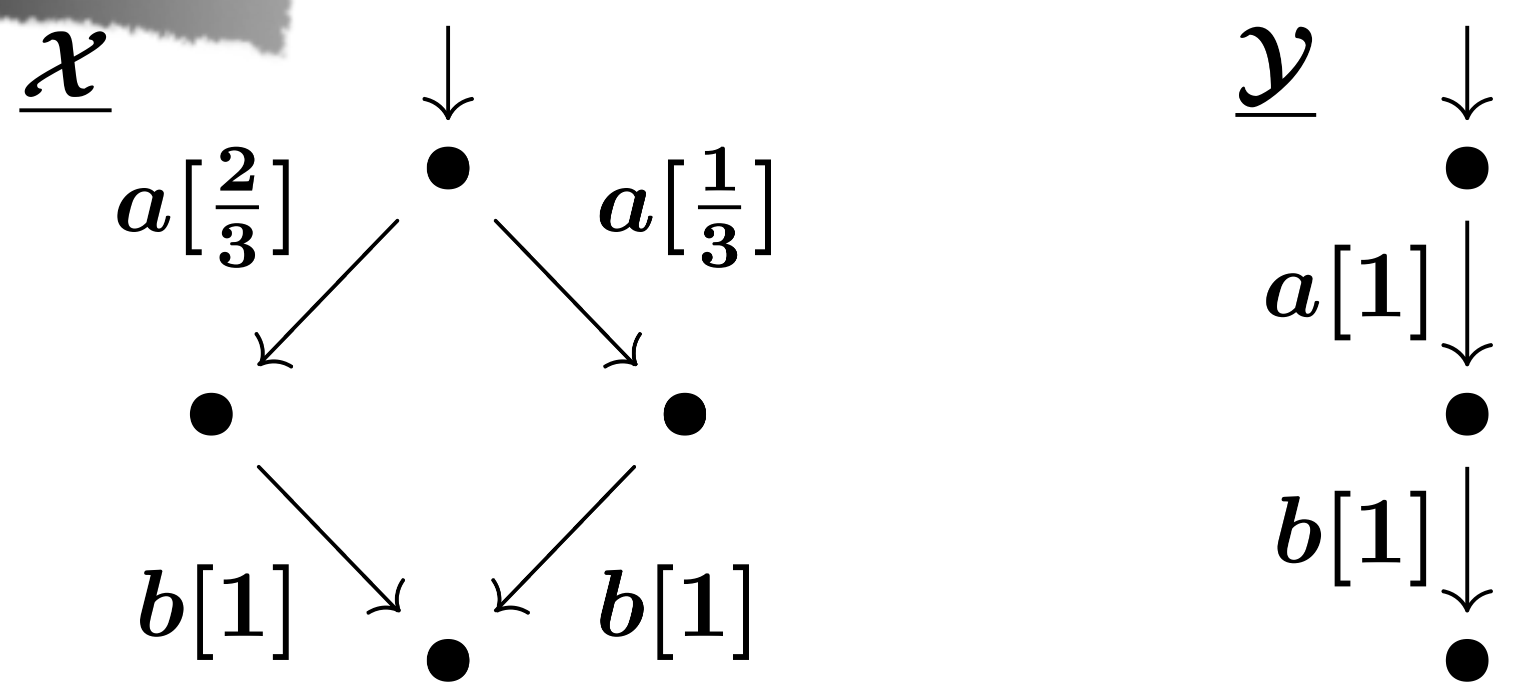

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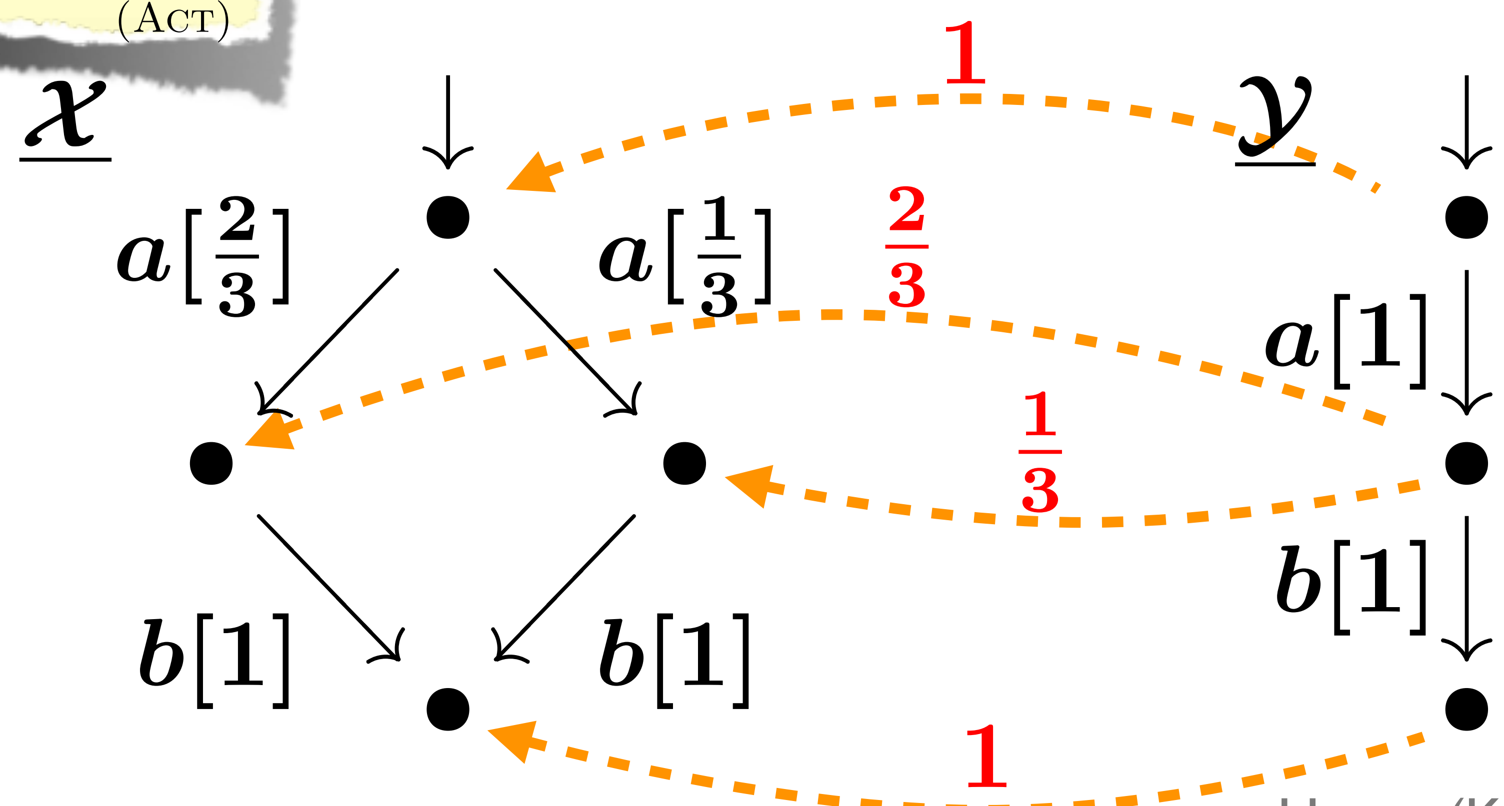

Hasuo (Kyoto, JP)

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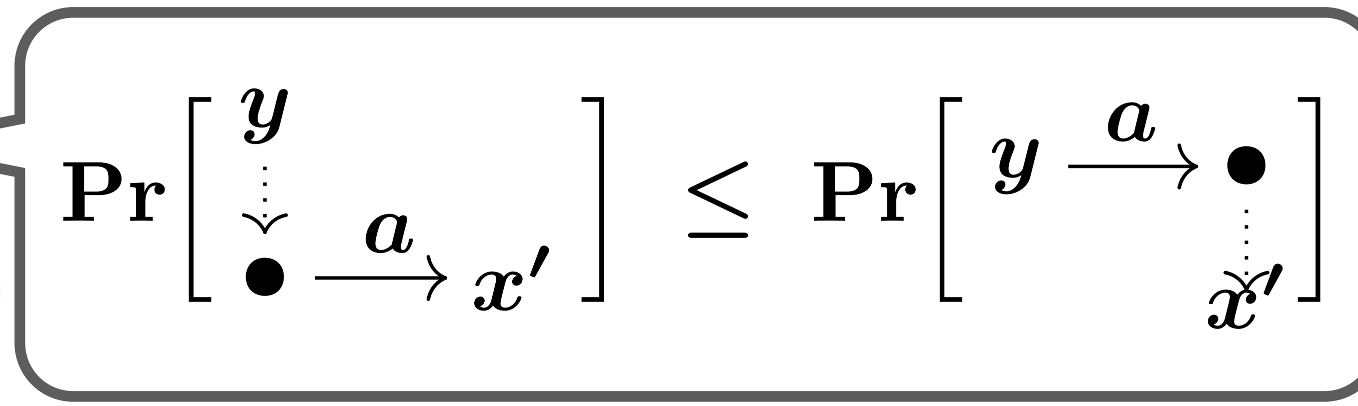

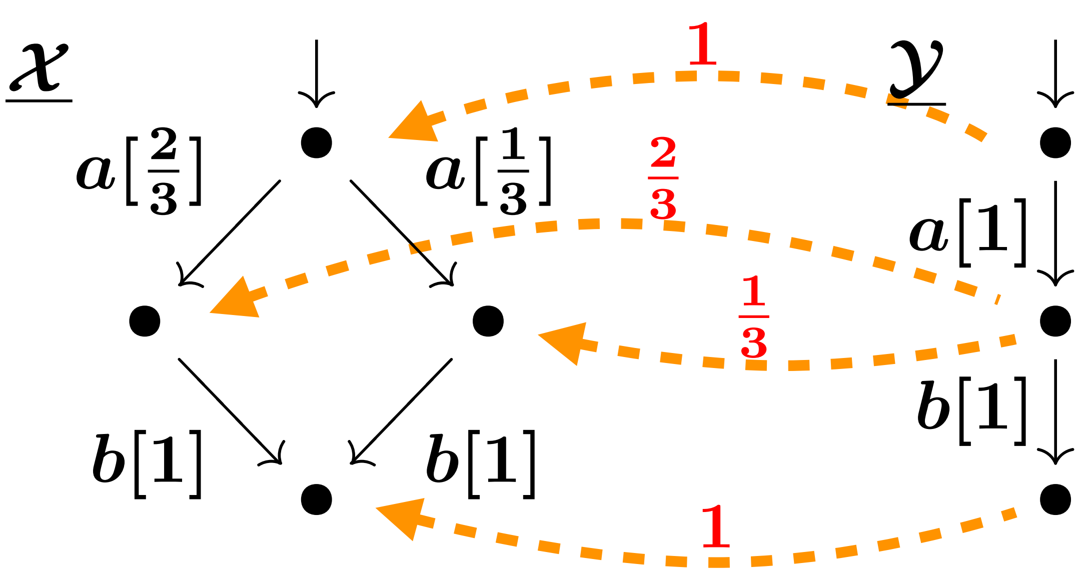

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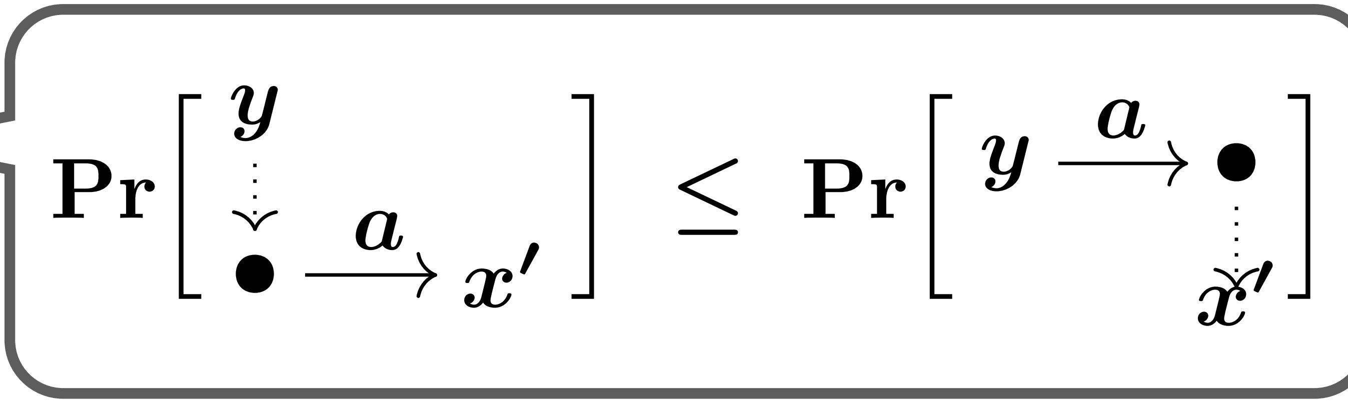
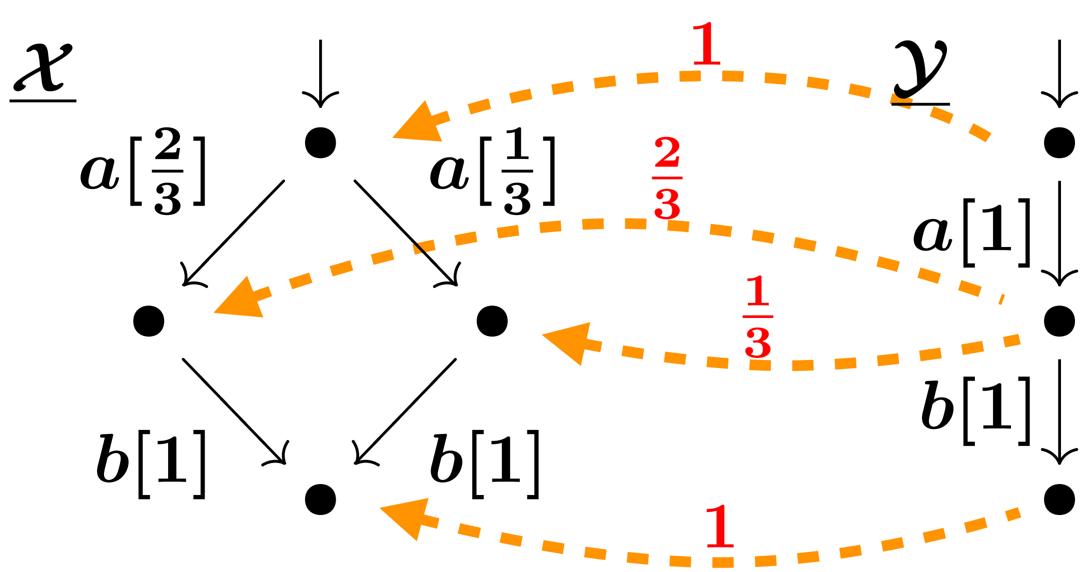
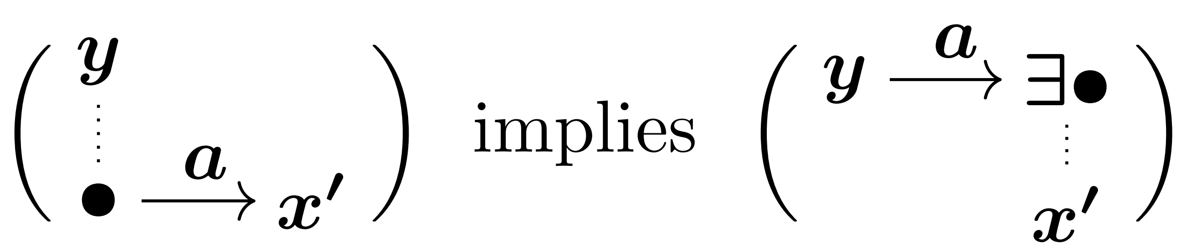




Cf.



Hasuo (Kyoto, JP)

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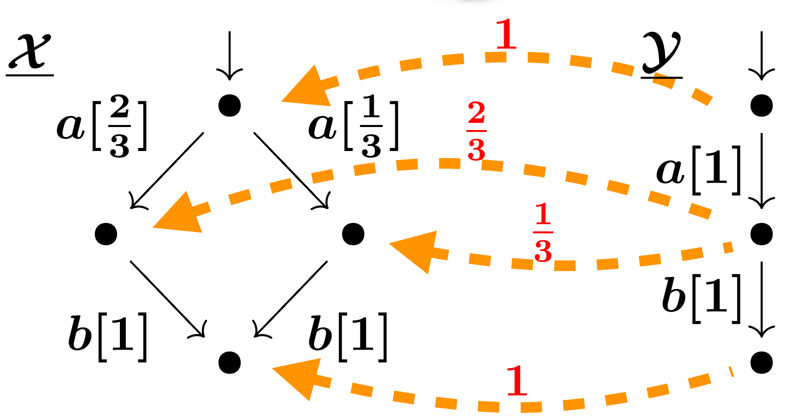

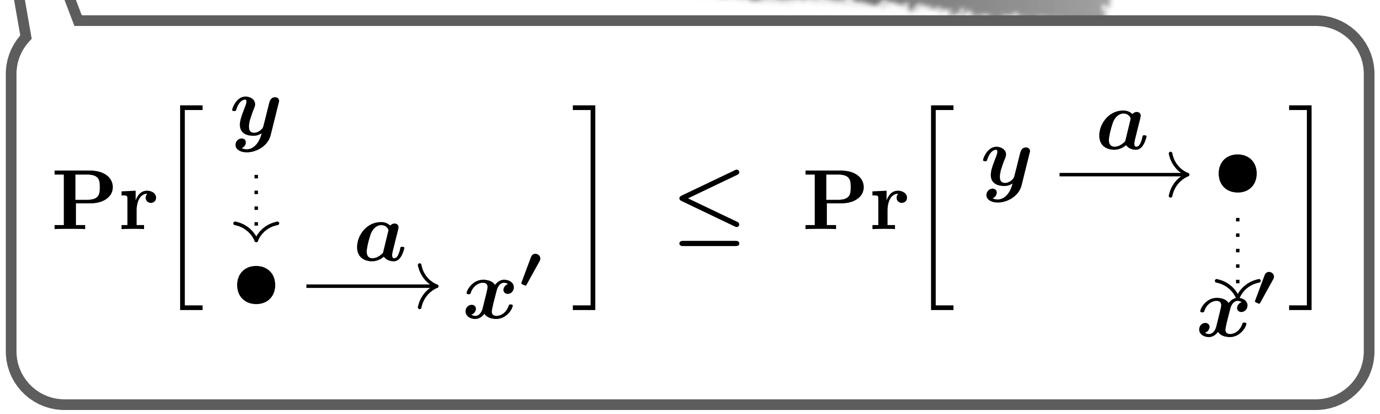

Hasuo (Kyoto, JP)

How Are These "The Same"?

Kleisli Arrow

$$\frac{X \longrightarrow\!\!\!\!\!\!\!\!\!+\;\;\;\; Y \quad \mathcal{P}\text{-Kleisli arrow}}{X \longrightarrow \mathcal{P}Y \quad \text{function}}$$

Kleisli Arrow

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- X, Y : sets
- $\mathcal{P}$: powerset opr.
- *non-deterministic function*

Kleisli Arrow

$$\frac{\boldsymbol{X} \longrightarrow\!\!\!\!\!\!\!\!\!+\;\;\; \boldsymbol{Y} \quad \mathcal{P}\text{-Kleisli arrow}}{\boldsymbol{X} \longrightarrow \mathcal{P}\boldsymbol{Y} \quad \text{function}}$$

$$\boldsymbol{x} \longmapsto \{\boldsymbol{y}, \boldsymbol{y}'\}$$

- X, Y : sets
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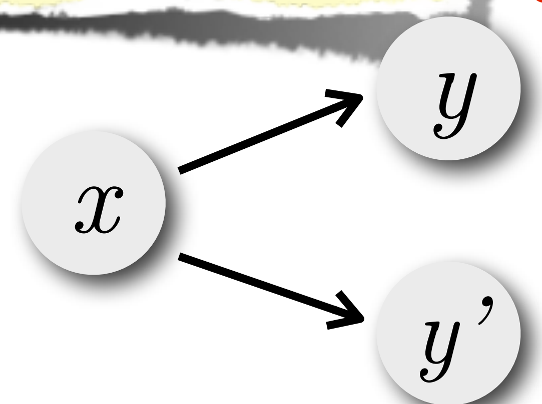

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$$\frac{X \longrightarrow\!\!\!\!\!\!\!+\;\; Y \quad \mathcal{D}\text{-Kleisli arrow}}{X \longrightarrow \mathcal{D}Y \quad \text{function}}$$

- $\mathcal{D}$: subdistribution opr.

$$\mathcal{D}X = \left\{ d : X \to [0,1] \;\middle|\; \sum_x d(x) \le 1 \right\}$$

- *probabilistic function*

Kleisli Arrow

$$\frac{X \longrightarrow\!\!\!\!\!\!\!+\;\; Y \quad \mathcal{P}\text{-Kleisli arrow}}{X \longrightarrow \mathcal{P}Y \quad \text{function}}$$

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- *non-deterministic function*

$$\frac{X \longrightarrow\!\!\!\!\!\!\!+\;\; Y \quad \mathcal{D}\text{-Kleisli arrow}}{X \longrightarrow \mathcal{D}Y \quad \text{function}}$$

$$x \longmapsto \begin{bmatrix} y \mapsto \frac{1}{3} \\ y' \mapsto \frac{2}{3} \end{bmatrix}$$

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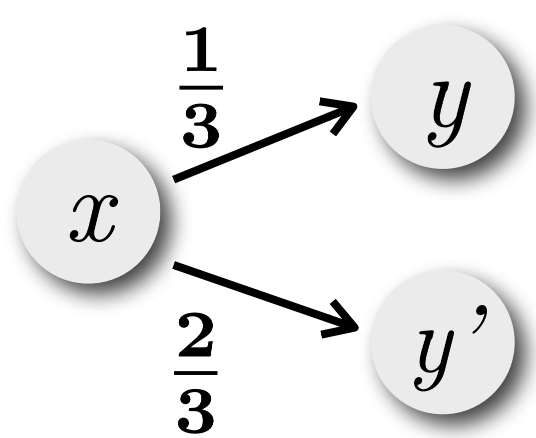
- *probabilistic function*

$$x \longmapsto \left[\begin{array}{l} y \mapsto \frac{1}{3} \\ y' \mapsto \frac{2}{3} \end{array} \right]$$



Hasuo (Kyoto, JP)

Kleisli Category

- Kleisli arrows form a *category*
  - identity
  - composition

$$\frac{X \overset{f}{\longrightarrow\!\!\!\!\!\!+\;\;} Y \quad Y \overset{g}{\longrightarrow\!\!\!\!\!\!+\;\;} Z}{X \overset{g \odot f}{\longrightarrow\!\!\!\!\!\!+\;\;} Z}$$

$$\frac{X \longrightarrow\!\!\!\!\!\!+\;\; Y \quad \mathcal{P}\text{-Kleisli arrow}}{X \longrightarrow \mathcal{P}Y \quad \text{function}}$$

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- Kleisli arrows form a *category*
  - identity
  - composition

$$\frac{X \xrightarrow{f} Y \quad Y \xrightarrow{g} Z}{X \xrightarrow{g \odot f} Z}$$

- A standard construction in category theory
  - $\mathcal{P}$, $\mathcal{D}$: *monads*. Cf. "Effect" monads in Haskell [Wadler, Moggi]
  - In this work: for "branching"
    - non-determinism, probability, weighted, quantum, ...

Kleisli Coalgebraic Modeling of Systems

An LTS
(w/o initial state)

$$\begin{array}{c} \mathcal{P}(\mathsf{Ac} \times X) \\ {\scriptstyle c}\uparrow \\ X \end{array}$$

Hasuo (Kyoto, JP)

Kleisli Coalgebraic Modeling of Systems

An LTS
(w/o initial state)

$$\begin{array}{c} \mathcal{P}(\mathsf{Ac} \times X) \\ c \uparrow \\ X \end{array}$$

$$\begin{array}{c} \{(a, x') \mid x \xrightarrow{a} x'\} \\ \uparrow \\ x \end{array}$$

Kleisli Coalgebraic Modeling of Systems

An LTS
(w/o initial state)

$$= \begin{array}{c} \mathcal{P}(\mathsf{Ac} \times X) \\ c \uparrow \\ X \end{array} = \begin{array}{c} \mathsf{Ac} \times X \\ c \uparrow_{\mathcal{P}} \\ X \end{array}$$

$$\begin{array}{c} \{(a, x') \mid x \xrightarrow{a} x'\} \\ \uparrow \\ x \end{array}$$

Hasuo (Kyoto, JP)

Kleisli Coalgebraic Modeling of Systems

An LTS
(w/o initial state)

$$\begin{array}{c}\mathcal{P}(\mathsf{Ac} \times X) \\ c\uparrow \\ X\end{array} \qquad \begin{array}{c}\mathsf{Ac} \times X \\ c\uparrow_{\mathcal{P}} \\ X\end{array}$$

$$\begin{array}{c}\{(a, x') \mid x \xrightarrow{a} x'\} \\ \uparrow \\ x\end{array}$$

A GPA
(w/o initial state)

$$\begin{array}{c}\mathcal{D}(\mathsf{Ac} \times X) \\ c\uparrow \\ X\end{array}$$

Hasuo (Kyoto, JP)

Kleisli Coalgebraic Modeling of Systems

An LTS
(w/o initial state)

$$\begin{array}{c} \mathcal{P}(\mathsf{Ac} \times X) \\ {\scriptstyle c}\uparrow \\ X \end{array} \qquad\qquad \begin{array}{c} \mathsf{Ac} \times X \\ {\scriptstyle c}\Uparrow_{\mathcal{P}} \\ X \end{array}$$

$$\begin{array}{c} \{(a, x') \mid x \xrightarrow{a} x'\} \\ \uparrow \\ x \end{array}$$

A GPA
(w/o initial state)

$$\begin{array}{c} \mathcal{D}(\mathsf{Ac} \times X) \\ {\scriptstyle c}\uparrow \\ X \end{array} \qquad\qquad \begin{array}{c} \mathsf{Ac} \times X \\ {\scriptstyle c}\Uparrow_{\mathcal{D}} \\ X \end{array}$$

Kleisli Coalgebraic Modeling of Systems

An LTS
(w/o initial state)

$$\begin{array}{c} \mathcal{P}(\mathsf{Ac} \times X) \\ c \uparrow \\ X \end{array} \qquad \begin{array}{c} \mathsf{Ac} \times X \\ c \uparrow_{\mathcal{P}} \\ X \end{array}$$

$$\begin{array}{c} \{(a, x') \mid x \xrightarrow{a} x'\} \\ \uparrow \\ x \end{array}$$

A GPA
(w/o initial state)

$$\begin{array}{c} \mathcal{D}(\mathsf{Ac} \times X) \\ c \uparrow \\ X \end{array} \qquad \begin{array}{c} \mathsf{Ac} \times X \\ c \uparrow_{\mathcal{D}} \\ X \end{array}$$

A (branching) system as a *Kleisli coalgebra*

$$\begin{array}{c} FX \\ c \uparrow \\ X \end{array}$$

Kleisli Simulation

Definition.
A *forward Kleisli simulation*

$$\text{from} \quad \begin{array}{c} \mathsf{Ac} \times X \\ c \uparrow \\ X \end{array} \quad \text{to} \quad \begin{array}{c} \mathsf{Ac} \times Y \\ d \uparrow \\ Y \end{array}$$

is a Kleisli arrow

$$f : Y \rightarrow\!\!\!\!\!+\, X$$

such that $\quad c \odot f \sqsubseteq (\mathsf{Ac} \times f) \odot d, \quad$ that is

$$\begin{tikzcd} \mathsf{Ac} \times X & & \mathsf{Ac} \times Y \arrow[ll, "\mathsf{Ac} \times f"'] \\ X \arrow[u, "c"] & \sqsubseteq & Y \arrow[ll, "f"] \arrow[u, "d"'] \end{tikzcd}$$

Kleisli Simulation

Definition.
A *forward Kleisli simulation*

Branching system
(LTS, GPA, ...)

$$\text{from} \quad \begin{array}{c} \mathsf{Ac} \times X \\ c \uparrow \\ X \end{array} \qquad \text{to} \quad \begin{array}{c} \mathsf{Ac} \times Y \\ d \uparrow \\ Y \end{array}$$

is a Kleisli arrow

$$f : Y \to\!\!\!\!\!+ X$$

such that $\quad c \odot f \sqsubseteq (\mathsf{Ac} \times f) \odot d, \quad$ that is

$$\begin{tikzcd} \mathsf{Ac} \times X & & \mathsf{Ac} \times Y \arrow[ll, "\mathsf{Ac} \times f"'] \\ X \arrow[u, "c"] \arrow[urr, phantom, "\sqsubseteq"] & & Y \arrow[ll, "f"] \arrow[u, "d"'] \end{tikzcd}$$

Kleisli Simulation

Branching system
(LTS, GPA, ...)

Definition.
A *forward Kleisli simulation*

$$\text{from}\quad \begin{array}{c}\mathsf{Ac} \times X \\ c\uparrow \\ X\end{array} \qquad \text{to}\quad \begin{array}{c}\mathsf{Ac} \times Y \\ d\uparrow \\ Y\end{array}$$

is a Kleisli arrow

$$f \;:\; Y \longrightarrow\!\!\!\!\!\!\!\!\;\; + \; X$$

such that $\quad c \odot f \sqsubseteq (\mathsf{Ac} \times f) \odot d,\quad$ that is

$$\begin{tikzcd} \mathsf{Ac} \times X & & \mathsf{Ac} \times Y \arrow[ll, "\mathsf{Ac} \times f"'] \\ X \arrow[u, "c"] \arrow[rr, phantom, "\sqsubseteq" near start, shift left=6] & & Y \arrow[ll, "f"] \arrow[u, "d"'] \end{tikzcd}$$

"Simulating"
(*more* behavior)

"Simulated"
(*less* behavior)

Kleisli Simulation

Definition.
A *forward Kleisli simulation*

$$\text{from}\quad \begin{tikzcd} \mathsf{Ac} \times X \\ X \arrow[u, "c", "\shortmid"'] \end{tikzcd} \quad\text{to}\quad \begin{tikzcd} \mathsf{Ac} \times Y \\ Y \arrow[u, "d", "\shortmid"'] \end{tikzcd}$$

Branching system
(LTS, GPA, ...)

is a Kleisli arrow

$$f \;:\; Y \rightarrow\!\!\!\!\!+\; X$$

such that $\quad c \odot f \sqsubseteq (\mathsf{Ac} \times f) \odot d, \quad$ that is

$$\begin{tikzcd} \mathsf{Ac} \times X & & \mathsf{Ac} \times Y \arrow[ll, "\mathsf{Ac} \times f"', "\shortmid"] \\ X \arrow[u, "c", "\shortmid"'] \arrow[urr, phantom, "\sqsubseteq"] & & Y \arrow[ll, "f", "\shortmid"'] \arrow[u, "d"', "\shortmid"] \end{tikzcd}$$

“Simulating”
(*more* behavior)

“Simulated”
(*less* behavior)

Natural order for
$\mathcal{P}$ / $\mathcal{D}$

Kleisli Simulation

Definition.
A fo

Branching system

$$\dfrac{\dfrac{Y \xrightarrow[\mathcal{P}]{} X}{Y \longrightarrow \mathcal{P}X, \text{ function}}}{R \subseteq X \times Y, \text{ relation}} \qquad \dfrac{Y \xrightarrow[\mathcal{D}]{} X}{Y \longrightarrow \mathcal{D}X, \text{ function}}$$

is a Kleisli arrow

$$f : Y \to X$$

such that $\quad c \odot f \sqsubseteq (\mathsf{Ac} \times f) \odot d, \quad$ that is

$$\begin{tikzcd}
\mathsf{Ac} \times X & & \mathsf{Ac} \times Y \arrow[ll, "\mathsf{Ac} \times f"'] \\
X \arrow[u, "c"] & \sqsubseteq & Y \arrow[ll, "f"] \arrow[u, "d"']
\end{tikzcd}$$

"Simulating"
(*more* behavior)

"Simulated"
(*less* behavior)

Natural order for
$\mathcal{P}$ / $\mathcal{D}$

Hasuo (Kyoto, JP)

is a Kleisli arrow

$$\boldsymbol{f} \;:\; \boldsymbol{Y} \looparrowright \boldsymbol{X}$$

such that $\quad \boldsymbol{c} \odot \boldsymbol{f} \sqsubseteq (\mathsf{\textbf{Ac}} \times \boldsymbol{f}) \odot \boldsymbol{d}, \quad$ that is

$$\begin{tikzcd}
\mathsf{\textbf{Ac}} \times \boldsymbol{X} & \mathsf{\textbf{Ac}} \times \boldsymbol{Y} \arrow[l, "\mathsf{\textbf{Ac}} \times \boldsymbol{f}"'] \\
\boldsymbol{X} \arrow[u, "\boldsymbol{c}"] \arrow[ur, phantom, "\sqsubseteq"] & \boldsymbol{Y} \arrow[l, "\boldsymbol{f}"] \arrow[u, "\boldsymbol{d}"']
\end{tikzcd}$$

$$
\begin{tikzcd}
\mathsf{Ac} \times X \arrow[d, phantom] & & \mathsf{Ac} \times Y \arrow[ll, "\mathsf{Ac} \times f"'] \\
X \arrow[u, "c"] \arrow[urr, phantom, "\sqsubseteq"] & & Y \arrow[ll, "f"] \arrow[u, "d"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbf{Ac} \times X & \mathbf{Ac} \times Y \arrow[l, "\mathbf{Ac} \times f"', "|" marking] \\
X \arrow[u, "c", "|" marking] & Y \arrow[l, "f", "|" marking] \arrow[u, "d"', "|" marking] \arrow[ul, phantom, "\sqsubseteq"]
\end{tikzcd}
$$

y

$$
\begin{tikzcd}
\mathsf{Ac} \times X & & \mathsf{Ac} \times Y \arrow[ll, "\mathsf{Ac} \times f"'] \\
X \arrow[u, "c"] \arrow[urr, phantom, "\sqsubseteq"] & & Y \arrow[ll, "f"] \arrow[u, "d"']
\end{tikzcd}
$$

$$\{x \mid y \overset{R}{\dashrightarrow} x\} \longleftarrow\!\shortmid\; y$$

$$\left\{ (a, x') \;\middle|\; \begin{array}{l} y \\ \downarrow \\ \bullet \xrightarrow{a} x' \end{array} \right\}$$

$$\begin{tikzcd}
\mathsf{Ac} \times X \arrow[d, phantom] & & \mathsf{Ac} \times Y \arrow[ll, "\mathsf{Ac} \times f"'] \\
X \arrow[u, "c"] \arrow[rr, phantom, "\sqsubseteq" description, shift left=4] & & Y \arrow[ll, "f"] \arrow[u, "d"']
\end{tikzcd}$$

$$\{ x \mid y \overset{R}{\dashrightarrow} x \}$$

$$y$$

$$\{(a, y') \mid y \xrightarrow{a} y'\}$$

$$\left\{ (a, x') \;\middle|\; \begin{array}{l} y \\ \downarrow \\ \bullet \xrightarrow{a} x' \end{array} \right\}$$

$$\begin{tikzcd}
\mathsf{Ac} \times X & & \mathsf{Ac} \times Y \arrow[ll, "\mathsf{Ac} \times f"'] \\
X \arrow[u, "c"] \arrow[rr, phantom, "\sqsubseteq"] & & Y \arrow[u, "d"'] \arrow[ll, "f"]
\end{tikzcd}$$

$$\{x \mid y \overset{R}{\dashrightarrow} x\} \longmapsfrom y$$

$$
\begin{tikzcd}[column sep=large]
 & \left\{ (a,x') \;\middle|\; \begin{matrix} y & \xrightarrow{a} & \bullet \\ & & \downarrow \\ & & x' \end{matrix} \right\} & & \{(a,y') \mid y \xrightarrow{a} y'\} \arrow[lll, mapsto, start anchor=west, end anchor=east, shift left=0, phantom] \arrow[ll, mapsto] \\
\left\{ (a,x') \;\middle|\; \begin{matrix} y & & \\ \downarrow & & \\ \bullet & \xrightarrow{a} & x' \end{matrix} \right\} & \mathsf{Ac} \times X & & \mathsf{Ac} \times Y \arrow[ll, "\mathsf{Ac} \times f"', "+" marking] \\
 & X \arrow[u, "c", "+" marking] & \sqsubseteq & Y \arrow[ll, "f", "+" marking] \arrow[u, "d"', "+" marking] \\
\{x \mid y \overset{R}{\dashrightarrow} x\} \arrow[uu, mapsto] & & & y \arrow[lll, mapsto] \arrow[uuu, mapsto]
\end{tikzcd}
$$

$$\left\{ (a,x') \;\middle|\; \begin{array}{l} y \\ \vdots \\ \bullet \xrightarrow{a} x' \end{array} \right\} \subseteq \left\{ (a,x') \;\middle|\; \begin{array}{r} y \xrightarrow{a} \bullet \\ \vdots \\ x' \end{array} \right\} \longmapsfrom \{(a,y') \mid y \xrightarrow{a} y'\}$$

$$\begin{tikzcd}
\mathsf{Ac} \times X \arrow[rr, phantom, "\sqsubseteq" near end, shift right=6ex] & & \mathsf{Ac} \times Y \arrow[ll, "\mathsf{Ac} \times f"', "+" marking] \\
X \arrow[u, "c", "+" marking] & & Y \arrow[ll, "f", "+" marking] \arrow[u, "d"', "+" marking]
\end{tikzcd}$$

$$\{x \mid y \overset{R}{\dashrightarrow} x\} \longmapsfrom y$$

Hasuo (Kyoto, JP)

$$
\begin{tikzcd}
\mathsf{Ac} \times X & & \mathsf{Ac} \times Y \arrow[ll, "\mathsf{Ac} \times f"'] \\
X \arrow[u, "c"] & \sqsubseteq & Y \arrow[u, "d"'] \arrow[ll, "f"]
\end{tikzcd}
$$

$$
y \mapsto \{x \mid y \overset{R}{\dashrightarrow} x\} \mapsto \left\{ (a, x') \;\middle|\; \begin{array}{l} y \\ \vdots \\ \bullet \xrightarrow{a} x' \end{array} \right\}
\subseteq
\left\{ (a, x') \;\middle|\; \begin{array}{r} y \xrightarrow{a} \bullet \\ \vdots \\ x' \end{array} \right\} \leftarrow \{(a, y') \mid y \xrightarrow{a} y'\} \leftarrow y
$$

For each $y \in Y$, $a \in \mathsf{Ac}$ and $x' \in X$,

$$
\left(\begin{array}{l} y \\ \vdots \\ \bullet \xrightarrow{a} x' \end{array} \right) \text{ implies } \left(\begin{array}{r} y \xrightarrow{a} \exists \bullet \\ \vdots \\ x' \end{array} \right)
$$

Kleisli Simulation

Definition.
A *forward Kleisli simulation*

$$\text{from} \quad \begin{array}{c} \mathsf{Ac} \times X \\ c \uparrow \\ X \end{array} \qquad \text{to} \quad \begin{array}{c} \mathsf{Ac} \times Y \\ d \uparrow \\ Y \end{array}$$

is a Kleisli arrow

$$f : Y \rightarrow\!\!\!\!\!+ X$$

such that $\quad c \odot f \sqsubseteq (\mathsf{Ac} \times f) \odot d, \quad$ that is

$$\begin{array}{ccc} \mathsf{Ac} \times X & \xleftarrow{\ \mathsf{Ac} \times f\ } & \mathsf{Ac} \times Y \\ c \uparrow & \sqsubseteq & \uparrow d \\ X & \xleftarrow[\ f\ ]{} & Y \end{array}$$

is a Kleisli arrow

$$f \;:\; Y \nrightarrow X$$

such that $\quad c \odot f \sqsubseteq (\mathsf{Ac} \times f) \odot d, \quad$ that is

$$\begin{tikzcd}
\mathsf{Ac} \times X & & \mathsf{Ac} \times Y \arrow[ll, "\mathsf{Ac} \times f"'] \\
X \arrow[u, "c"] & & Y \arrow[ll, "f"] \arrow[u, "d"'] \arrow[ull, phantom, "\sqsubseteq"]
\end{tikzcd}$$

$$
\begin{tikzcd}
\mathsf{Ac} \times X & & \mathsf{Ac} \times Y \arrow[ll, "\mathsf{Ac} \times f"', "|"] \\
X \arrow[u, "c", "|"'] \arrow[urr, phantom, "\sqsubseteq"] & & Y \arrow[ll, "f", "|"'] \arrow[u, "d"', "|"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathsf{Ac} \times X & & \mathsf{Ac} \times Y \arrow[ll, "\mathsf{Ac} \times f"'] \\
X \arrow[u, "c"] & & Y \arrow[ll, "f"] \arrow[u, "d"'] \arrow[llu, phantom, "\sqsubseteq"]
\end{tikzcd}
$$

y

$$\left[\begin{array}{rcl} (a, x') & \mapsto & \sum_{x} \mathbf{Pr}[y \dashrightarrow x] \cdot \mathbf{Pr}[x \xrightarrow{a} x'] \\ & & = \mathbf{Pr}\left[\begin{array}{l} y \\ \vdots \\ \bullet \xrightarrow{a} x' \end{array}\right] \end{array}\right]$$

$$\uparrow$$

$$\left[\, x \mapsto \mathbf{Pr}[y \dashrightarrow x] \,\right]$$

$$\begin{tikzcd}
\mathsf{Ac} \times X & & \mathsf{Ac} \times Y \arrow[ll, "\mathsf{Ac} \times f"'] \\
X \arrow[u, "c"] & \sqsubseteq & Y \arrow[ll, "f"] \arrow[u, "d"']
\end{tikzcd}$$

$$\longleftarrow\!\!\mid \; \boxed{y}$$

Hasuo (Kyoto, JP)

$$\left[\ (a,x') \ \mapsto\ \sum_{y'} \Pr[y \xrightarrow{a} y'] \cdot \Pr[y' \dashrightarrow x'] = \Pr\left[\begin{array}{l} y \xrightarrow{a} \exists\bullet \\ \qquad\quad\ \vdots \\ \qquad\quad\ x' \end{array}\right]\ \right]$$

$$\left[\ (a,y') \ \mapsto\ \Pr[y \xrightarrow{a} y']\ \right]$$

$$\left[\ (a,x') \ \mapsto\ \sum_{x} \Pr[y \dashrightarrow x] \cdot \Pr[x \xrightarrow{a} x'] = \Pr\left[\begin{array}{l} y \\ \vdots \\ \bullet \xrightarrow{a} x' \end{array}\right]\ \right]$$

$$\begin{tikzcd}
\mathsf{Ac} \times X & & \mathsf{Ac} \times Y \arrow[ll, "\mathsf{Ac} \times f"'] \\
X \arrow[u, "c"] & & Y \arrow[ll, "f"] \arrow[u, "d"'] \arrow[ull, phantom, "\sqsubseteq"]
\end{tikzcd}$$

$$\left[\ x \ \mapsto\ \Pr[y \dashrightarrow x]\ \right] \qquad\qquad y$$

$$\left[\begin{array}{rcl} (a,x') & \mapsto & \sum_{y'} \Pr[y \xrightarrow{a} y'] \cdot \Pr[y' \dashrightarrow x'] \\ & & = \Pr\left[\begin{array}{l} y \xrightarrow{a} \exists\bullet \\ \qquad\quad \vdots \\ \qquad\quad x' \end{array}\right] \end{array}\right] \longleftarrow\!\!\shortmid \left[\, (a,y') \mapsto \Pr[y \xrightarrow{a} y'] \,\right]$$

$$\left[\begin{array}{rcl} (a,x') & \mapsto & \sum_{x} \Pr[y \dashrightarrow x] \cdot \Pr[x \xrightarrow{a} x'] \\ & & = \Pr\left[\begin{array}{l} y \\ \vdots \\ \bullet \xrightarrow{a} x' \end{array}\right] \end{array}\right]$$

$$\begin{tikzcd}
\mathsf{Ac} \times X & & \mathsf{Ac} \times Y \arrow[ll, "\mathsf{Ac} \times f"'] \\
X \arrow[u, "c"] \arrow[urr, phantom, "\sqsubseteq"] & & Y \arrow[ll, "f"] \arrow[u, "d"']
\end{tikzcd}$$

$$\left[\, x \mapsto \Pr[y \dashrightarrow x] \,\right] \longleftarrow\!\!\shortmid \; y$$

$$\left[\begin{array}{l}(a,x') \;\mapsto\; \displaystyle\sum_{y'} \mathbf{Pr}[y \xrightarrow{a} y'] \cdot \mathbf{Pr}[y' \dashrightarrow x'] \\ \qquad = \mathbf{Pr}\left[\begin{array}{ccc} y & \xrightarrow{a} & \exists\bullet \\ & & \vdots \\ & & x' \end{array}\right]\end{array}\right] \longleftarrow\!\shortmid \left[\,(a,y') \mapsto \mathbf{Pr}[y \xrightarrow{a} y']\,\right]$$

$$\leq$$

$$\left[\begin{array}{l}(a,x') \;\mapsto\; \displaystyle\sum_{x} \mathbf{Pr}[y \dashrightarrow x] \cdot \mathbf{Pr}[x \xrightarrow{a} x'] \\ \qquad = \mathbf{Pr}\left[\begin{array}{ccc} y & & \\ \vdots & & \\ \bullet & \xrightarrow{a} & x' \end{array}\right]\end{array}\right]$$

$$\begin{tikzcd}
\mathsf{Ac} \times X & & \mathsf{Ac} \times Y \arrow[ll, "\mathsf{Ac} \times f"'] \\
X \arrow[u, "c"] & \sqsubseteq & Y \arrow[ll, "f"] \arrow[u, "d"']
\end{tikzcd}$$

$$\left[\, x \mapsto \mathbf{Pr}[y \dashrightarrow x] \,\right] \longleftarrow\!\shortmid \; y$$

For each $y \in Y$, $a \in \mathsf{Ac}$ and $x' \in X$,

$$\mathbf{Pr}\left[\begin{array}{ccc} y & & \\ \vdots & & \\ \bullet & \xrightarrow{a} & x' \end{array}\right] \;\leq\; \mathbf{Pr}\left[\begin{array}{ccc} y & \xrightarrow{a} & \bullet \\ & & \vdots \\ & & x' \end{array}\right]$$

Kleisli Simulation for Probabilistic LTS

Definition.
A *forward simulation* from (X, x_0, c) to (Y, y_0, d) is a function

$$f : Y \longrightarrow \mathcal{D}X$$

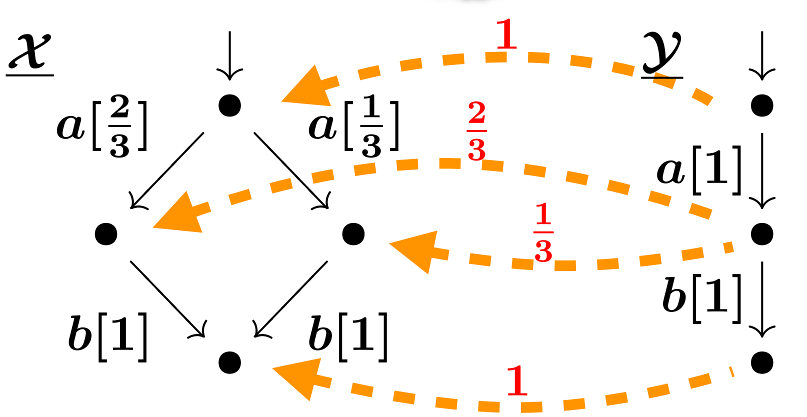
“delegation function”

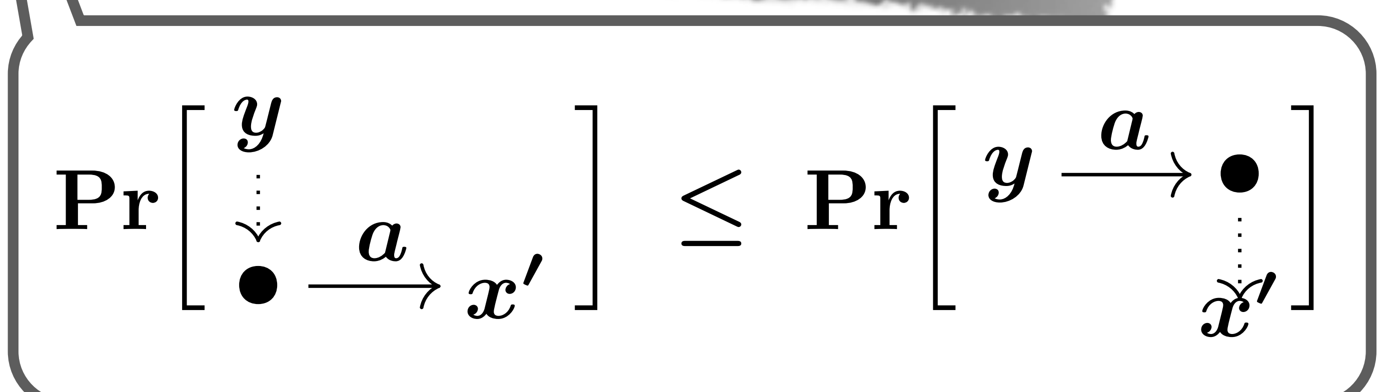
such that

$$f(y_0)(x_0) = 1 \qquad \text{(INIT)}$$

$$\sum_{x \in X} f(y)(x) \cdot c(x)(a, x') \leq \sum_{y' \in Y} d(y)(a, y') \cdot f(y')(x')$$

for each $y \in Y$, $a \in \mathsf{Ac}$ and $x' \in X$ (ACT)





Hasuo (Kyoto, JP)

Four Variations

forward

$$
\begin{tikzcd}
FX & FY \arrow[l, "Ff"', "|" marking] \\
X \arrow[u, "c", "-" marking] & Y \arrow[l, "f", "|" marking] \arrow[u, "d"', "-" marking] \arrow[ul, phantom, "\sqsubseteq"]
\end{tikzcd}
$$

backward

$$
\begin{tikzcd}
FX \arrow[r, "Fb", "|" marking] & FY \\
X \arrow[u, "c", "-" marking] \arrow[r, "b"', "|" marking] \arrow[ur, phantom, "\sqsubseteq"] & Y \arrow[u, "d"', "-" marking]
\end{tikzcd}
$$

forward-backward

$$
\begin{tikzcd}
FX & FU \arrow[l, "Ff"', "|" marking] \arrow[r, "Fb", "|" marking] & FY \\
X \arrow[u, "c", "-" marking] & U \arrow[l, "f", "|" marking] \arrow[u, "e", "-" marking] \arrow[r, "b"', "|" marking] \arrow[ul, phantom, "\sqsubseteq"] & Y \arrow[u, "d"', "-" marking] \arrow[ul, phantom, "\sqsubseteq"]
\end{tikzcd}
$$

backward-forward

$$
\begin{tikzcd}
FX \arrow[r, "Fb", "|" marking] & FU & FY \arrow[l, "Ff"', "|" marking] \\
X \arrow[u, "c", "-" marking] \arrow[r, "b"', "|" marking] \arrow[ur, phantom, "\sqsubseteq"] & U \arrow[u, "e", "-" marking] \arrow[ur, phantom, "\sqsubseteq"] & Y \arrow[l, "f", "|" marking] \arrow[u, "d"', "-" marking]
\end{tikzcd}
$$

Four Variations

forward

$$\begin{tikzcd} FX & & FY \arrow[ll, "Ff"', "\shortmid"{marking}] \\ X \arrow[u, "c", "\shortmid"{marking}] & \sqsubseteq & Y \arrow[u, "d"', "\shortmid"{marking}] \arrow[ll, "f", "\shortmid"{marking}] \end{tikzcd}$$

backward

$$\begin{tikzcd} FX \arrow[rr, "Fb", "\shortmid"{marking}] & & FY \\ X \arrow[u, "c", "\shortmid"{marking}] \arrow[rr, "b"', "\shortmid"{marking}] & \sqsubseteq & Y \arrow[u, "d"', "\shortmid"{marking}] \end{tikzcd}$$

forward-backward

$$\begin{tikzcd} FX & FU \arrow[l, "Ff"', "\shortmid"{marking}] \arrow[r, "Fb", "\shortmid"{marking}] & FY \\ X \arrow[u, "c", "\shortmid"{marking}] \arrow[ur, phantom, "\sqsubseteq"] & U \arrow[u, "e", "\shortmid"{marking}] \arrow[l, "f", "\shortmid"{marking}] \arrow[r, "b"', "\shortmid"{marking}] \arrow[ur, phantom, "\sqsubseteq"] & Y \arrow[u, "d"', "\shortmid"{marking}] \end{tikzcd}$$

Intermediate system

backward-forward

$$\begin{tikzcd} FX \arrow[r, "Fb", "\shortmid"{marking}] & FU & FY \arrow[l, "Ff"', "\shortmid"{marking}] \\ X \arrow[u, "c", "\shortmid"{marking}] \arrow[r, "b"', "\shortmid"{marking}] \arrow[ur, phantom, "\sqsubseteq"] & U \arrow[u, "e", "\shortmid"{marking}] \arrow[ur, phantom, "\sqsubseteq"] & Y \arrow[u, "d"', "\shortmid"{marking}] \arrow[l, "f", "\shortmid"{marking}] \end{tikzcd}$$

Intermediate system

Four Variations

forward

$$\begin{tikzcd} FX & FY \arrow[l, "Ff"'] \\ X \arrow[u, "c"] \arrow[ur, phantom, "\sqsubseteq"] & Y \arrow[l, "f"] \arrow[u, "d"'] \end{tikzcd}$$

backward

$$\begin{tikzcd} FX \arrow[r, "Fb"] & FY \\ X \arrow[u, "c"] \arrow[r, "b"'] \arrow[ur, phantom, "\sqsubseteq"] & Y \arrow[u, "d"'] \end{tikzcd}$$

forward-backward

$$\begin{tikzcd} FX & FU \arrow[l, "Ff"'] \arrow[r, "Fb"] & FY \\ X \arrow[u, "c"] \arrow[ur, phantom, "\sqsubseteq"] & U \arrow[l, "f"] \arrow[u, "e"] \arrow[r, "b"'] \arrow[ur, phantom, "\sqsubseteq"] & Y \arrow[u, "d"'] \end{tikzcd}$$

Intermediate system

"Simulated"
(*less* behavior)

backward-forward

$$\begin{tikzcd} FX \arrow[r, "Fb"] & FU & FY \arrow[l, "Ff"'] \\ X \arrow[u, "c"] \arrow[r, "b"'] \arrow[ur, phantom, "\sqsubseteq"] & U \arrow[u, "e"] \arrow[ur, phantom, "\sqsubseteq"] & Y \arrow[l, "f"] \arrow[u, "d"'] \end{tikzcd}$$

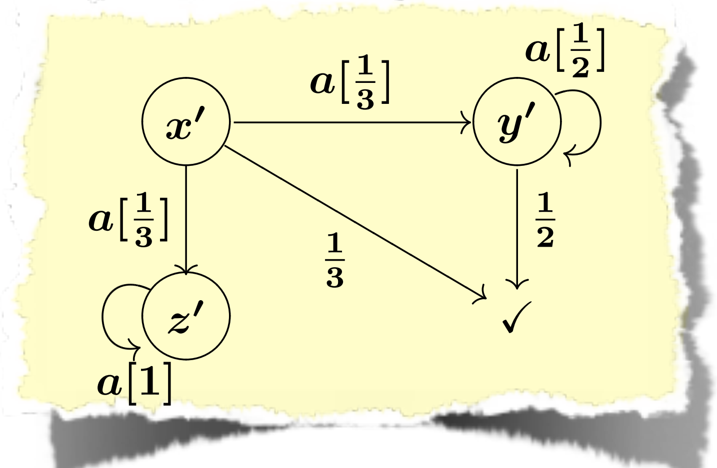
Intermediate system

"Simulating"
(*more* behavior)

Generic Soundness Theorem

[H., CONCUR'06]

- Trace semantics:



NB. We need explicit termination: see paper.

$$\mathbf{tr}(x') = \left[\begin{array}{llll} \langle\rangle \mapsto \frac{1}{3}, & a \mapsto \frac{1}{3}\cdot\frac{1}{2}, & a^2 \mapsto \frac{1}{3}\cdot\frac{1}{2}\cdot\frac{1}{2}, & \cdots \\ a^n \mapsto \frac{1}{3}\cdot\left(\frac{1}{2}\right)^n, & \cdots & & \end{array}\right]$$

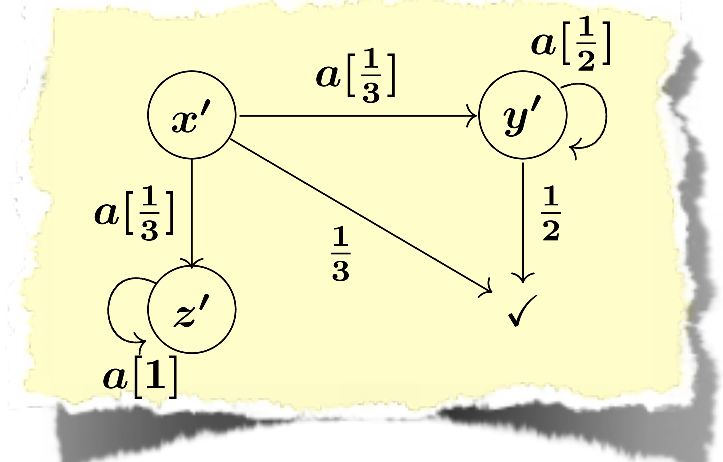
- Thm. ∃ Kleisli simulation ⇒ trace inclusion

Generic Soundness Theorem

[H., CONCUR'06]

- Trace semantics:

  NB. We need explicit termination: see paper.



$$\mathsf{tr}(x') = \left[\begin{array}{l} \langle\rangle \mapsto \frac{1}{3},\quad a \mapsto \frac{1}{3}\cdot\frac{1}{2},\quad a^2 \mapsto \frac{1}{3}\cdot\frac{1}{2}\cdot\frac{1}{2},\quad \cdots \\ a^n \mapsto \frac{1}{3}\cdot\left(\frac{1}{2}\right)^n,\quad \cdots \end{array}\right]$$

- <u>Thm.</u> ∃ Kleisli simulation ⇒ trace inclusion

- Proof uses *generic trace semantics* via final coalgebra

  [H.-Jacobs-Sokolova, LMCS'07]

$$\begin{tikzcd} FX \arrow[r, dashed, "F(\mathsf{tr}(c))"] & FZ \\ X \arrow[u, "c"] \arrow[r, dashed, "\mathsf{tr}(c)"'] & Z \arrow[u, "\text{final}"'] \end{tikzcd} \quad \text{in } \mathcal{K}\ell(\mathcal{D})$$

“trace semantics”

$$\begin{tikzcd} FX \arrow[r] \arrow[rr, dashed, bend left] & FY \arrow[r, dashed] & FZ \\ X \arrow[u] \arrow[r] \arrow[rr, dashed, bend right] \arrow[ur, phantom, "\sqsupseteq"] & Y \arrow[u] \arrow[r, dashed] \arrow[ur, phantom, "="] & Z \arrow[u, "\cong"', "\text{final}"] \end{tikzcd}$$

Kleisli Simulation: Summary

$$
\begin{tikzcd}
FX & FY \arrow[l, "Ff"'] \\
X \arrow[u, "c"] \arrow[ur, phantom, "\sqsubseteq"] & Y \arrow[l, "f"] \arrow[u, "d"']
\end{tikzcd}
$$

Kleisli Simulation: Summary

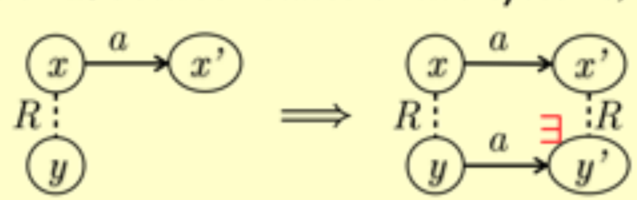
$$\begin{tikzcd} FX & FY \arrow[l, "Ff"'] \\ X \arrow[u, "c"] \arrow[ur, phantom, "\sqsubseteq"] & Y \arrow[l, "f"] \arrow[u, "d"'] \end{tikzcd}$$

$T = \mathcal{P}$
$F = \mathsf{Ac} \times \_$

$T = \mathcal{D}$
$F = \mathsf{Ac} \times \_$

Forward simulation ($T = \mathcal{P}$)

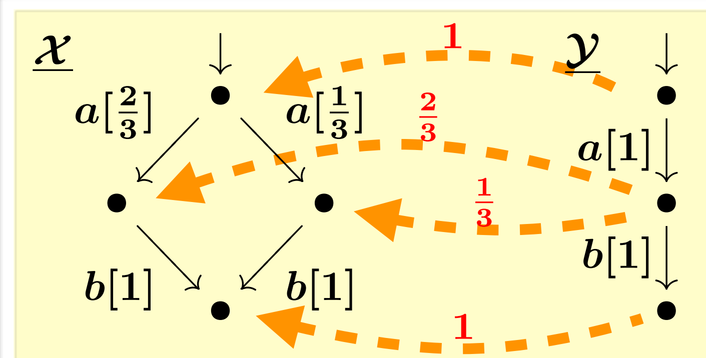
A relation R between states of two systems, s.t.



Soundness theorem ($T = \mathcal{P}$)

Existence of fwd./bwd. simulation
⇒ trace incl.

Forward simulation ($T = \mathcal{D}$)



Soundness theorem ($T = \mathcal{D}$)

Existence of fwd./bwd. simulation
⇒ trace incl.

Kleisli Simulation: Summary

- Uniform definition for a variety of systems

<u>fwd.</u>
$$
\begin{tikzcd}
FX & FY \arrow[l, "Ff"'] \arrow[dl, phantom, "\sqsubseteq"] \\
X \arrow[u, "c"] & Y \arrow[l, "f"] \arrow[u, "d"']
\end{tikzcd}
$$

<u>bwd.</u>
$$
\begin{tikzcd}
FX \arrow[r, "Fb"] & FY \\
X \arrow[u, "c"] \arrow[r, "b"'] \arrow[ur, phantom, "\sqsubseteq"] & Y \arrow[u, "d"']
\end{tikzcd}
$$

- esp.: non-det. & probability

Kleisli Simulation: Summary

- Uniform definition for a variety of systems

$$
\underline{\text{fwd.}} \quad
\begin{tikzcd}
FX & FY \arrow[l, "Ff"', "|" marking] \\
X \arrow[u, "c", "|" marking] \arrow[ur, phantom, "\sqsubseteq"] & Y \arrow[l, "f", "|" marking] \arrow[u, "d"', "|" marking]
\end{tikzcd}
\qquad
\underline{\text{bwd.}} \quad
\begin{tikzcd}
FX \arrow[r, "Fb", "|" marking] & FY \\
X \arrow[u, "c", "|" marking] \arrow[r, "b"', "|" marking] \arrow[ur, phantom, "\sqsubseteq"] & Y \arrow[u, "d"', "|" marking]
\end{tikzcd}
$$

  - esp.: non-det. & probability

- Generic soundness theorem:
  ∃ simulation ⇒ trace inclusion

Kleisli Simulation: Summary

- Uniform definition for a variety of systems

fwd.
$$\begin{tikzcd} FX & FY \arrow[l, "Ff"'] \\ X \arrow[u, "c"] \arrow[ur, phantom, "\sqsubseteq"] & Y \arrow[l, "f"] \arrow[u, "d"'] \end{tikzcd}$$

bwd.
$$\begin{tikzcd} FX \arrow[r, "Fb"] & FY \\ X \arrow[u, "c"] \arrow[r, "b"'] \arrow[ur, phantom, "\sqsubseteq"] & Y \arrow[u, "d"'] \end{tikzcd}$$

  - esp.: non-det. & probability

- Generic soundness theorem:
  ∃ simulation ⇒ trace inclusion

- Has been applied to verif. of *probabilistic anonymity*
  [H.-Kawabe-Sakurada, TCS'10]

III Characterization Results

Characterization

- <u>Thm.</u> A Jonsson-Larsen simulation R is a fwd.-bwd. Kleisli simulation, by

$$
\begin{tikzcd}[column sep=huge]
\mathbf{Ac}\times X & \mathbf{Ac}\times R \arrow[l, "\shortmid" marking] \arrow[r, "\shortmid" marking] & \mathbf{Ac}\times Y \\
X \arrow[u, "c", "\shortmid" marking] \arrow[ur, phantom, "\sqsubseteq"] & R \arrow[l, "J\pi_1", "\shortmid" marking] \arrow[r, "J\pi_2"', "\shortmid" marking] \arrow[u, "\Delta", "\shortmid" marking] \arrow[ur, phantom, "\sqsubseteq"] & Y \arrow[u, "d"', "\shortmid" marking]
\end{tikzcd}
$$

- <u>Cor.</u> Soundness of JL-simulation

Characterization

- <u>Thm.</u> A Jonsson-Larsen simulation R is a fwd.-bwd. Kleisli simulation, by

weight function

$$
\begin{tikzcd}
\mathbf{Ac} \times X & & \mathbf{Ac} \times R \arrow[ll, "|" marking] \arrow[rr, "|" marking] & & \mathbf{Ac} \times Y \\
X \arrow[u, "c"] \arrow[urr, phantom, "\sqsubseteq"] & & R \arrow[ll, "|" marking, "J\pi_1"'] \arrow[rr, "|" marking, "J\pi_2"'] \arrow[u, "\Delta"] \arrow[urr, phantom, "\sqsubseteq"] & & Y \arrow[u, "d"']
\end{tikzcd}
$$

simulation relation

- <u>Cor.</u> Soundness of JL-simulation

Characterization

- <u>Thm.</u> A Jonsson-Larsen simulation R is a fwd.-bwd. Kleisli simulation, by

weight function

$$
\begin{tikzcd}
\mathbf{Ac} \times X \arrow[d, phantom, "\sqsubseteq" {xshift=12ex}] & \mathbf{Ac} \times R \arrow[l, "|" marking] \arrow[r, "|" marking] & \mathbf{Ac} \times Y \\
X \arrow[u, "|" marking, "c"] & R \arrow[u, "|" marking, "\Delta"] \arrow[l, "|" marking, "J\pi_1"] \arrow[r, "|" marking, "J\pi_2"'] \arrow[ur, phantom, "\sqsubseteq"] & Y \arrow[u, "|" marking, "d"']
\end{tikzcd}
$$

projection simulation relation projection

- <u>Cor.</u> Soundness of JL-simulation

Characterization

- <u>Thm.</u> A Jonsson-Larsen simulation R is a fwd.-bwd. Kleisli simulation, by

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$$
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X \arrow[u, "c", "|" marking] \arrow[ur, phantom, "\sqsubseteq"] & R \arrow[l, "J\pi_1", "|" marking] \arrow[r, "J\pi_2"', "|" marking] \arrow[u, "\Delta", "|" marking] \arrow[ur, phantom, "\sqsubseteq"] & Y \arrow[u, "d"', "|" marking]
\end{tikzcd}
$$

projection

simulation relation

projection

- <u>Cor.</u> Soundness of JL-simulation

For Purely Probabilistic Systems...

coalgebraic

Kleisli simulation

[H., CONCUR'06]

fwd.
$$\begin{tikzcd} FX & FY \arrow[l, "Ff"'] \\ X \arrow[u, "c"] \arrow[ur, phantom, "\sqsubseteq"] & Y \arrow[l, "f"] \arrow[u, "d"'] \end{tikzcd}$$

bwd.
$$\begin{tikzcd} FX \arrow[r, "Fb"] & FY \\ X \arrow[u, "c"] \arrow[r, "b"'] \arrow[ur, phantom, "\sqsubseteq"] & Y \arrow[u, "d"'] \end{tikzcd}$$

specializes

Hughes-Jacobs simulation

[Hughes-Jacobs, TCS'04]

$$\begin{tikzcd} BX & BR \arrow[l, "B\pi_1"'] \arrow[r, "B\pi_2"] & BY \\ X \arrow[u, "c"] \arrow[ur, phantom, "\sqsubseteq"] & R \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] \arrow[u, "r"] \arrow[ur, phantom, "\sqsubseteq"] & Y \arrow[u, "d"'] \end{tikzcd}$$

specializes

Jonsson-Larsen simulation for prob. sys.

[Jonsson-Larsen, LICS'91]

via *weight function*

$u \backslash v$	$\perp$	$\cdots\ y'\ \cdots$
$\perp$	$\lesseqgtr 0$	≥ 0
$\vdots$ x' $\vdots$	0	$\begin{cases} \geq 0 & (x' R y') \\ 0 & (\text{o.w.}) \end{cases}$

A Few Words on Hughes-Jacobs Simulation

- Initial success of coalgebra:
  generic definition of *bisimulation*

- HJ-simulation:
  modified coalgebraic bisimulation
  - HJ-simulation: function-based
  - Kleisli simulation: Kleisli arrow-based

- For non-det./prob. systems:
  HJ is Kleisli

Hughes-Jacobs simulation

[Hughes-Jacobs, TCS'04]

$$\begin{tikzcd} BX & BR \arrow[l, "B\pi_1"'] \arrow[r, "B\pi_2"] & BY \\ X \arrow[u, "c"] \arrow[ur, phantom, "\sqsubseteq"] & R \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] \arrow[u, "r"] \arrow[ur, phantom, "\sqsubseteq"] & Y \arrow[u, "d"'] \end{tikzcd}$$

Conclusions & Future Work

- *Kleisli simulation*: new simulation notion for purely probabilistic systems
  - Mathematical generalization of non-deterministic notion
  - Not a relation, but a "delegation" function
  - Generic soundness
  - Jonsson-Larsen simulation as a special case

- Other probabilistic systems? E.g. Stochastic CFG
- Other branching? E.g. quantum channels
- Algorithmic aspects

Conclusions & Future Work

- *Kleisli simulation*: new simulation notion for purely probabilistic systems
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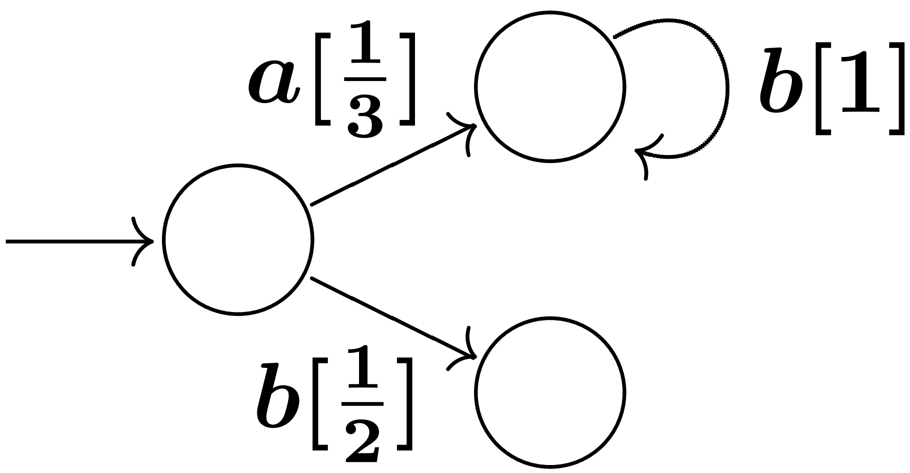
- Other probabilistic systems? E.g. Stochastic CFG
- Other branching? E.g. quantum channels
- Algorithmic aspects

Thank you for your attention!

Ichiro Hasuo (RIMS, Kyoto U.)
http://www.kurims.kyoto-u.ac.jp/~ichiro

The Following Slides Are for Backup

Kleisli Simulation for Probabilistic LTS



Definition. (GPA)
A *generative probabilistic automaton (GPA)* is

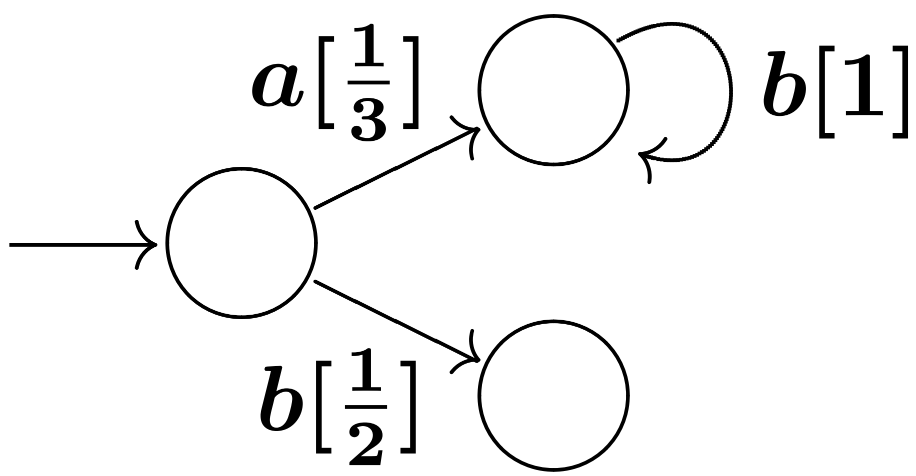
$$(X, x_0, c)$$

where

- X is a *state space*;
- $x_0 \in X$ is an *initial state*;
- c is a *transition function* $c : X \to \mathcal{D}(\mathsf{Ac} \times X)$.

N.B. without explicit termination, for simplicity

Kleisli Simulation for Probabilistic LTS



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$\mathcal{D}X$

$$= \left\{ d : X \to [0,1] \;\middle|\; \sum_{x} d(x) \leq 1 \right\}$$

N.B. without explicit termination, for simplicity

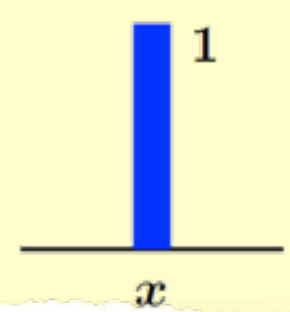
Identity Kleisli Arrow

- "No branching"
- Non-deterministic: *singleton*

$$
\begin{array}{ll}
X \xrightarrow{\ \eta_X\ } X & \mathcal{P}\text{-Kleisli arrow} \\
\hline\hline
X \longrightarrow \mathcal{P}X & \text{function} \\
x \longmapsto \{x\} &
\end{array}
$$

- Probabilistic: *pointmass/Dirac distribution*

$$
\begin{array}{ll}
X \xrightarrow{\ \eta_X\ } X & \mathcal{D}\text{-Kleisli arrow} \\
\hline\hline
X \longrightarrow \mathcal{D}X & \text{function} \\
x \longmapsto \begin{bmatrix} x \mapsto 1 & \\ x' \mapsto 0 & (x' \neq x) \end{bmatrix} &
\end{array}
$$



Composition of Kleisli Arrows

$$\frac{X \overset{f}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Y \qquad Y \overset{g}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Z}{X \overset{g \odot f}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Z}$$

Composition of Kleisli Arrows

$$\frac{X \overset{f}{\longrightarrow\!\!\!\!\!\!\!\!\!\!+\;\;\;\;} Y \quad Y \overset{g}{\longrightarrow\!\!\!\!\!\!\!\!\!\!+\;\;\;\;} Z}{X \overset{g\odot f}{\longrightarrow\!\!\!\!\!\!\!\!\!\!+\;\;\;\;} Z}$$

that is

$$\frac{X \xrightarrow{f} \mathcal{P}Y \quad Y \xrightarrow{g} \mathcal{P}Z}{X \xrightarrow{g\odot f} \mathcal{P}Z}$$

Composition of Kleisli Arrows

$$\frac{X \overset{f}{\longrightarrow\!\!\!\!\!\!\!+}\; Y \quad Y \overset{g}{\longrightarrow\!\!\!\!\!\!\!+}\; Z}{X \overset{g \odot f}{\longrightarrow\!\!\!\!\!\!\!+}\; Z}$$

Composition of Kleisli Arrows

$$\frac{X \xrightarrow{f} Y \quad Y \xrightarrow{g} Z}{X \xrightarrow{g \odot f} Z}$$

Composition of Kleisli Arrows

$$\frac{X \xrightarrow{f} Y \qquad Y \xrightarrow{g} Z}{X \xrightarrow{g \odot f} Z}$$

$$\begin{tikzcd}
 & & \mathcal{P}\mathcal{P}Z \arrow[d, "\bigcup"] \\
 & \mathcal{P}Y \arrow[ur, "\mathcal{P}g"] & \mathcal{P}Z \\
X \arrow[ur, "f"] & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & \{\,\{z, z'\}, \{z''\}\,\} \arrow[d, mapsto] \\
 & \{y, y'\} \arrow[ur, mapsto] & \{z, z', z''\} \\
x \arrow[ur, mapsto] & &
\end{tikzcd}$$

Composition of Kleisli Arrows

$$\frac{X \xrightarrow{f} Y \qquad Y \xrightarrow{g} Z}{X \xrightarrow{g \odot f} Z}$$

$$\begin{tikzcd}
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x \arrow[ur, mapsto] & &
\end{tikzcd}$$

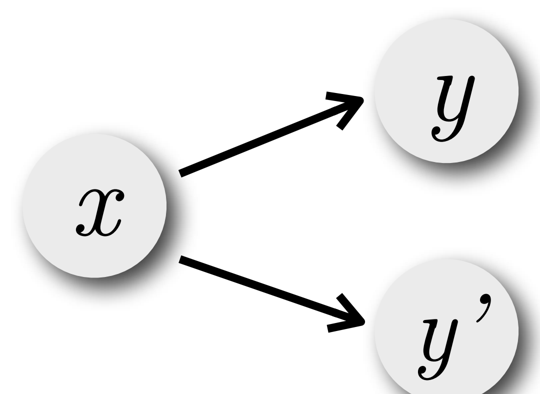
x

Composition of Kleisli Arrows

$$\frac{X \xrightarrow{f} Y \quad Y \xrightarrow{g} Z}{X \xrightarrow{g \odot f} Z}$$

$$\begin{tikzcd}
 & & \mathcal{P}\mathcal{P}Z \arrow[d, "\bigcup"] \\
 & \mathcal{P}Y \arrow[ur, "\mathcal{P}g"] & \mathcal{P}Z \\
X \arrow[ur, "f"] & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & \{\, \{z, z'\}, \{z''\} \,\} \arrow[d, mapsto] \\
 & \{y, y'\} \arrow[ur, mapsto] & \{z, z', z''\} \\
x \arrow[ur, mapsto] & &
\end{tikzcd}$$



Composition of Kleisli Arrows

$$\frac{X \xrightarrow{\;f\;} Y \qquad Y \xrightarrow{\;g\;} Z}{X \xrightarrow{\;g \odot f\;} Z}$$

$$\begin{tikzcd}
 & & \mathcal{P}\mathcal{P}Z \arrow[d, "\bigcup"] \\
 & \mathcal{P}Y \arrow[ur, "\mathcal{P}g"] & \mathcal{P}Z \\
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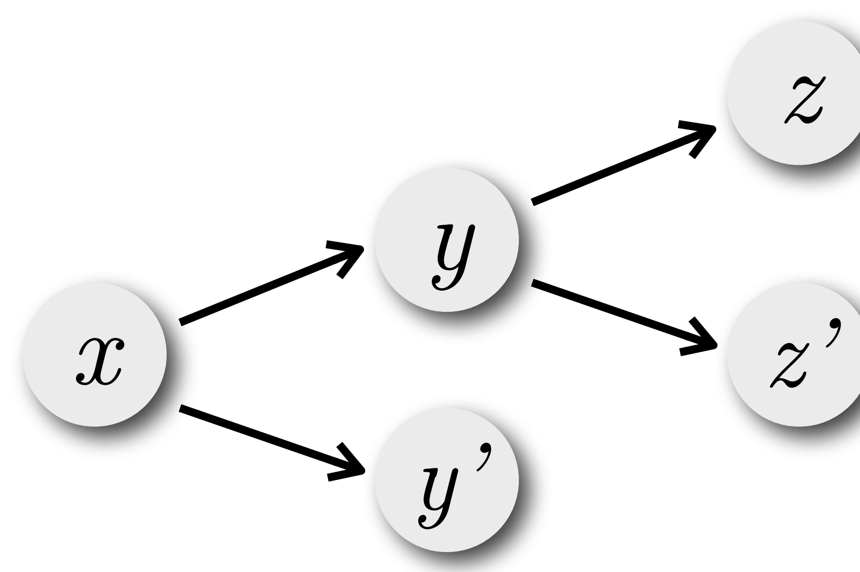
 & & \{\, \{z, z'\}, \{z''\} \,\} \arrow[d, mapsto] \\
 & \{y, y'\} \arrow[ur, mapsto] & \{z, z', z''\} \\
x \arrow[ur, mapsto] & &
\end{tikzcd}$$

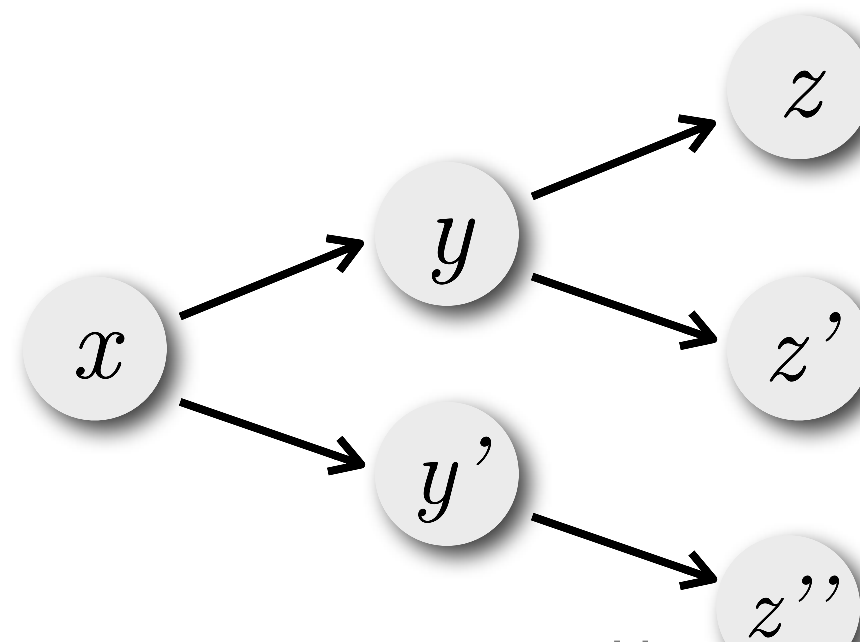


Composition of Kleisli Arrows

$$\frac{X \overset{f}{\longmapsto} Y \quad Y \overset{g}{\longmapsto} Z}{X \overset{g \odot f}{\longmapsto} Z}$$

$$\begin{tikzcd}
 & & & \mathcal{P}\mathcal{P}Z \arrow[d, "\bigcup"] \\
 & & \mathcal{P}Y \arrow[ur, "\mathcal{P}g"] & \mathcal{P}Z \\
X \arrow[urr, "f"] & & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & \{\,\{z, z'\}, \{z''\}\,\} \arrow[d, mapsto] \\
 & \{y, y'\} \arrow[ur, mapsto] & \{z, z', z''\} \\
x \arrow[ur, mapsto] & &
\end{tikzcd}$$

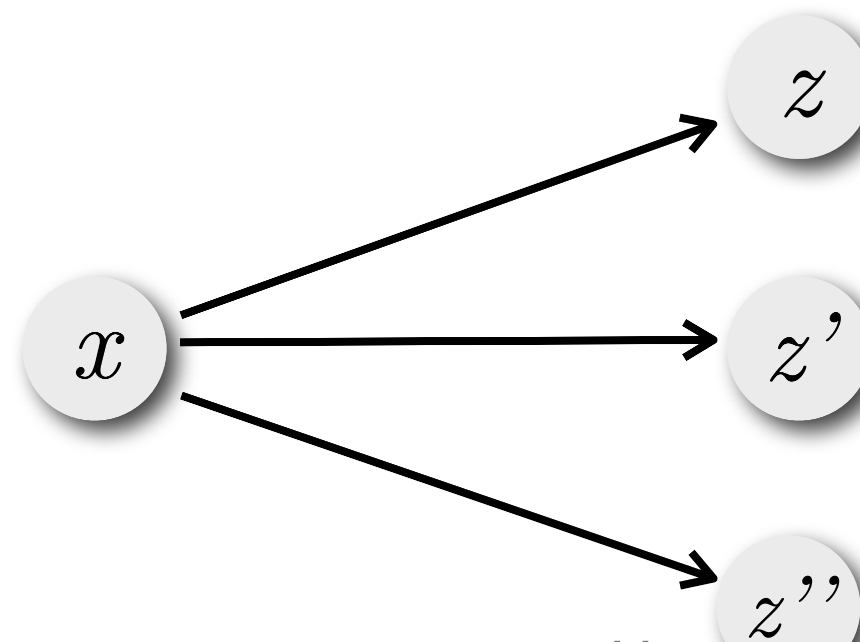


Composition of Kleisli Arrows

$$\frac{X \xrightarrow{\;f\;} Y \qquad Y \xrightarrow{\;g\;} Z}{X \xrightarrow{\;g \odot f\;} Z}$$

$$X \xrightarrow{f} \mathcal{P}Y \xrightarrow{\mathcal{P}g} \mathcal{P}\mathcal{P}Z \xrightarrow{\bigcup} \mathcal{P}Z$$

$$x \mapsto \{y, y'\} \mapsto \{\, \{z, z'\}, \{z''\} \,\} \mapsto \{z, z', z''\}$$



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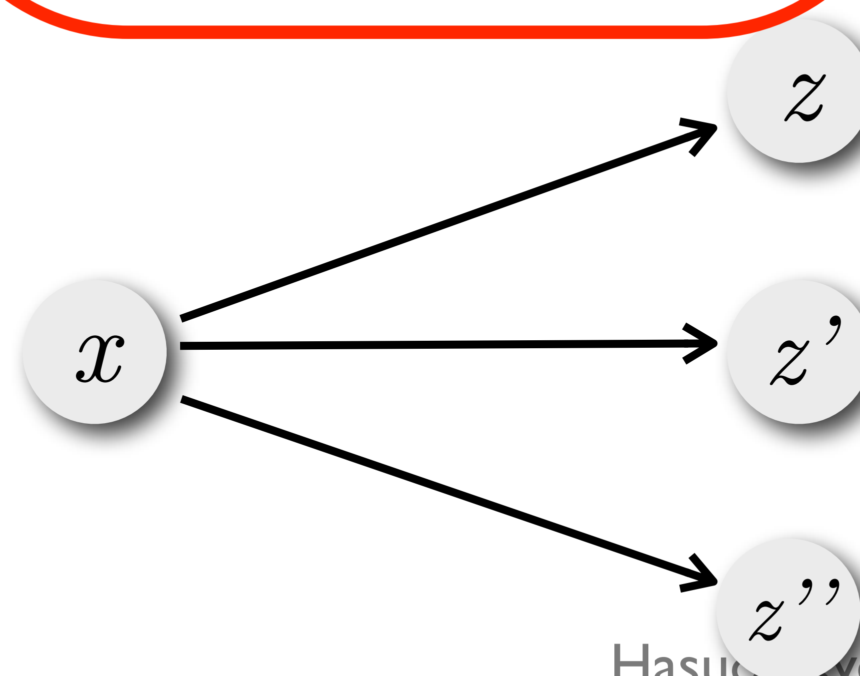
$$\frac{X \overset{f}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Y \qquad Y \overset{g}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Z}{X \overset{g \odot f}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Z}$$

$$\begin{tikzcd}
X \arrow[r, "f"] & \mathcal{P}Y \arrow[r, "\mathcal{P}g"] & \mathcal{P}\mathcal{P}Z \arrow[d, "\bigcup"] \\
 & & \mathcal{P}Z
\end{tikzcd}$$

$$\begin{tikzcd}
x \arrow[r, mapsto] & \{y, y'\} \arrow[r, mapsto] & \{\,\{z, z'\}, \{z''\}\,\} \arrow[d, mapsto] \\
 & & \{z, z', z''\}
\end{tikzcd}$$

unfolding internal branching
(relevant to trace sem., soundness)



Composition of Kleisli Arrows

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Hasuo (Kyoto, JP)

Composition of Kleisli Arrows

$$\frac{X \overset{f}{\longrightarrow\!\!\!\!\!\!\!+\;\;\;} Y \qquad Y \overset{g}{\longrightarrow\!\!\!\!\!\!\!+\;\;\;} Z}{X \overset{g \odot f}{\longrightarrow\!\!\!\!\!\!\!+\;\;\;} Z}$$

$$\begin{tikzcd}
 & & \mathcal{D}\mathcal{D}Z \arrow[d, "\mu"] \\
 & \mathcal{D}Y \arrow[ur, "\mathcal{D}g"] & \mathcal{D}Z \\
X \arrow[ur, "f"] & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & \left[\begin{array}{r} \left[\begin{array}{r} z \mapsto \frac{1}{2} \\ z' \mapsto \frac{1}{2} \end{array}\right] \mapsto \frac{2}{3} \\ [z \mapsto 1] \mapsto \frac{1}{3} \end{array}\right] \arrow[d, mapsto] \\
 & \left[\begin{array}{r} y \mapsto \frac{2}{3} \\ y' \mapsto \frac{1}{3} \end{array}\right] \arrow[ur, mapsto] & \left[\begin{array}{r} z \mapsto \frac{1}{3} \\ z' \mapsto \frac{1}{3} \\ z'' \mapsto \frac{1}{3} \end{array}\right] \\
x \arrow[ur, mapsto] & &
\end{tikzcd}$$

Composition of Kleisli Arrows

$$\frac{X \xrightarrow{f} Y \qquad Y \xrightarrow{g} Z}{X \xrightarrow{g \odot f} Z}$$

$$\begin{tikzcd}
 & & \mathcal{D}\mathcal{D}Z \arrow[d, "\mu"] \\
 & \mathcal{D}Y \arrow[ur, "\mathcal{D}g"] & \mathcal{D}Z \\
X \arrow[ur, "f"] & &
\end{tikzcd}$$

$$x \mapsto \begin{bmatrix} y \mapsto \frac{2}{3} \\ y' \mapsto \frac{1}{3} \end{bmatrix} \mapsto \begin{bmatrix} \begin{bmatrix} z \mapsto \frac{1}{2} \\ z' \mapsto \frac{1}{2} \end{bmatrix} \mapsto \frac{2}{3} \\ [z \mapsto 1] \mapsto \frac{1}{3} \end{bmatrix} \mapsto \begin{bmatrix} z \mapsto \frac{1}{3} \\ z' \mapsto \frac{1}{3} \\ z'' \mapsto \frac{1}{3} \end{bmatrix}$$

x

Composition of Kleisli Arrows

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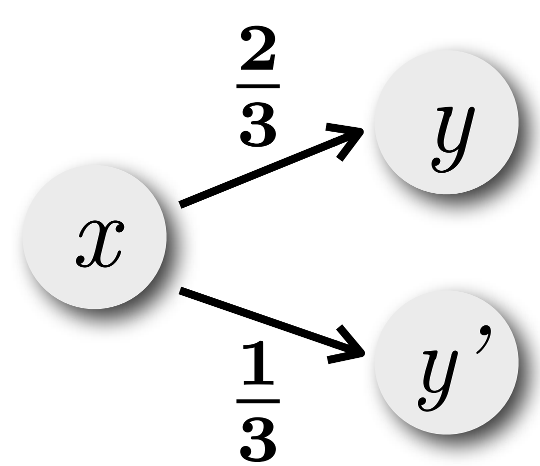
$$x \mapsto \begin{bmatrix} y \mapsto \frac{2}{3} \\ y' \mapsto \frac{1}{3} \end{bmatrix} \mapsto \begin{bmatrix} \begin{bmatrix} z \mapsto \frac{1}{2} \\ z' \mapsto \frac{1}{2} \end{bmatrix} \mapsto \frac{2}{3} \\ [z \mapsto 1] \mapsto \frac{1}{3} \end{bmatrix} \mapsto \begin{bmatrix} z \mapsto \frac{1}{3} \\ z' \mapsto \frac{1}{3} \\ z'' \mapsto \frac{1}{3} \end{bmatrix}$$

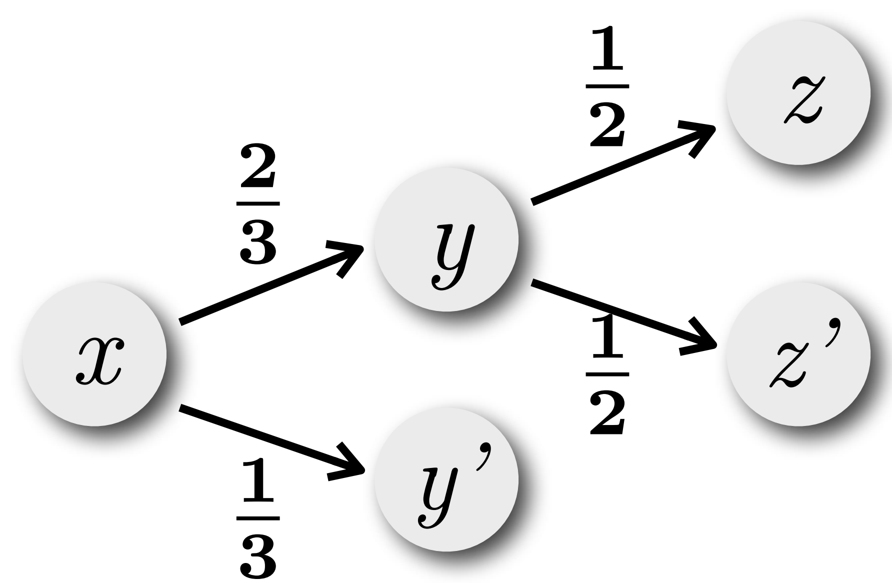


Composition of Kleisli Arrows

$$\frac{X \xrightarrow{f} Y \quad Y \xrightarrow{g} Z}{X \xrightarrow{g \odot f} Z}$$

$$X \xrightarrow{f} \mathcal{D}Y \xrightarrow{\mathcal{D}g} \mathcal{D}\mathcal{D}Z \xrightarrow{\mu} \mathcal{D}Z$$

$$x \mapsto \begin{bmatrix} y \mapsto \frac{2}{3} \\ y' \mapsto \frac{1}{3} \end{bmatrix} \mapsto \begin{bmatrix} \begin{bmatrix} z \mapsto \frac{1}{2} \\ z' \mapsto \frac{1}{2} \end{bmatrix} \mapsto \frac{2}{3} \\ [z \mapsto 1] \mapsto \frac{1}{3} \end{bmatrix} \mapsto \begin{bmatrix} z \mapsto \frac{1}{3} \\ z' \mapsto \frac{1}{3} \\ z'' \mapsto \frac{1}{3} \end{bmatrix}$$

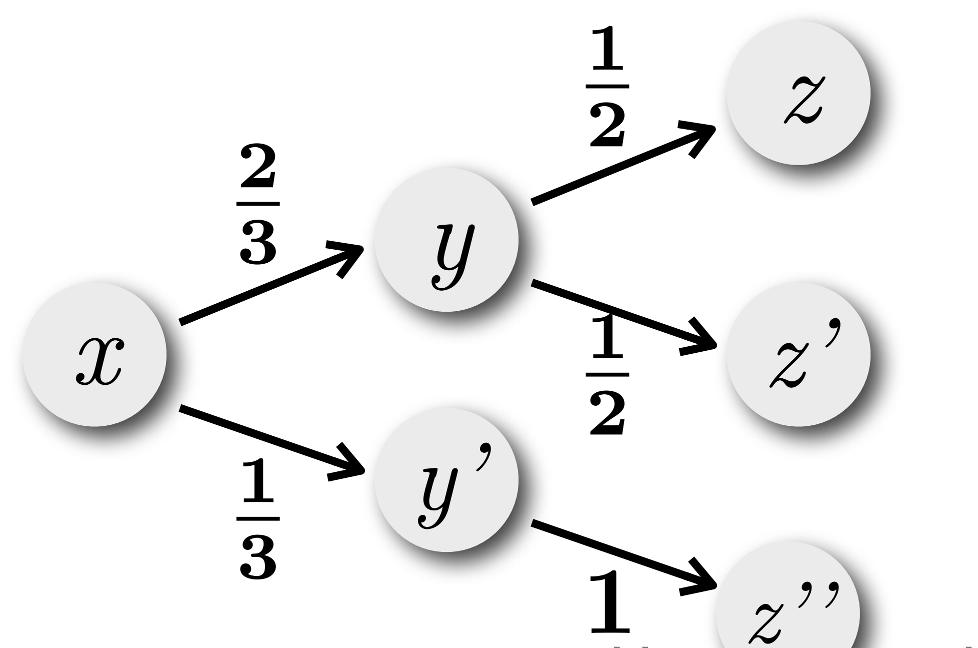


Composition of Kleisli Arrows

$$\frac{X \overset{f}{\nrightarrow} Y \qquad Y \overset{g}{\nrightarrow} Z}{X \overset{g \odot f}{\nrightarrow} Z}$$

$$X \xrightarrow{f} \mathcal{D}Y \xrightarrow{\mathcal{D}g} \mathcal{D}\mathcal{D}Z \xrightarrow{\mu} \mathcal{D}Z$$

$$x \mapsto \begin{bmatrix} y \mapsto \frac{2}{3} \\ y' \mapsto \frac{1}{3} \end{bmatrix} \mapsto \begin{bmatrix} \begin{bmatrix} z \mapsto \frac{1}{2} \\ z' \mapsto \frac{1}{2} \end{bmatrix} \mapsto \frac{2}{3} \\ [z \mapsto 1] \mapsto \frac{1}{3} \end{bmatrix} \mapsto \begin{bmatrix} z \mapsto \frac{1}{3} \\ z' \mapsto \frac{1}{3} \\ z'' \mapsto \frac{1}{3} \end{bmatrix}$$

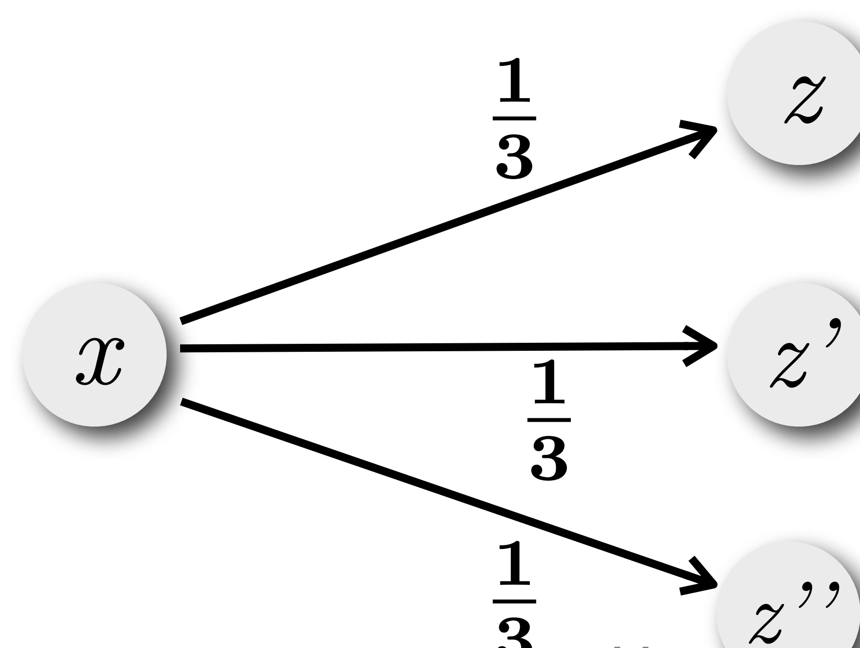


Composition of Kleisli Arrows

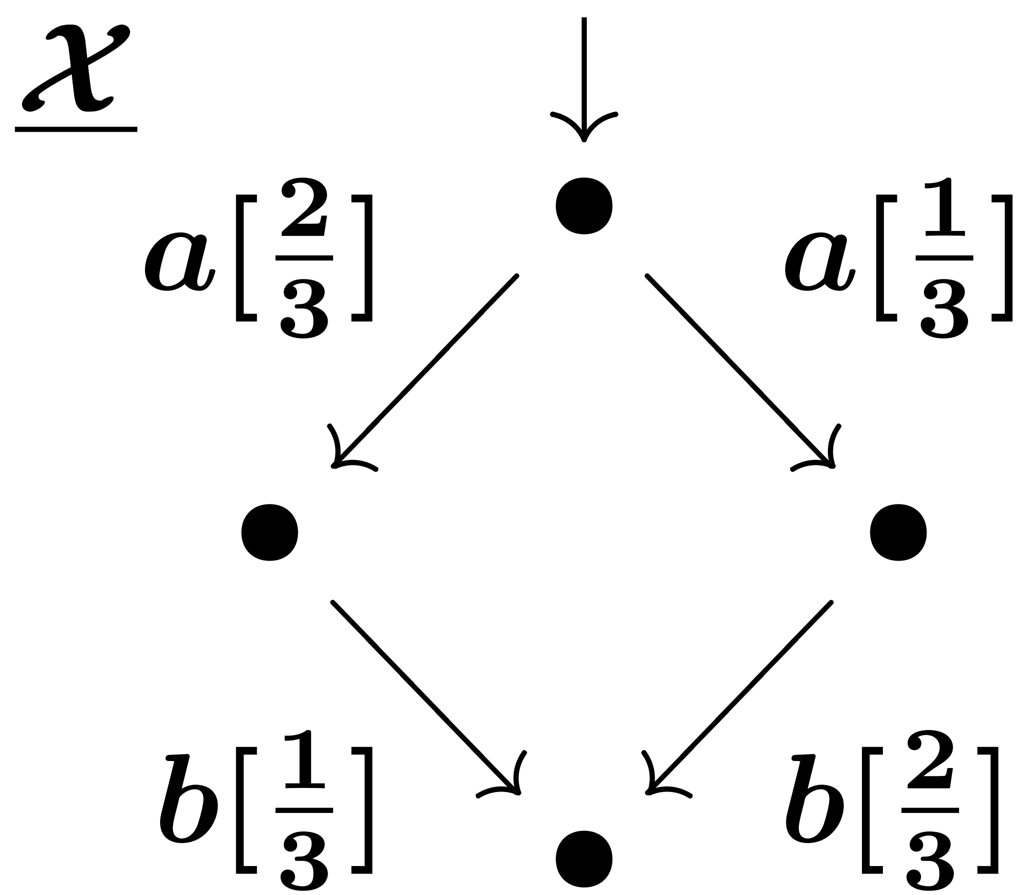
$$\frac{X \xrightarrow{f} Y \qquad Y \xrightarrow{g} Z}{X \xrightarrow{g \odot f} Z}$$

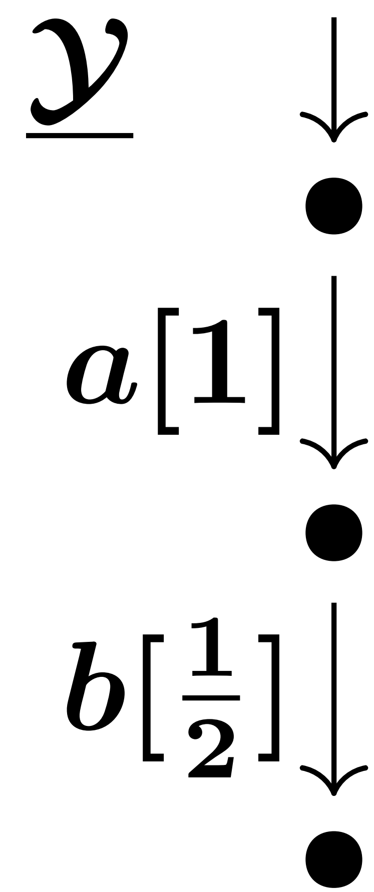
$$X \xrightarrow{f} \mathcal{D}Y \xrightarrow{\mathcal{D}g} \mathcal{D}\mathcal{D}Z \xrightarrow{\mu} \mathcal{D}Z$$

$$x \mapsto \begin{bmatrix} y \mapsto \frac{2}{3} \\ y' \mapsto \frac{1}{3} \end{bmatrix} \mapsto \begin{bmatrix} \begin{bmatrix} z \mapsto \frac{1}{2} \\ z' \mapsto \frac{1}{2} \end{bmatrix} \mapsto \frac{2}{3} \\ [z \mapsto 1] \mapsto \frac{1}{3} \end{bmatrix} \mapsto \begin{bmatrix} z \mapsto \frac{1}{3} \\ z' \mapsto \frac{1}{3} \\ z'' \mapsto \frac{1}{3} \end{bmatrix}$$

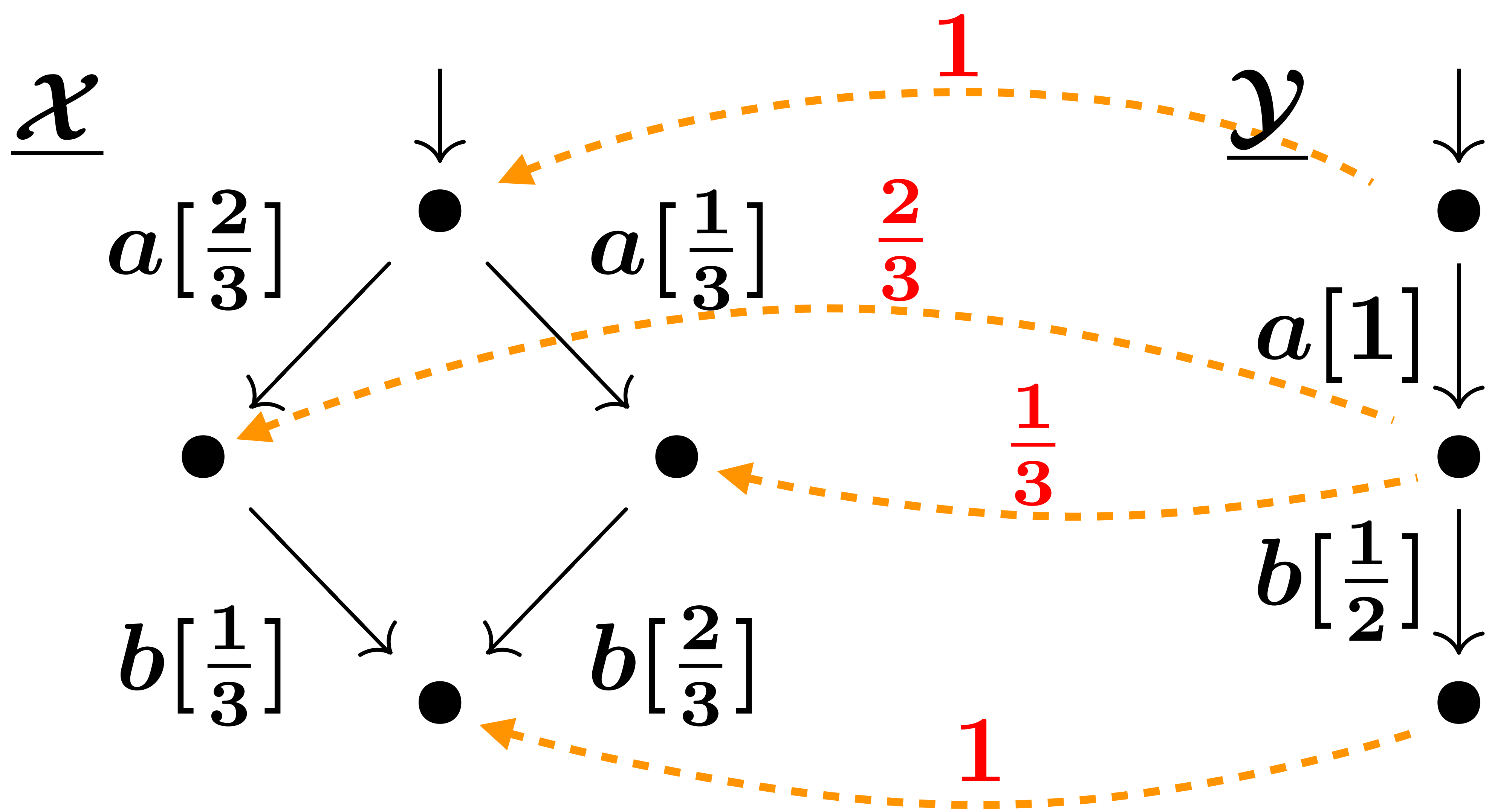


Example


$\underline{\mathcal{X}}$
$a[\frac{2}{3}]$
$a[\frac{1}{3}]$
$b[\frac{1}{3}]$
$b[\frac{2}{3}]$

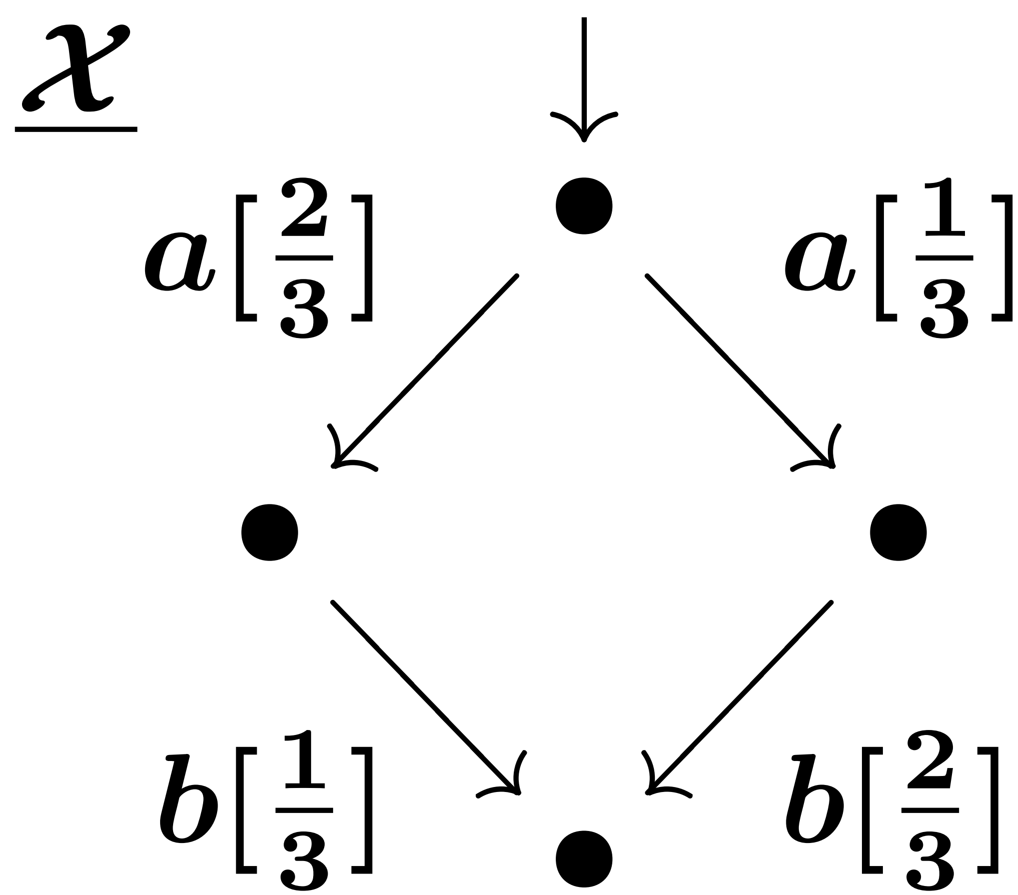

$\underline{\mathcal{Y}}$
$a[1]$
$b[\frac{1}{2}]$

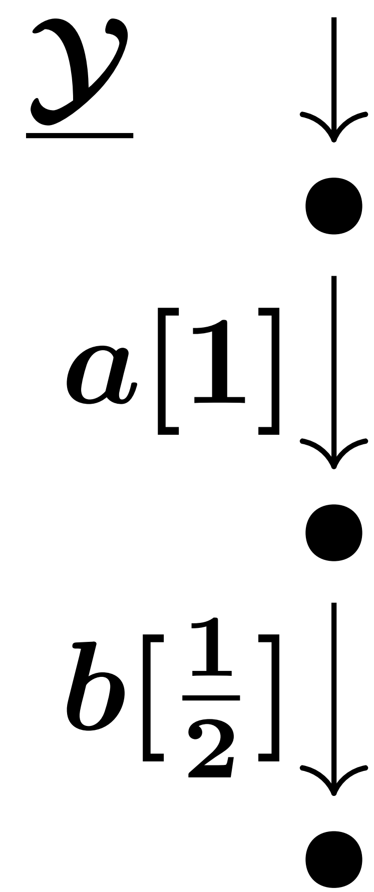
Example



- There is a fwd. Kleisli simulation

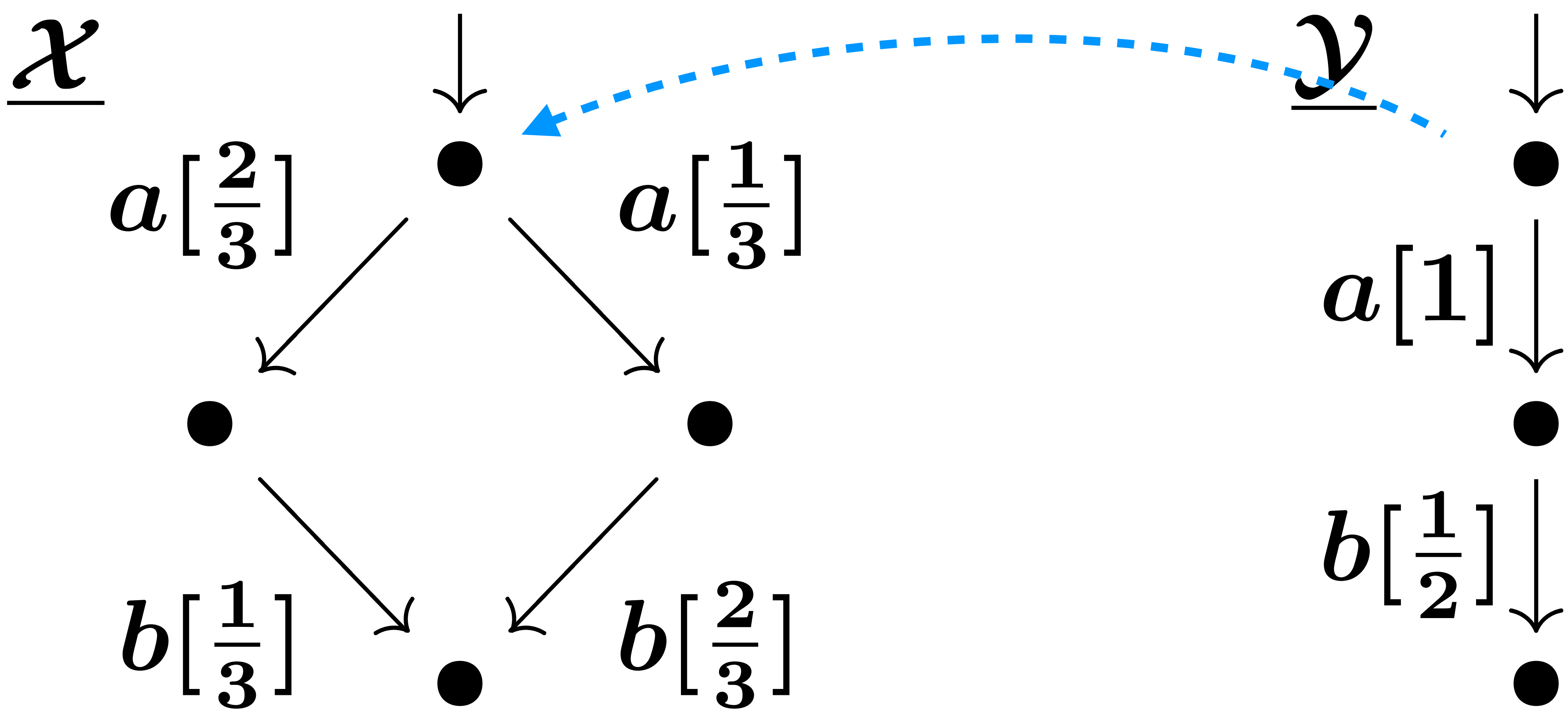
Example





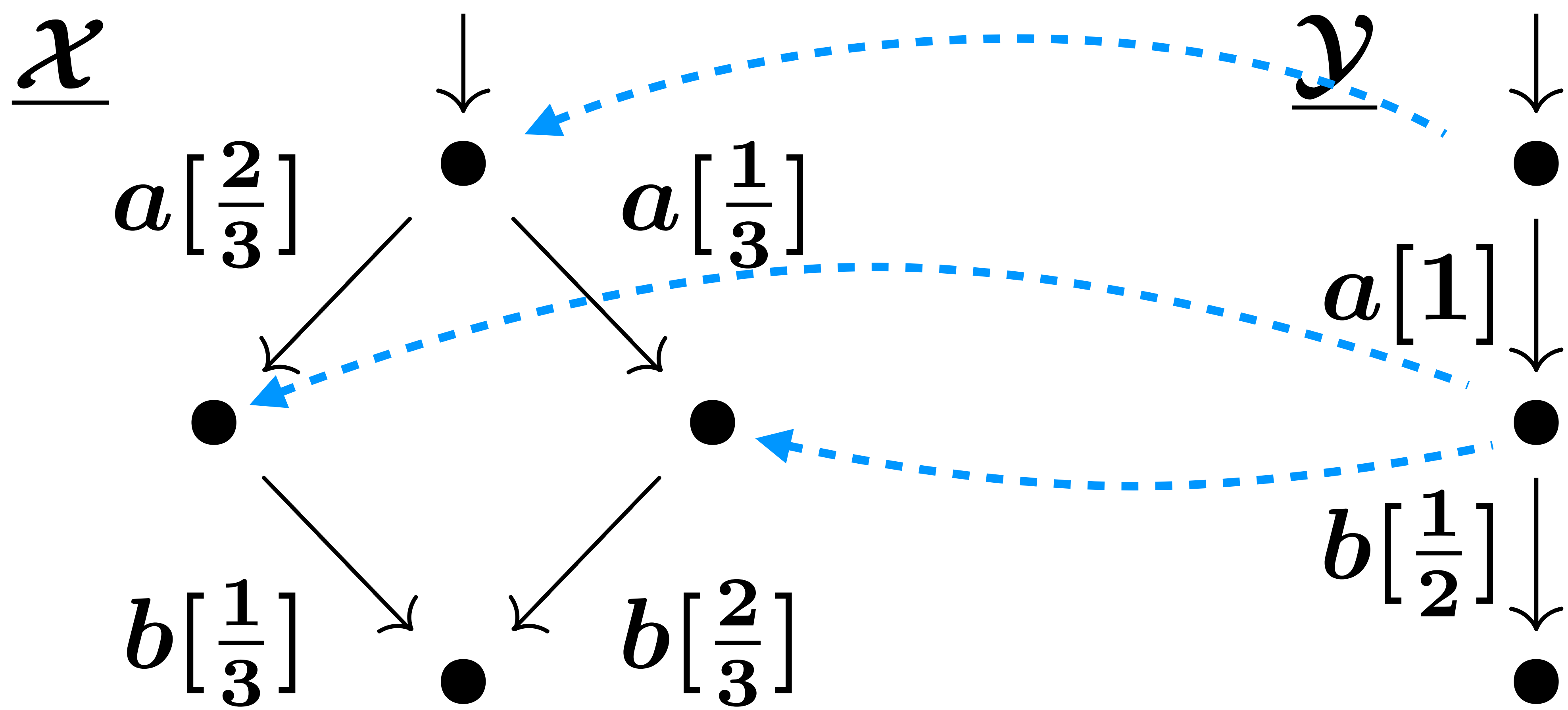
- There is a fwd. Kleisli simulation
- No Jonsson-Larsen simulation!

Example



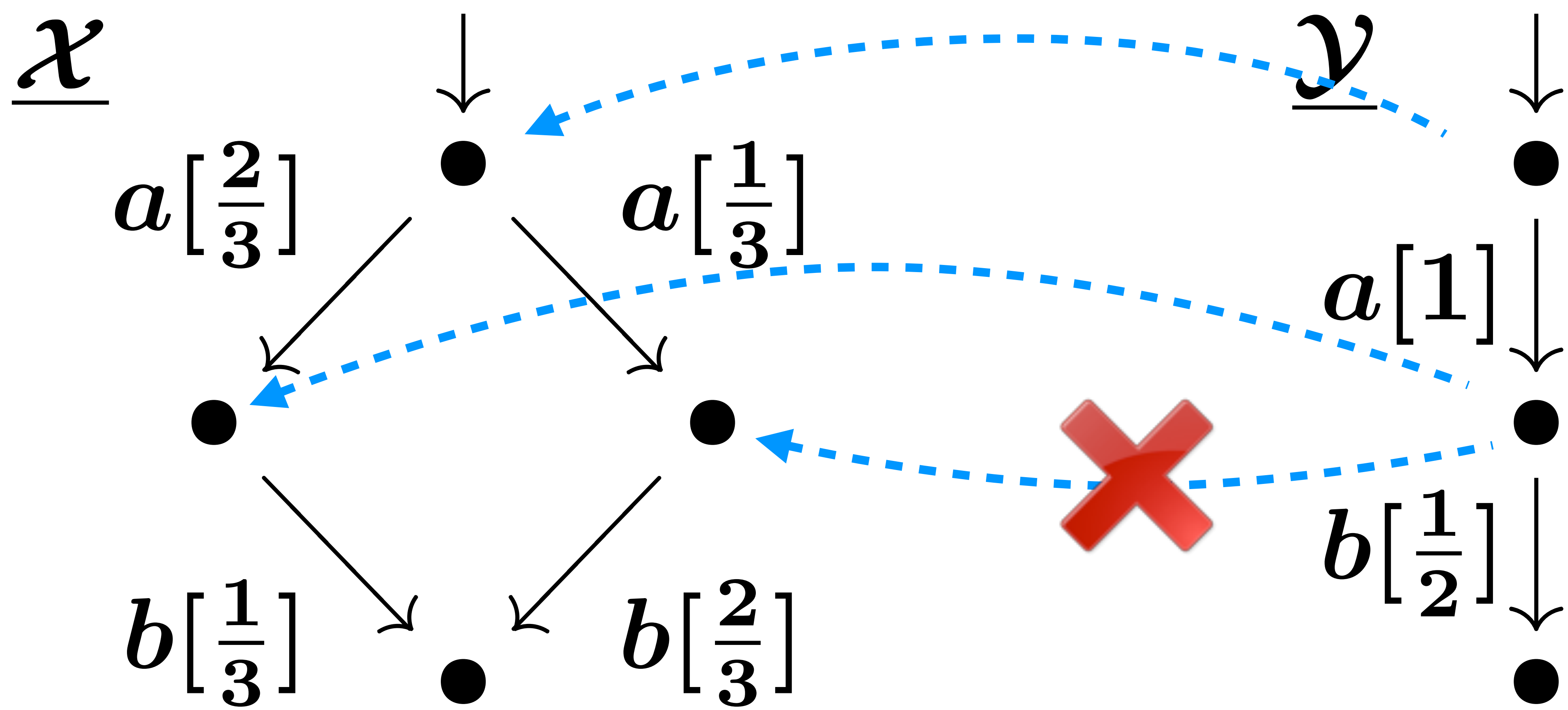
- There is a fwd. Kleisli simulation
- No Jonsson-Larsen simulation!

Example



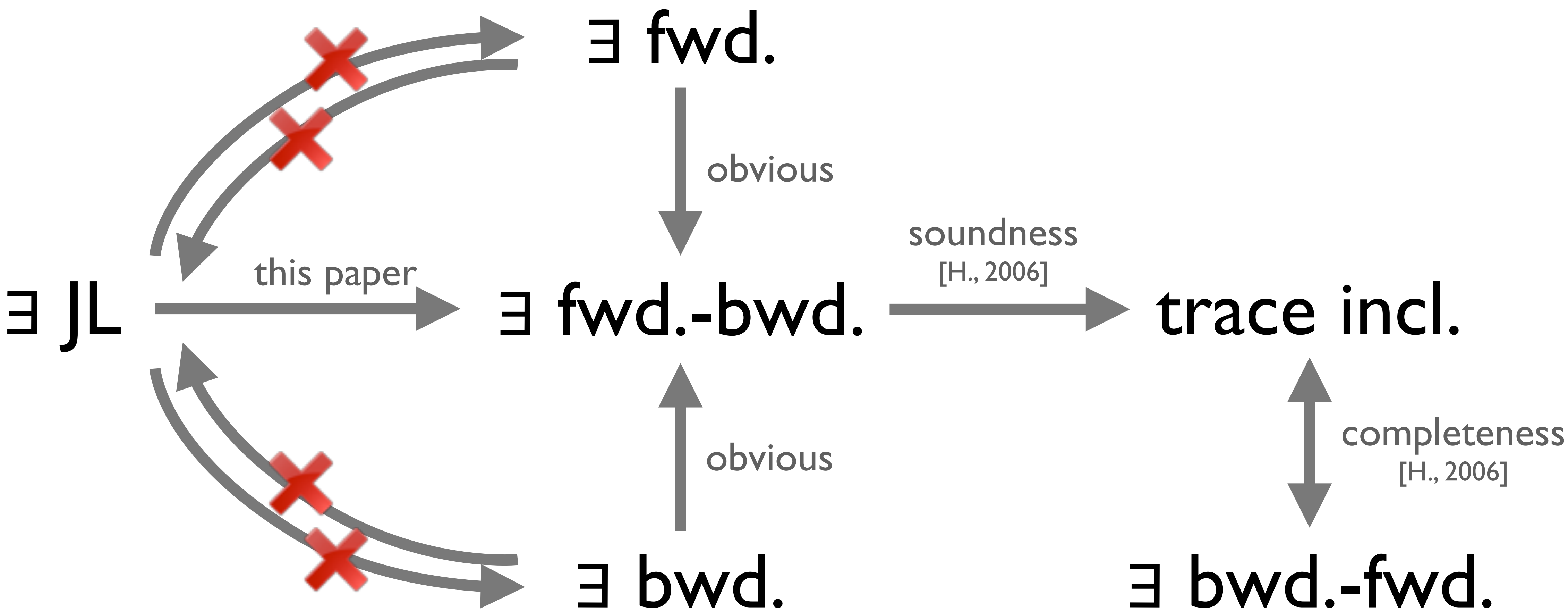
- There is a fwd. Kleisli simulation
- No Jonsson-Larsen simulation!

Example



- There is a fwd. Kleisli simulation
- No Jonsson-Larsen simulation!

Comparison


∃ fwd.
obvious
∃ JL
this paper
∃ fwd.-bwd.
soundness
[H., 2006]
trace incl.
obvious
completeness
[H., 2006]
∃ bwd.
∃ bwd.-fwd.