The Microcosm Principle and Concurrency in Coalgebras

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1-slide review of coalgebra/coinduction

Theory of coalgebras

...eory of state-based systems

in **Sets** : bisimilarity
in Kleisli: trace semantics
[Hasuo,Jacobs,Sokolova LMCS´07]

	categorically	
...stem	coalgebra	$$\begin{tikzcd} FX \\ X \arrow[u] \end{tikzcd}$$
...ehavior- ...reserving map	morphism of coalgebras	$$\begin{tikzcd} FX \arrow[r, "Ff"] & FY \\ X \arrow[u] \arrow[r, "f"'] & Y \arrow[u] \end{tikzcd}$$
behavior	**coinduction** (via final coalgebra)	$$\begin{tikzcd} FX \arrow[r, dashed] & FZ \\ X \arrow[u, "c"] \arrow[r, dashed, "\mathsf{beh}(c)"'] & Z \arrow[u, "\cong"', "\text{final}"] \end{tikzcd}$$

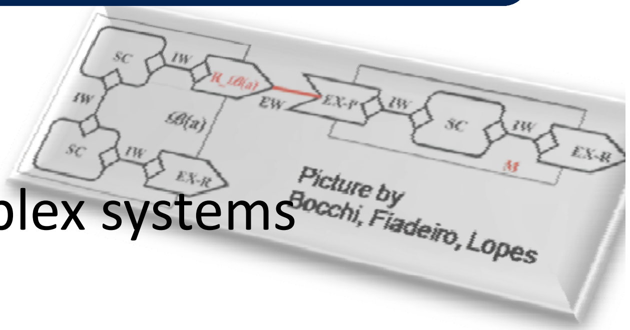
Concurrency

$C \,||\, D$

running *C* and *D* in parallel

is **everywhere**

- computer networks
- multi-core processors
- modular, component-based design of complex systems



is **hard to get right**

- so easy to get into *deadlocks*
- exponentially growing complexity
- cf. Edward Lee. *Making Concurrency Mainstream.*
    - Invited talk at CONCUR 2006.

Compositionality

aids **compositional verification**

Behavior of *C* || *D*
is determined by
behavior of *C* and behavior of *D*

Conventional presentation

$$\mathcal{C}_1 \sim \mathcal{C}_2 \quad \text{and} \quad \mathcal{D}_1 \sim \mathcal{D}_2 \quad \Longrightarrow \quad \mathcal{C}_1 \parallel \mathcal{D}_1 \sim \mathcal{C}_2 \parallel \mathcal{D}_2$$

$\sim$ **behavioral equivalence**

- bisimilarity
- trace equivalence
- ...

„bisimilarity is a congruence“

Compositionality in coalg

operation on

NFAs

$\|$: $\mathbf{Coalg}_F \times \mathbf{Coalg}_F \rightarrow \mathbf{Coalg}_F$

o composing coalgebras/systems

FZ

$\cong \uparrow$ final

eh(c) $\dashrightarrow Z$

- "Co raic compositionality"

$$\mathsf{beh}\left(\begin{array}{c} FX \\ c\uparrow \\ X \end{array} \,\Big\|\, \begin{array}{c} FY \\ d\uparrow \\ Y \end{array}\right) = \mathsf{beh}\left(\begin{array}{c} FX \\ c\uparrow \\ X \end{array}\right) \Big\| \,\mathsf{beh}\left(\begin{array}{c} FY \\ d\uparrow \\ Y \end{array}\right)$$

operation on regular languages

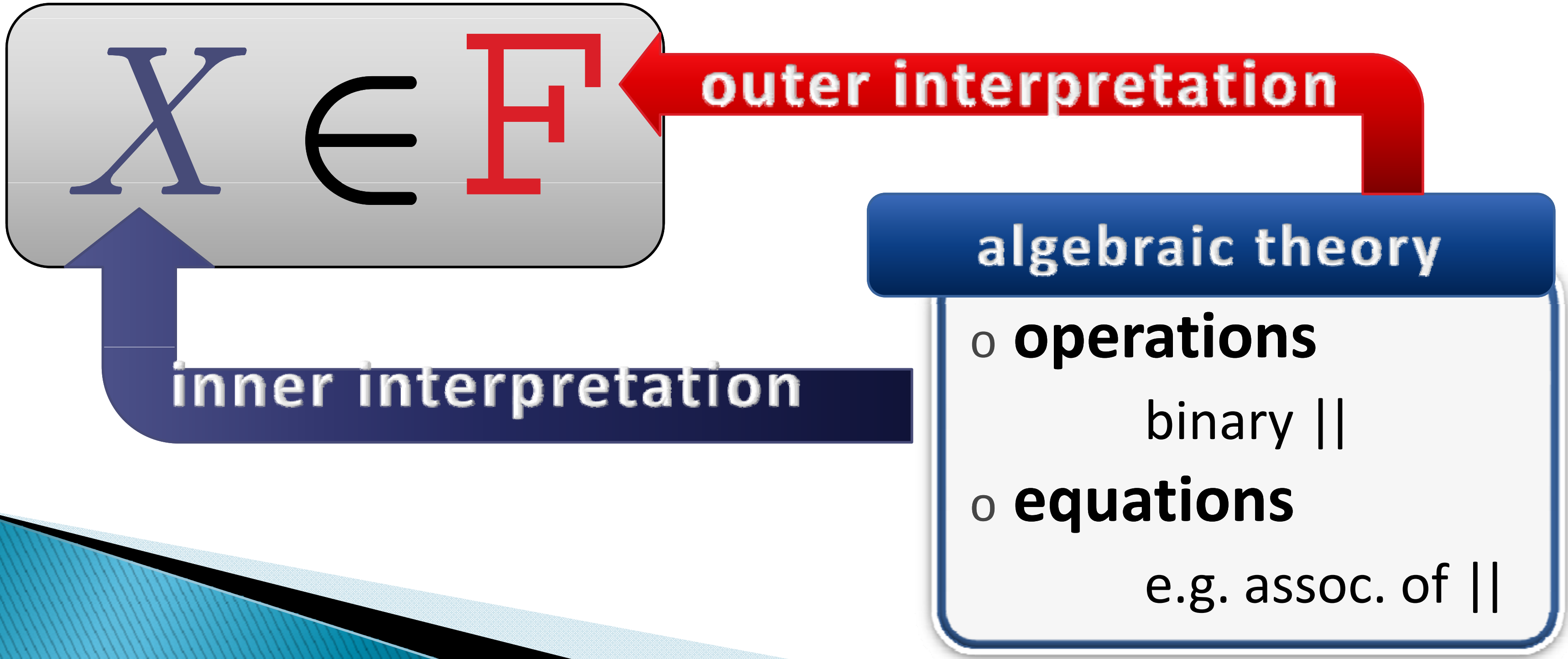
$\|$: $Z \times Z \rightarrow Z$

o composing behavior

Nested algebraic structures: *the microcosm principle*

$$
\begin{array}{ccccc}
\mathrm{Coalg}_F & \times & \mathrm{Coalg}_F & \xrightarrow{\;\|\;} & \mathrm{Coalg}_F \\
Z & \times & Z & \xrightarrow{\;\|\;} & Z
\end{array}
$$

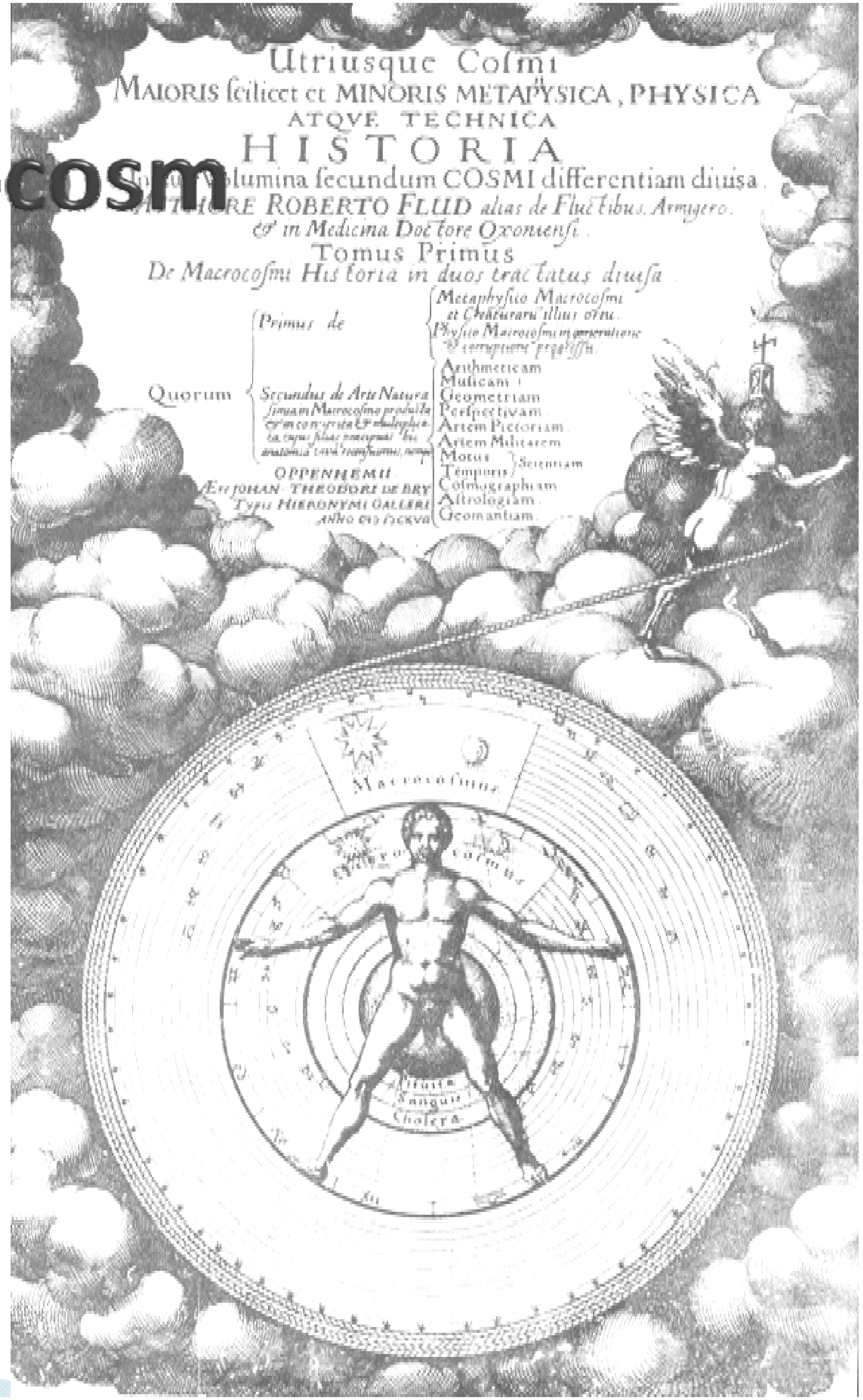
with $\left(\begin{array}{c} FZ \\ \cong\uparrow \text{final} \\ Z \end{array}\right) \in \mathrm{Coalg}_F$



Microcosm in macrocosm

We name this principle the *microcosm principle, after the theory, common in pre-modern correlative cosmologies,* that every feature of the microcosm (e.g. the human soul) corresponds to some feature of the macrocosm.

John Baez & James Dolan
Higher-Dimensional Algebra III:
n-Categories and the Algebra of Opetopes
Adv. Math. 1998



The microcosm principle: you may have seen it

monoid in a monoidal category

monoidal cat. $\mathbb{C}$		monoid $M \in \mathbb{C}$
$\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ $I \in \mathbb{C}$	mult. unit	$M \otimes M \xrightarrow{m} M$ $I \xrightarrow{e} M$
$I \otimes X \cong X \cong X \otimes I$	unit law	$\begin{array}{ccccc} M & \to & M \otimes M & \leftarrow & M \\ & \searrow & & \swarrow & \\ & & M & & \end{array}$
$(X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z)$	assoc. law	$\begin{array}{ccc} M \otimes M \otimes M & \to & M \otimes M \\ M \otimes M & \longrightarrow & M \end{array}$

inner depends on outer

Formalizing the microcosm principle

What do we mean by
"**microcosm principle**"?
i.e. mathematical definition of such nested models?

Answer

inner model
as lax natural trans.

$\mathbb{L}$

algebraic theory
as *Lawvere theory*

outer model
as prod.-pres. functor

Outline

concurrency/ compositionali

generic compositionality theorem

for arbitrary algebraic theory

microcosm for concurrency
(|| and ||)

parallel composition
via **sync** nat. trans.

2-categorical formulation

$$\begin{tikzcd} \mathbb{L} \arrow[rr, bend left, "\mathbf{1}"] \arrow[rr, "\mathbb{C}"'] \arrow[rr, phantom, bend left=20, "\Downarrow X"] & & \mathbf{CAT} \end{tikzcd}$$

Parallel composition of coalgebras

via

sync$_{X,Y}$: $FX \otimes FY \rightarrow F(X \otimes Y)$

Part 1

Parallel

bifunctor $\mathbf{Coalg}_F \times \mathbf{Coalg}_F \rightarrow \mathbf{Coalg}_F$

usually denoted by $\otimes$ (tensor)

$$\mathrm{beh}\left(\begin{pmatrix} FX \\ \uparrow c \\ X \end{pmatrix} \otimes \begin{pmatrix} FY \\ \uparrow d \\ Y \end{pmatrix}\right) = \mathrm{beh}\left(\begin{pmatrix} FX \\ \uparrow c \\ X \end{pmatrix}\right) \,\|\, \mathrm{beh}\left(\begin{pmatrix} FY \\ \uparrow d \\ Y \end{pmatrix}\right)$$

Theorem

- If
  - the base category $\mathcal{C}$ has a tensor

    $$\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$$

  - and $F : \mathcal{C} \rightarrow \mathcal{C}$ comes with natural transformation

    $$\mathbf{sync}_{X,Y} : FX \otimes FY \rightarrow F(X \otimes Y)$$

- then we have

  $$\otimes : \mathbf{Coalg}_F \times \mathbf{Coalg}_F \rightarrow \mathbf{Coalg}_F$$

$\otimes$: $\mathbf{Coalg}_F \times \mathbf{Coalg}_F \rightarrow \mathbf{Coalg}_F$

F with sync

lifting

$\otimes$: $\mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C}$

Parallel composition via sync

$$\begin{pmatrix} FX \\ c \uparrow \\ X \end{pmatrix} \otimes \begin{pmatrix} FY \\ d \uparrow \\ Y \end{pmatrix} = \begin{array}{c} F(X \otimes Y) \\ \uparrow \mathsf{sync}_{X,Y} \\ FX \otimes FY \\ \uparrow c \otimes d \\ X \otimes Y \end{array}$$

$\otimes$ on base category

different **sync** → different $\otimes$

⊗ : Coalg_F x Coalg_F → Coalg_F

Examples of sync : $FX \otimes FY \to F(X \otimes Y)$

F with sync

lifting

x : Sets x Sets → Sets

- **CSP-style** (Hoare) $$a.P \parallel a.Q \xrightarrow{a} P \parallel Q$$

$$\begin{array}{ccc} \mathcal{P}_{\mathrm{fin.}}(\Sigma \times X) \times \mathcal{P}_{\mathrm{fin.}}(\Sigma \times Y) & \xrightarrow{\mathrm{sync}_{X,Y}} & \mathcal{P}_{\mathrm{fin.}}(\Sigma \times (X \times Y)) \\ (S,T) & \longmapsto & \{\, (a,(x,y)) \mid (a,x) \in S \wedge (a,y) \in T \,\} \end{array}$$

- **CCS-style** (Milner) $$a.P \parallel \overline{a}.Q \xrightarrow{\tau} P \parallel Q$$
  Assuming $\Sigma = \{a, a', \dots\} + \{\overline{a}, \overline{a'}, \dots\} + \{\tau\}$

$$\begin{array}{ccc} \mathcal{P}_{\mathrm{fin.}}(\Sigma \times X) \times \mathcal{P}_{\mathrm{fin.}}(\Sigma \times Y) & \xrightarrow{\mathrm{sync}_{X,Y}} & \mathcal{P}_{\mathrm{fin.}}(\Sigma \times (X \times Y)) \\ (S,T) & \longmapsto & \{\, (\tau,(x,y)) \mid (a,x) \in S \wedge (\overline{a},y) \in T \,\} \end{array}$$

***C* = Sets, *F* = $P_{\mathrm{fin}}(\Sigma$ x _)**

F-coalgebra = LTS

Inner composition

$$\mathrm{beh}\left(\begin{matrix} FX \\ c\big\uparrow \\ X \end{matrix} \otimes \begin{matrix} FY \\ d\big\uparrow \\ Y \end{matrix}\right) = \mathrm{beh}\left(\begin{matrix} FX \\ c\big\uparrow \\ X \end{matrix}\right) \parallel \mathrm{beh}\left(\begin{matrix} FY \\ d\big\uparrow \\ Y \end{matrix}\right)$$

- $\parallel$ "composition of states/*behavior*"

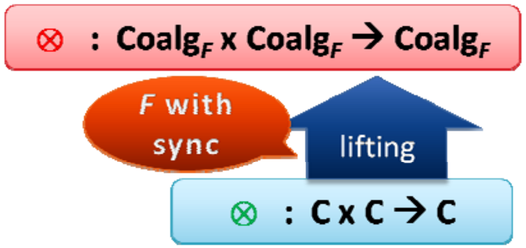
arises by **coinduction**

$$\begin{tikzcd} F(Z\otimes Z) \arrow[r, dashed] & FZ \\ Z\otimes Z \arrow[u, "\zeta\otimes\zeta"] \arrow[r, dashed, "\parallel"'] & Z \arrow[u, "\zeta"', "\text{final}"] \end{tikzcd}$$

Compositionality theorem

Assume
- $\mathcal{C}$ has tensor $\otimes$
- F has $\mathbf{sync}_{X,Y} : FX \otimes FY \to F(X \otimes Y)$
- there is a final coalgebra $Z \to FZ$

1. $\otimes$ by

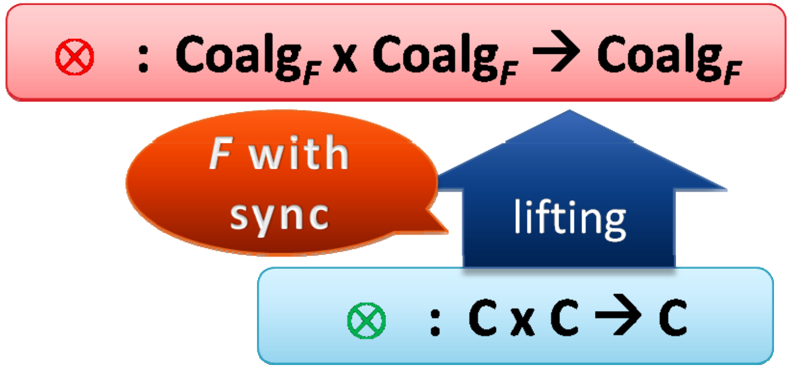


2. $||$ by

$$
\begin{tikzcd}
F(Z \otimes Z) \arrow[r, dashed] & FZ \\
Z \otimes Z \arrow[u, "\zeta \otimes \zeta"] \arrow[r, dashed, "||"'] & Z \arrow[u, "\zeta\ \text{final}"']
\end{tikzcd}
$$

3.
$$
\mathbf{beh}\left(\begin{matrix} FX \\ c\uparrow \\ X \end{matrix} \otimes \begin{matrix} FY \\ d\uparrow \\ Y \end{matrix}\right) = \mathbf{beh}\left(\begin{matrix} FX \\ c\uparrow \\ X \end{matrix}\right) \, \| \, \mathbf{beh}\left(\begin{matrix} FY \\ d\uparrow \\ Y \end{matrix}\right)
$$

- by finality
- yields: $\mathcal{C}_1 \sim \mathcal{C}_2 \quad \text{and} \quad \mathcal{D}_1 \sim \mathcal{D}_2 \quad \Longrightarrow \quad \mathcal{C}_1 \| \mathcal{D}_1 \sim \mathcal{C}_2 \| \mathcal{D}_2$



Equational properties

- When is

$$\otimes \;:\; \mathrm{Coalg}_F \times \mathrm{Coalg}_F \rightarrow \mathrm{Coalg}_F$$

  associative?

- **Answ**

commutativity?

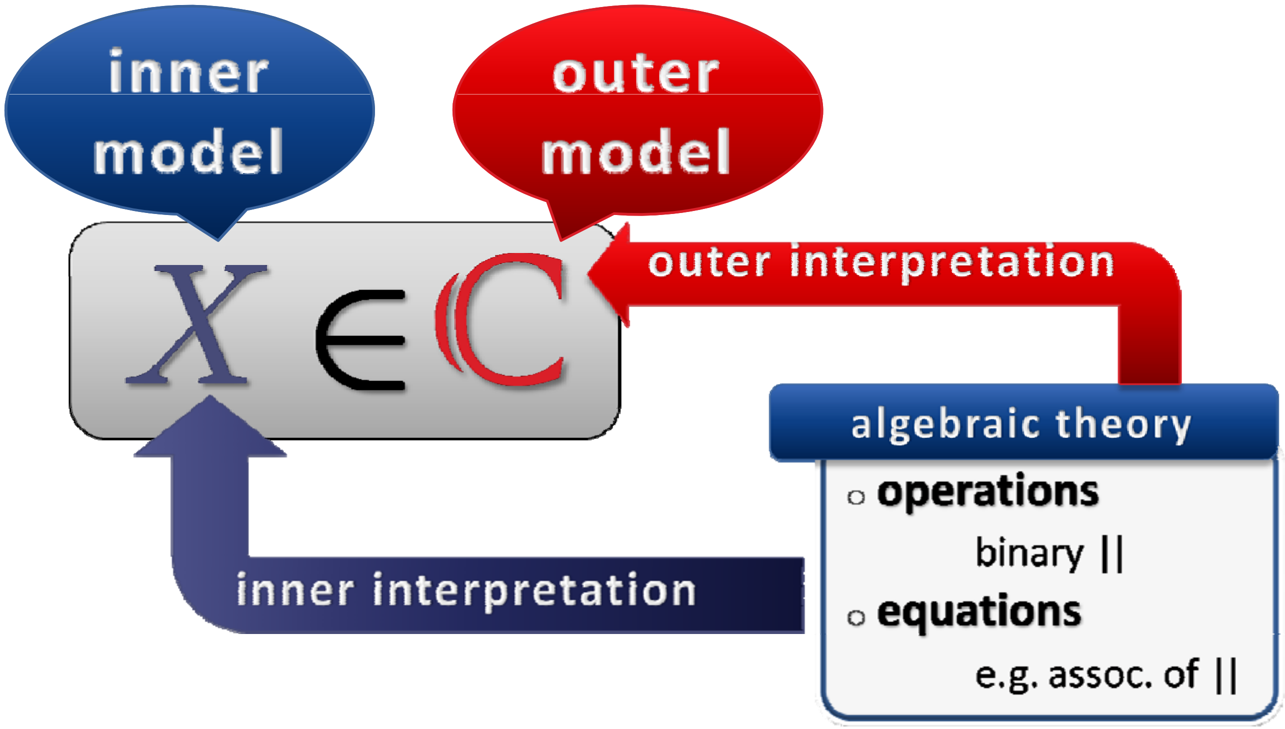
arbitrary algebraic theory?

for arbitrary algebraic theory

2-categorical formulation of the microcosm principle

Part 2

Microcosm principle (Baez & Dolan)



examples

- **monoid** in **monoidal category**
- **final coalg.** in **Coalg$_F$** with $\otimes$
- **reg. lang.** vs. **NFAs**

What is precisely "microcosm principle"?

i.e. mathematical definition of such nested models?

Lawvere theory $\mathbb{L}$

a category representing an algebraic theory

Definition

A **Lawvere theory** $\mathbb{N}$ is a small category s.t.

- ***L***'s objects are natural numbers
- ***L*** has finite products

Lawvere theory

other arrows:

- projections $2 \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} 1$
- composed terms $3 \xrightarrow{\mathbf{m}(\mathbf{m}(\pi_1, \pi_2), \pi_3)} 1$

algebraic theory	as category $\mathbb{L}$
operations **m** (binary) **e** (nullary)	as **arrows** $2 \xrightarrow{\mathbf{m}} 1$ $\quad$ $0 \xrightarrow{\mathbf{e}} 1$
equations assoc. of **m** unit law	as **commuting diagrams**

$$
\begin{tikzcd}
3 \arrow[r, "\mathbf{m} \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times \mathbf{m}"'] & 2 \arrow[d, "\mathbf{m}"] \\
2 \arrow[r, "\mathbf{m}"'] & 1
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[r, "\langle e, \mathrm{id} \rangle"] \arrow[dr, "\mathrm{id}"'] & 2 \arrow[d, "\mathbf{m}"] & 1 \arrow[l, "\langle \mathrm{id}, e \rangle"'] \arrow[dl, "\mathrm{id}"] \\
& 1 &
\end{tikzcd}
$$

Models for Lawvere theory $\mathbb{L}$

Standard: set-theoretic model

- a set with $\mathbb{N}$-structure ➔ **N-set**

$$\mathbb{L} \xrightarrow{X} \mathbf{Sets}$$

(product-preserving)

$$\begin{array}{ccc} \mathbf{2} & & X^2 \\ \downarrow \mathbf{m} & \longmapsto & \downarrow [\mathbf{m}] \\ \mathbf{1} & & X \end{array}$$

binary opr. on X

what about nested models?

$X \in \mathbb{F}$

Outer model: $\mathbb{L}$-category

outer model

- a **category** with N-structure ➔ **N-category**

$$\mathbb{L} \xrightarrow{\mathbb{C}} \mathbf{Cat} \qquad \text{(product-preserving)}$$

$$\begin{array}{c} \mathbf{2} \\ \downarrow \mathbf{m} \\ \mathbf{1} \end{array} \quad \longmapsto \quad \begin{array}{c} \mathbb{C}^{\mathbf{2}} \\ \downarrow [\mathbf{m}] = \otimes \\ \mathbb{C} \end{array}$$

NB. our focus is on **strict** alg. structures

Standard: set-theoretic model

- a set with $\mathbb{L}$-structure ➔ **$\mathbb{L}$-set**

$$\mathbb{L} \xrightarrow{X} \mathrm{Sets} \qquad \text{product-preserving}$$

$$\begin{array}{c} \mathbf{2} \\ \downarrow \mathbf{m} \\ \mathbf{1} \end{array} \quad \longmapsto \quad \begin{array}{c} X^{\mathbf{2}} \\ \downarrow [\mathbf{m}] \\ X \end{array}$$

binary opr. on X

Inner model: $\mathbb{L}$-object

Definition

Given an $\mathbb{N}$-category $\mathbb{E}$.

an **$\mathbb{N}$-object** X in it

is a lax natural transformation compatible with products.

inner alg. str.
by
mediating 2-cells

components

$$X_0 : 1 \xrightarrow{\;!\;} 1$$

$$X_1 : 1 \xrightarrow{\;X\;} \mathbb{C}$$

$$X_2 : 1 \xrightarrow{\;(X,X)\;} \mathbb{C}^2$$

$$\vdots$$

X: carrier obj.

$$\frac{X \in \mathbb{C}}{1 \xrightarrow{X} \mathbb{C}}$$

lax naturality

$\boxed{\text{In } \mathbb{L}}$

$$\begin{tikzcd} 2 \arrow[d, "m"] \\ 1 \end{tikzcd}$$

$\boxed{\text{In Cat}}$

$$\begin{tikzcd} 1 \arrow[r, "{(X,X)}"] \arrow[d, equal] & \mathbb{C}^2 \arrow[d, "\otimes"] \\ 1 \arrow[r, "X"'] & \mathbb{C} \end{tikzcd}$$

$$\frac{\qquad\qquad}{X \otimes X \xrightarrow{X_m} X \quad \text{in } \mathbb{C}}$$

lax $\mathbb{L}$-functor
= F with sync

- $\mathbb{E}$: $\mathbb{N}$-category
- F: $\mathbb{E}$ → $\mathbb{E}$, lax $\mathbb{N}$-functor

→ $\mathbf{Coalg}_F$ is an $\mathbb{N}$-category

lax L-functor?

$$\begin{tikzcd} \mathbf{L} \arrow[r, bend left, "\mathbb{C}"] \arrow[r, bend right] & \mathbf{Cat} \end{tikzcd}$$

lax natur. trans.

Generalizes

$\otimes$: $\mathbf{Coalg}_F$ x $\mathbf{Coalg}_F$ → $\mathbf{Coalg}_F$

F with sync

lifting

$\otimes$: C x C → C

sync$_{X,Y}$

$\mathbb{C}$

- $\mathbb{E}$: $\mathbb{N}$-category
- $Z \in \mathbb{E}$, final object

→ Z is an $\mathbb{N}$-object

⋮

Generalizes

$$\begin{tikzcd} F(Z \otimes Z) \arrow[rr, dashed] & & FZ \\ Z \otimes Z \arrow[u, "\zeta \otimes \zeta"] \arrow[rr, dashed, "\|"'] & & Z \arrow[u, "\zeta", "\text{final}"'] \end{tikzcd}$$

Generic compositionality theorem

Assume

- ***C*** is an ***L***-category
- *F* : ***C*** → ***C*** is a lax ***L***-functor
- there is a final coalgebra *Z* → *FZ*

1. $\mathbf{Coalg}_F$ is an $\mathbb{N}$-category
2. $Z \to FZ$ is an $\mathbb{N}$-object
3. the ***behavior*** functor

$$\mathrm{Coalg}_F \xrightarrow{\mathbf{beh}} \mathbb{C}/Z$$

$$\begin{pmatrix} FX \\ c\uparrow \\ X \end{pmatrix} \longmapsto \left(X \xrightarrow{\mathrm{beh}(c)} Z\right)$$

by coinduction

$$\begin{tikzcd} FX \arrow[r, dashed] & FZ \\ X \arrow[u, "c"] \arrow[r, dashed, "\mathrm{beh}(c)"'] & Z \arrow[u, "\cong"', "\mathrm{final}"] \end{tikzcd}$$

is a (strict) $\mathbb{N}$-functor

subsumes $\mathrm{beh}\begin{pmatrix} FX \\ c\uparrow \\ X \end{pmatrix} \bullet \begin{pmatrix} FY \\ d\uparrow \\ Y \end{pmatrix} = \mathrm{beh}\begin{pmatrix} FX \\ c\uparrow \\ X \end{pmatrix} \mid \mathrm{beh}\begin{pmatrix} FY \\ d\uparrow \\ Y \end{pmatrix}$

Equational properties

associative

$\otimes$: $\mathrm{Coalg}_F \times \mathrm{Coalg}_F \rightarrow \mathrm{Coalg}_F$

lifting

associative

$\otimes$: $C \times C \rightarrow C$

F with "associative" sync

$$\begin{tikzcd}
FX \otimes (FY \otimes FZ) \arrow[r, "FX \otimes \mathsf{sync}"] \arrow[d, "\mathrm{id}"] & FX \otimes F(Y \otimes Z) \arrow[r, "\mathsf{sync}"] & F(X \otimes (Y \otimes Z)) \arrow[d, "\mathrm{id}"] \\
(FX \otimes FY) \otimes FZ \arrow[r, "\mathsf{sync} \otimes FZ"'] & F(X \otimes Y) \otimes FZ \arrow[r, "\mathsf{sync}"'] & F((X \otimes Y) \otimes Z)
\end{tikzcd}$$

Equational properties, generally

- equations are built-in in ***L***
  - as
$$\begin{tikzcd} 3 \arrow[r, "\mathbf{m} \times \mathbf{id}"] \arrow[d, "\mathbf{id} \times \mathbf{m}"'] & 2 \arrow[d, "\mathbf{m}"] \\ 2 \arrow[r, "\mathbf{m}"'] & 1 \end{tikzcd}$$
- how about „assoc“ of **sync**?
$$\begin{tikzcd} FX \otimes (FY \otimes FZ) \arrow[r, "FX \otimes \mathsf{sync}"] \arrow[d, "\mathrm{id}"] & FX \otimes F(Y \otimes Z) \arrow[r, "\mathsf{sync}"] & F(X \otimes (Y \otimes Z)) \arrow[d, "\mathrm{id}"] \\ (FX \otimes FY) \otimes FZ \arrow[r, "\mathsf{sync} \otimes FZ"'] & F(X \otimes Y) \otimes FZ \arrow[r, "\mathsf{sync}"'] & F((X \otimes Y) \otimes Z) \end{tikzcd}$$
  - automatic
    via “coherence condition“
$$\begin{tikzcd} 1 \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\swarrow X_{\mathsf{b \circ a}}"] & \mathbb{C}^l \arrow[d, "[\![\mathsf{b \circ a}]\!]"] \\ 1 \arrow[r] & \mathbb{C}^n \end{tikzcd} = \begin{tikzcd} 1 \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\swarrow X_{\mathsf{a}}"] & \mathbb{C}^l \arrow[d, "[\![\mathsf{a}]\!]"] \\ 1 \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\swarrow X_{\mathsf{b}}"] & \mathbb{C}^k \arrow[d, "[\![\mathsf{b}]\!]"] \\ 1 \arrow[r] & \mathbb{C}^n \end{tikzcd}$$

***L*-structure**
on Coalg$_F$

***F* : lax *L*-functor**

lifting

<u>in Mon</u>
$$\begin{tikzcd} & 3 \arrow[dl, "\mathsf{id \times m}"'] \arrow[dr, "\mathsf{m \times id}"] & \\ 2 \arrow[dr, "\mathsf{m}"'] & & 2 \arrow[dl, "\mathsf{m}"] \\ & 1 & \end{tikzcd}$$

<u>in CAT</u>
$$\begin{tikzcd} 1 \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\swarrow X_{\mathsf{id \times m}}"] & \mathbb{C}^3 \arrow[d, "[\![\mathsf{id \times m}]\!]"] \\ 1 \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\swarrow X_{\mathsf{m}}"] & \mathbb{C}^2 \arrow[d, "[\![\mathsf{m}]\!]"] \\ 1 \arrow[r] & \mathbb{C} \end{tikzcd} \overset{(*)}{=} \begin{tikzcd} 1 \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\swarrow X_{\mathsf{m \circ (id \times m)}} = X_{\mathsf{m \circ (m \times id)}}"] & \mathbb{C}^3 \arrow[d] \\ 1 \arrow[r] & \mathbb{C}^1 \end{tikzcd} \overset{(*)}{=} \begin{tikzcd} 1 \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\swarrow X_{\mathsf{m \times id}}"] & \mathbb{C}^3 \arrow[d, "[\![\mathsf{m \times id}]\!]"] \\ 1 \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\swarrow X_{\mathsf{m}}"] & \mathbb{C}^2 \arrow[d, "[\![\mathsf{m}]\!]"] \\ 1 \arrow[r] & \mathbb{C} \end{tikzcd}$$

***L*-structure**
on C

Related work: bialgebras

- Related to the study of ***bialgebraic structures***
  [Turi-Plotkin, Bartels, Klin, …]
  - Algebraic structures on coalgebras

- In the current work:
  - ***Equations***, not only *operations*, are also an integral part
  - Algebraic structures are ***nested***, ***higher-dimensional***

Future work

- "Pseudo" algebraic structures
  - **monoidal** category (cf. **strictly monoidal** category)
  - equations hold **up-to-isomorphism**
  - ***L*** **→ CAT**, product-preserving **pseudo**-functor?

- Microcosm principle for full GSOS

bialgebra	microcosm
$\Sigma B \Rightarrow B \Sigma$	current work
$\Sigma (B \times \mathrm{id}) \Rightarrow B T_\Sigma$ (for full GSOS)	??

Conclusion

concurrency/
compositionali

for arbitrary algebraic theory

generic compositionality theorem

microcosm for concurrency
($\|$ and $\|$)

parallel composition
via **sync** nat. trans.

2-categorical formulation

$$
\begin{tikzcd}
\mathbb{L} \arrow[rr, bend left, "\mathbf{1}"] \arrow[rr, "\mathbb{C}"'] \arrow[rr, phantom, bend left=20, "\Downarrow X"] & & \mathbf{CAT}
\end{tikzcd}
$$

Thanks for your attention!

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Conclusion

- **<u>Microcosm principle</u>** :
  same algebraic structure
  - on a **category** ***C*** and
  - on an **object** $X \in C$

- 2-categorical formulation:

inner model

$$
\begin{tikzcd}
\mathbb{L} \arrow[r, bend left=60, "\mathbf{1}"{name=U}] \arrow[r, "\mathbb{C}"'{name=D}] \arrow[Rightarrow, from=U, to=D, phantom, "\Downarrow X"] & \mathbf{CAT}
\end{tikzcd}
$$

algebraic theory

outer model

- ***Concurrency in coalgebras*** as a CS example

Thank you for your attention!

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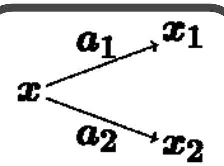
Behavior by coinduction: example

Take $F = \mathcal{P}_{\mathrm{fin}}(\Sigma \times \_)$ in **Sets.**

- System as coalgebra:

$$\begin{array}{ccc} \mathcal{P}_{\mathrm{fin}}(\Sigma \times X) & \qquad & \{(a_1, x_1),\ (a_2, x_2)\} \\ {\scriptstyle c}\uparrow & & \uparrow \\ X & & x \end{array}$$



- Behavior by coinduction:

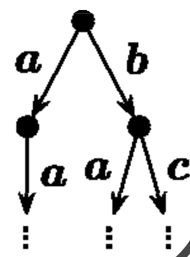
$$\begin{tikzcd} FX \arrow[r, dashed] & FZ \\ X \arrow[u, "c"] \arrow[r, dashed, "\mathsf{beh}(c)"'] & Z \arrow[u, "\cong"', "\text{final}"] \end{tikzcd}$$

$x \longmapsto$ (process graph of x)

the set of
- finitely branching
- edges labeled with Σ
- infinite-depth

trees,

such as



  - in **Sets:** ***bisimilarity***
  - in certain Kleisli categories: ***trace equivalence***

    [Ha… Jacobs,Sokolova,CMCS'06]

Examples of sync : $FX \otimes FY \rightarrow F(X \otimes Y)$

- **Note:**

  Asynchronous/interleaving compositions don't fit in this framework

  - such as

    $$a.P \parallel Q \xrightarrow{a} P \parallel Q$$

  - We have to use, instead of F, the ***cofree comonad*** on F

Lawvere theory

- Presentation of an algebraic theory as a category:
  - <u>objects</u>: 0, 1, 2, 3, … “***arities***”
  - <u>arrows</u>: “***terms*** (in a context)”

$$\begin{tikzcd} 2 \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & 1 \end{tikzcd} \qquad \begin{tikzcd} 2 \arrow[r, shift left, "x_1, x_2 \vdash x_1"] \arrow[r, shift right, "x_1, x_2 \vdash x_2"'] & 1 \end{tikzcd}$$ *projections*

$$2 \xrightarrow{\mathbf{m}} 1 \qquad 2 \xrightarrow{x_1, x_2 \vdash \mathbf{m}(x_1, x_2)} 1$$ *operation*

$$3 \xrightarrow{\mathbf{m}(\mathbf{m}(\pi_1, \pi_2), \pi_3)} 1 \qquad 3 \xrightarrow{x_1, x_2, x_3 \vdash m(m(x_1, x_2), x_3)} 1$$ *composed term*

  - <u>commuting diagrams</u> are understood as “***equations***”

$$\begin{tikzcd} 1 \arrow[r, "\langle e, \mathrm{id} \rangle"] \arrow[dr, "\mathrm{id}"'] & 2 \arrow[d, "\mathrm{m}"] & 1 \arrow[l, "\langle \mathrm{id}, e \rangle"'] \arrow[dl, "\mathrm{id}"] \\ & 1 & \end{tikzcd}$$ ~ unit law

$$\begin{tikzcd} 3 \arrow[r, "\Vert \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times \Vert"'] & 2 \arrow[d, "\Vert"] \\ 2 \arrow[r, "\Vert"'] & 1 \end{tikzcd}$$ ~ assoc law

  - arises from
    - (single-sorted) algebraic specification (Σ, E) as the *syntactic category*
    - FP-sketch

Outline

- In a coalgebraic study of ***concurrency***,

- ***Nested*** algebraic structures
  - on a **category** $\boldsymbol{C}$ and
  - on an **object** $X \in \boldsymbol{C}$

  arise naturally (**microcosm principle**)

- Our contributions:
  - Syntactic formalization of microcosm principle
  - 2-categorical formalization with Lawvere theories
  - Application to coalgebras:
    - generic compositionality theorem

Generic soundness result

- A Lawvere theory ***L*** is for
  - operations, and
  - ***equations*** (e.g. associativity, commutativity)

- **Coalg**$_F$ is an ***L***-category
  ➔ Parallel composition $\otimes$ is automatically **associative** (for example)
  - Ultimately, this is due to the ***coherence condition*** on the **lax *L*-functor** *F*

- **Possible application** :
  Study of *syntactic formats* that ensure associativity/commutativity (future work)

Microcosm principle for concurrency (|| and ||)

- “Parallel composition via **sync** nat. trans”
- compositionality theorem

The microcosm principle 2-categorically

$$\begin{tikzcd} \mathbb{L} \arrow[r, bend left=60, "\mathbf{1}"] \arrow[r, "\mathbb{C}"'] \arrow[r, phantom, shift left=3, "\Downarrow X"] & \mathbf{CAT} \end{tikzcd}$$

Back to concurrency

- Part 1 for **arbitrary algebraic theory**
- **Generic** compositionality theorem