Coalgebras and Higher-Order Computation: a GoI Approach

Ichiro Hasuo

University of Tokyo (JP)

東京大学
THE UNIVERSITY OF TOKYO

FSCD 2016, Porto, 24 Jun 2016

Outline

Coalgebra meets **higher-order computation** in **Geometry of Interaction** [Girard, LC'88]

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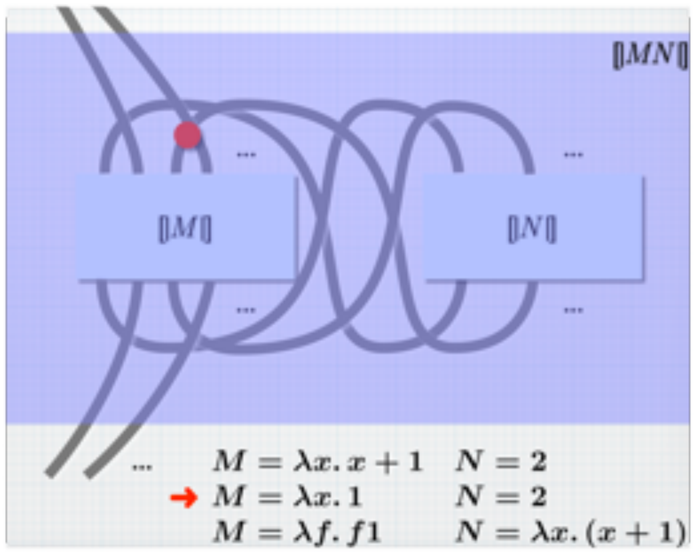

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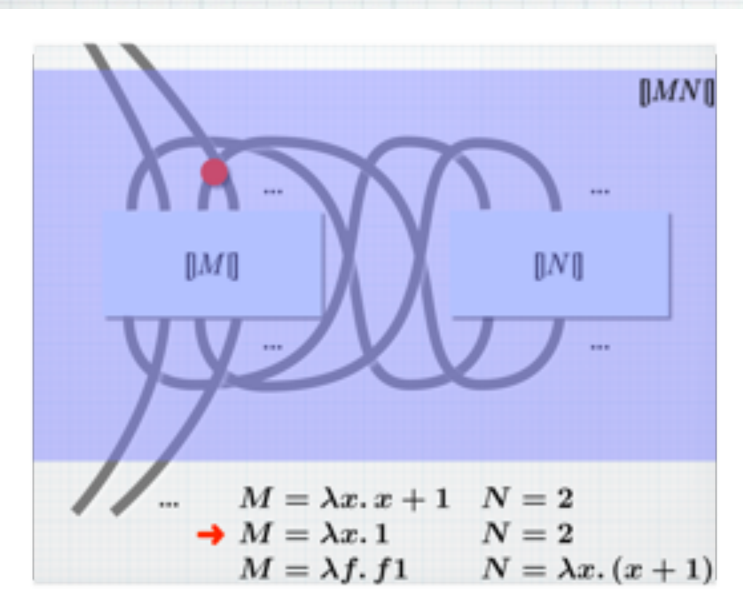

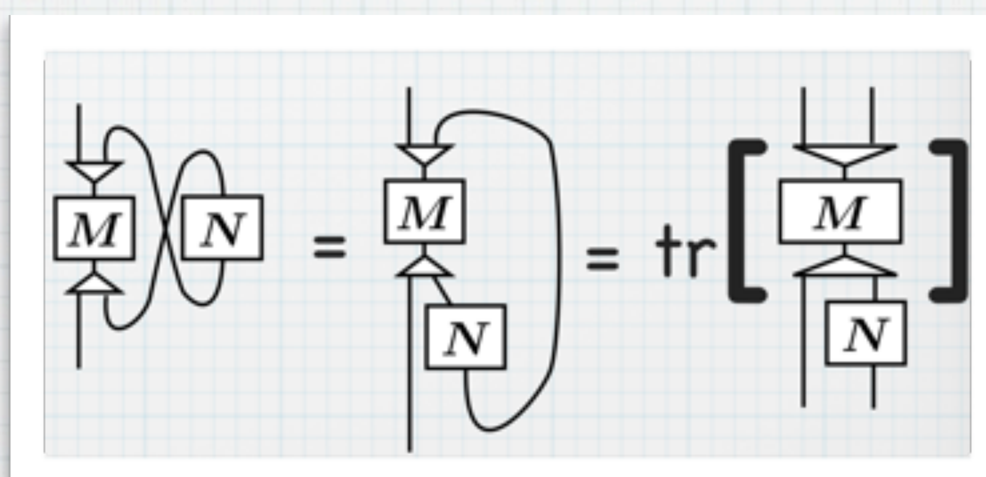

Categorical GoI

[Abramsky, Haghverdi & Scott, MSCS'02]

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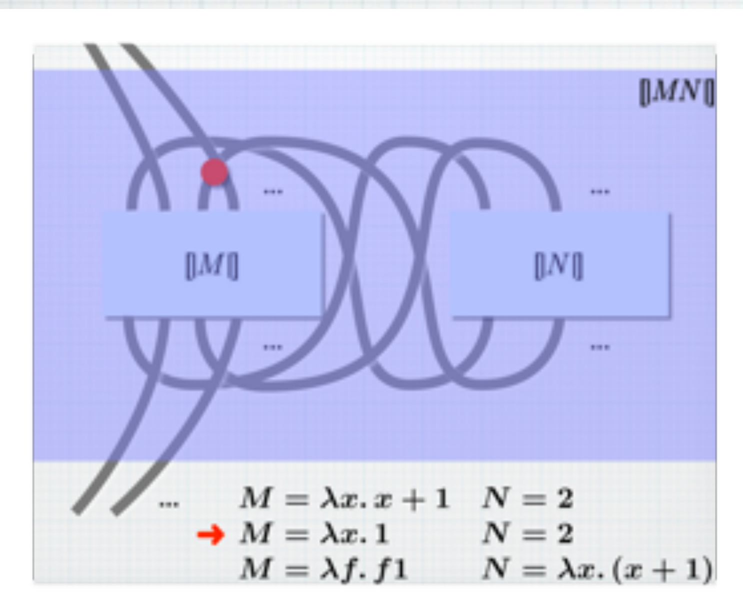
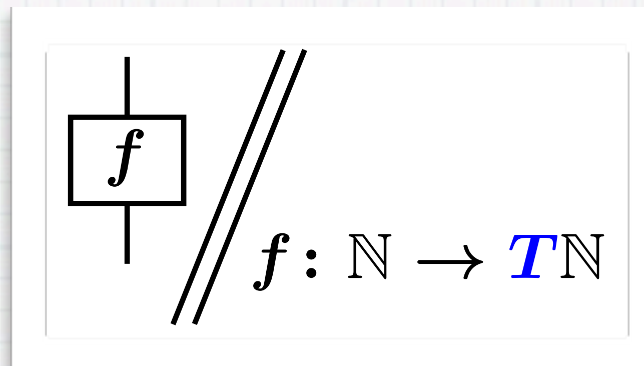
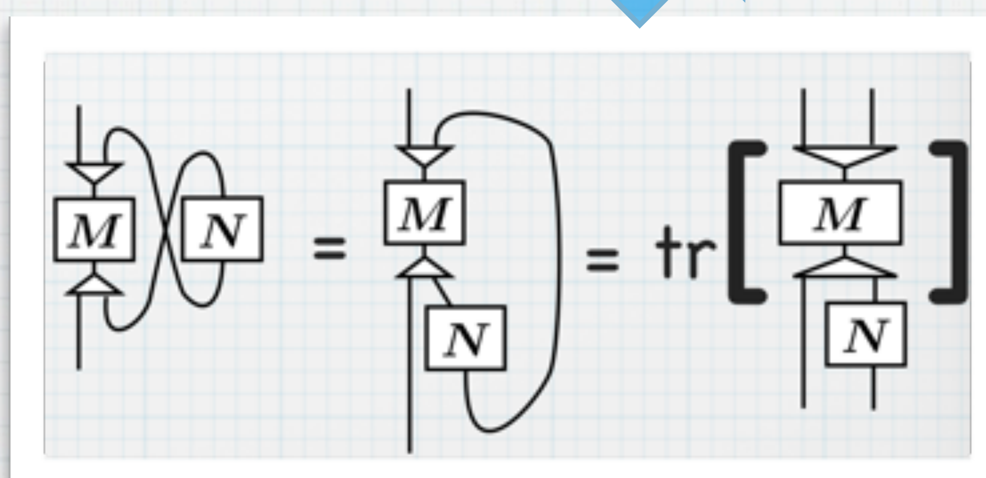




GoI w/ T-branching

[IH & Hoshino, LICS'11]



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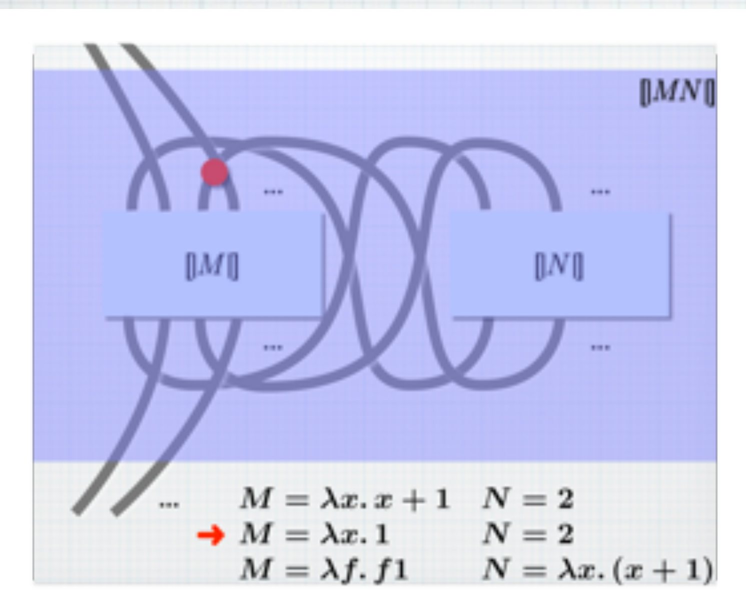

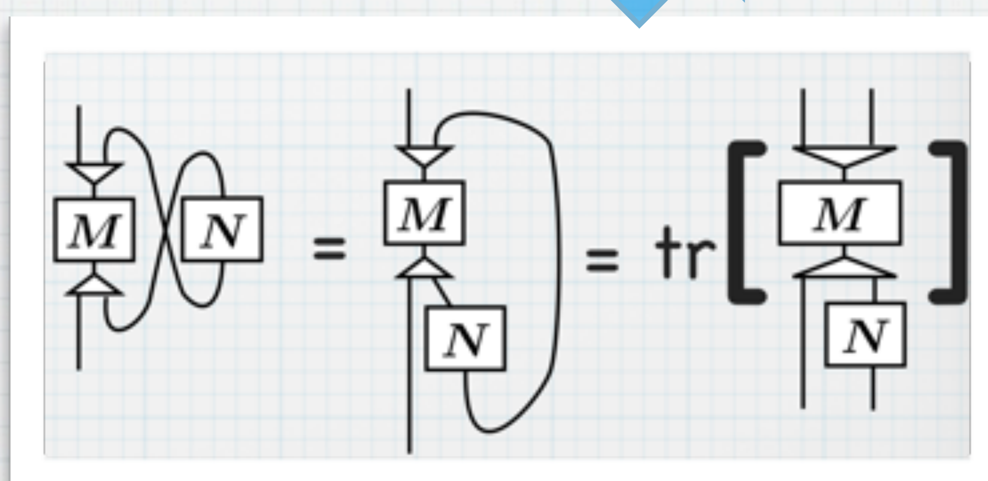

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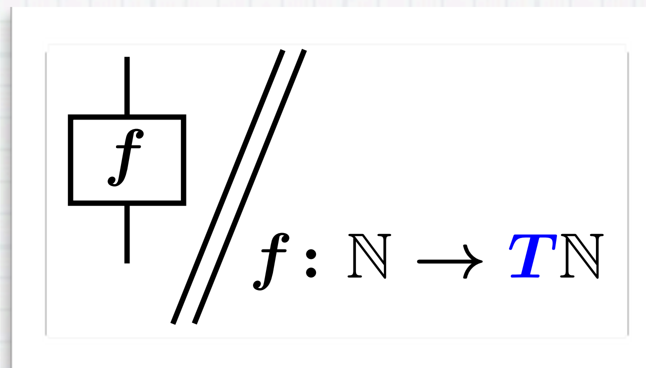

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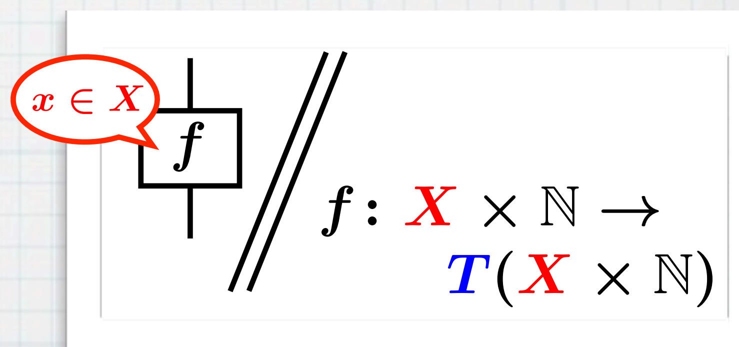

Memoryful GoI

[Hoshino, Muroya & IH, CSL-LICS'14 & POPL'16]

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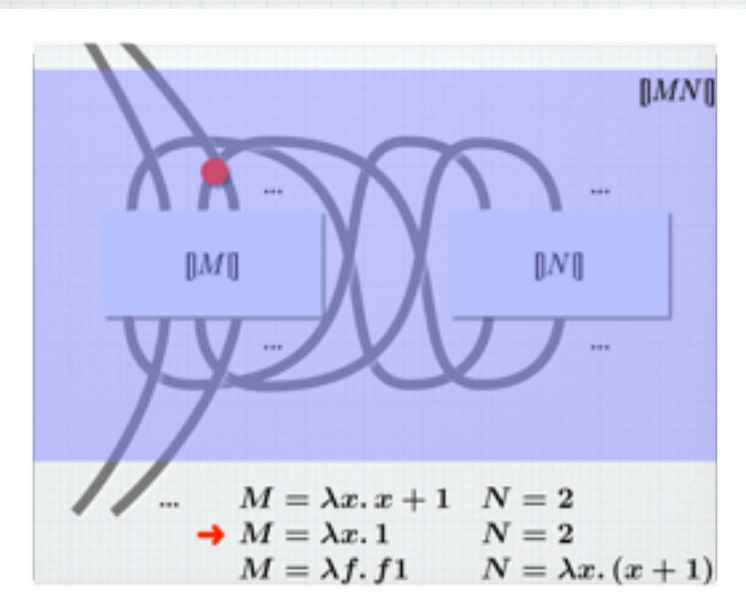

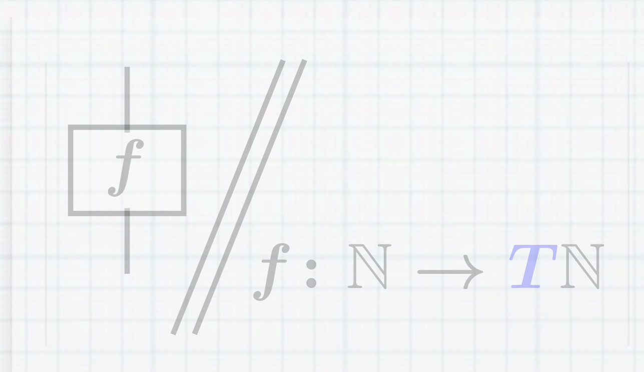

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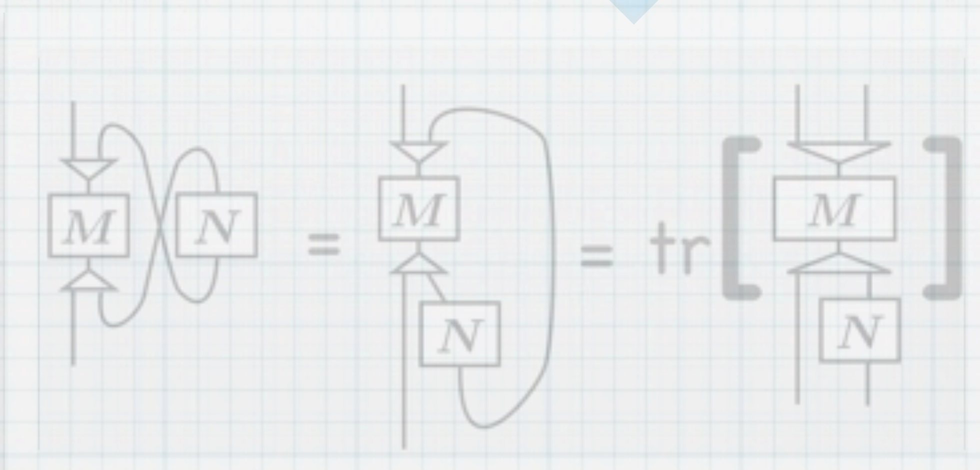

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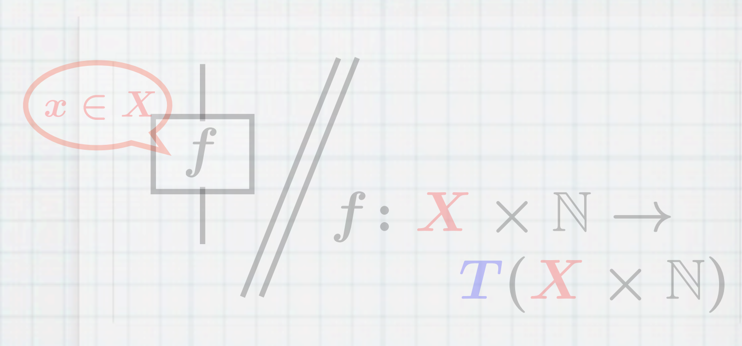

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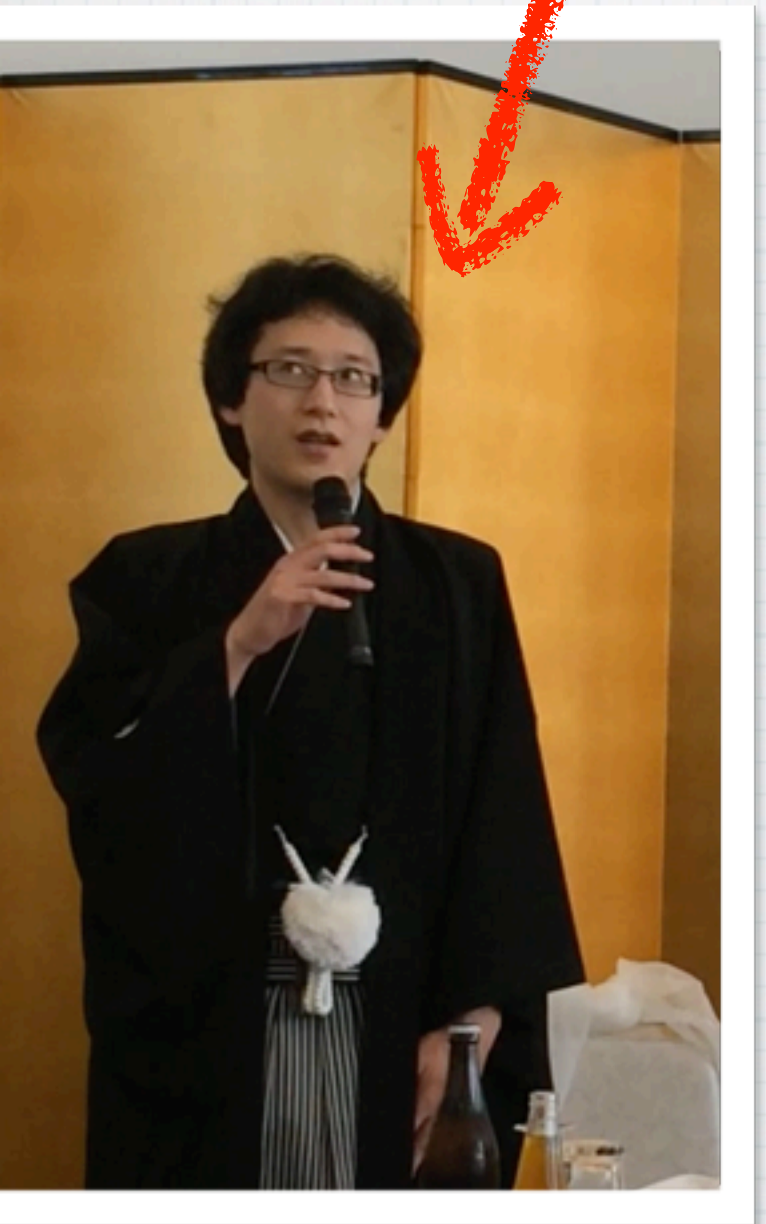

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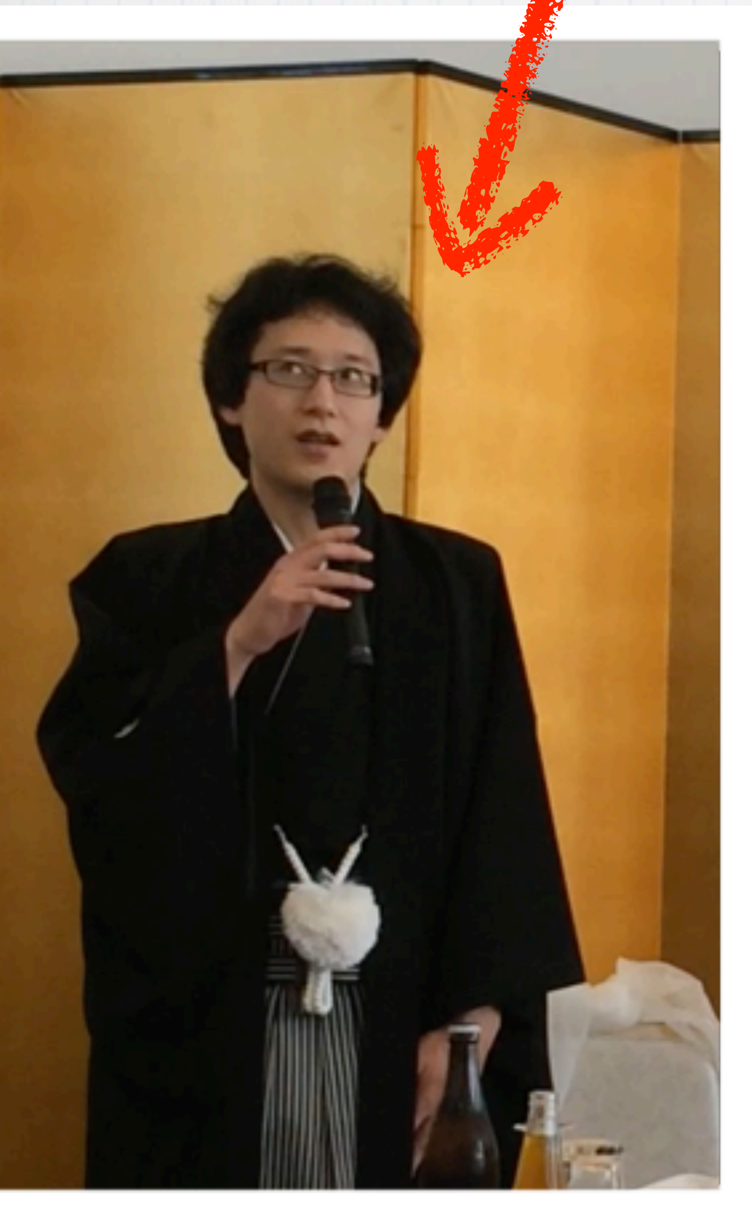

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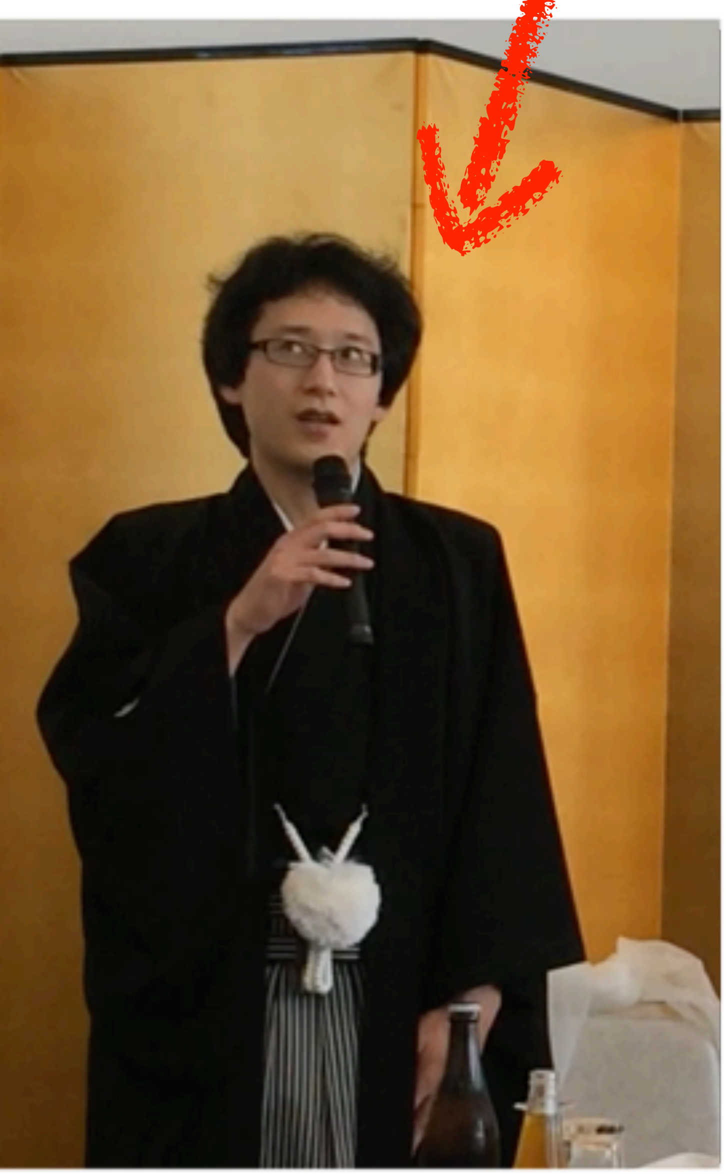

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Toshiki Kataoka

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References

* [LICS 2011] IH and Naohiko Hoshino. **Semantics of Higher-Order Quantum Computation via Geometry of Interaction.** (Extended ver. to appear in Annals Pure & Appl. Logic)
* [CSL-LICS 2014] Naohiko Hoshino, Koko Muroya and IH. **Memoryful Geometry of Interaction: From Coalgebraic Components to Algebraic Effects.**
* [POPL 2016] Koko Muroya, Naohiko Hoshino and IH. **Memoryful Geometry of Interaction II: Recursion and Adequacy.**
* [LOLA 2014] Koko Muroya, Toshiki Kataoka, IH and Naohiko Hoshino. **Compiling Effectful Terms to Transducers: Prototype Implementation of Memoryful Geometry of Interaction (Preliminary Report).**
* [Math. Str. in Comp. Sci. 2011] IH and Bart Jacobs. **Traces for Coalgebraic Components.**

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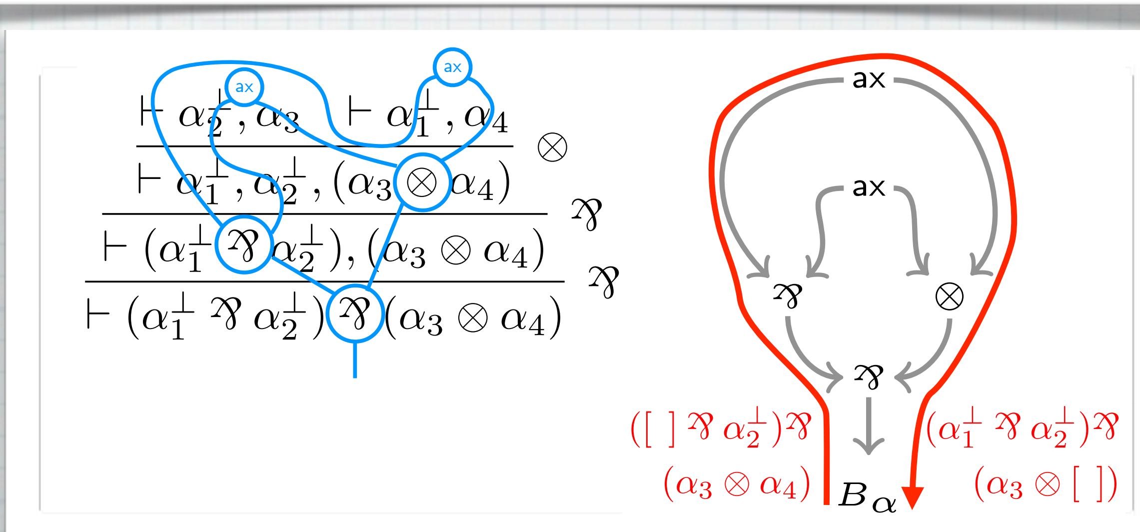
  [Mackie, POPL'95] [Pinto, TLCA'01] [Ghica et al., POPL'07, POPL'11, ICFP'11, ...]
* Two presentations:
  * (Operator-) Algebraic [Girard]
  * Token machines/ interaction abstract machines

    [Danos & Regnier, TCS'99] [Mackie, POPL'95]

$$\dfrac{\dfrac{\overline{\vdash A,\ A^\perp} \quad \overline{\vdash A^\perp,\ A}}{\vdash A,\ A^\perp,\ A^\perp \otimes A} \quad \dfrac{\overline{\vdash A,\ A^\perp}}{\vdash A ⅋ A^\perp}}{\vdash [A^\perp \otimes A],\ A,\ A^\perp} \qquad \Pi^\bullet = \begin{pmatrix} 0 & 0 & p & q \\ 0 & pq^* + qp^* & 0 & 0 \\ p^* & 0 & 0 & 0 \\ q^* & 0 & 0 & 0 \end{pmatrix} \quad \sigma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

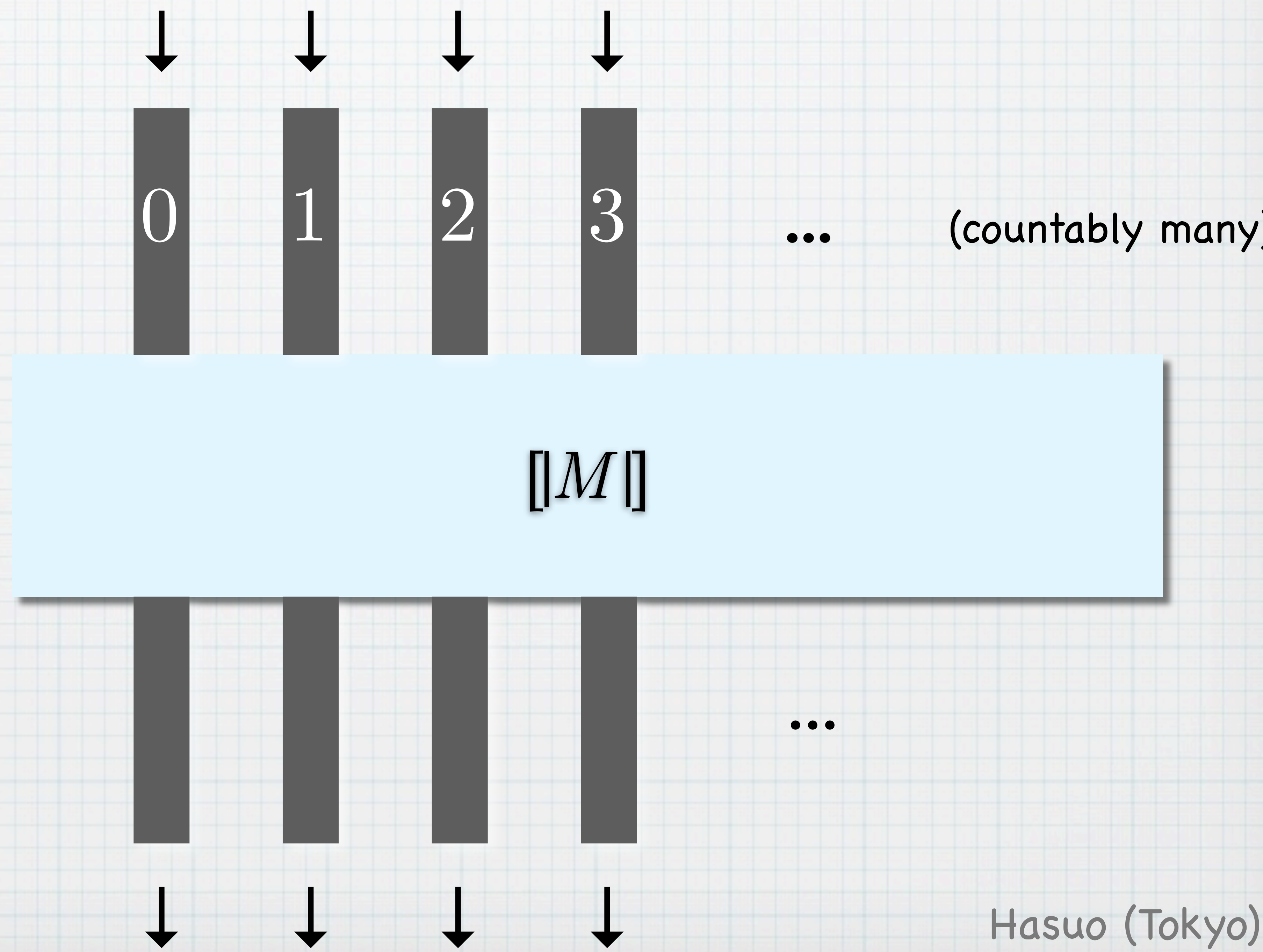
$$(p^*p = q^*q = 1,\ p^*q = q^*p = 0)$$



The GoI Animation

$$[\![M]\!] \;=\; (\, \mathbb{N} \rightharpoonup \mathbb{N}, \text{ a partial function}\,)$$

$$=\; \text{“piping”}$$



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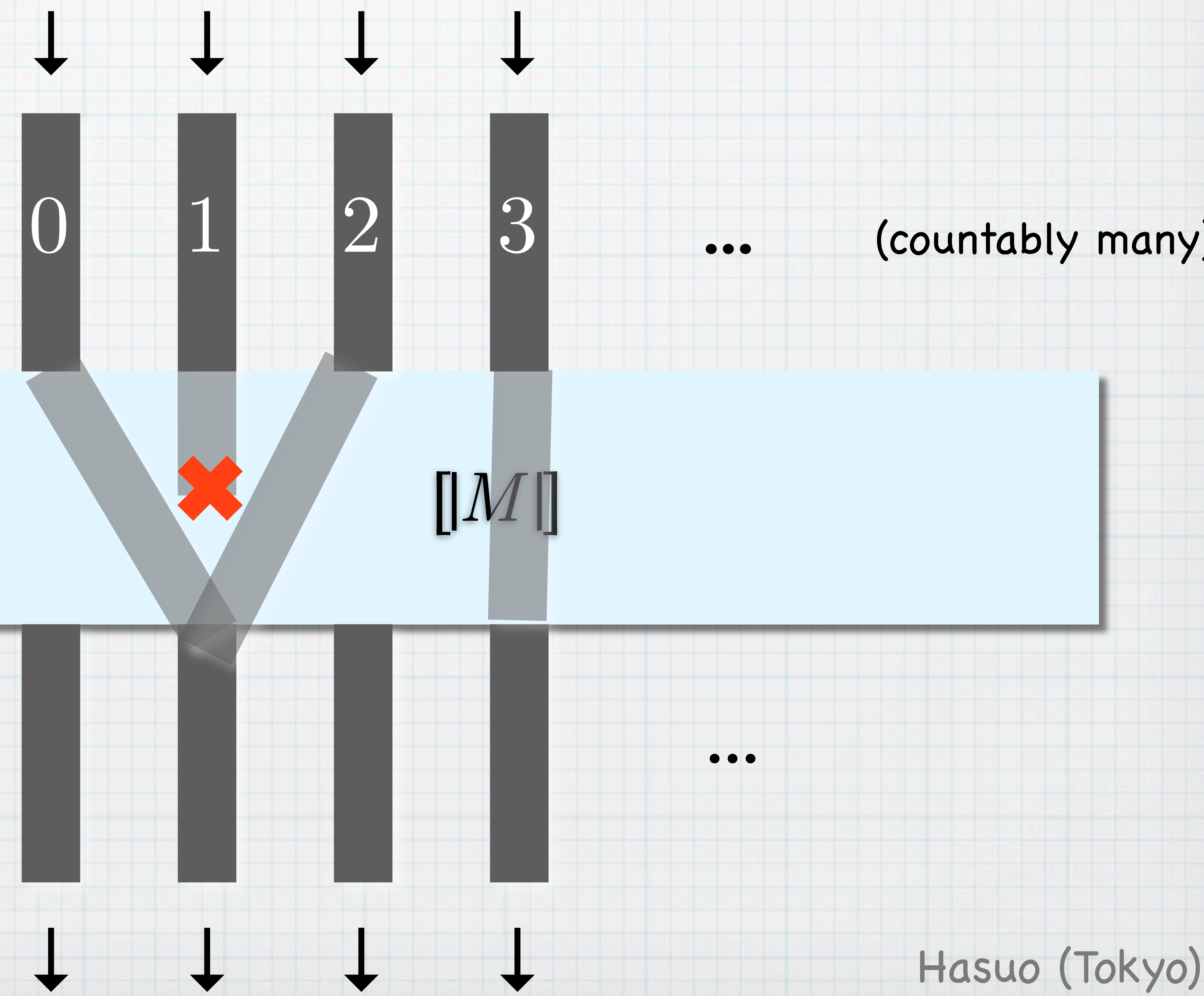

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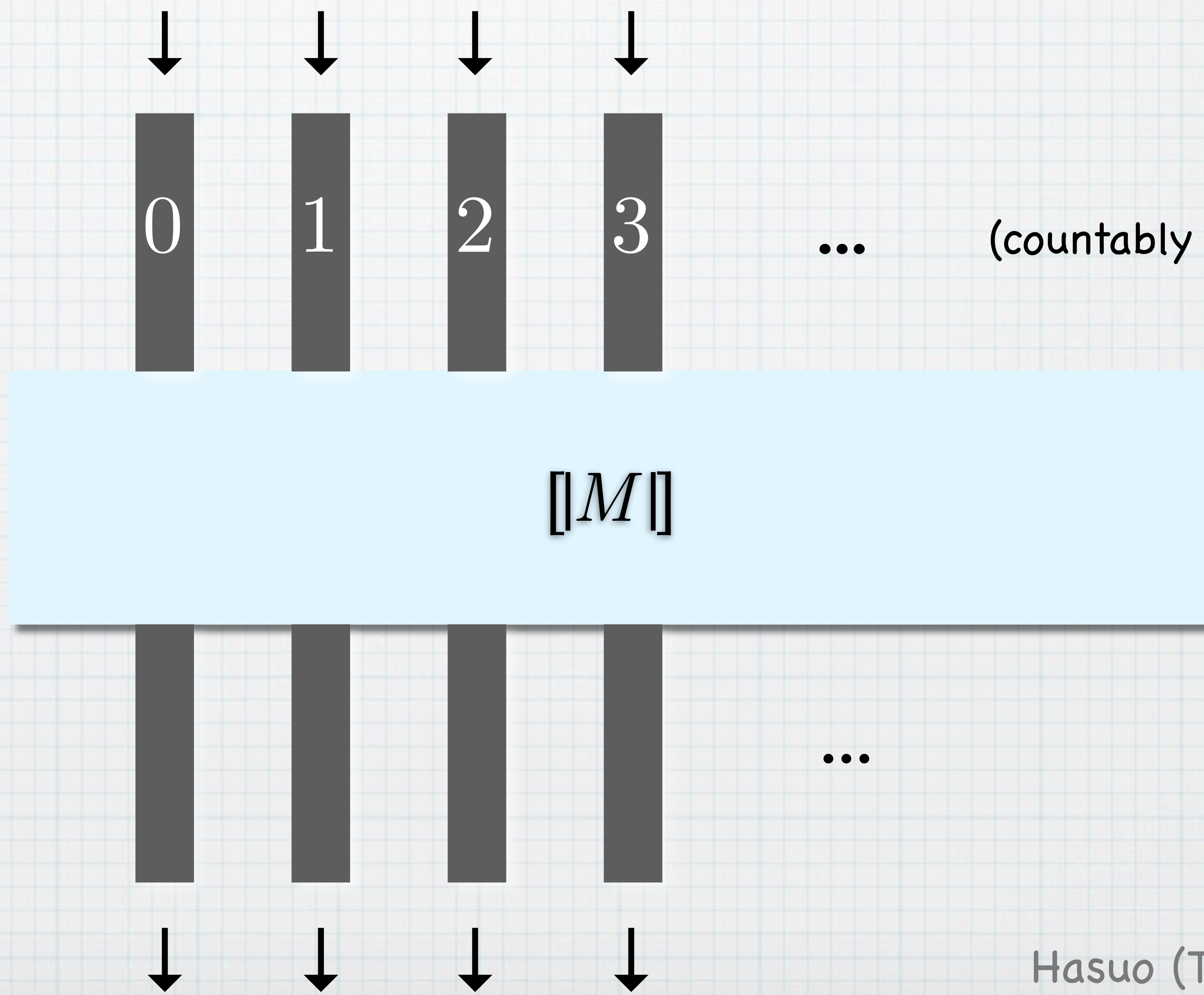

(countably many)

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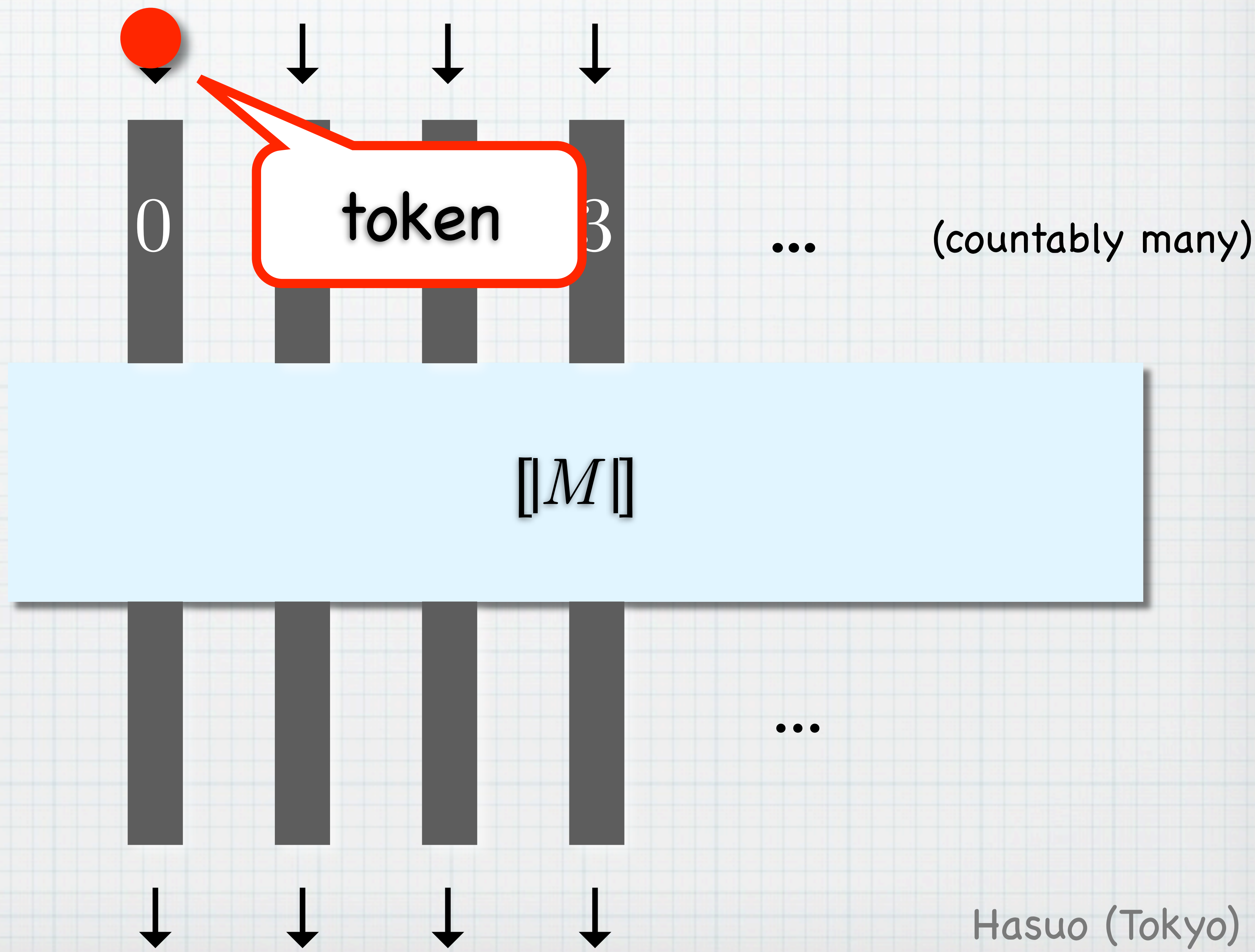

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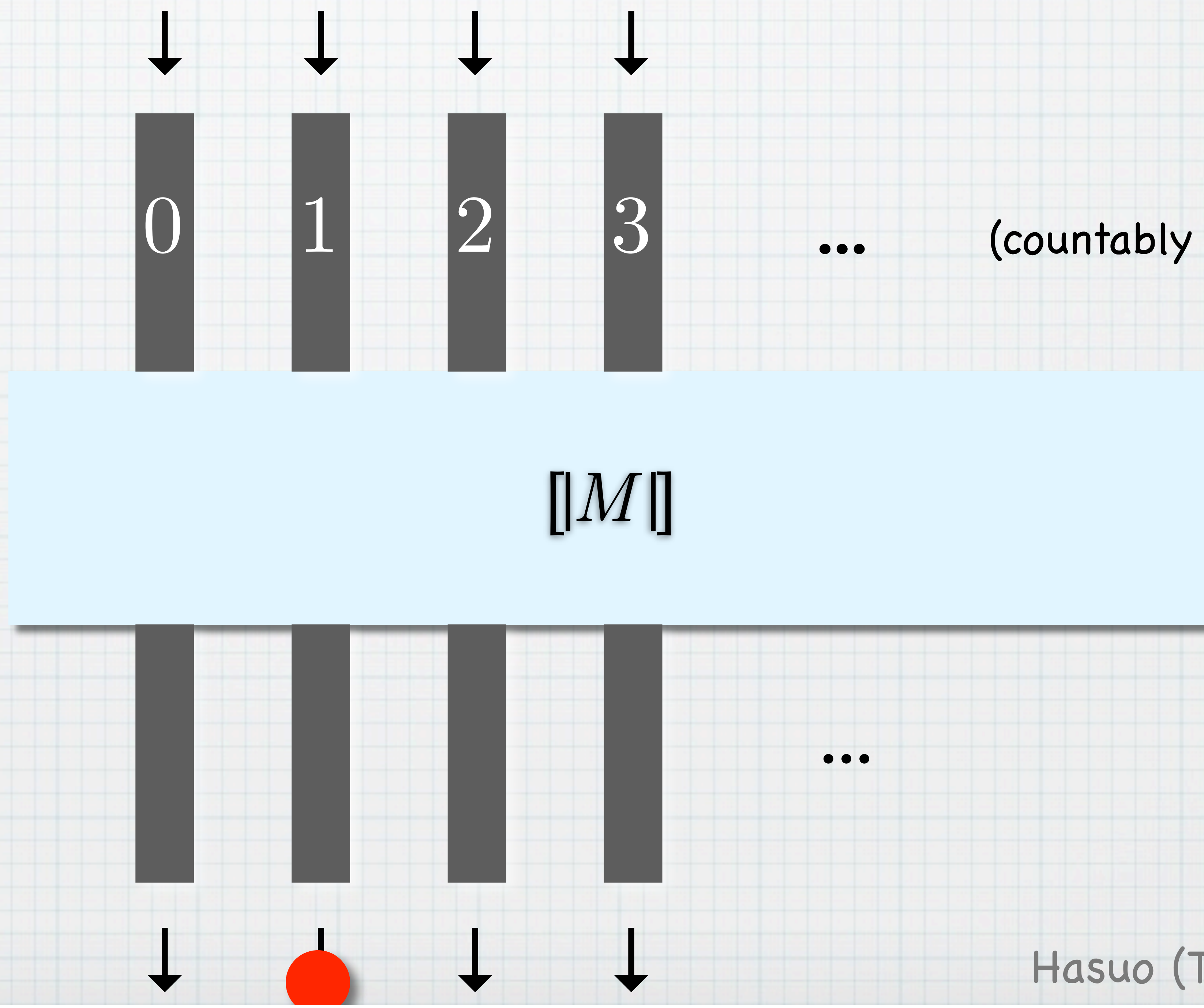

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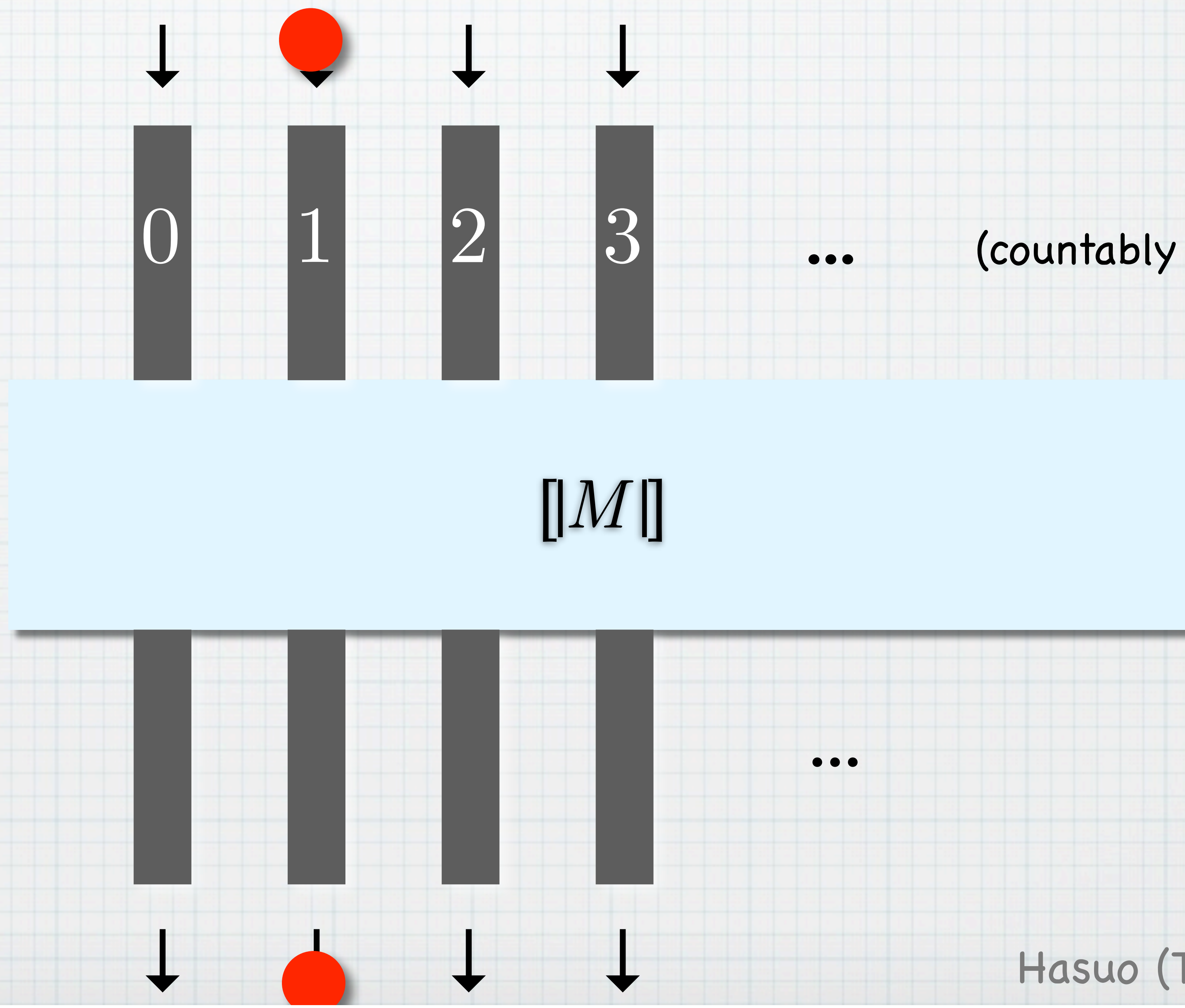

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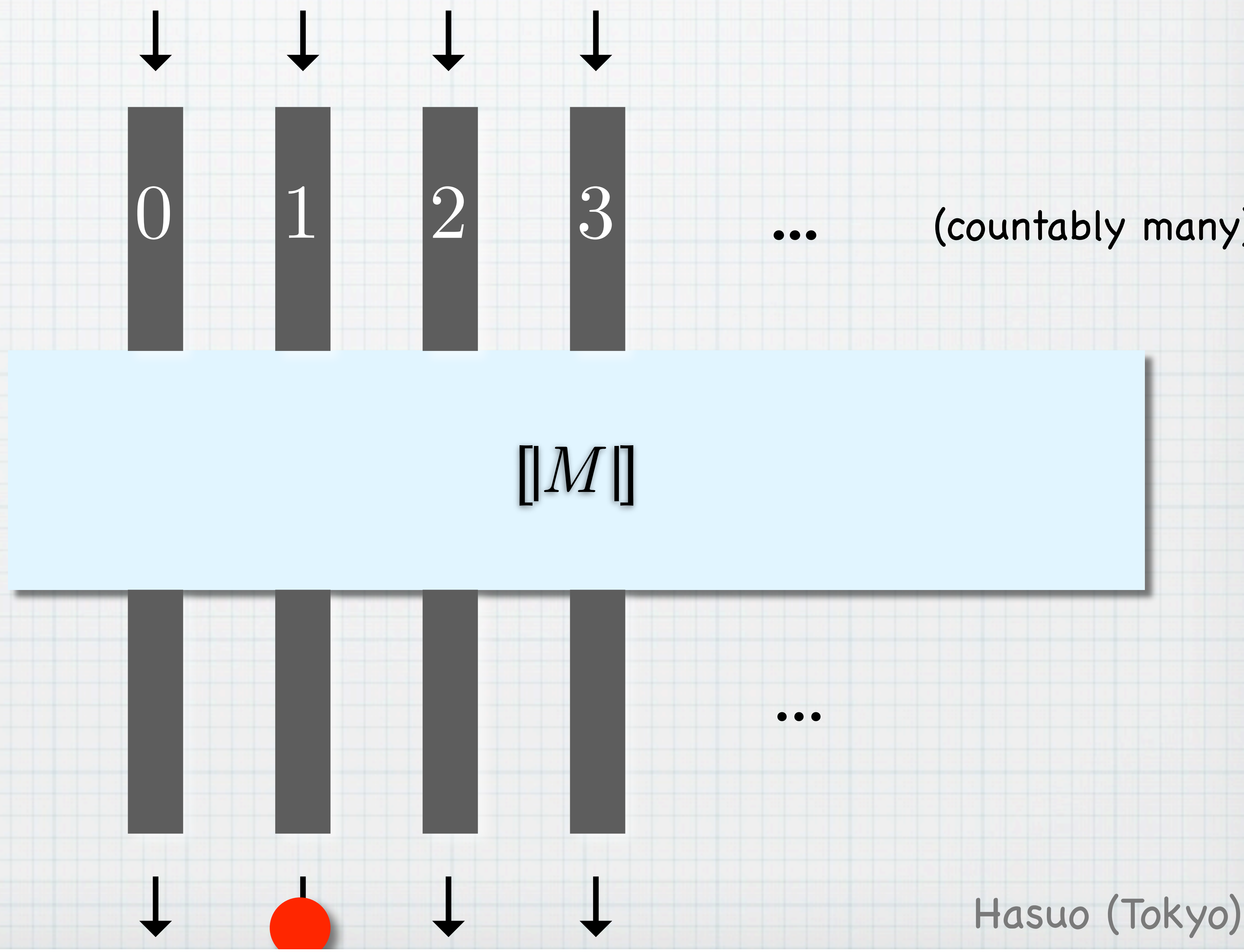
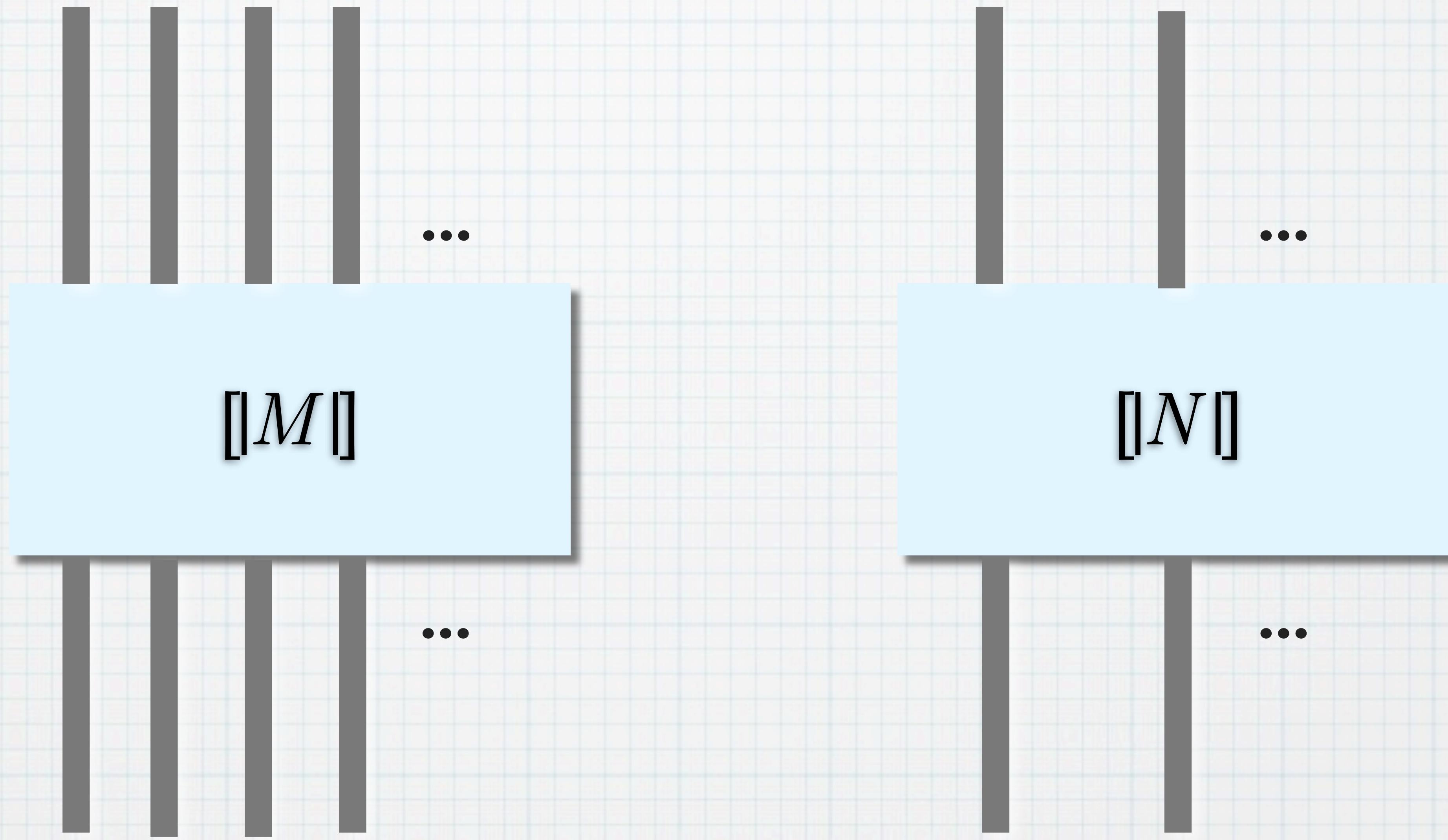


The GoI Animation

* Function application $[\![M N]\!]$
    * by "parallel composition + hiding"

$$[\![MN]\!] =$$



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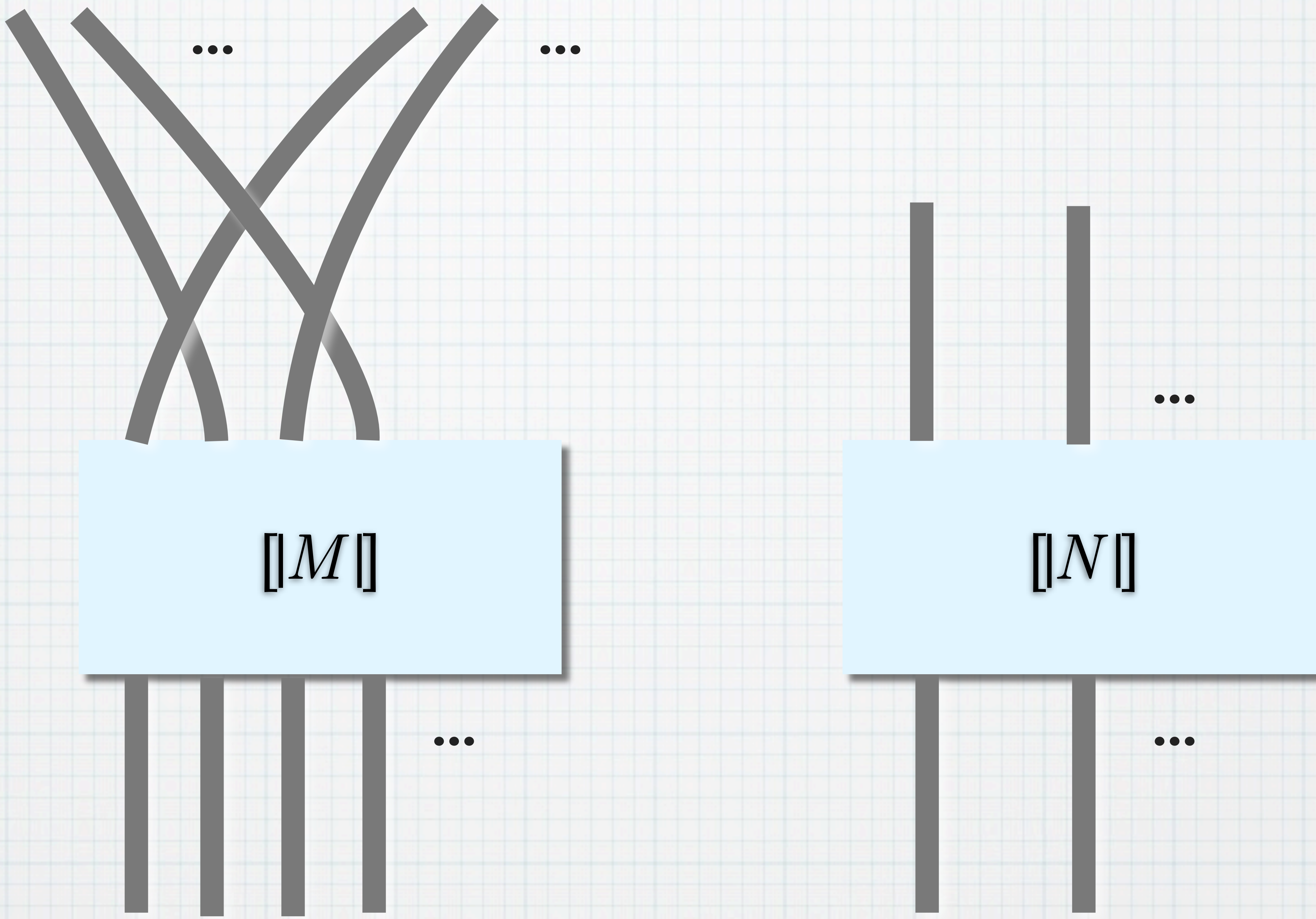

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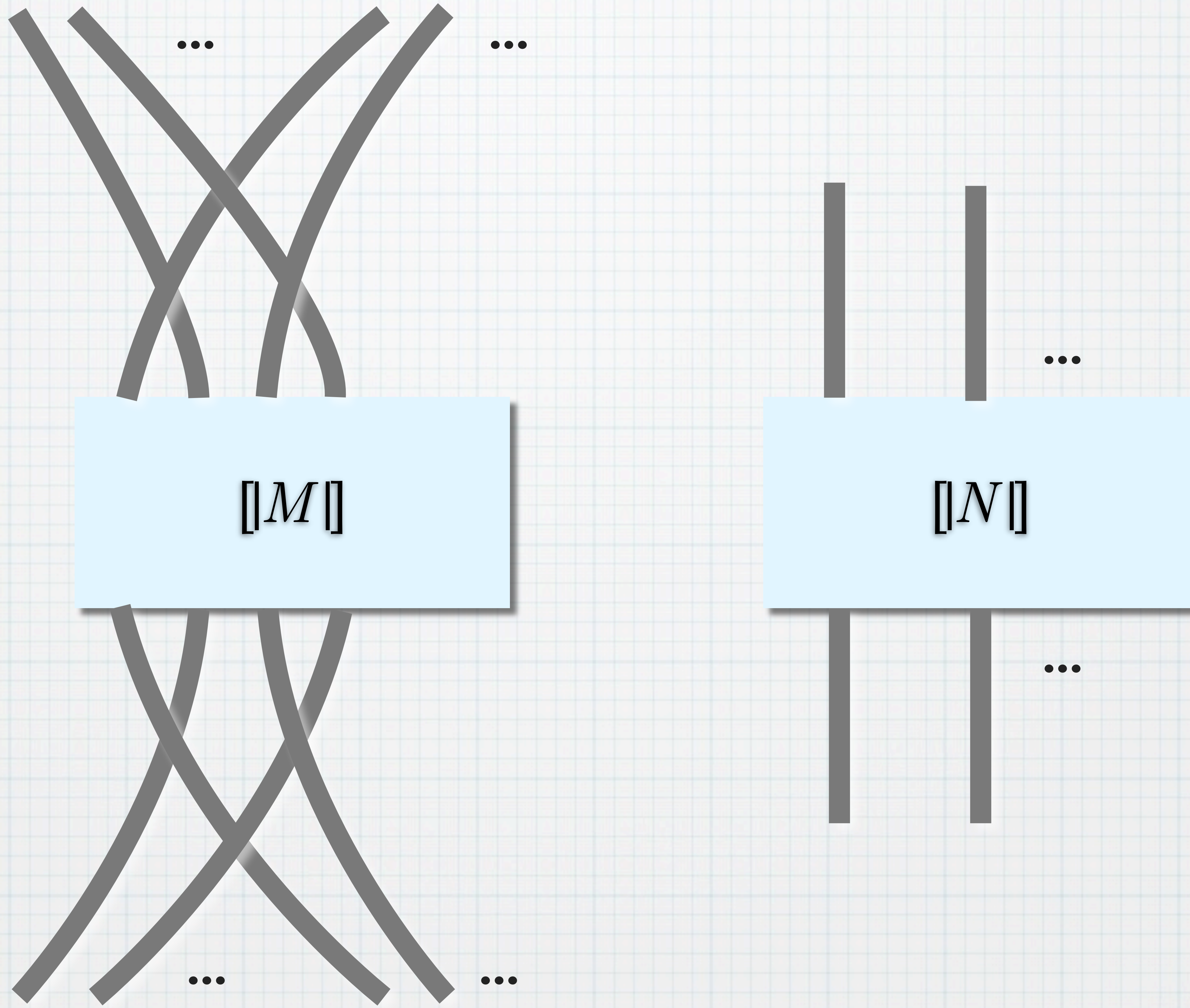
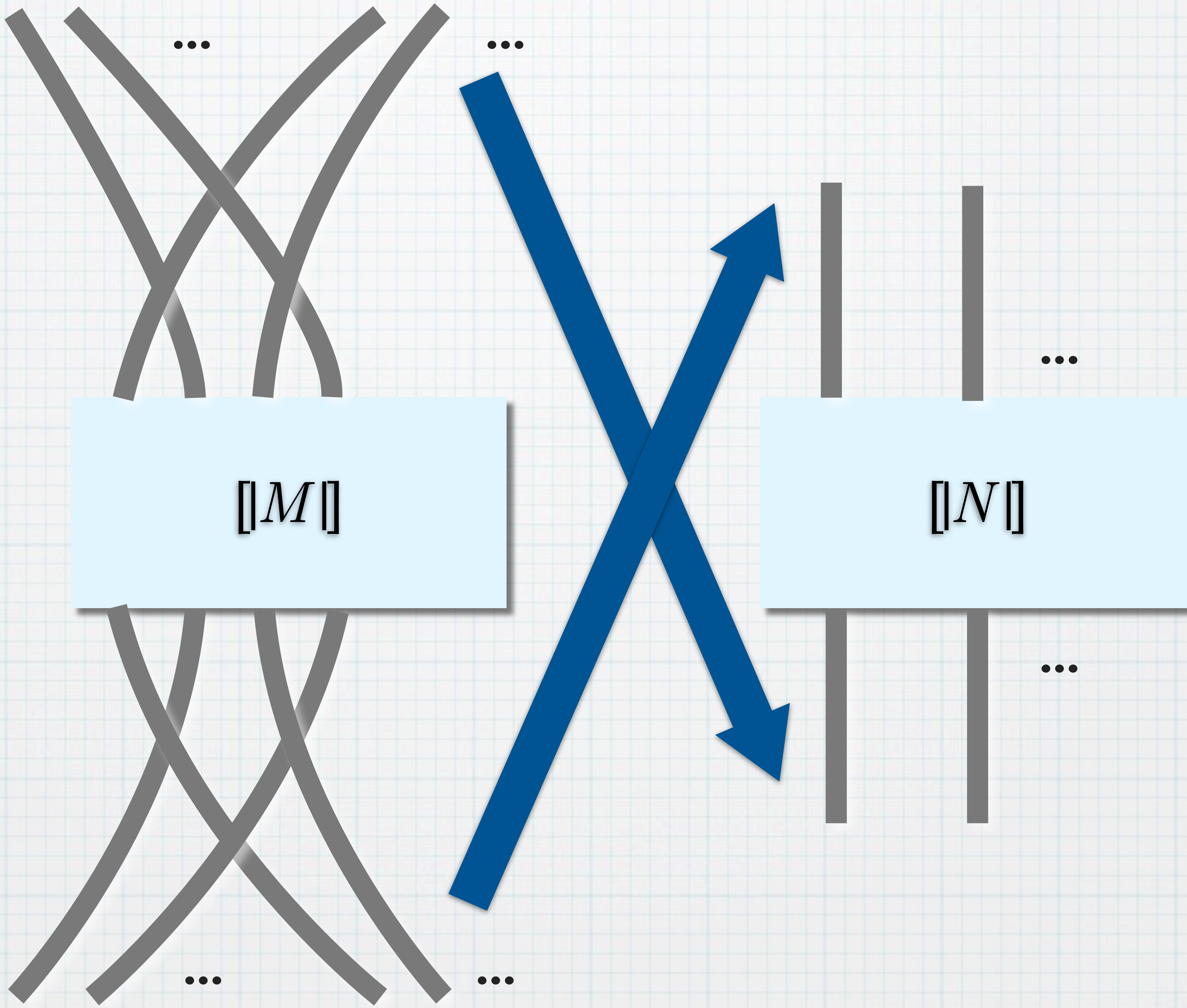


$[\![MN]\!]$

$=$



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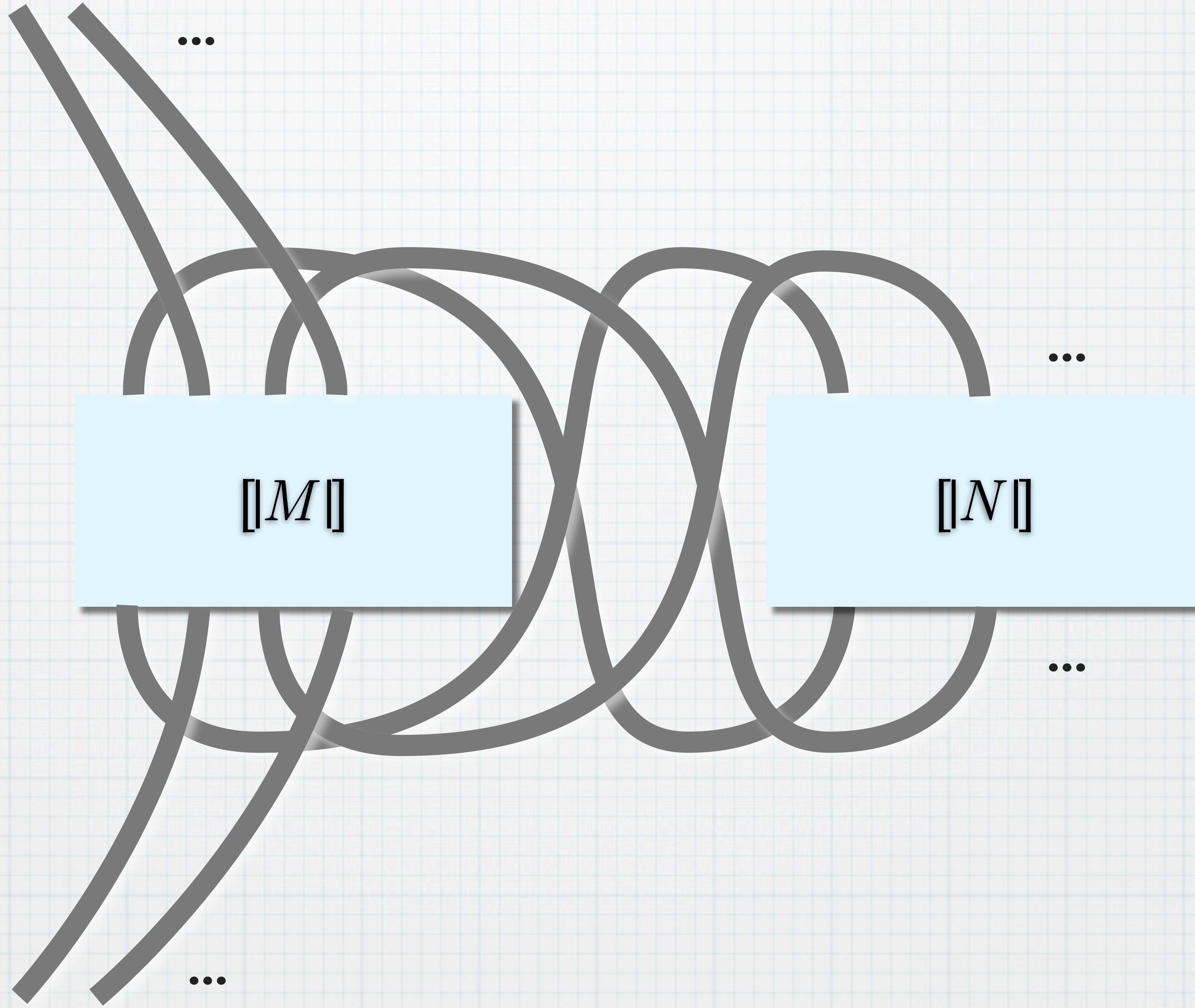

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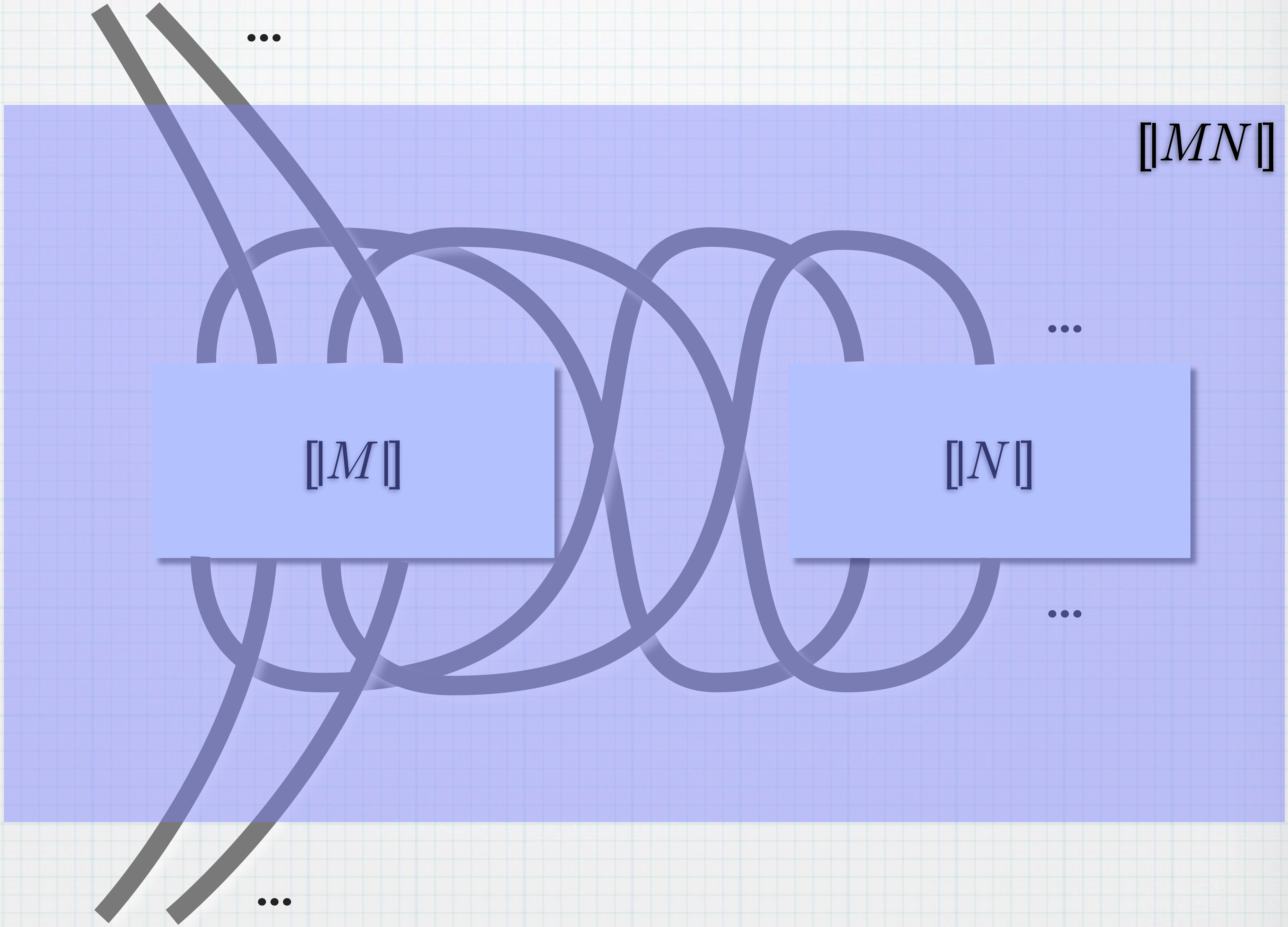

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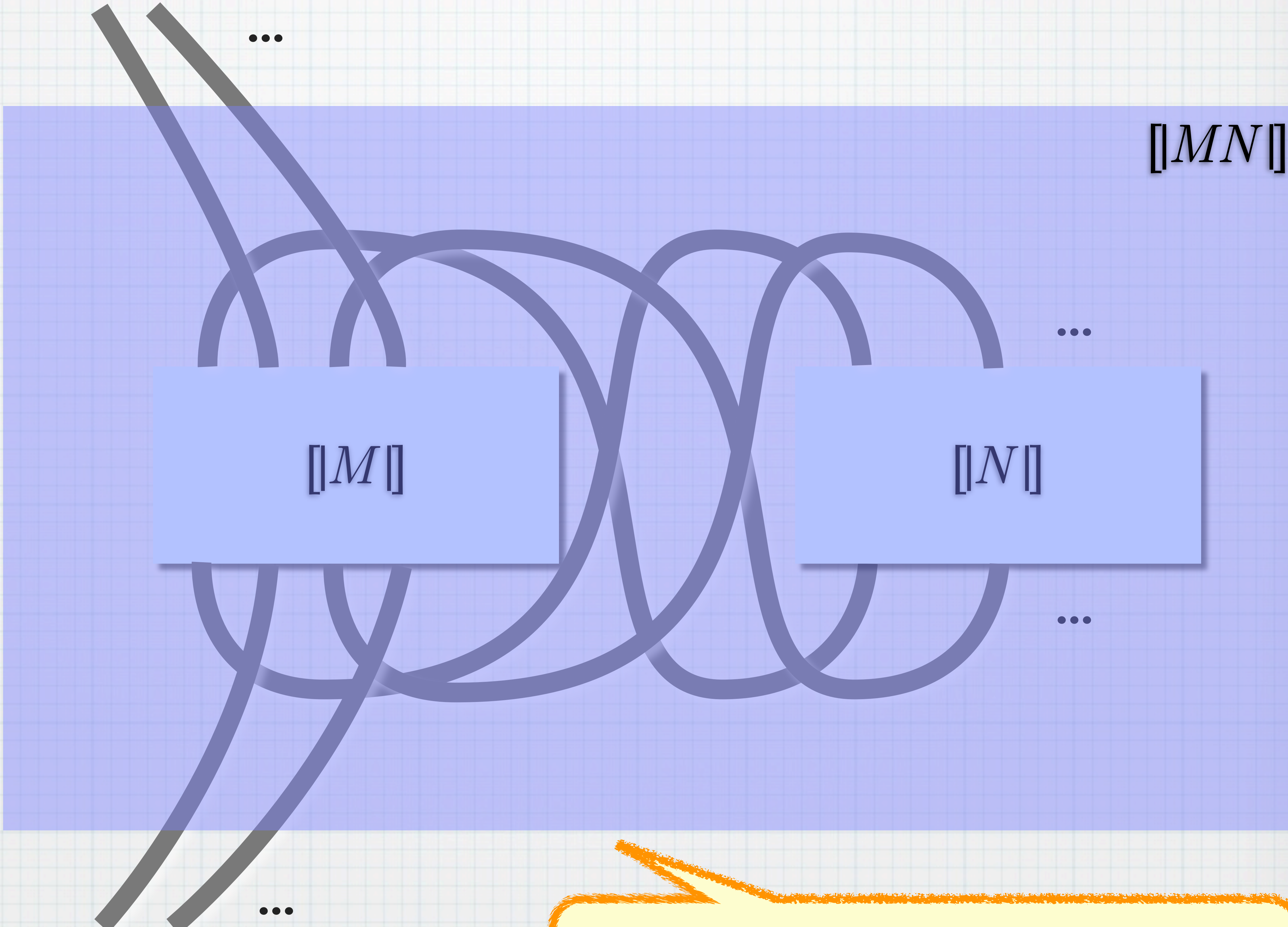

"parallel composition + hiding"
(cf. AJM games)

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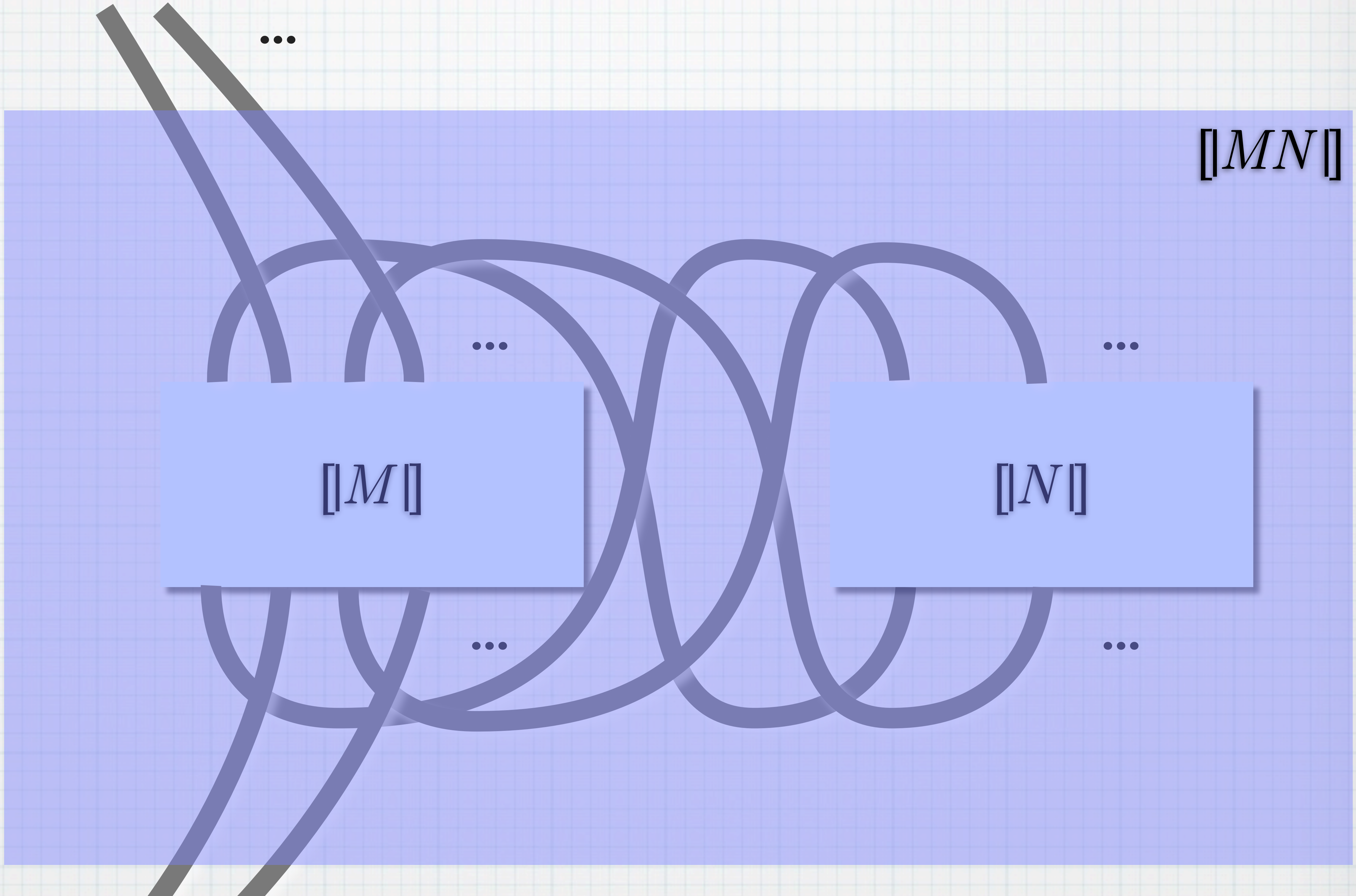

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$$\boldsymbol{M = \lambda x.\, 1 \quad N = 2}$$
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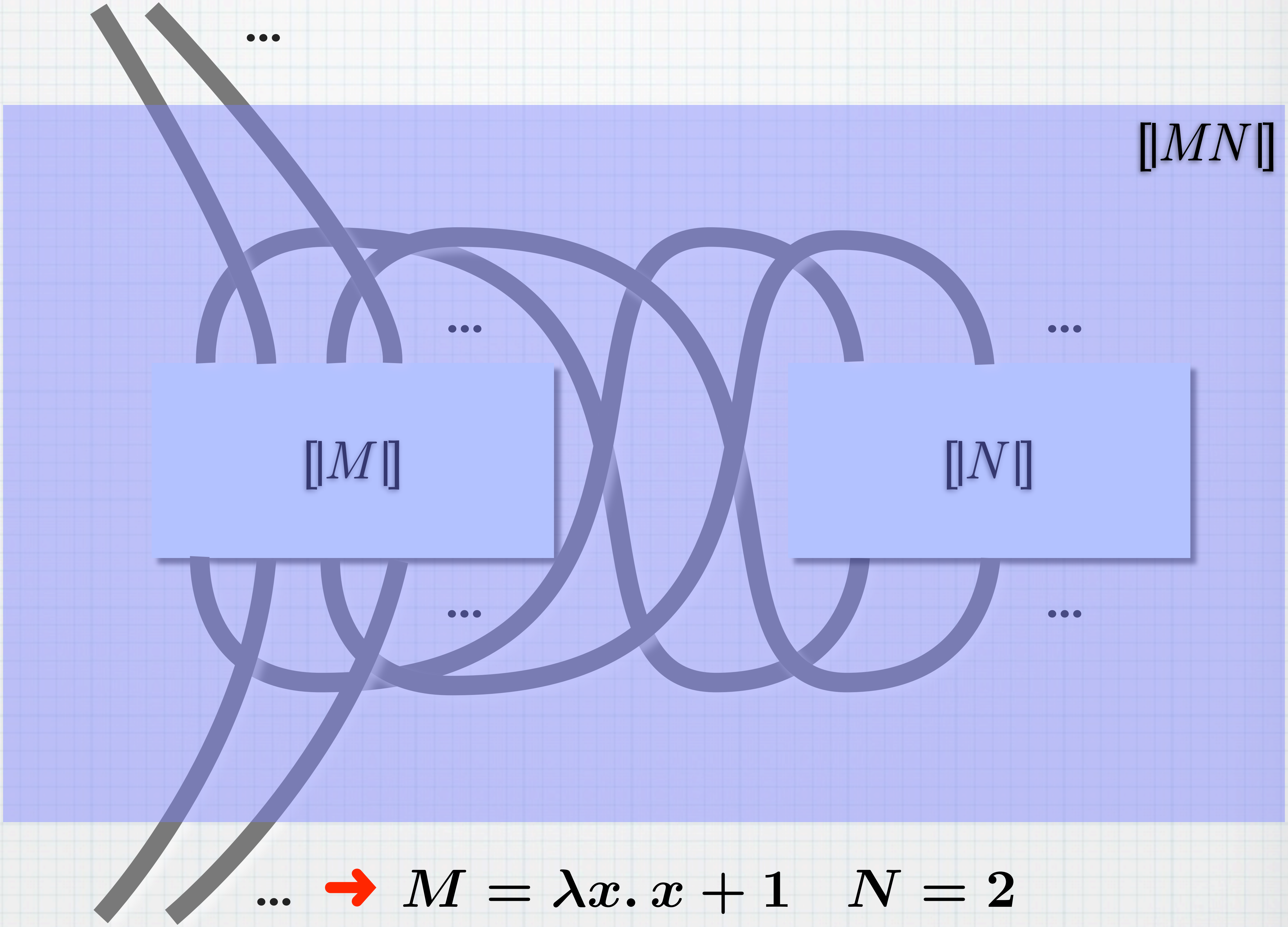
$$[\![MN]\!] =$$



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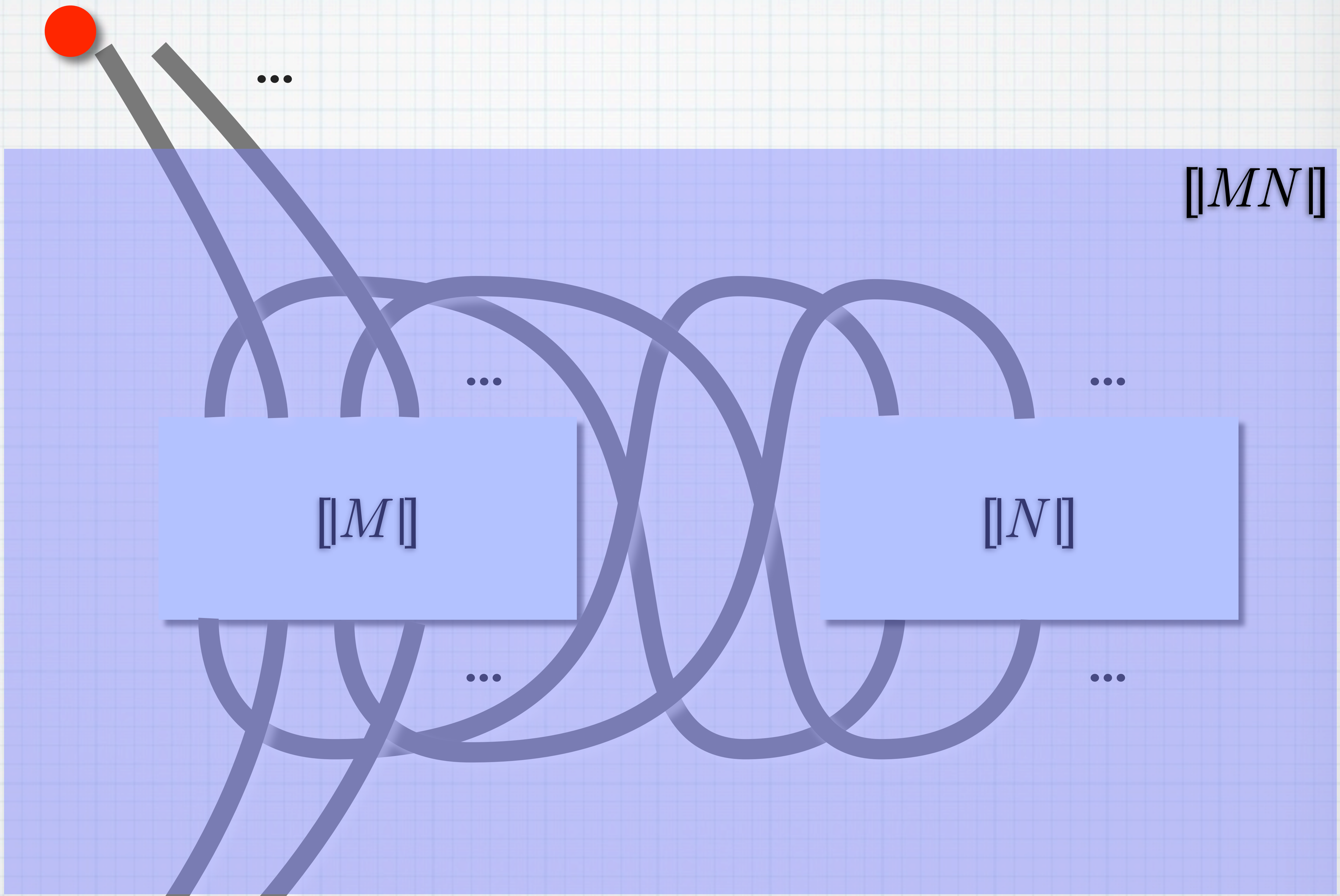

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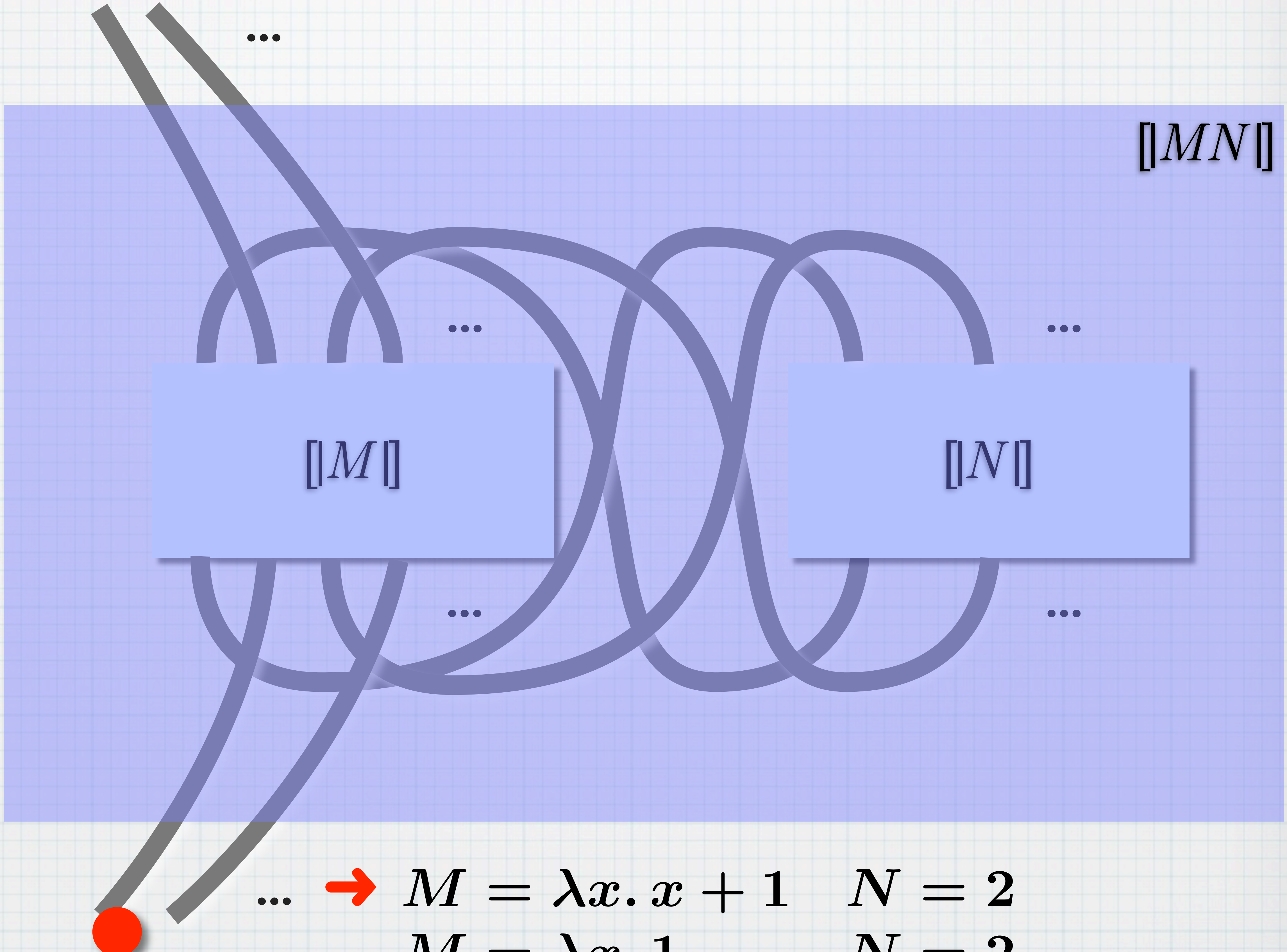
$=$



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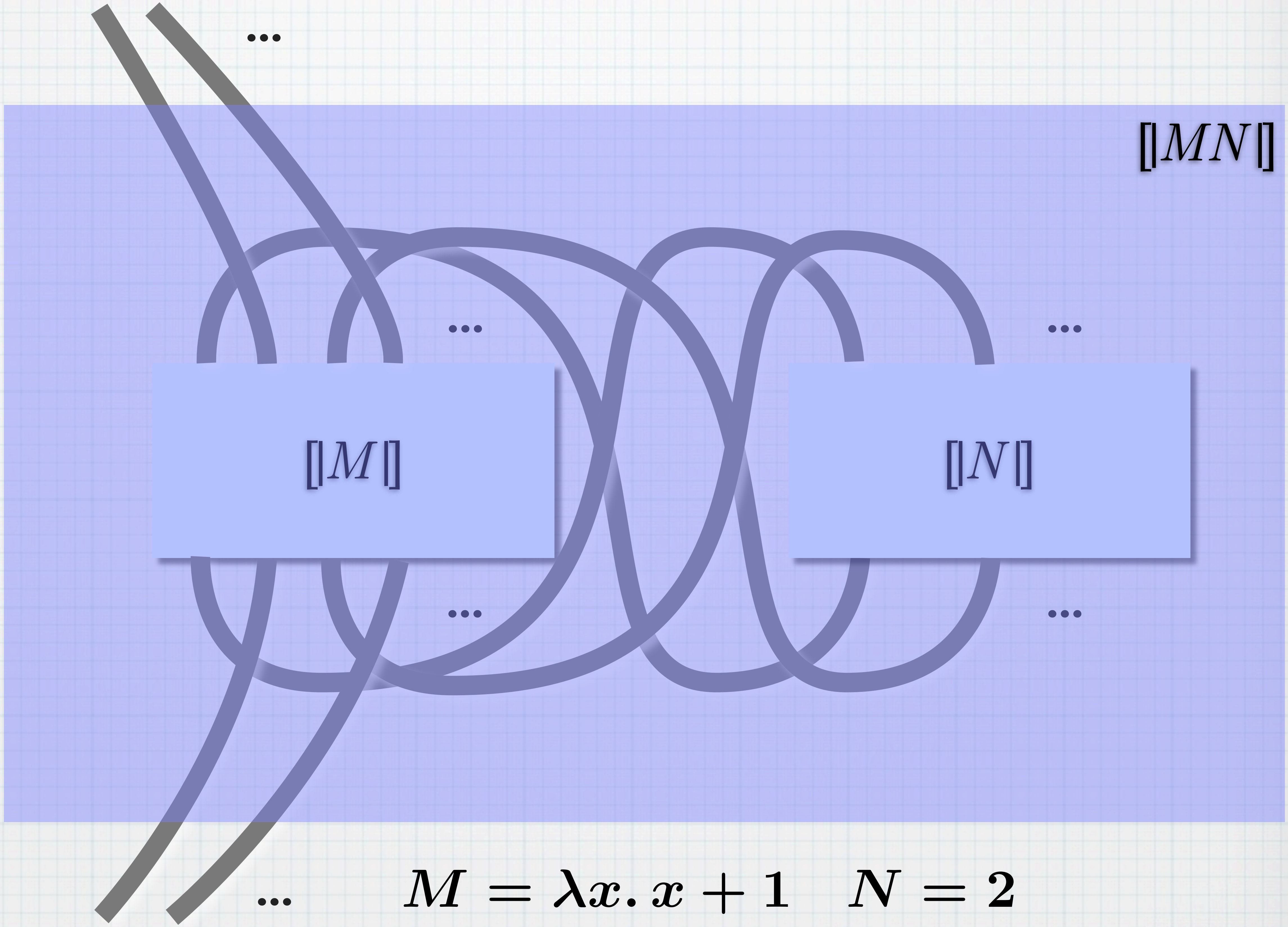

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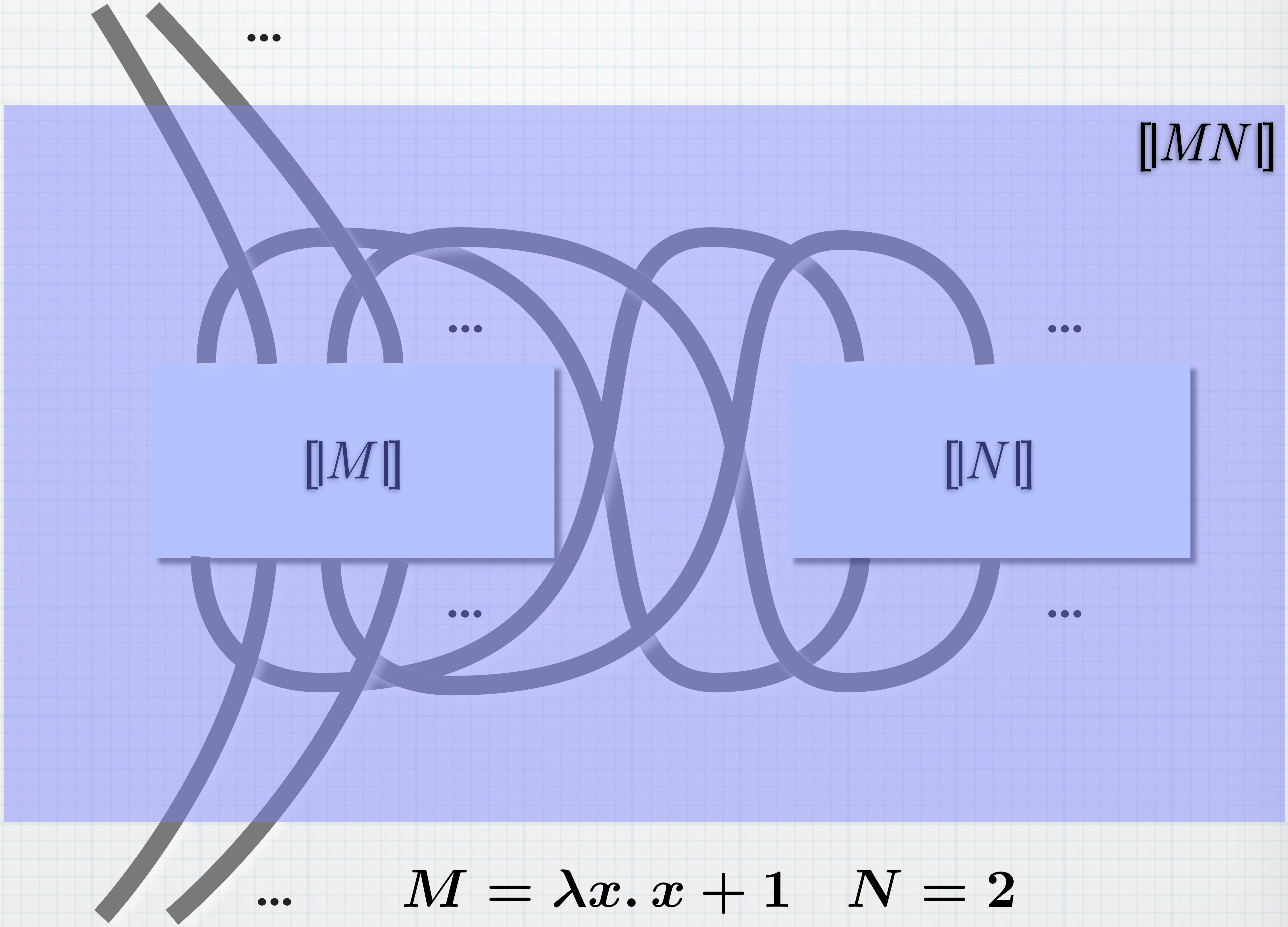

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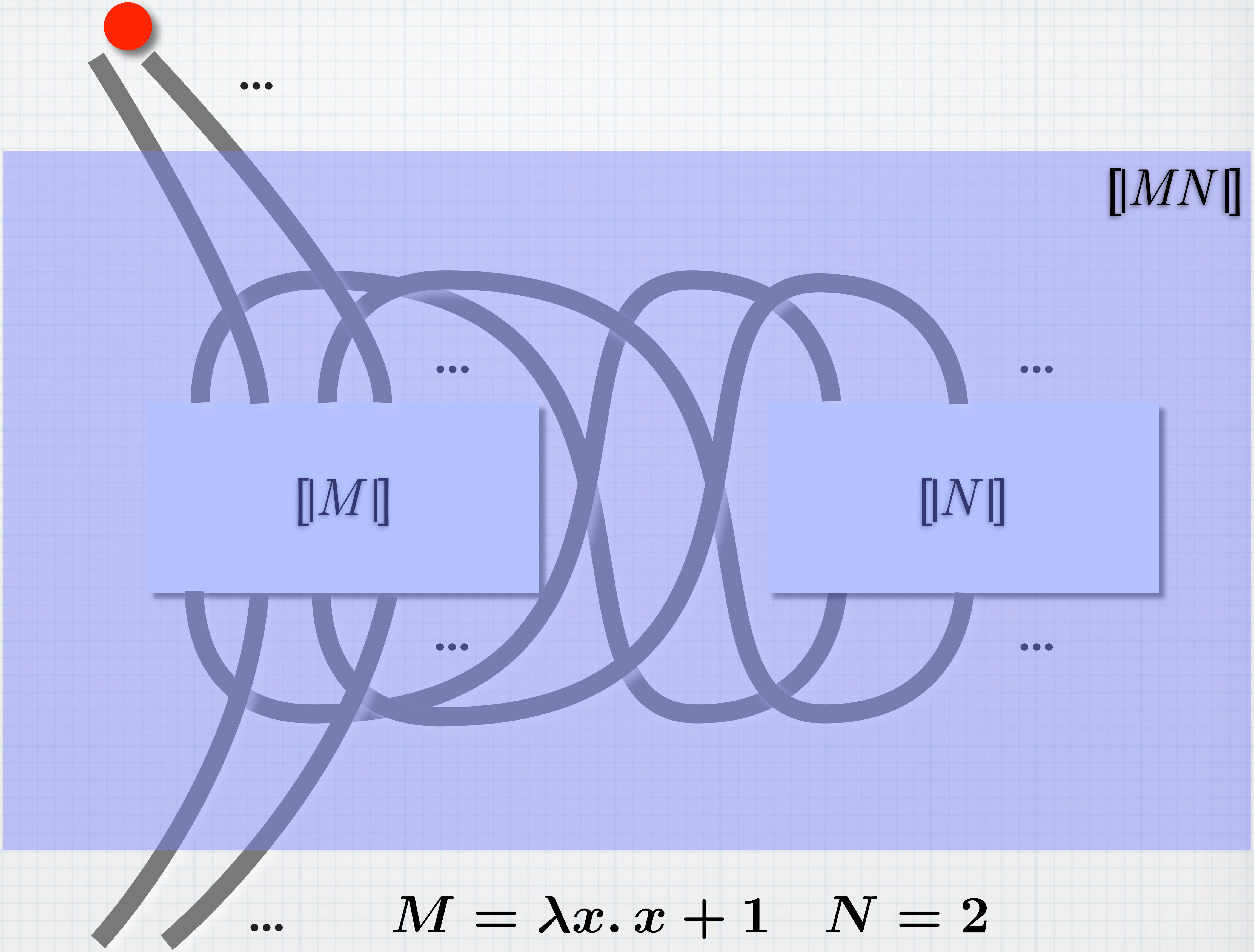

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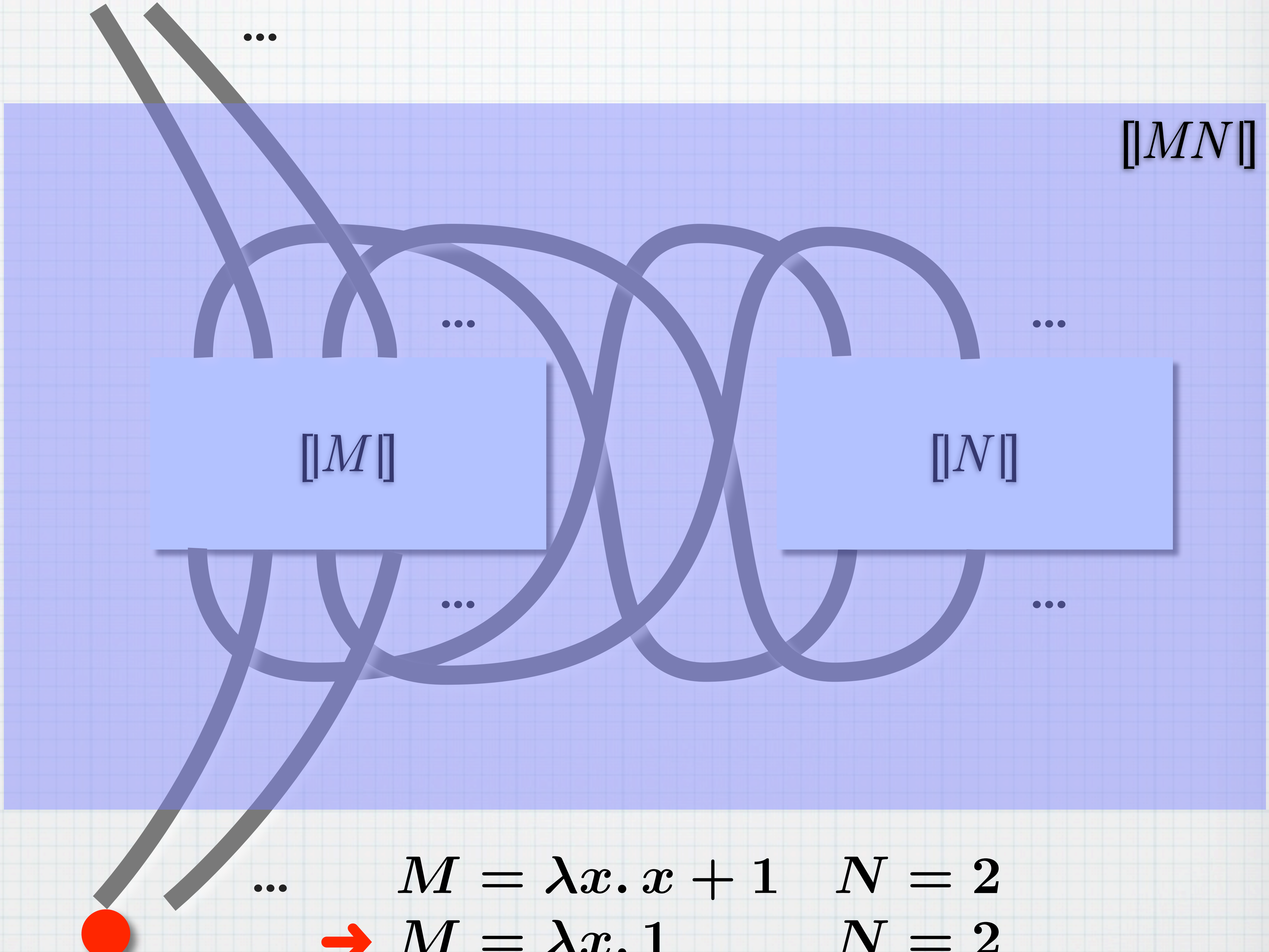

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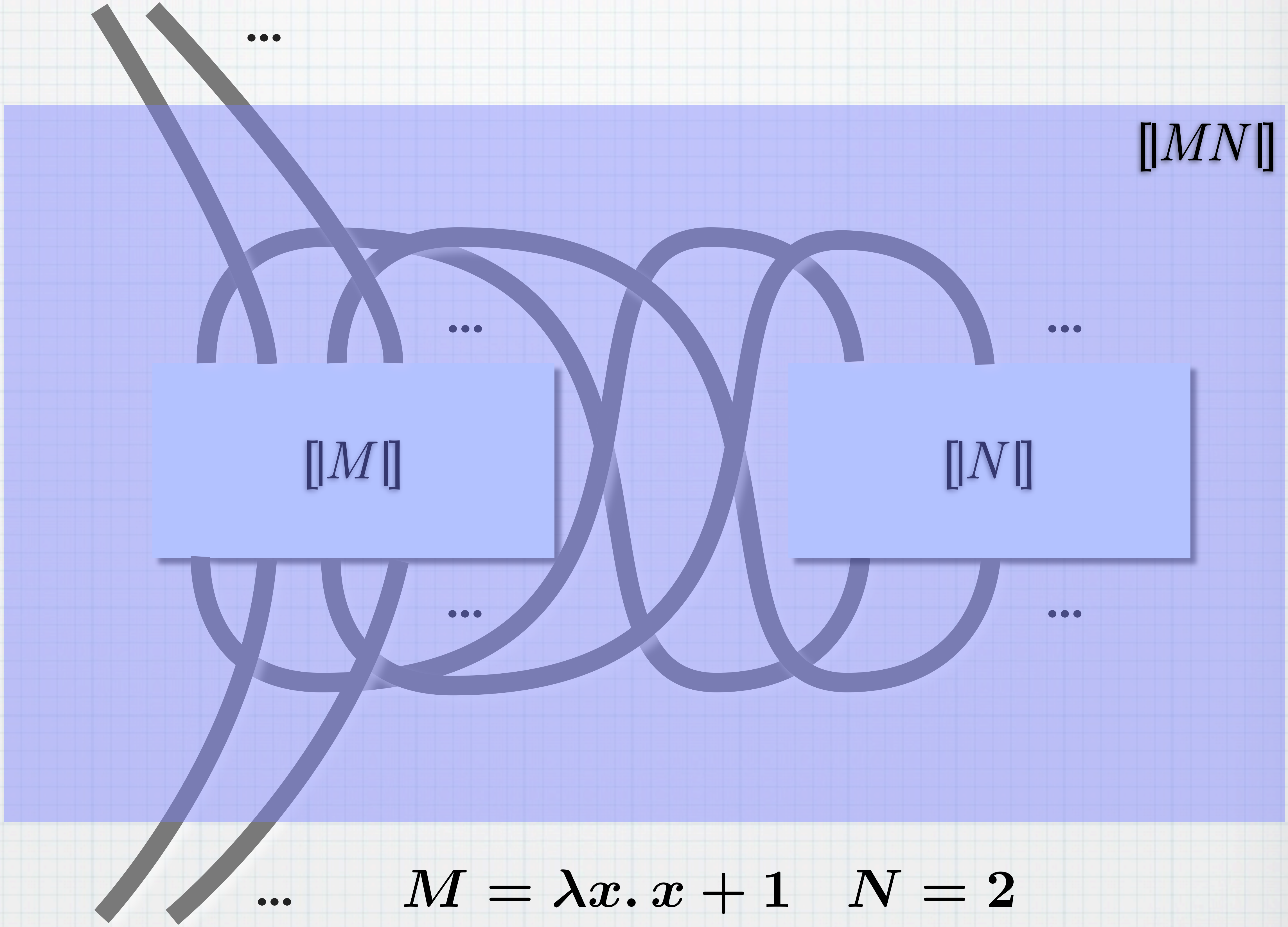

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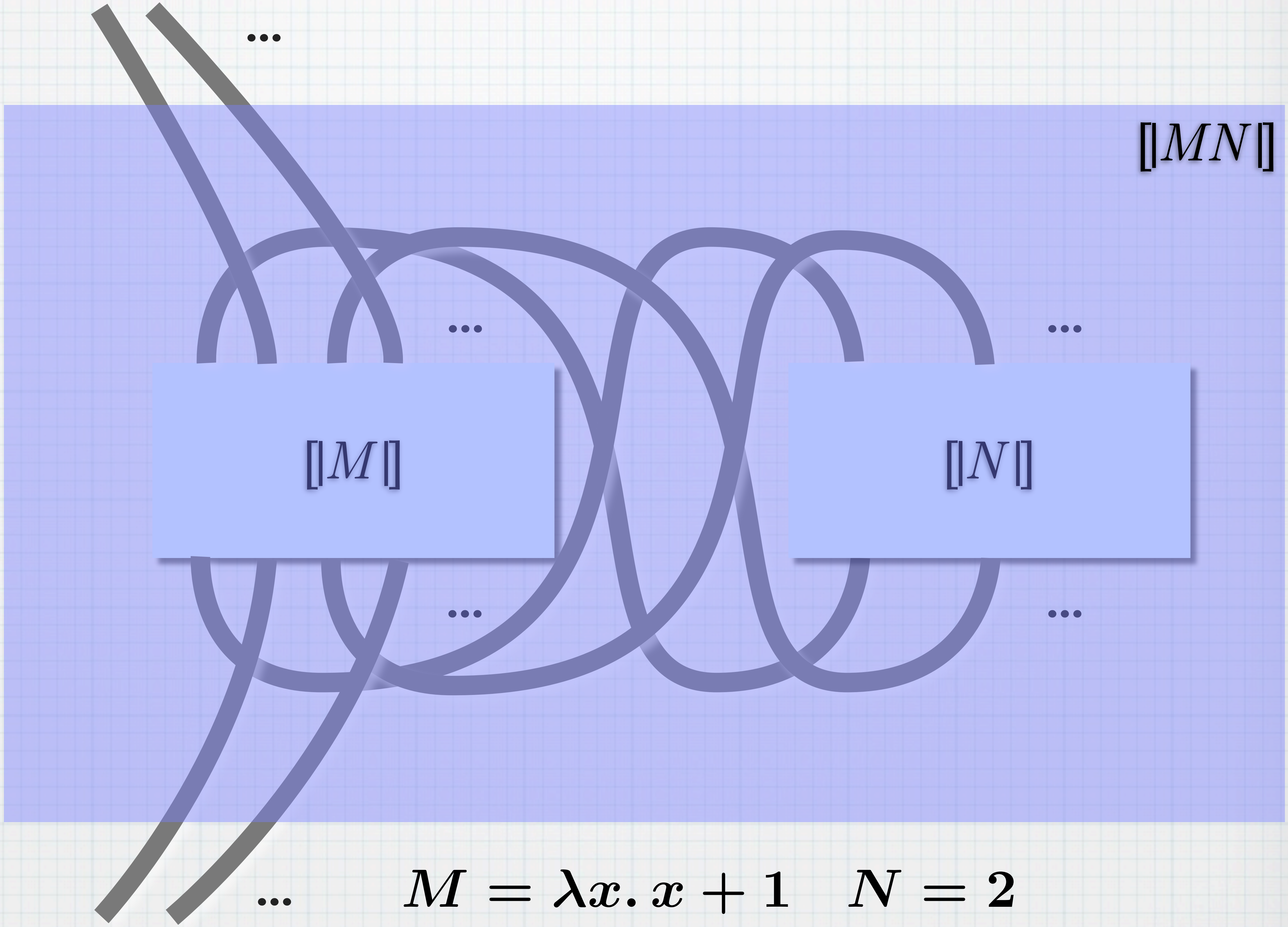

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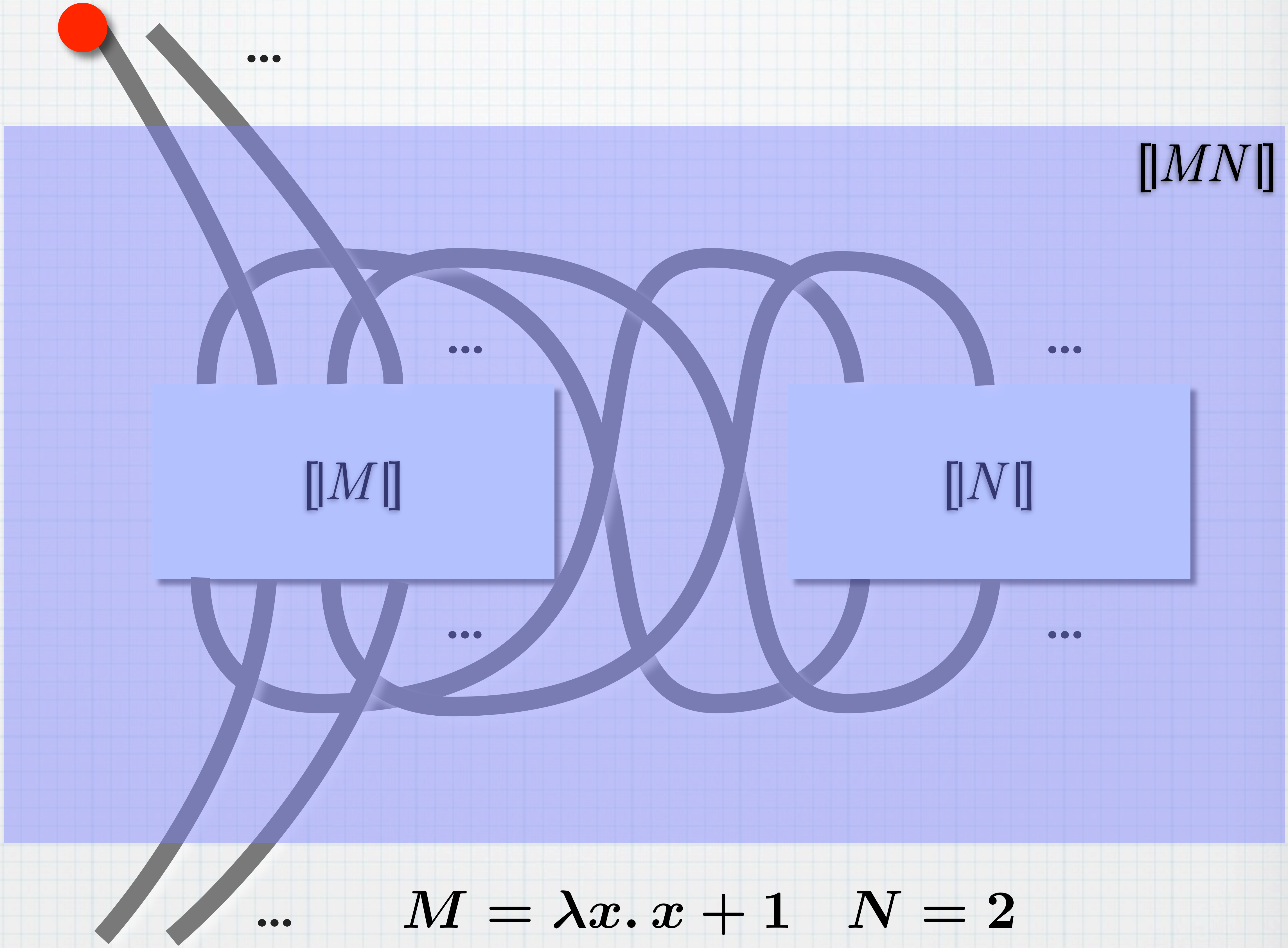

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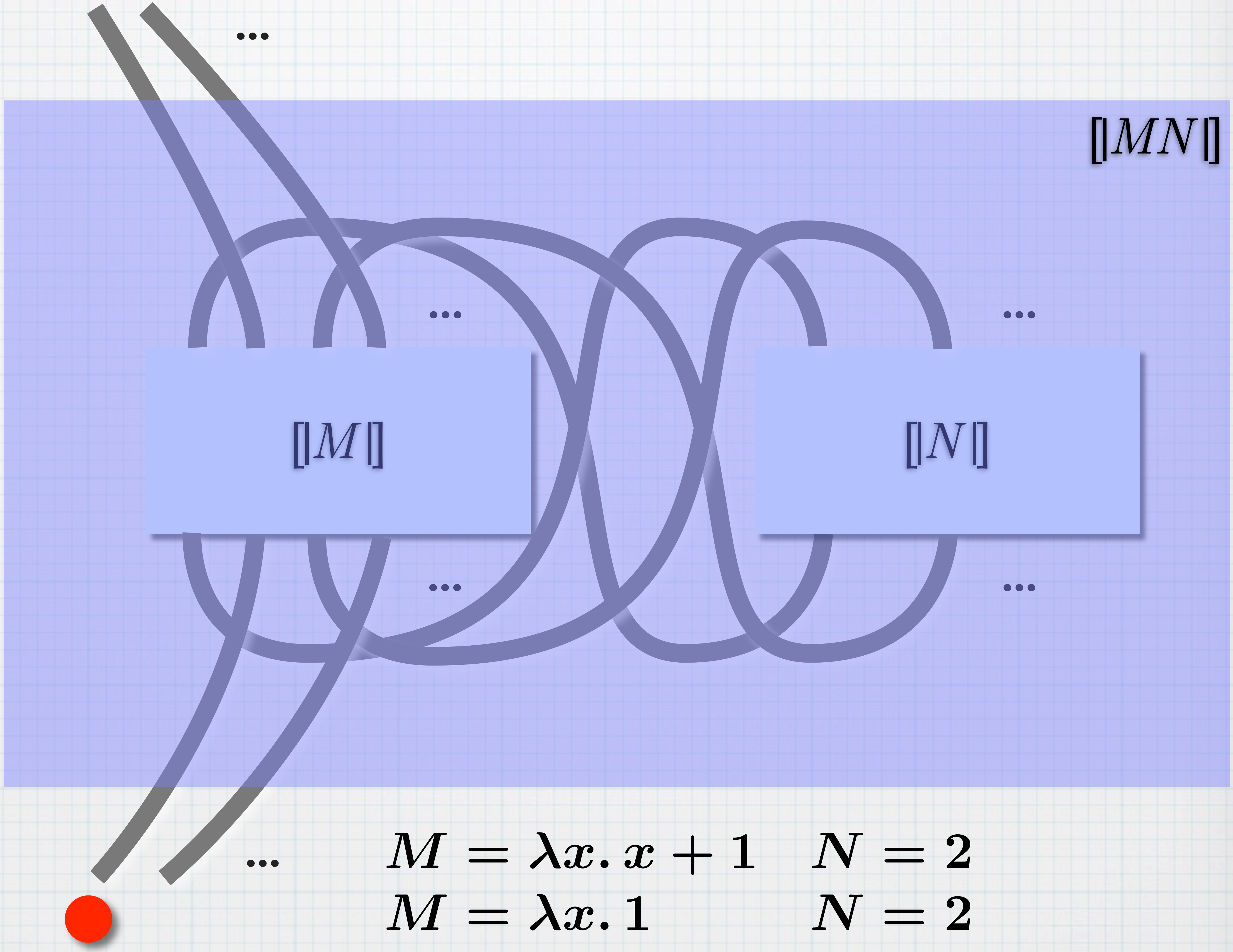

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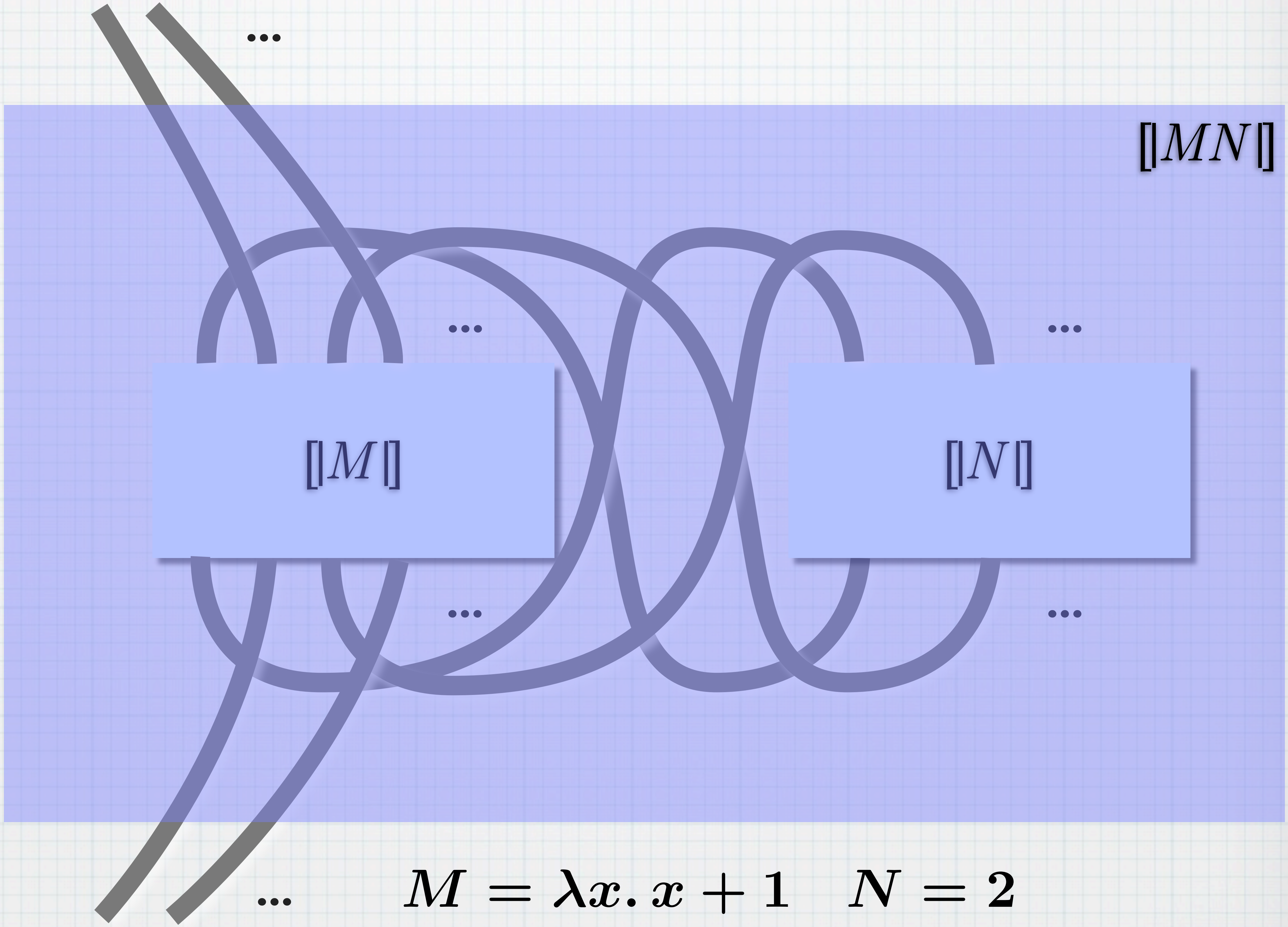

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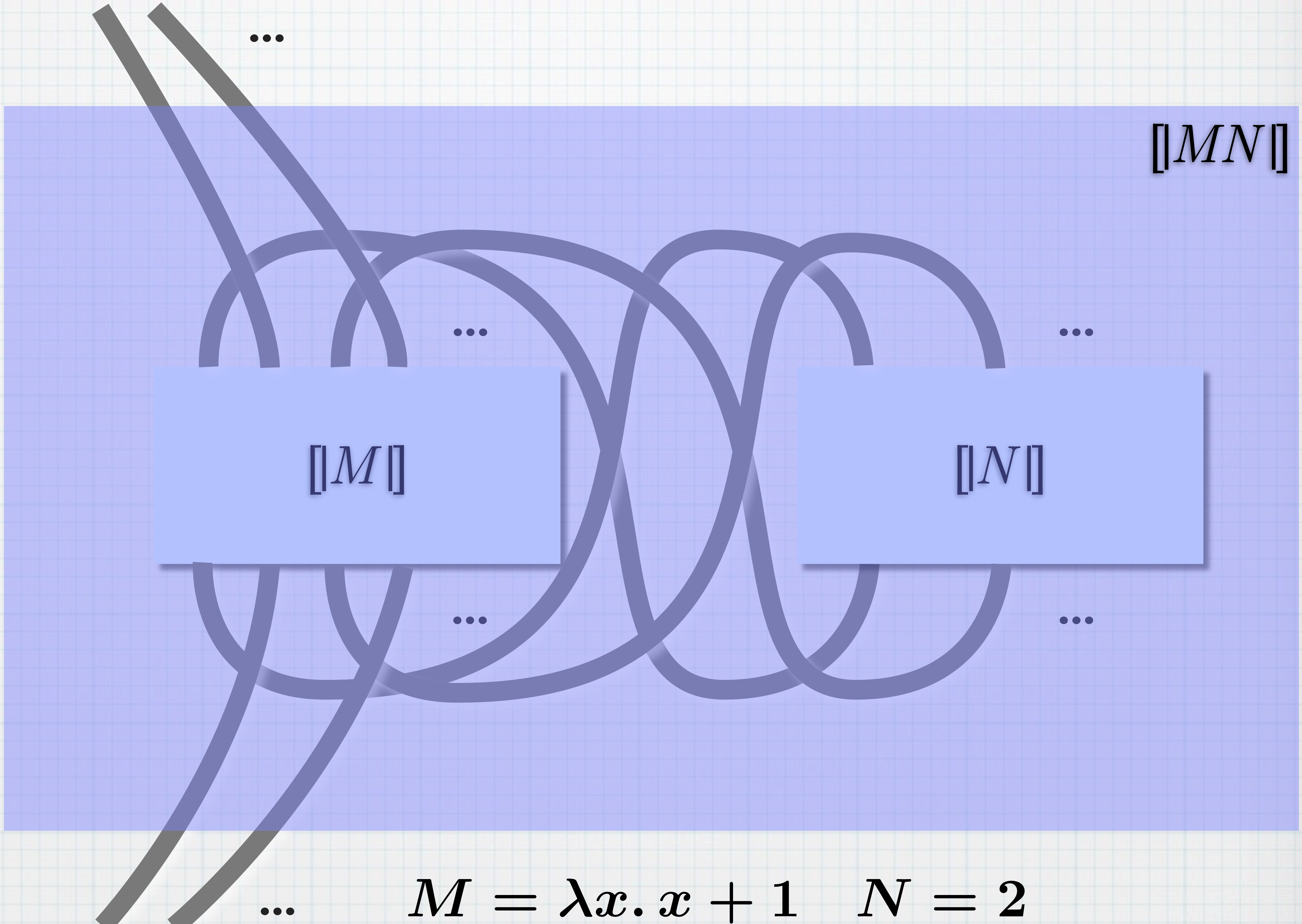

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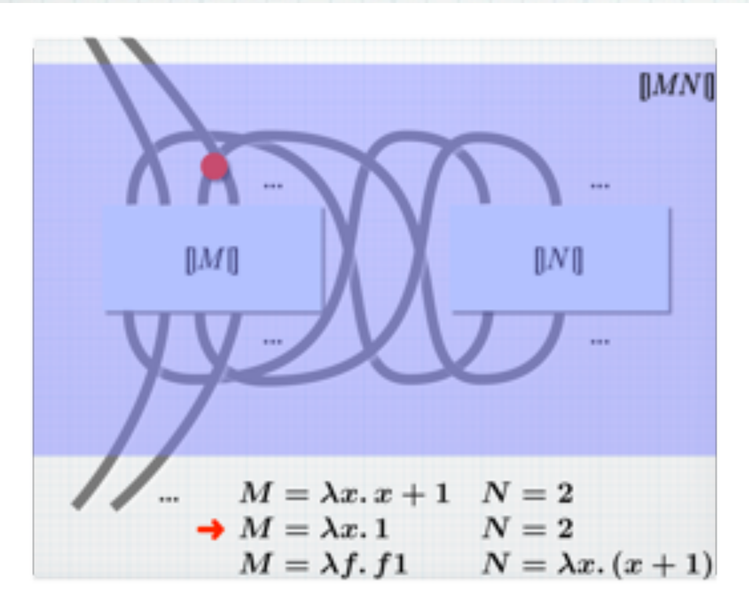

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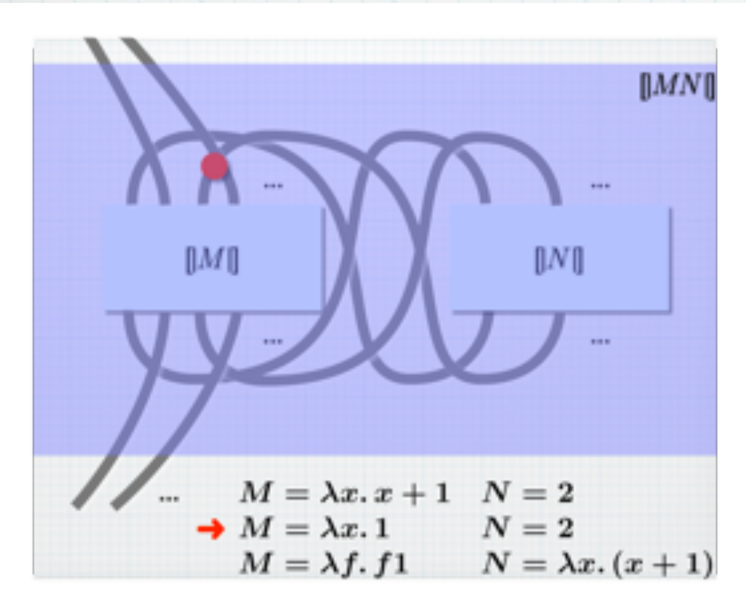
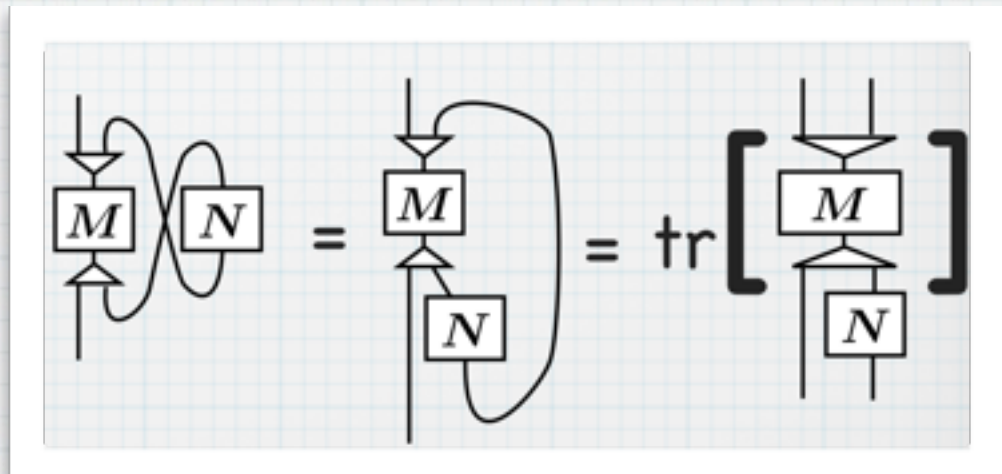
"GoI Animation"





Categorical GoI

[Abramsky, Haghverdi & Scott, MSCS'02]

Hasuo (Tokyo)

Categorical GoI

* Axiomatics of GoI in the categorical language
* Our main reference:
  * [AHS02] S. Abramsky, E. Haghverdi, and P. Scott, **Geometry of interaction and linear combinatory algebras**, Math. Str. Comp. Sci, 2002
  * Especially its technical report version (Oxford CL), since it's a bit more detailed
* See also:
  * IH and Naohiko Hoshino. **Semantics of Higher-Order Quantum Computation via Geometry of Interaction.** Extended ver. of [LICS'11], to appear in Annals Pure & Applied Logic. arxiv.org/abs/1605.05079

The Categorical GoI Workflow

Traced monoidal category $\mathbb{C}$

+ other constructs ➜ "GoI situation" [AHS02]

Categorical GoI [AHS02]

Linear combinatory algebra

Realizability

Linear category

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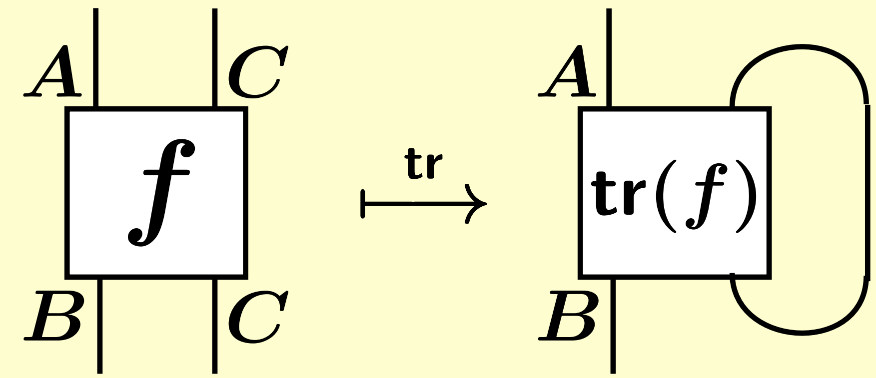

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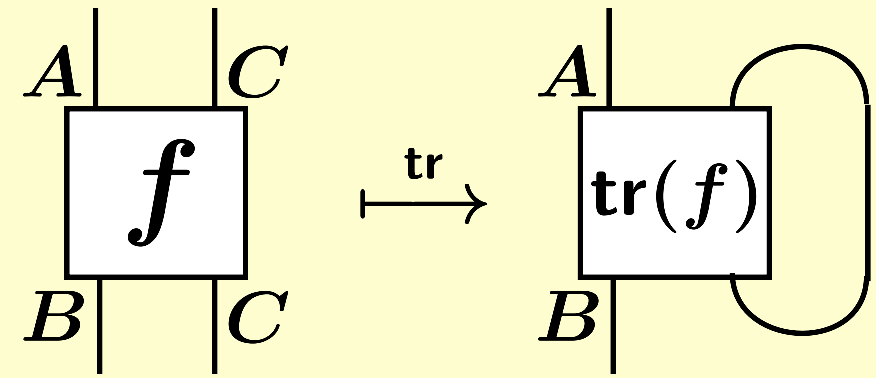

Categorical GoI [AHS02]

* Applicative str. + combinators
* Model of untyped calculus

Linear combinatory algebra

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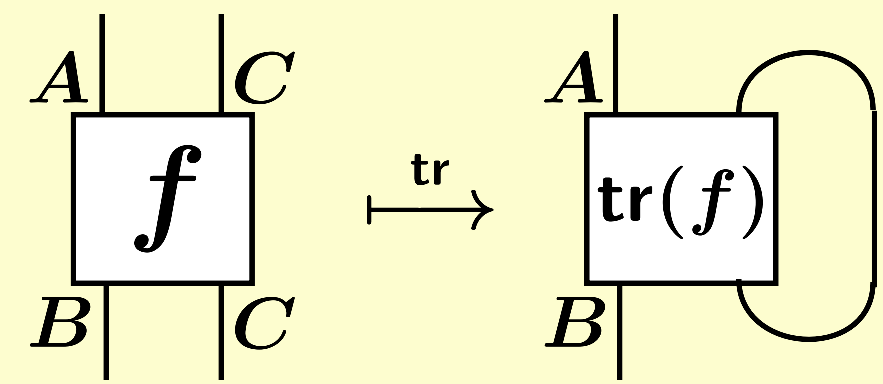

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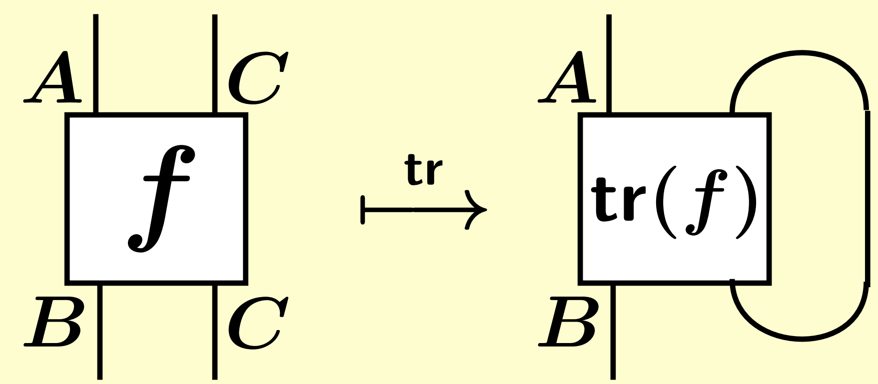

Categorical GoI [AHS02]

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Linear combinatory algebra

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* PER, ω-set, assembly, ...
* "Programming in untyped λ"

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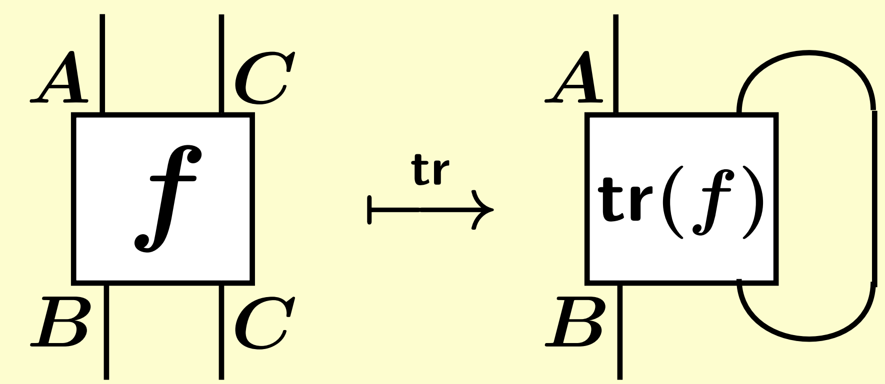

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Linear Combinatory Algebra (LCA)

Defn. (LCA)
A *linear combinatory algebra (LCA)* is a set A equipped with

- a binary operator (called an *applicative structure*)

$$\cdot \;:\; A^2 \longrightarrow A$$

- a unary operator

$$! \;:\; A \longrightarrow A$$

- (*combinators*) distinguished elements $\mathsf{B}, \mathsf{C}, \mathsf{I}, \mathsf{K}, \mathsf{W}, \mathsf{D}, \delta, \mathsf{F}$ satisfying

$\mathsf{B}xyz = x(yz)$	Composition, Cut
$\mathsf{C}xyz = (xz)y$	Exchange
$\mathsf{I}x = x$	Identity
$\mathsf{K}x\,!y = x$	Weakening
$\mathsf{W}x\,!y = x\,!y\,!y$	Contraction
$\mathsf{D}\,!x = x$	Dereliction
$\delta\,!x = !\,!x$	Comultiplication
$\mathsf{F}\,!x\,!y = !(xy)$	Monoidal functoriality

Here: $\cdot$ associates to the left; $\cdot$ is suppressed; and $!$ binds stronger than $\cdot$ does.

Hasuo (Tokyo)

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What
we want (outcome)

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* Model of
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Hasuo (Tokyo)

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* Model of
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* $a \in A \quad \approx$
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* Model of untyped linear λ

* $a \in A$ $\approx$ closed linear λ-term

* No **S** or **K** (linear!)

* Combinatory completeness: e.g.

$$\lambda xyz.\, zxy$$

designates an elem. of A

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What we use (ingredient)

GoI situation

Defn. (GoI situation [AHS02])
A *GoI situation* is a triple $(\mathbb{C}, \boldsymbol{F}, \boldsymbol{U})$ where

- $\mathbb{C} = (\mathbb{C}, \otimes, \boldsymbol{I})$ is a traced symmetric monoidal category (TSMC);
- $\boldsymbol{F} : \mathbb{C} \to \mathbb{C}$ is a traced symmetric monoidal functor, equipped with the following retractions (which are monoidal natural transformations).

$$\begin{array}{rcl}
\boldsymbol{e} : \boldsymbol{F}\boldsymbol{F} \lhd \boldsymbol{F} : \boldsymbol{e}' & & \text{Comultiplication} \\
\boldsymbol{d} : \mathbf{id} \lhd \boldsymbol{F} : \boldsymbol{d}' & & \text{Dereliction} \\
\boldsymbol{c} : \boldsymbol{F} \otimes \boldsymbol{F} \lhd \boldsymbol{F} : \boldsymbol{c}' & & \text{Contraction} \\
\boldsymbol{w} : \boldsymbol{K_I} \lhd \boldsymbol{F} : \boldsymbol{w}' & & \text{Weakening}
\end{array}$$

  Here $\boldsymbol{K_I}$ is the constant functor into the monoidal unit $\boldsymbol{I}$;
- $\boldsymbol{U} \in \mathbb{C}$ is an object (called *reflexive object*), equipped with the following retractions.

$$\begin{array}{c}
\boldsymbol{j} : \boldsymbol{U} \otimes \boldsymbol{U} \lhd \boldsymbol{U} : \boldsymbol{k} \\
\boldsymbol{I} \lhd \boldsymbol{U} \\
\boldsymbol{u} : \boldsymbol{F}\boldsymbol{U} \lhd \boldsymbol{U} : \boldsymbol{v}
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* **Monoidal category** $(\mathbb{C}, \otimes, \boldsymbol{I})$
  * **String diagrams**

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* Monoidal category $(\mathbb{C}, \otimes, I)$

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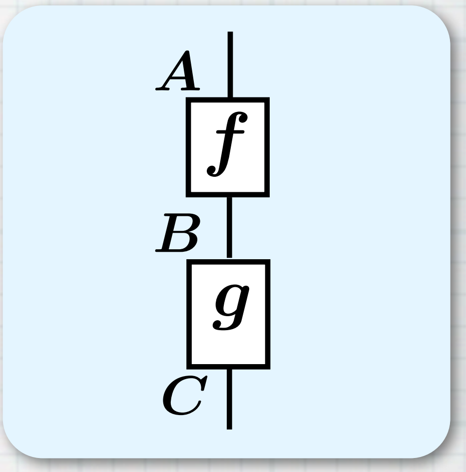
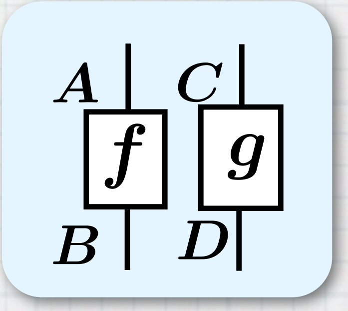
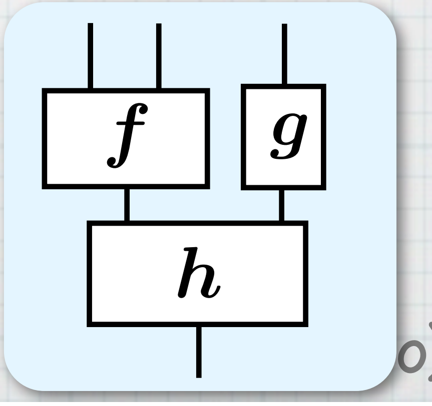
$$\frac{A \xrightarrow{f} B \quad B \xrightarrow{g} C}{A \xrightarrow{g \circ f} C}$$



$$\frac{A \xrightarrow{f} B \quad C \xrightarrow{g} D}{A \otimes C \xrightarrow{f \otimes g} B \otimes D}$$



$$h \circ (f \otimes g)$$



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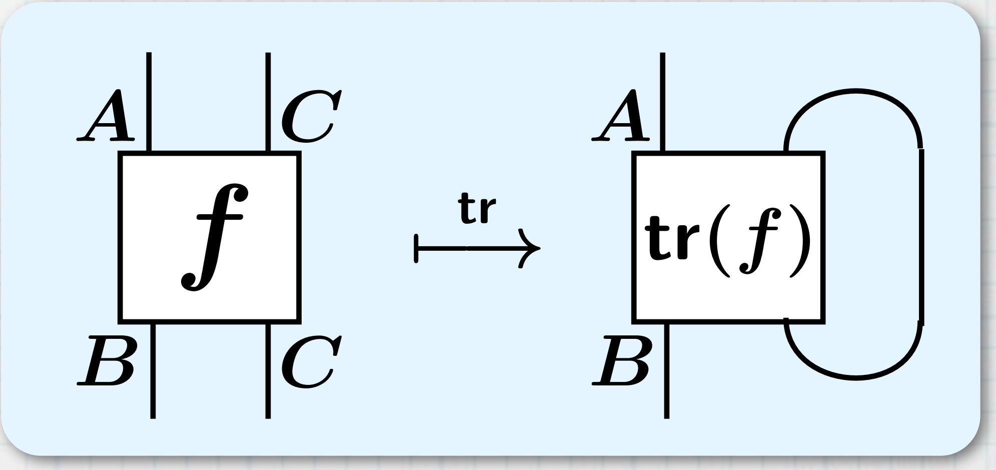
* **Traced** monoidal category

* "feedback"

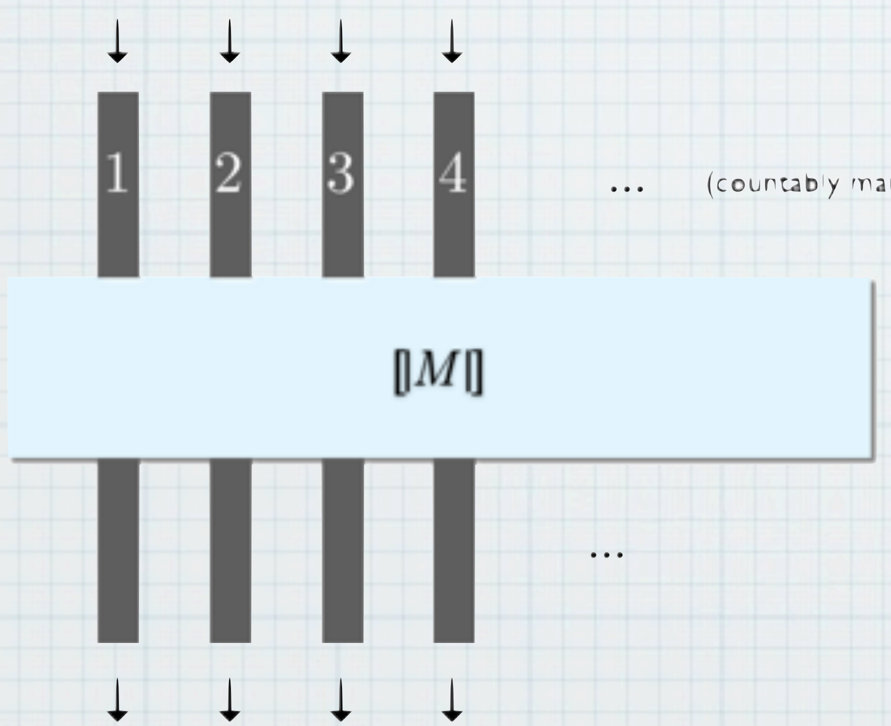
$$\frac{A \otimes C \xrightarrow{f} B \otimes C}{A \xrightarrow{\mathrm{tr}(f)} B}$$

that is

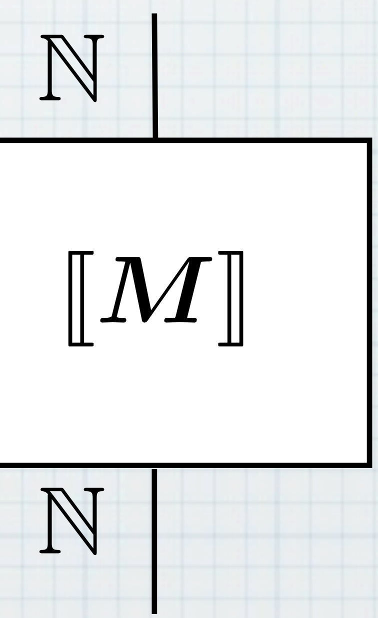


String Diagram vs. "Pipe Diagram"

* I use two ways of depicting partial functions $\mathbb{N} \rightharpoonup \mathbb{N}$



Pipe diagram



String diagram

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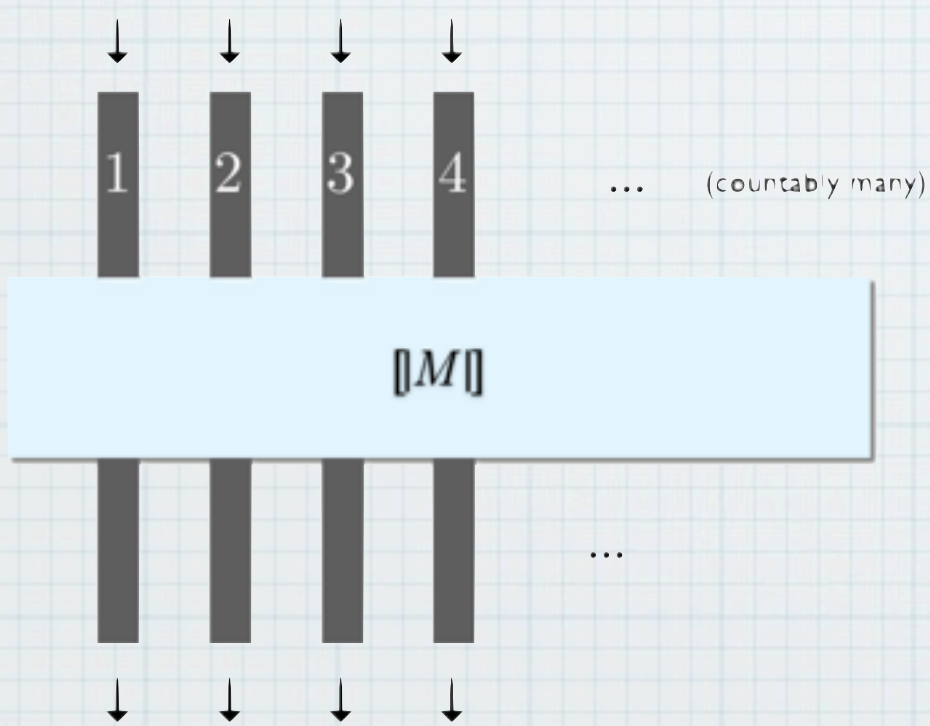
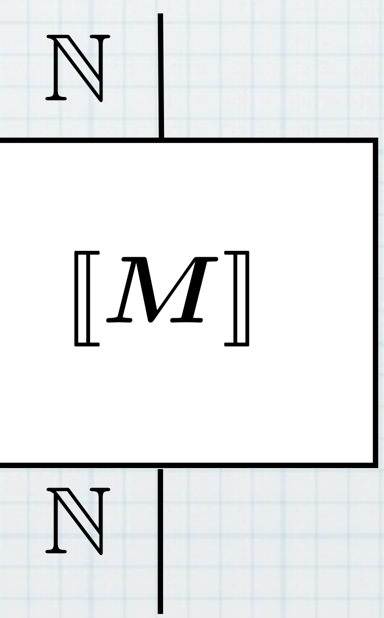


Pipe diagram

In the monoidal category $(\mathbf{Pfn}, +, 0)$

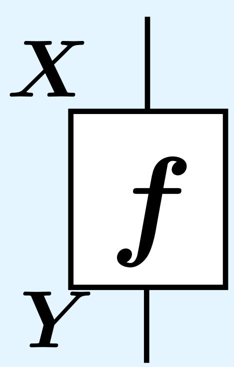


String diagram

Traced Sym. Monoidal Category $(\mathbf{Pfn}, +, 0)$

* Category $\mathbf{Pfn}$ of **partial functions**
  * Obj. A set X
  * Arr. A partial function

$$\frac{X \to Y \ \text{ in } \mathbf{Pfn}}{X \rightharpoonup Y, \ \text{partial function}}$$



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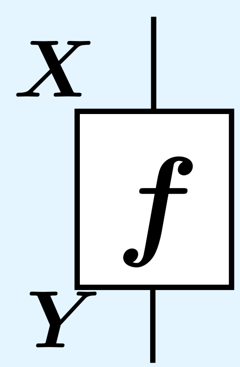

* is traced symmetric monoidal

Traced Sym. Monoidal Category

$$(\mathbf{Pfn}, +, 0)$$

* $$\dfrac{X + Z \xrightarrow{f} Y + Z \quad \text{in } \mathbf{Pfn}}{X \xrightarrow{\mathrm{tr}(f)} Y \quad \text{in } \mathbf{Pfn}}$$

  How?

*

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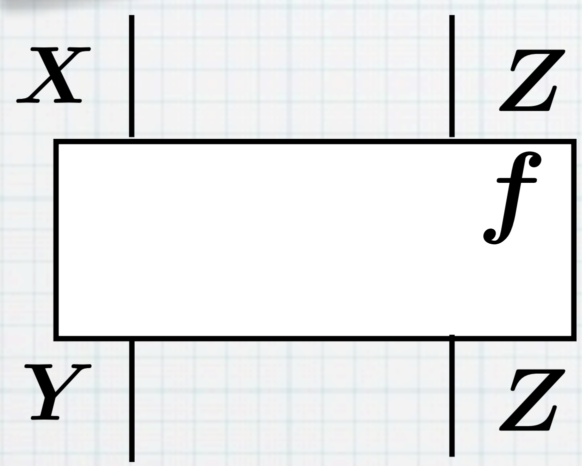
- 

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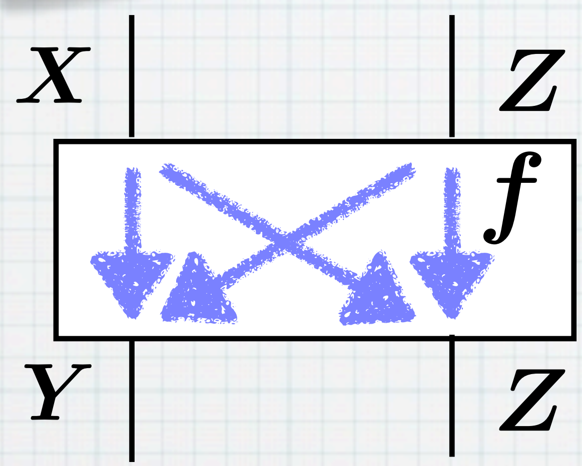
* 

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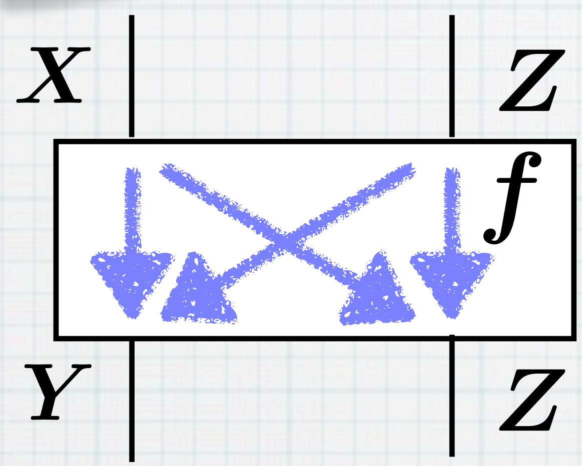
* 

$$f_{XY}(x) := \begin{cases} f(x) & \text{if } f(x) \in Y \\ \perp & \text{o.w.} \end{cases}$$

Similar for f_{XZ}, f_{ZY}, f_{ZZ}

Traced Sym. Monoidal Category
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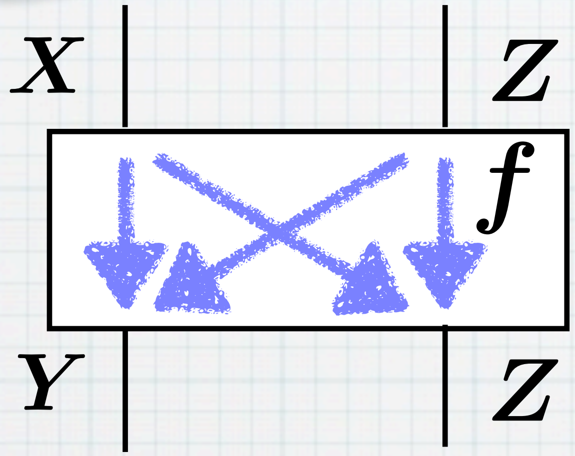
* 

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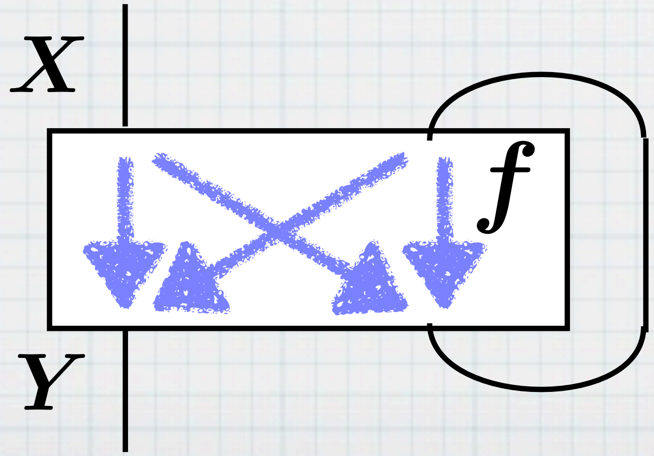
Similar for f_{XZ}, f_{ZY}, f_{ZZ}

* Trace operator:



Traced Sym. Monoidal Category

$(\mathbf{Pfn}, +, 0)$

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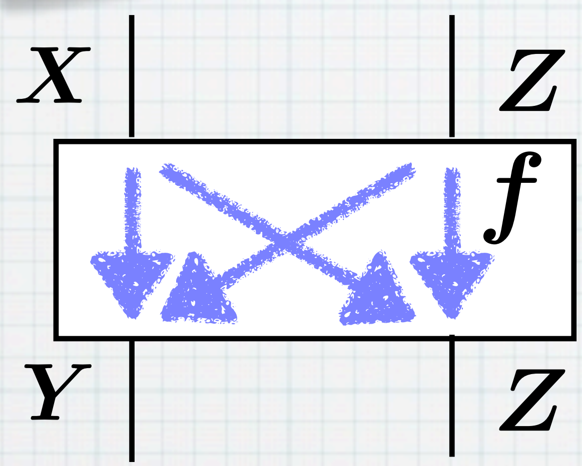
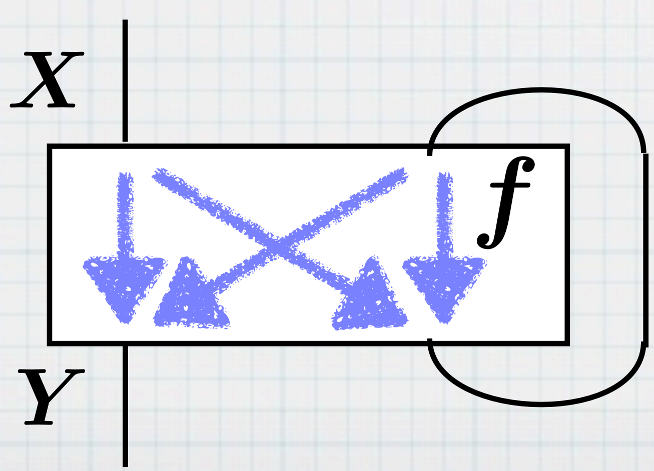
How?

* 

$$f_{XY}(x) := \begin{cases} f(x) & \text{if } f(x) \in Y \\ \bot & \text{o.w.} \end{cases}$$

Similar for f_{XZ}, f_{ZY}, f_{ZZ}

* Trace operator:



$$\mathsf{tr}(f) = f_{XY} \sqcup \left(\coprod_{n \in \mathbb{N}} f_{ZY} \circ (f_{ZZ})^n \circ f_{XZ} \right)$$

(Tokyo)

Traced Sym. Monoidal Category

$(\mathbf{Pfn}, +, 0)$

* $$\dfrac{X + Z \xrightarrow{f} Y + Z \quad \text{in } \mathbf{Pfn}}{X \xrightarrow{\mathsf{tr}(f)} Y \quad \text{in } \mathbf{Pfn}}$$

How?

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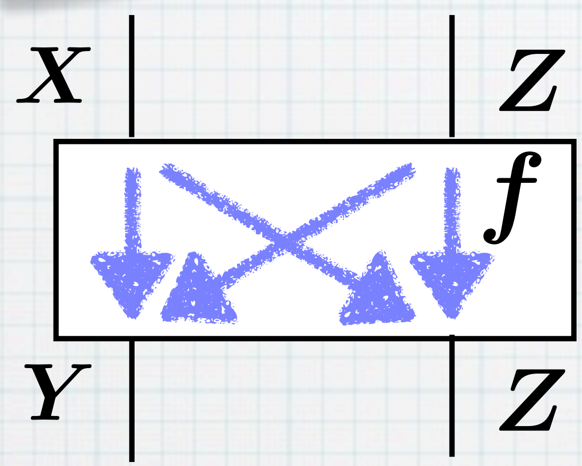

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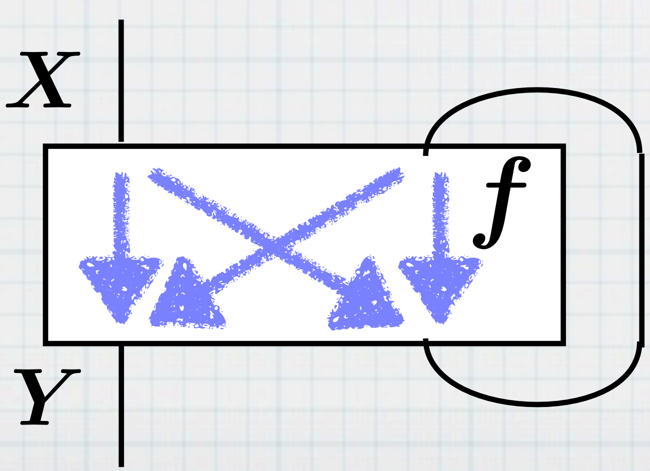
* **Execution formula** (Girard)
* Partiality is essential (infinite loop)

* Trace operator:



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(Tokyo)

GoI situation

Defn. (GoI situation [AHS02])
A *GoI situation* is a triple $(\mathbb{C}, F, U)$ where

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$$\begin{array}{rclcl} e & : & FF \lhd F & : & e' \\ d & : & \mathrm{id} \lhd F & : & d' \\ c & : & F \otimes F \lhd F & : & c' \\ w & : & K_I \lhd F & : & w' \end{array}$$

Comultiplication, Dereliction, Contraction, Weakening (respectively)

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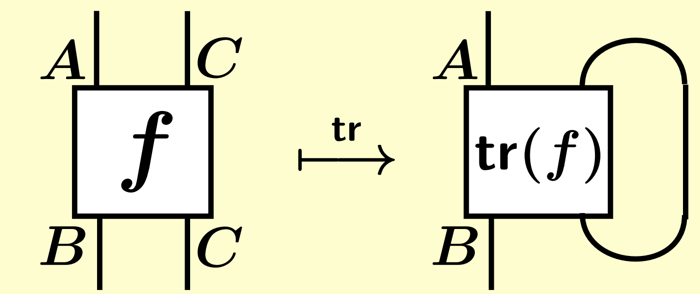
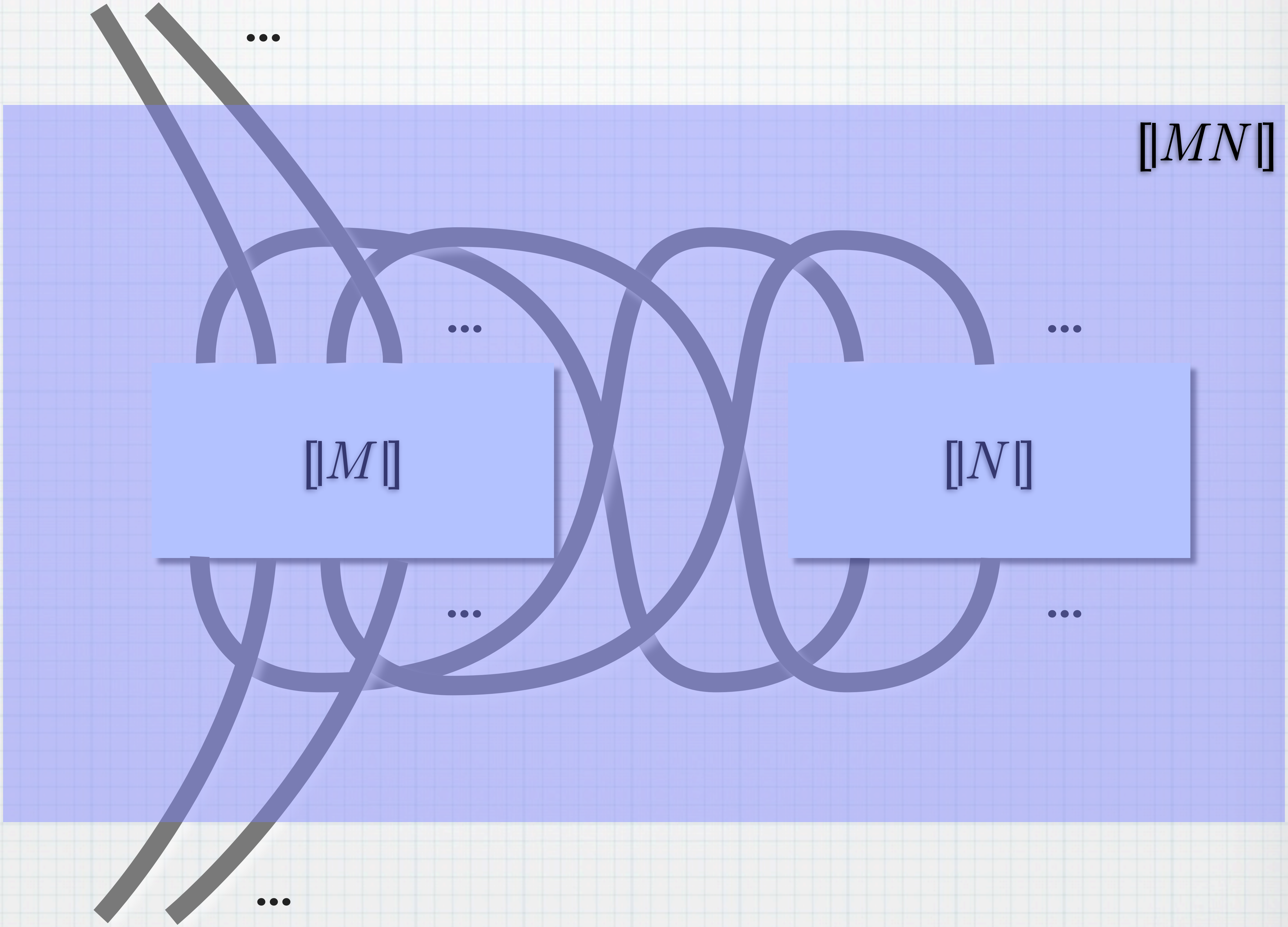
$$\begin{array}{rclcl} j & : & U \otimes U \lhd U & : & k \\ & & I \lhd U & & \\ u & : & FU \lhd U & : & v \end{array}$$

* Traced sym. monoidal cat.
  * Where one can "feedback"



  * Why for GoI?

$$[\![MN]\!] =$$



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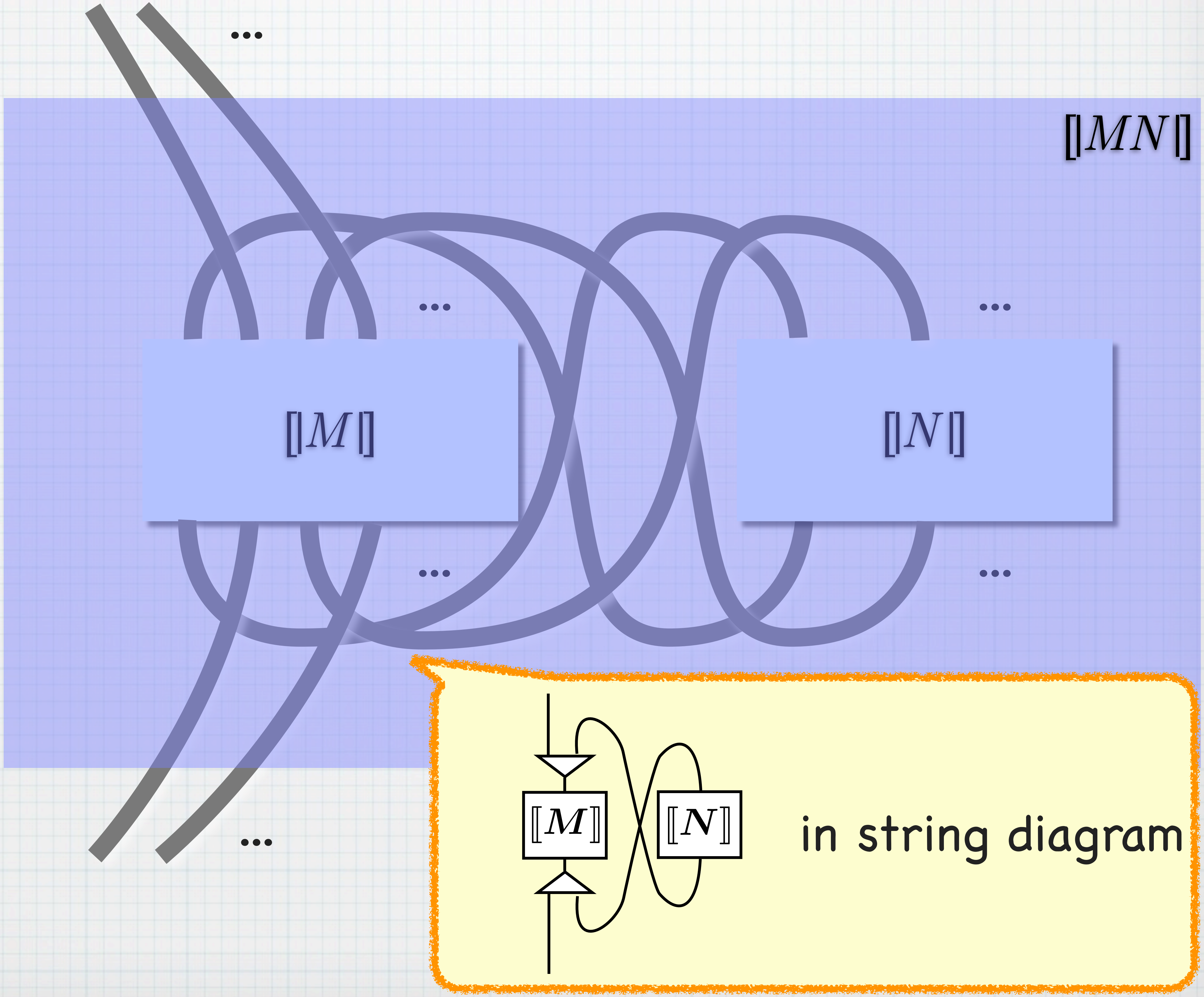

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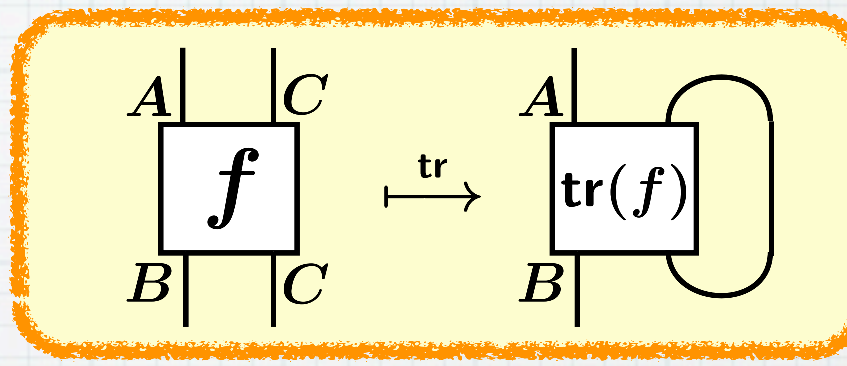

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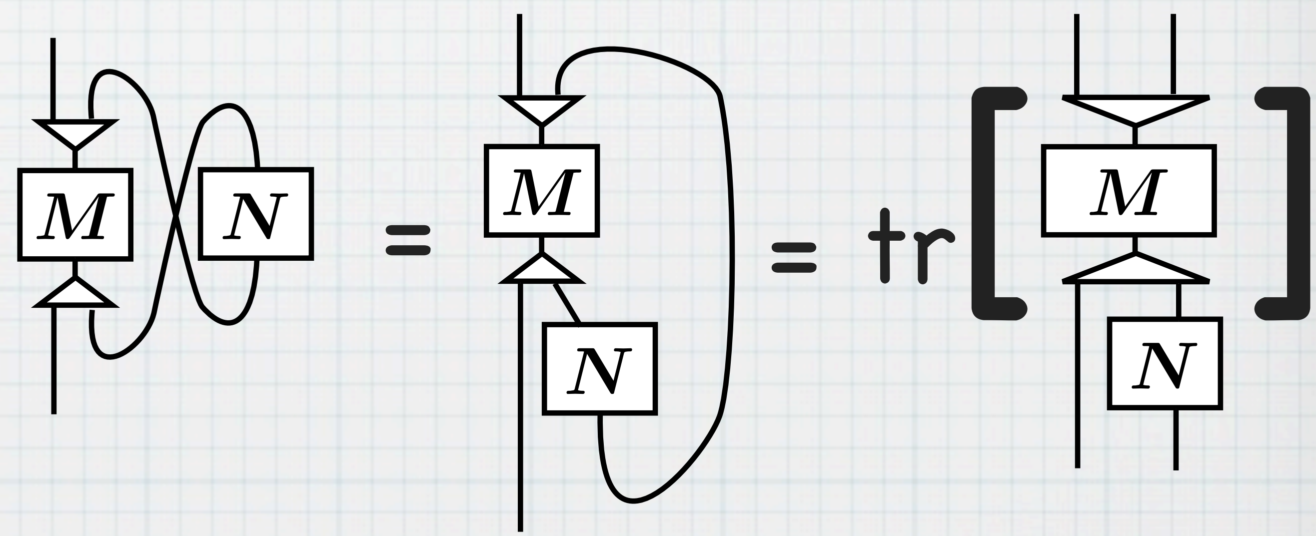

  - Leading example: **Pfn**

Hasuo (Tokyo)

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Defn. (Retraction)
A *retraction* from $\boldsymbol{X}$ to $\boldsymbol{Y}$,

$$\boldsymbol{f} : \boldsymbol{X} \lhd \boldsymbol{Y} : \boldsymbol{g} \ ,$$

is a pair of arrows

$$\begin{tikzcd} \boldsymbol{X} \arrow[loop left, "\mathbf{id}"] \arrow[r, bend left, "\boldsymbol{f}"] & \boldsymbol{Y} \arrow[l, bend left, "\boldsymbol{g}"] \end{tikzcd}$$

"embedding"

"projection"

such that $\boldsymbol{g} \circ \boldsymbol{f} = \mathbf{id}_{\boldsymbol{X}}$.

- Functor F
  - For obtaining $! : A \to A$

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- The reflexive object U
- Retr. $\begin{tikzcd} U \otimes U \arrow[r, bend left, "j"] & U \arrow[l, bend left, "k"] \end{tikzcd}$

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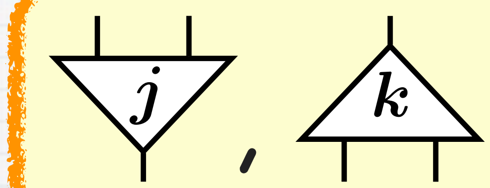

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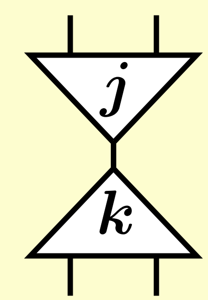

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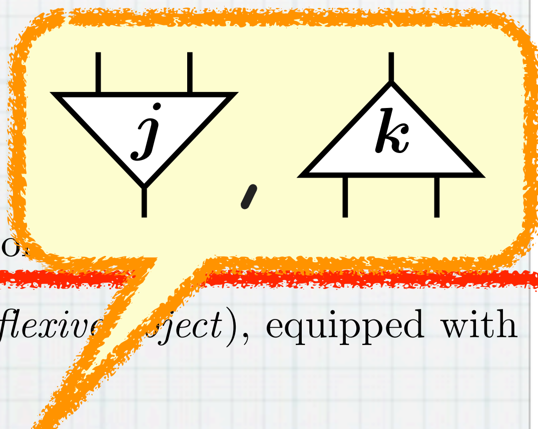

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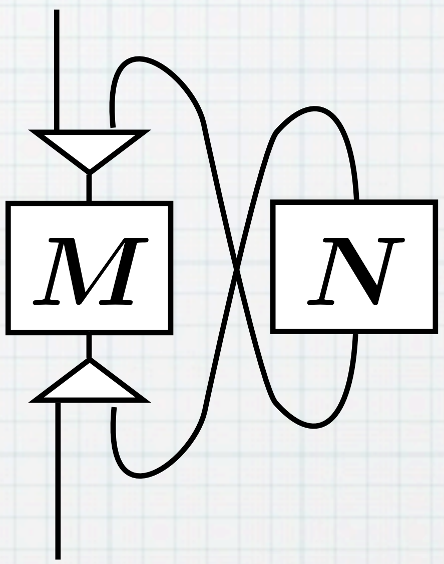

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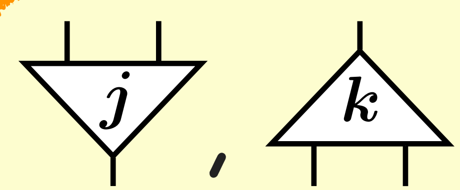

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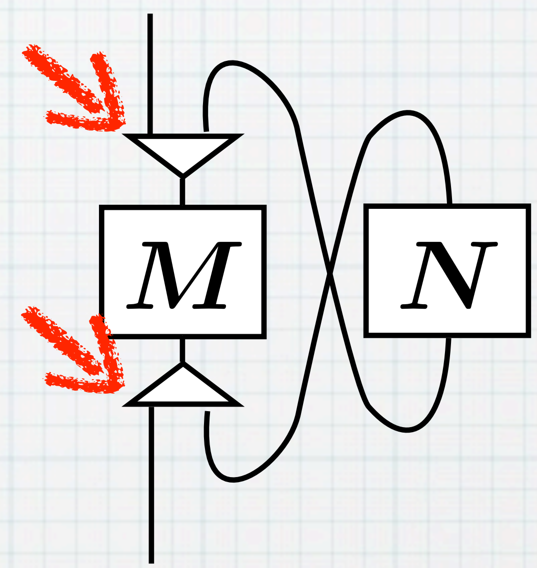

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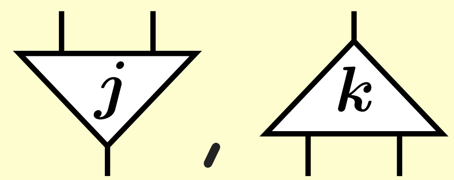

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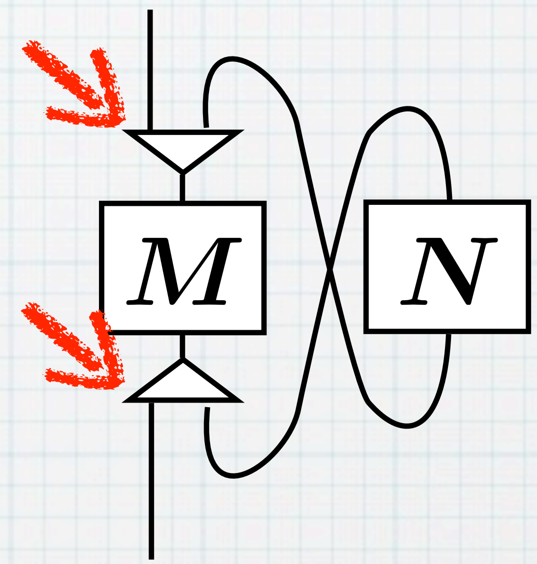

- Example in **Pfn**:

$$\mathbb{N} \in \mathbf{Pfn}, \text{ with}$$
$$\mathbb{N} + \mathbb{N} \cong \mathbb{N},$$
$$\mathbb{N} \cdot \mathbb{N} \cong \mathbb{N}$$

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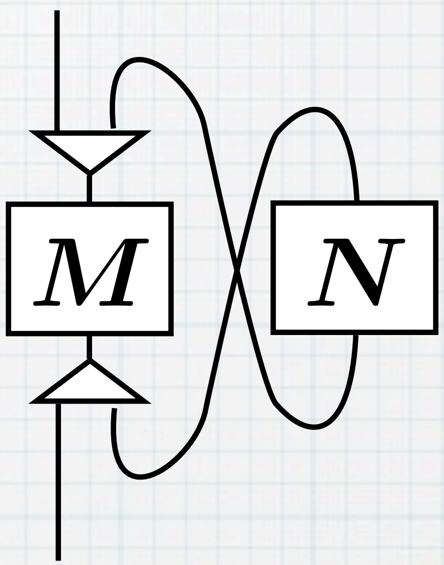

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$$(\mathbf{Pfn}, \mathbb{N} \cdot \_\_, \mathbb{N})$$

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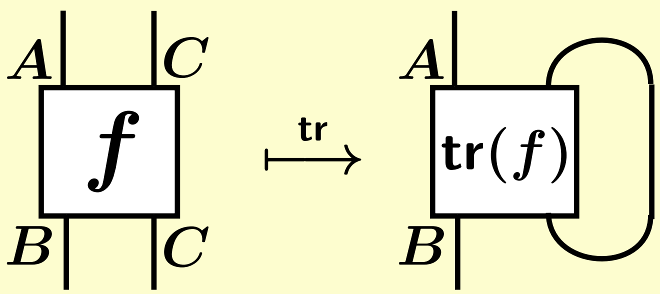

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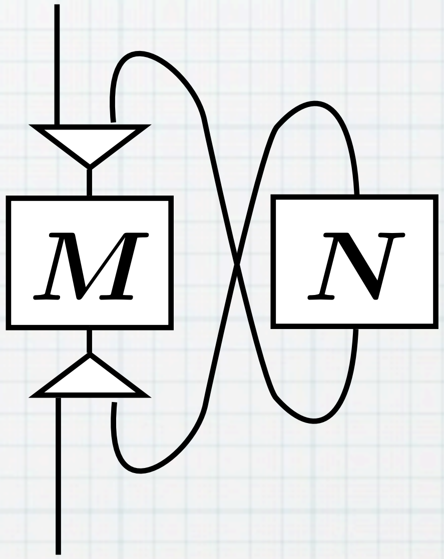

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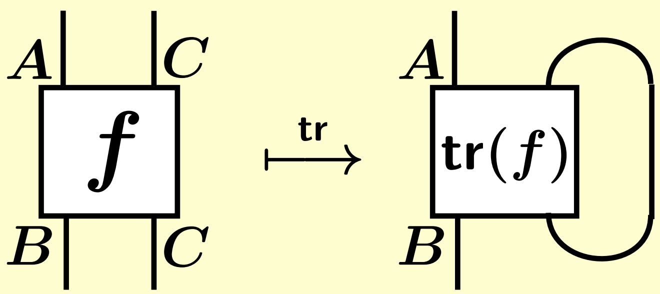

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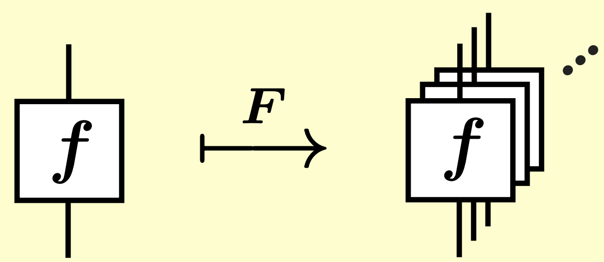

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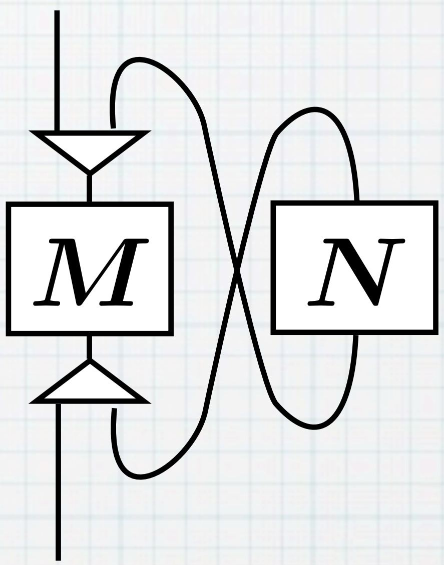

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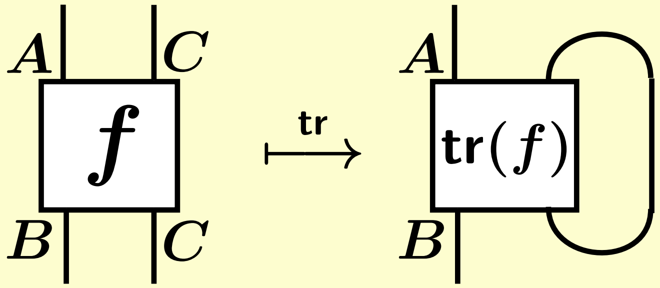

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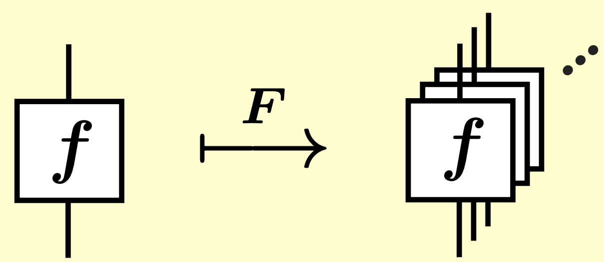

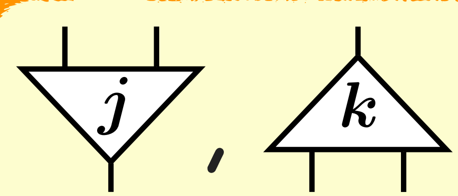

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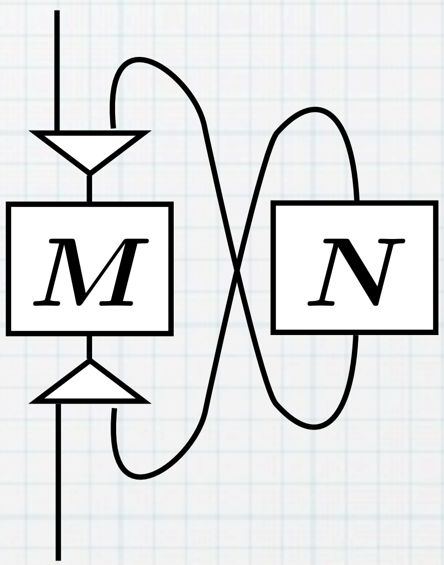
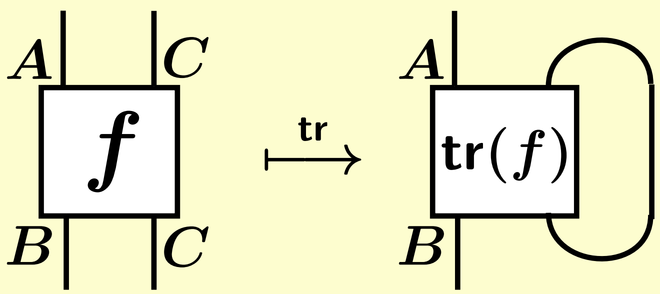


- Example:

$$(\mathbf{Pfn}, \mathbb{N} \cdot \_\,, \mathbb{N})$$

tuation: Summary



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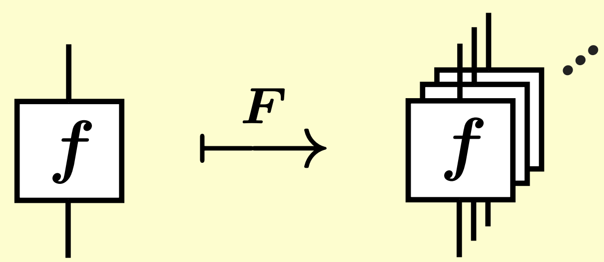
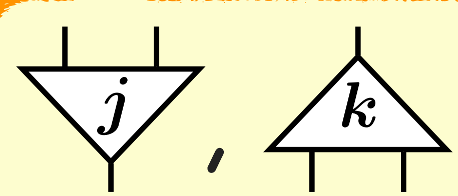
$$I \lhd U$$
$$u : FU \lhd U : v$$

For ! , via





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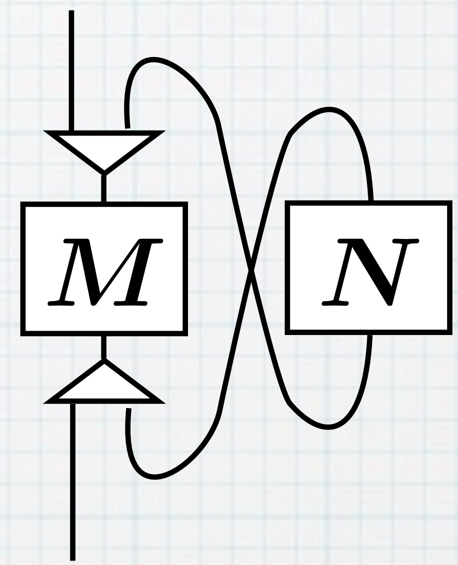

* Example:

$(\mathbf{Pfn}, \mathbb{N} \cdot \_\_, \mathbb{N})$

Hasuo (Tokyo)

Friday, June 24, 16

Categorical GoI: Constr. of an LCA

Thm. ([AHS02])
Given a GoI situation $(\mathbb{C}, \boldsymbol{F}, \boldsymbol{U})$, the homset

$$\mathbb{C}(\boldsymbol{U}, \boldsymbol{U})$$

carries a canonical LCA structure.

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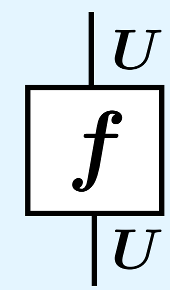

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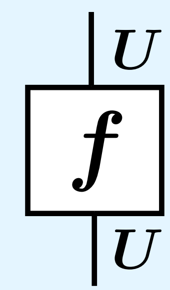
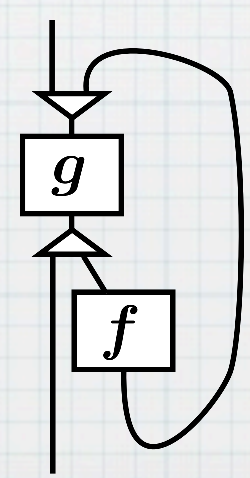
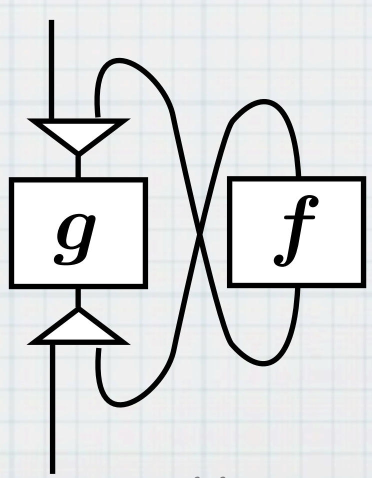


$\in \mathbb{C}(U, U)$

- $g \cdot f$

$$:= \mathbf{tr}\big(\, (U \otimes f) \circ k \circ g \circ j \,\big)$$

$=$ 

$=$ 

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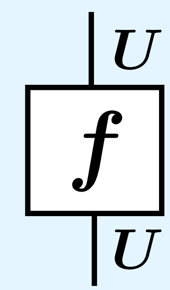
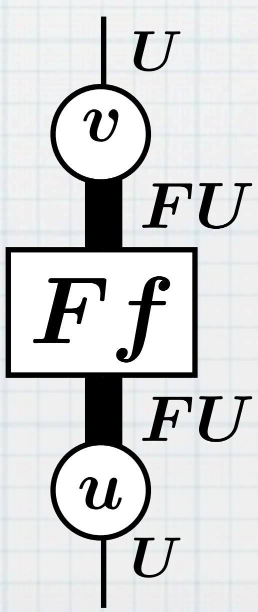


$\in \mathbb{C}(\boldsymbol{U}, \boldsymbol{U})$

* $!\boldsymbol{f} \;:=\; \boldsymbol{u} \circ \boldsymbol{F}\boldsymbol{f} \circ \boldsymbol{v}$

$=$



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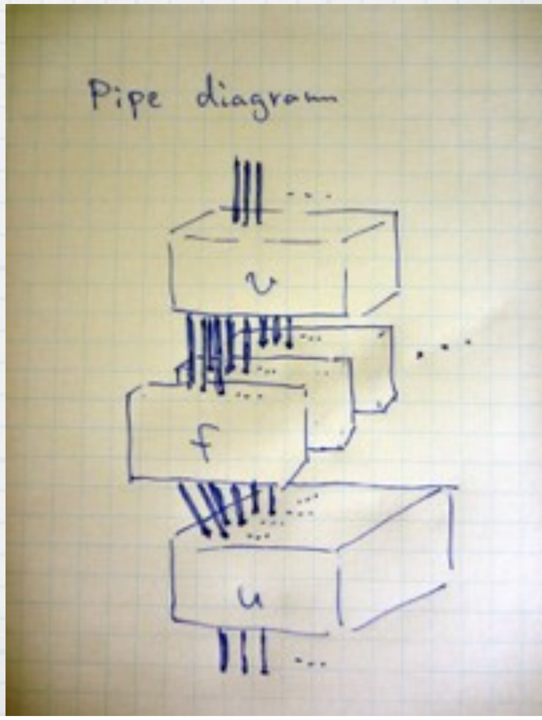

Categorical GoI: Constr. of an LCA

* Combinator $Bxyz = x(yz)$

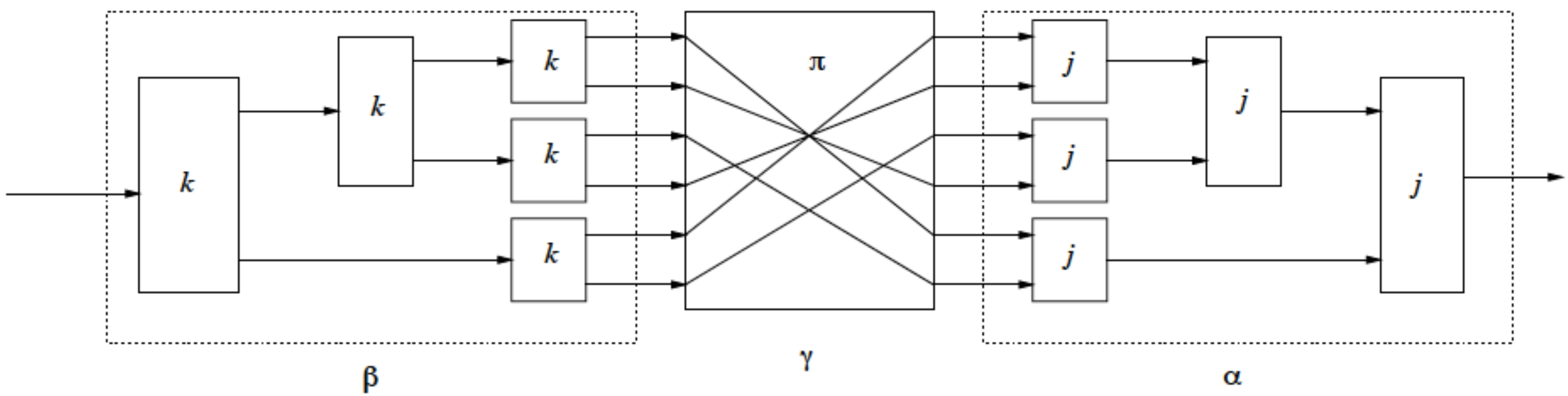

Figure 7: Composition Combinator B

from [AHS02]

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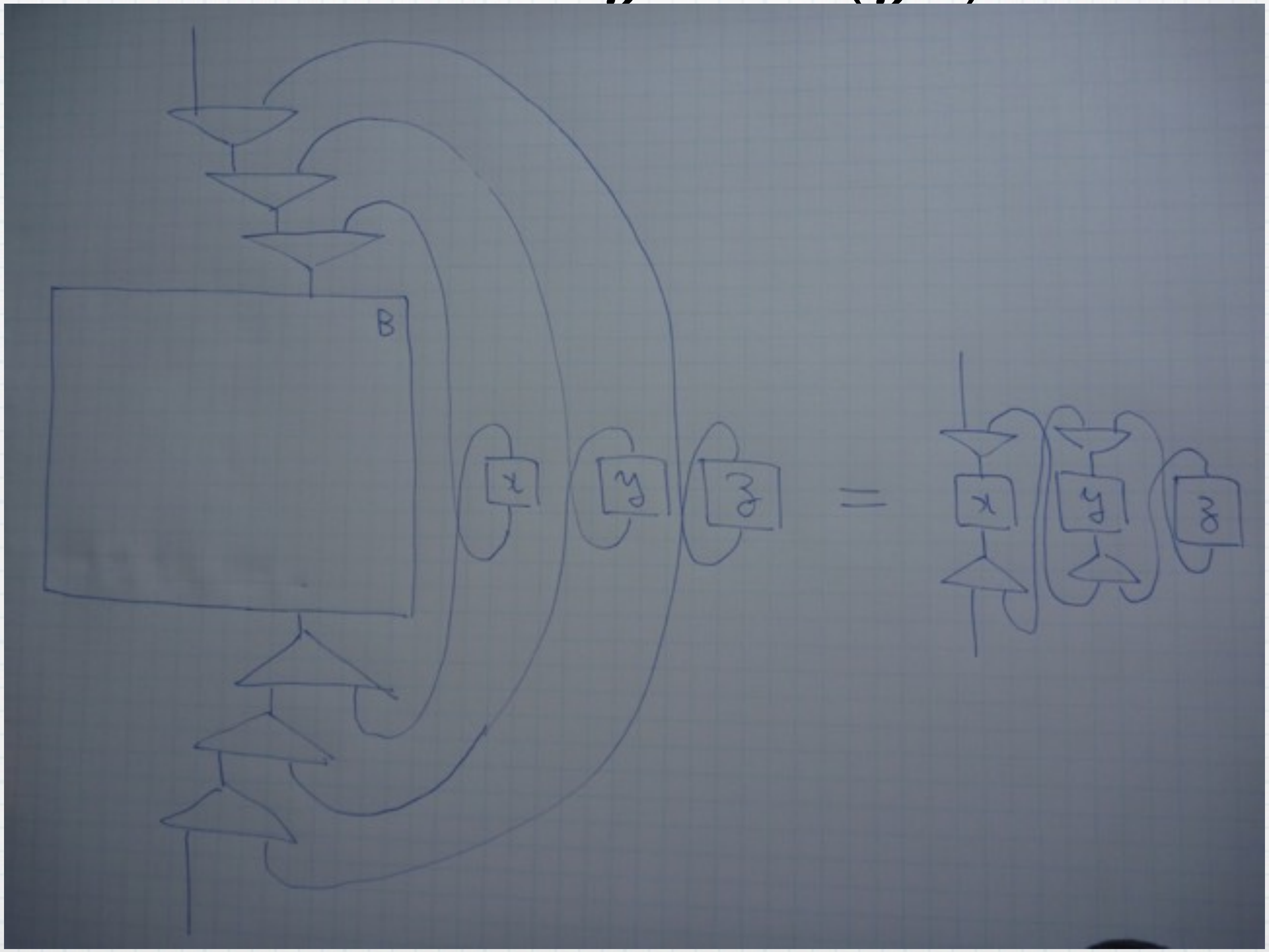
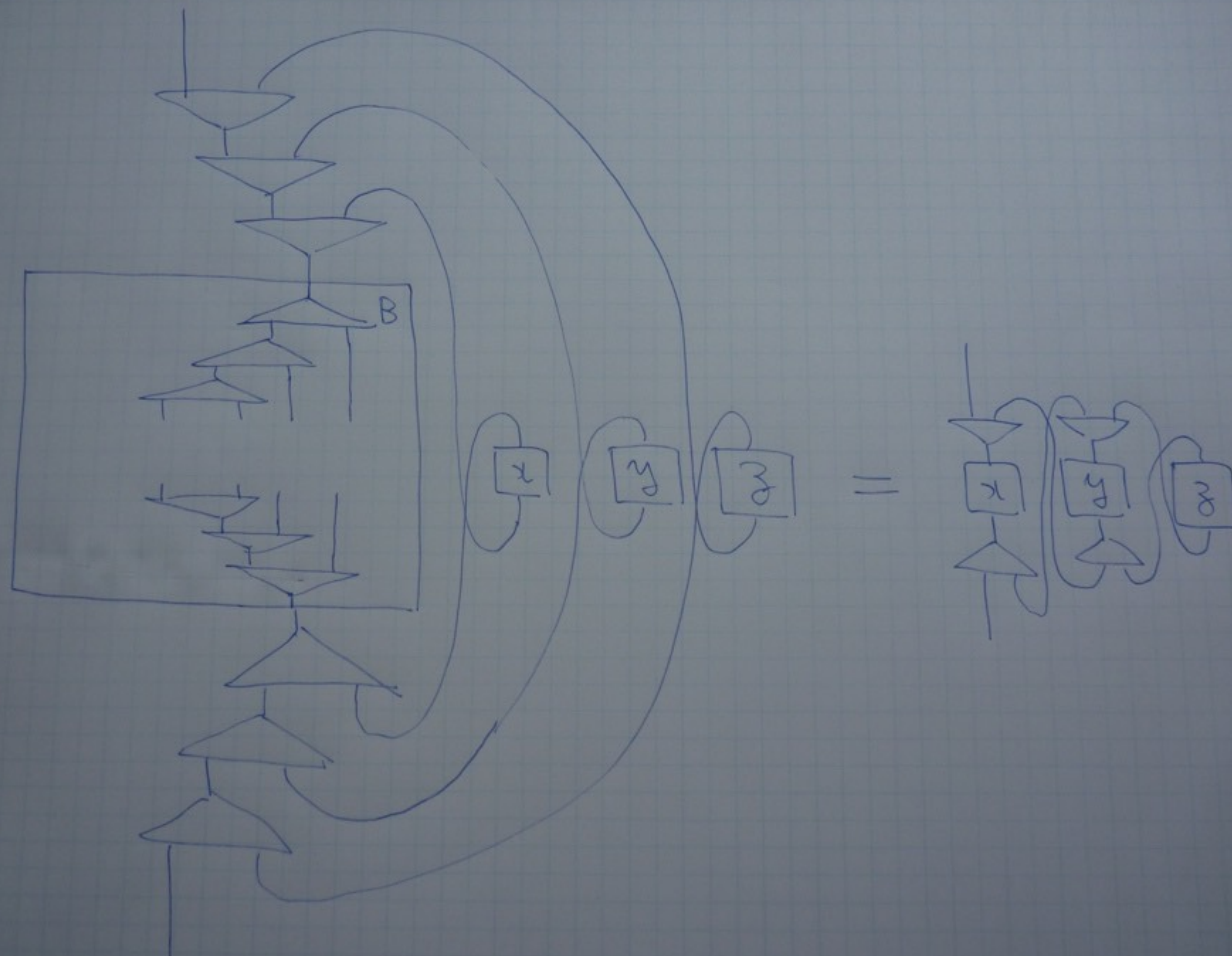
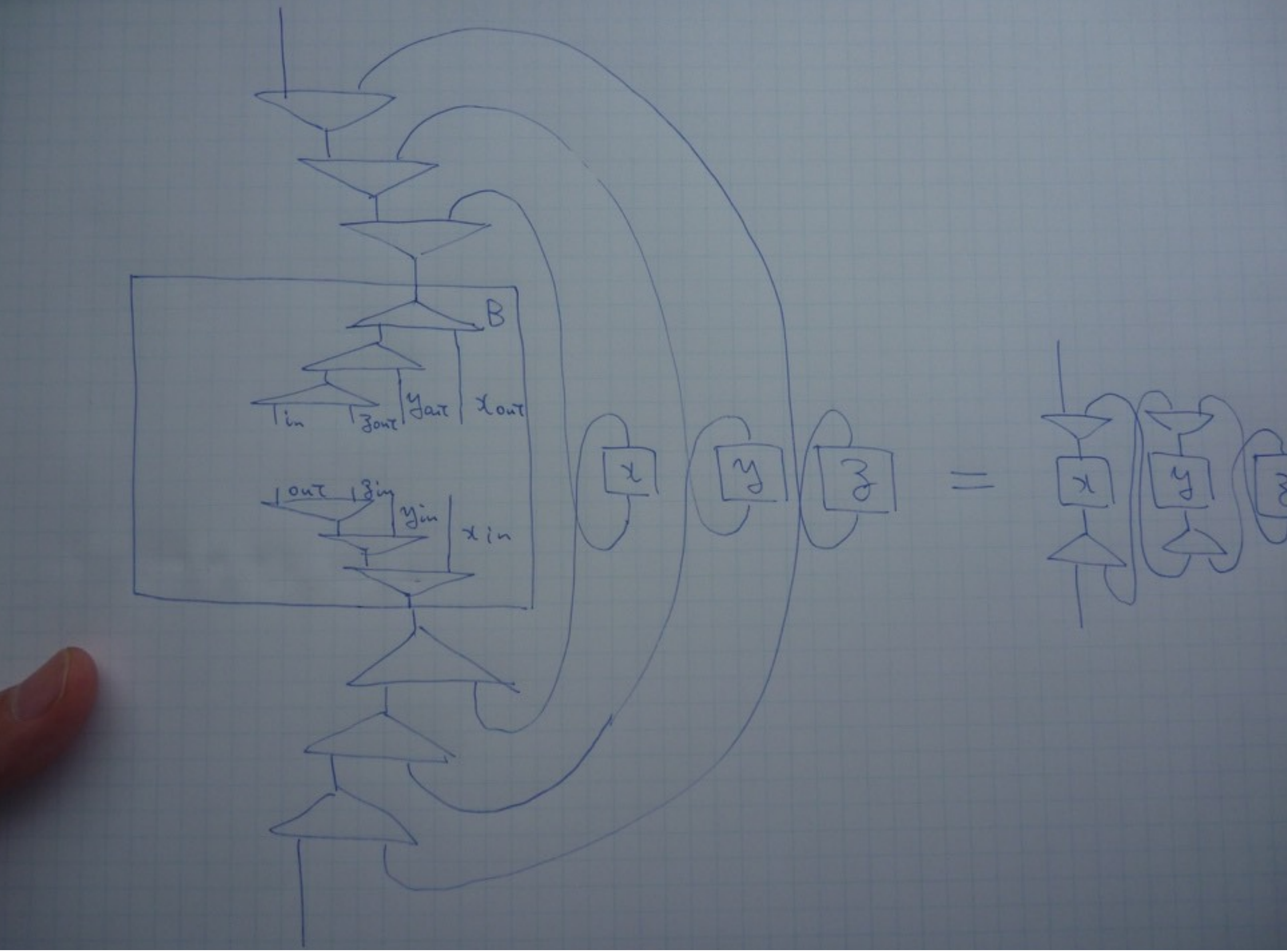
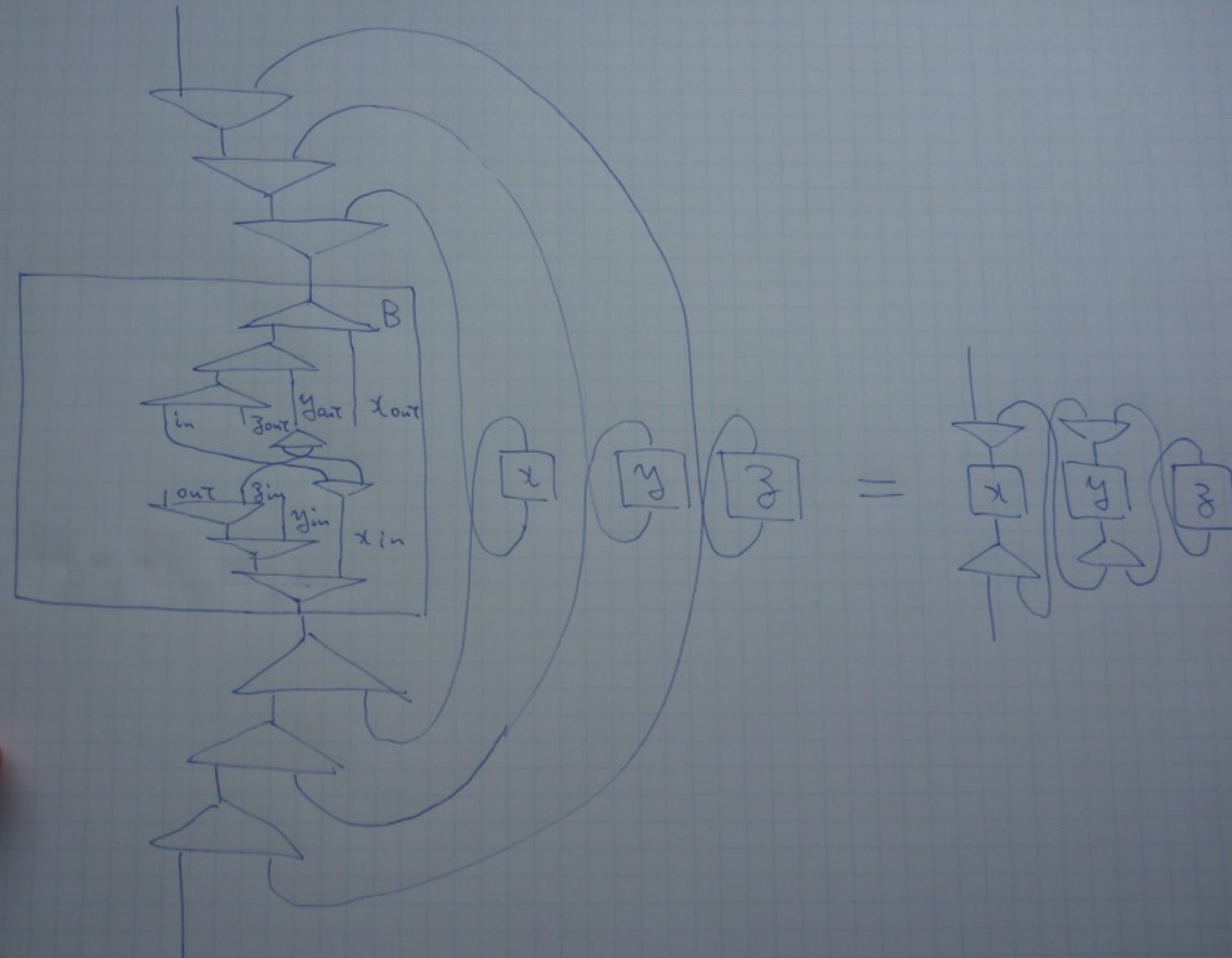
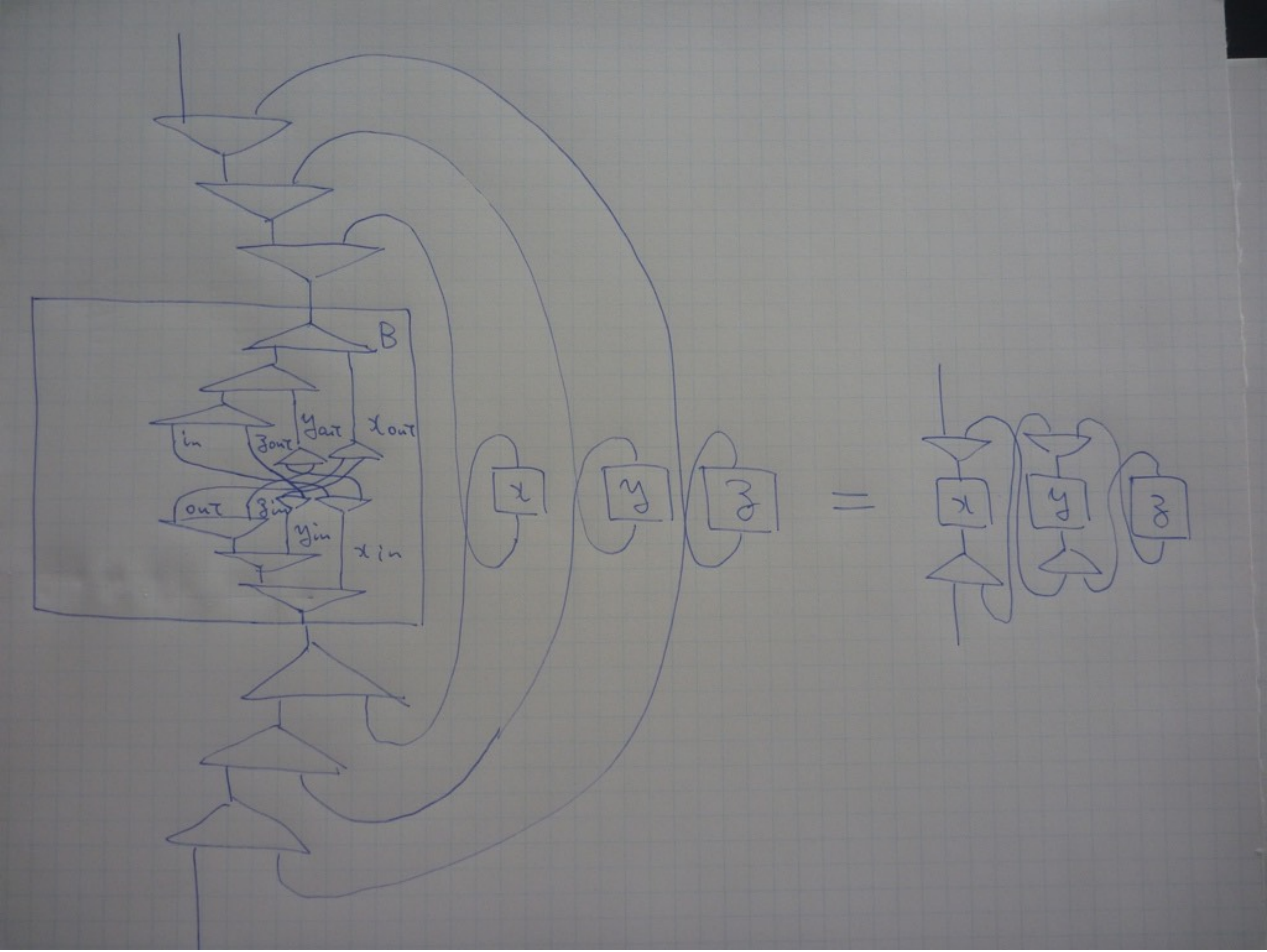



B
x
y
z
=
x
y
z



B
in
z out
y out
x out
out
z in
y in
x in
x
y
z
=
x
y


B
in
z_{out}
y_{out}
x_{out}
out
z_{in}
y_{in}
x_{in}
x
y
z
=
x
y
z



B
y_{out}
x_{out}
in
z_{out}
out
z_{in}
y_{in}
x_{in}
x
y
z
=
x
y
z

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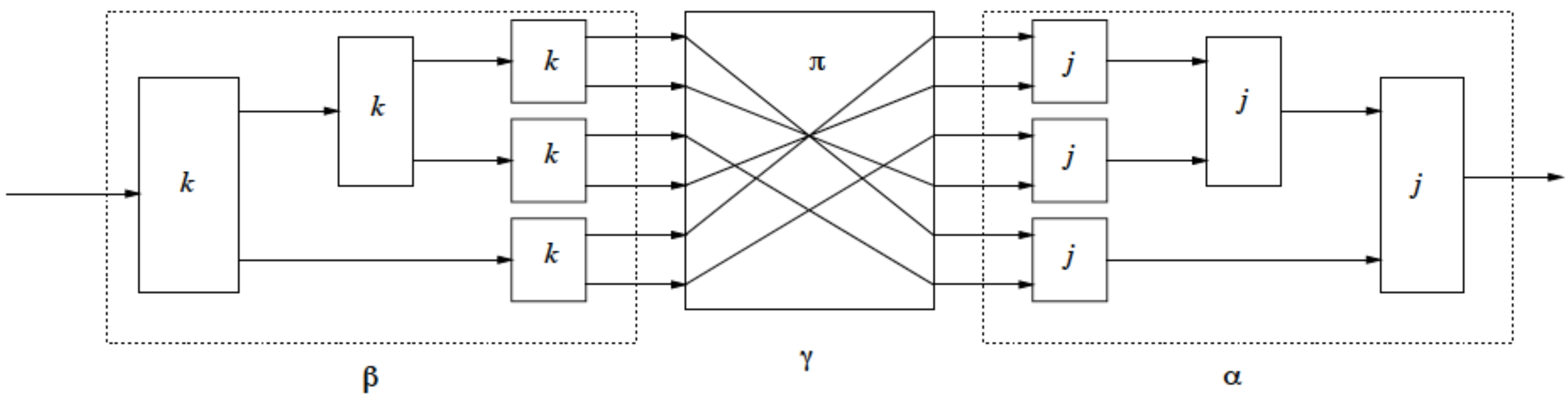

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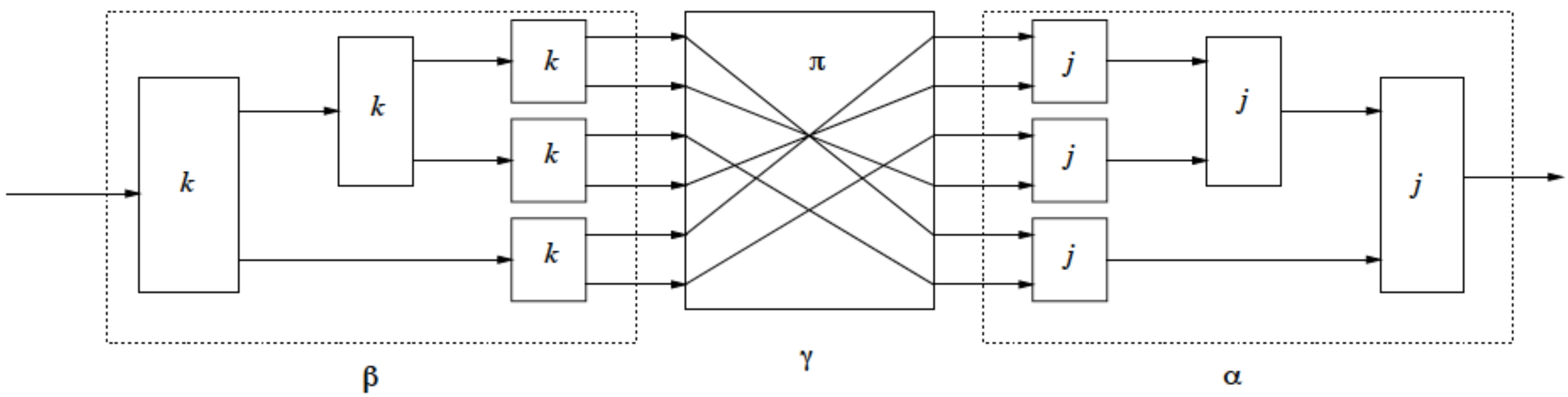

Figure 7: Composition Combinator B

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Nice dynamic interpretation of (linear) computation!!

Summary: Categorical GoI

Defn. (GoI situation [AHS02])
A *GoI situation* is a triple $(\mathbb{C}, F, U)$ where

- $\mathbb{C} = (\mathbb{C}, \otimes, I)$ is a traced symmetric monoidal category (TSMC);
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$$\begin{array}{rcll} e : FF \lhd F : e' & & & \text{Comultiplication} \\ d : \mathrm{id} \lhd F : d' & & & \text{Dereliction} \\ c : F \otimes F \lhd F : c' & & & \text{Contraction} \\ w : K_I \lhd F : w' & & & \text{Weakening} \end{array}$$

  Here K_I is the constant functor into the monoidal unit I;
- $U \in \mathbb{C}$ is an object (called *reflexive object*), equipped with the following retractions.

$$\begin{array}{c} j : U \otimes U \lhd U : k \\ I \lhd U \\ u : FU \lhd U : v \end{array}$$

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Given a GoI situation $(\mathbb{C}, F, U)$, the homset

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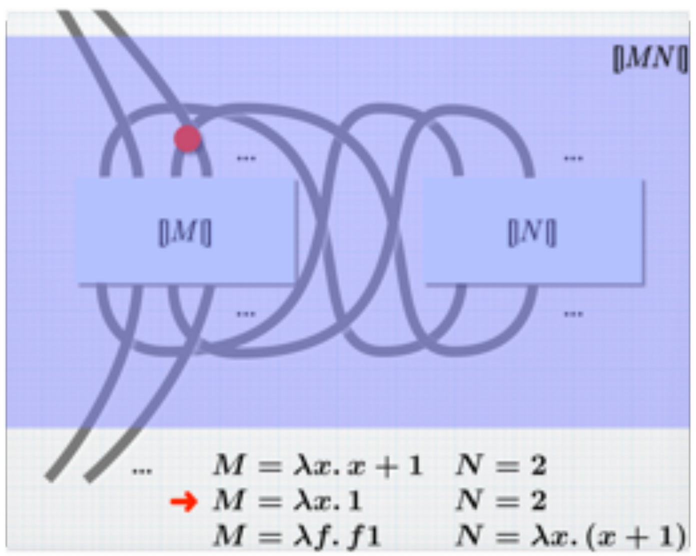
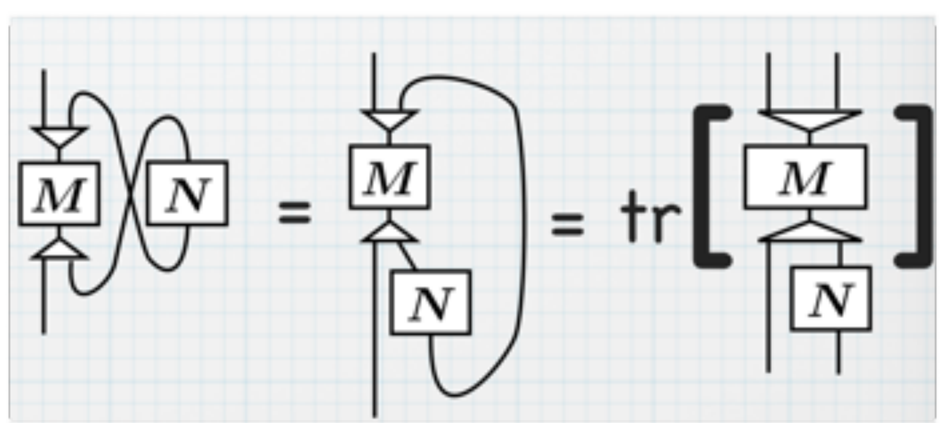
Hasuo (Tokyo)

Outline

Coalgebra meets **higher-order computation** in **Geometry of Interaction** [Girard, LC'88]

"GoI Animation"





Categorical GoI

[Abramsky, Haghverdi & Scott, MSCS'02]

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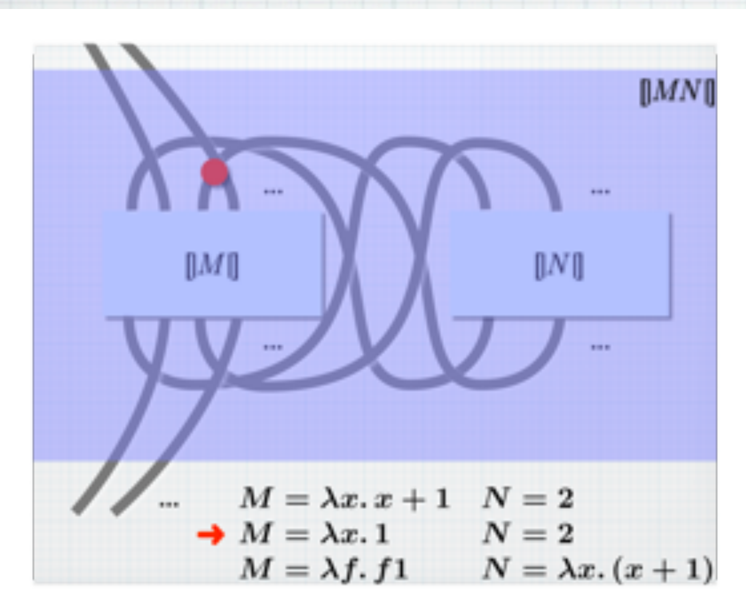
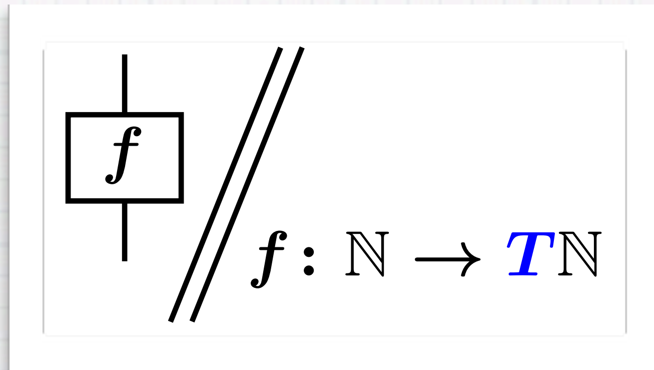
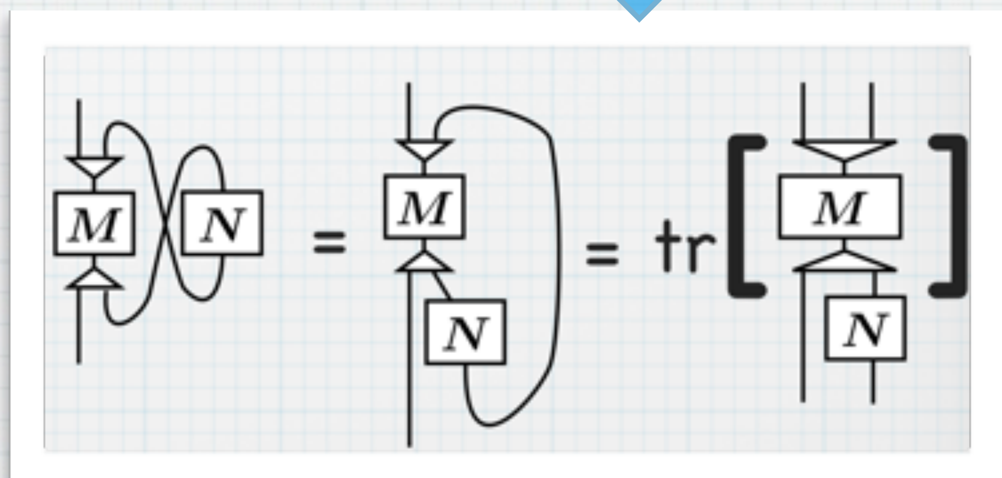




GoI w/ T-branching
[IH & Hoshino, LICS'11]



Categorical GoI
[Abramsky, Haghverdi & Scott, MSCS'02]

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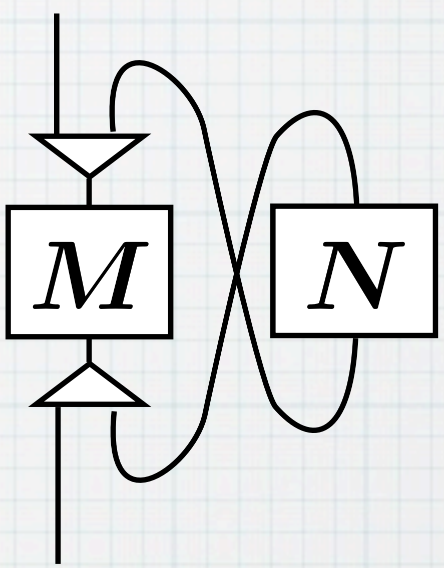
Why Categorical Generalization?: Examples Other Than Pfn [AHS02]

* Strategy: find a TSMC!

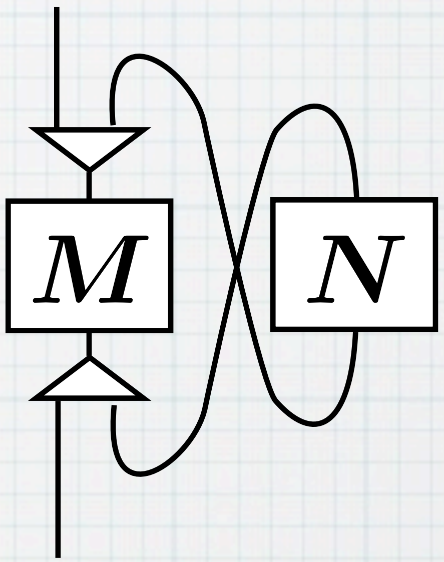


* "Wave-style" examples
  * $\otimes$ is Cartesian product(-like)
    * in which case,

      trace $\approx$ fixed point operator [Hasegawa/Hyland]
  * An example: $\left((\omega\text{-}\mathbf{Cpo}, \times, 1),\ (\_)^{\mathbb{N}},\ A^{\mathbb{N}} \right)$
  * (... less of a dynamic flavor)

Why Categorical Generalization?: Examples Other Than $\mathbf{Pfn}$ [AHS02]

- "Particle-style" examples
  - Obj. X∈C is set-like; ⊗ is coproduct-like
    - The GoI animation is valid
  - Examples:
    - Partial functions $\big(\,(\mathbf{Pfn}, +, \mathbf{0}),\ \mathbb{N}\cdot\_\,,\ \mathbb{N}\,\big)$
    - Binary relations $\big(\,(\mathbf{Rel}, +, \mathbf{0}),\ \mathbb{N}\cdot\_\,,\ \mathbb{N}\,\big)$
    - "Discrete stochastic relations" $\big(\,(\mathbf{DSRel}, +, \mathbf{0}),\ \mathbb{N}\cdot\_\,,\ \mathbb{N}\,\big)$



Why Categorical Generalization?: Examples Other Than Pfn [AHS02]

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$$\dfrac{\dfrac{X \to Y \text{ in } \mathbf{Pfn}}{X \rightharpoonup Y \text{, partial function}}}{X \to \mathcal{L}Y \text{ in } \mathbf{Sets}} \quad \text{where } \mathcal{L}Y = \{\bot\} + Y$$

* $\mathbf{Rel}$ (relations)

$$\dfrac{\dfrac{X \to Y \text{ in } \mathbf{Rel}}{R \subseteq X \times Y \text{, relation}}}{X \to \mathcal{P}Y \text{ in } \mathbf{Sets}} \quad \text{where } \mathcal{P} \text{ is the powerset monad}$$

* $\mathbf{DSRel}$

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Why Catego
Examples [AHS02]

Categories of sets and
(functions with different branching/partiality)

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(Potential) non-termination

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Non-determinism

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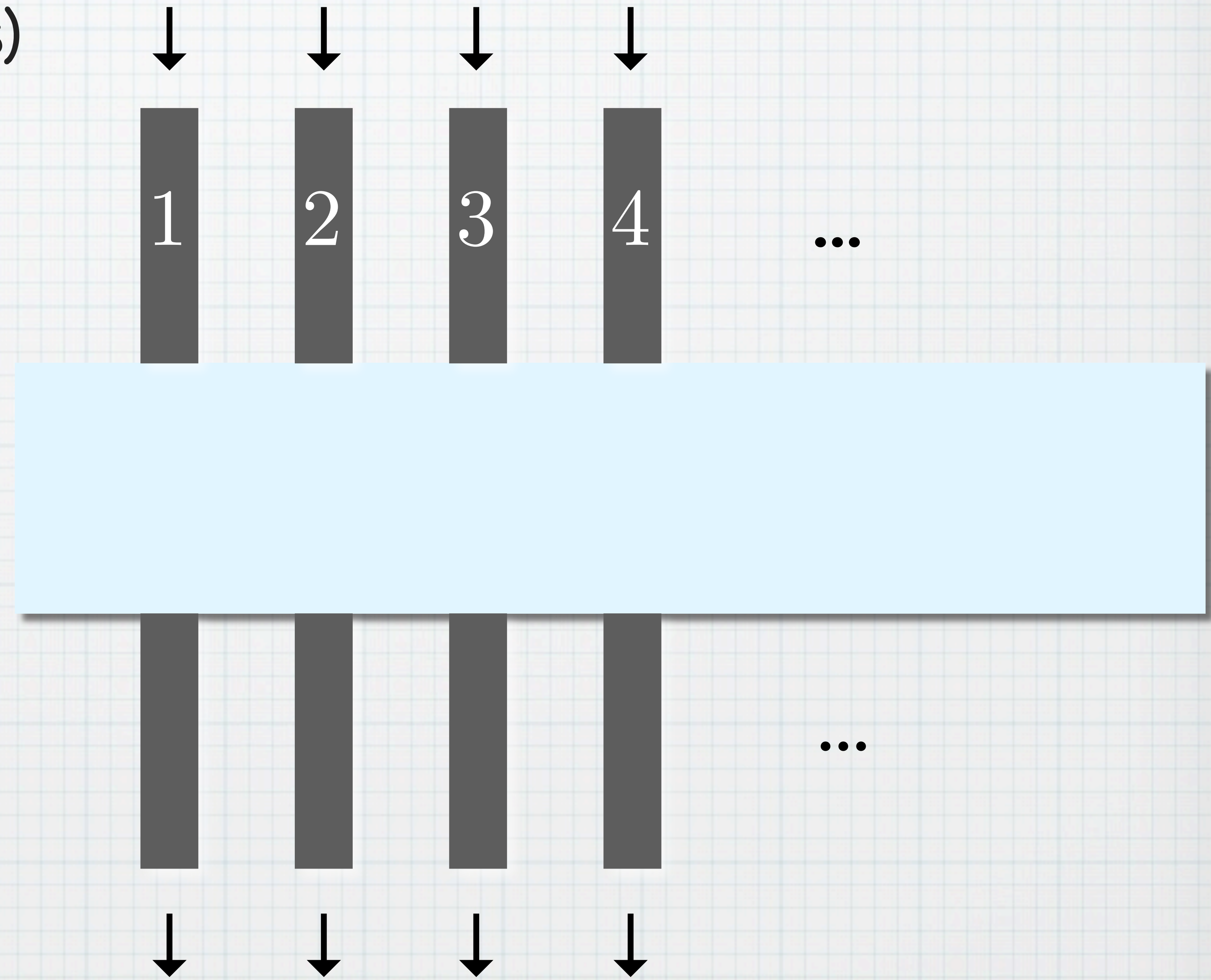
Probabilistic branching

Different Branching in The GoI Animation

- **Pfn** (partial functions)
  - Pipes can be stuck
- **Rel** (relations)
  - Pipes can branch
- **DSRel**
  - Pipes can branch probabilistically



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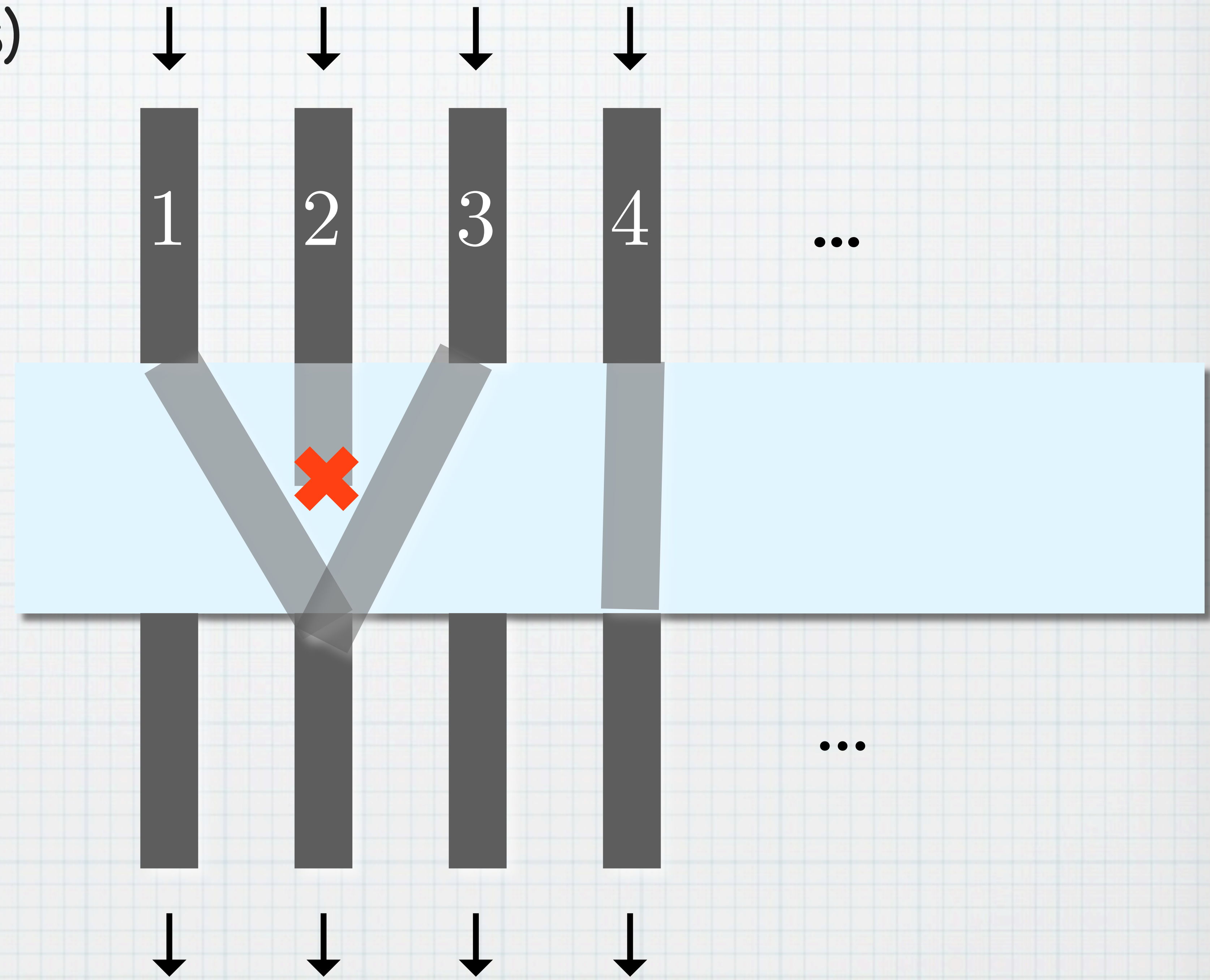

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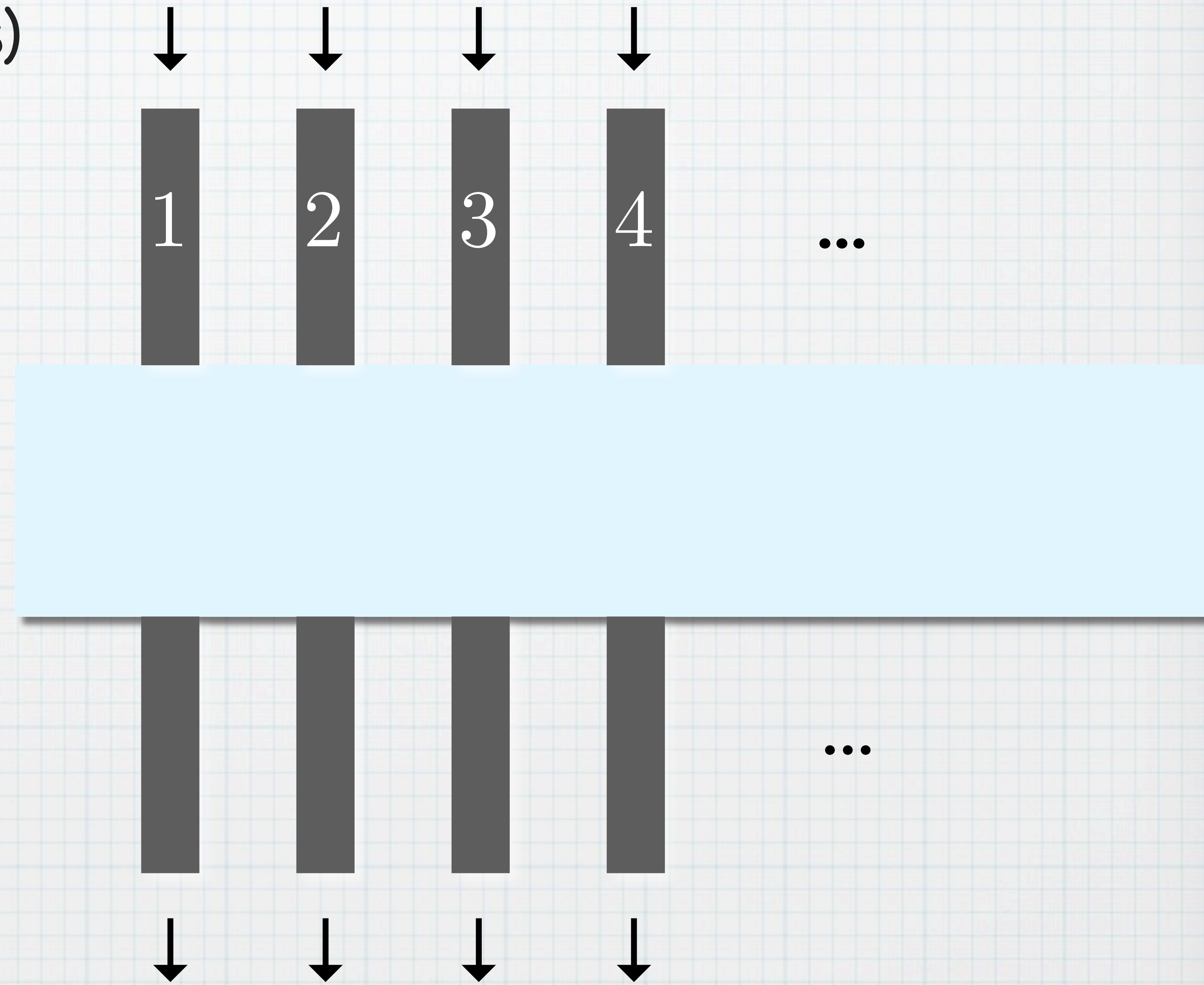

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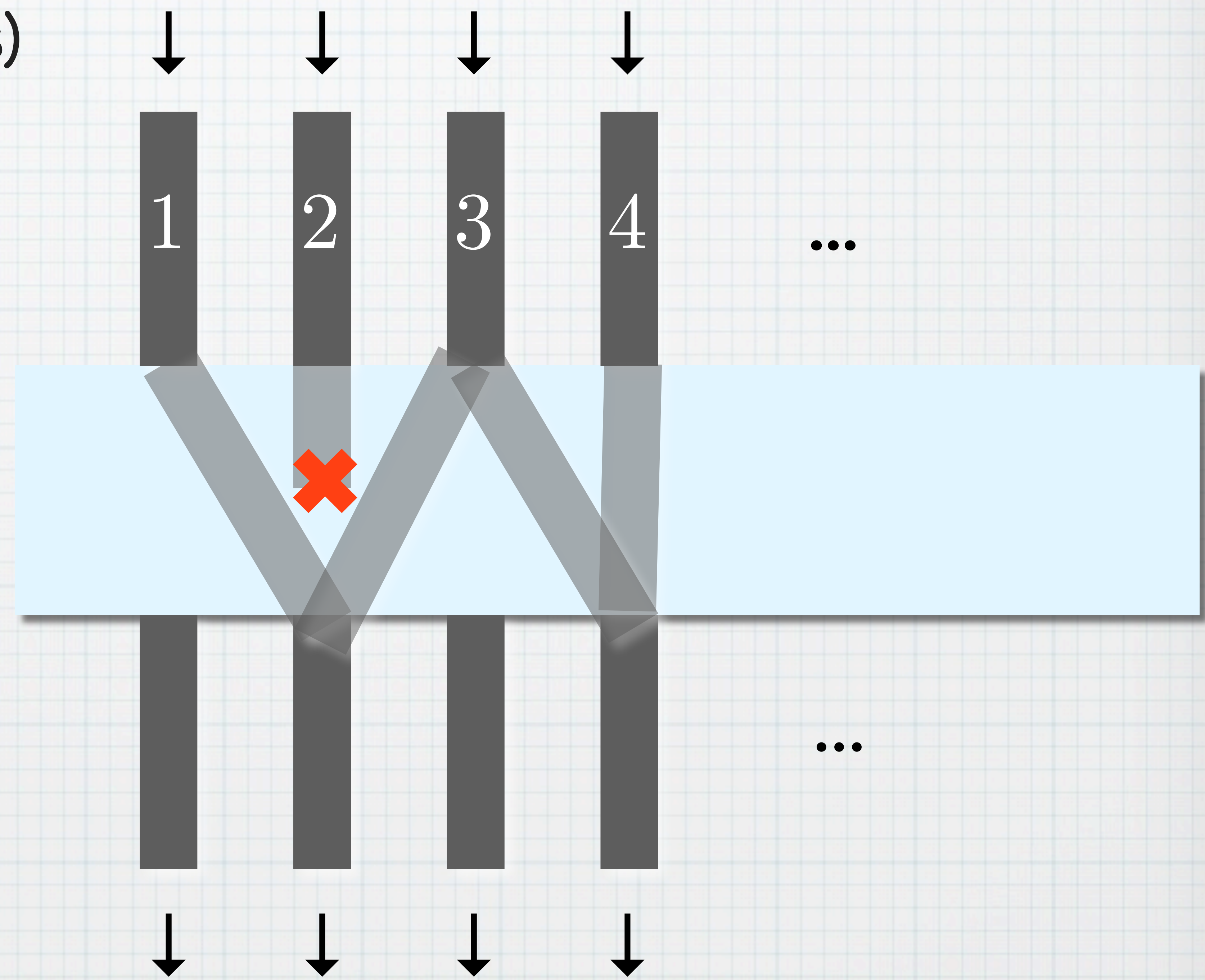

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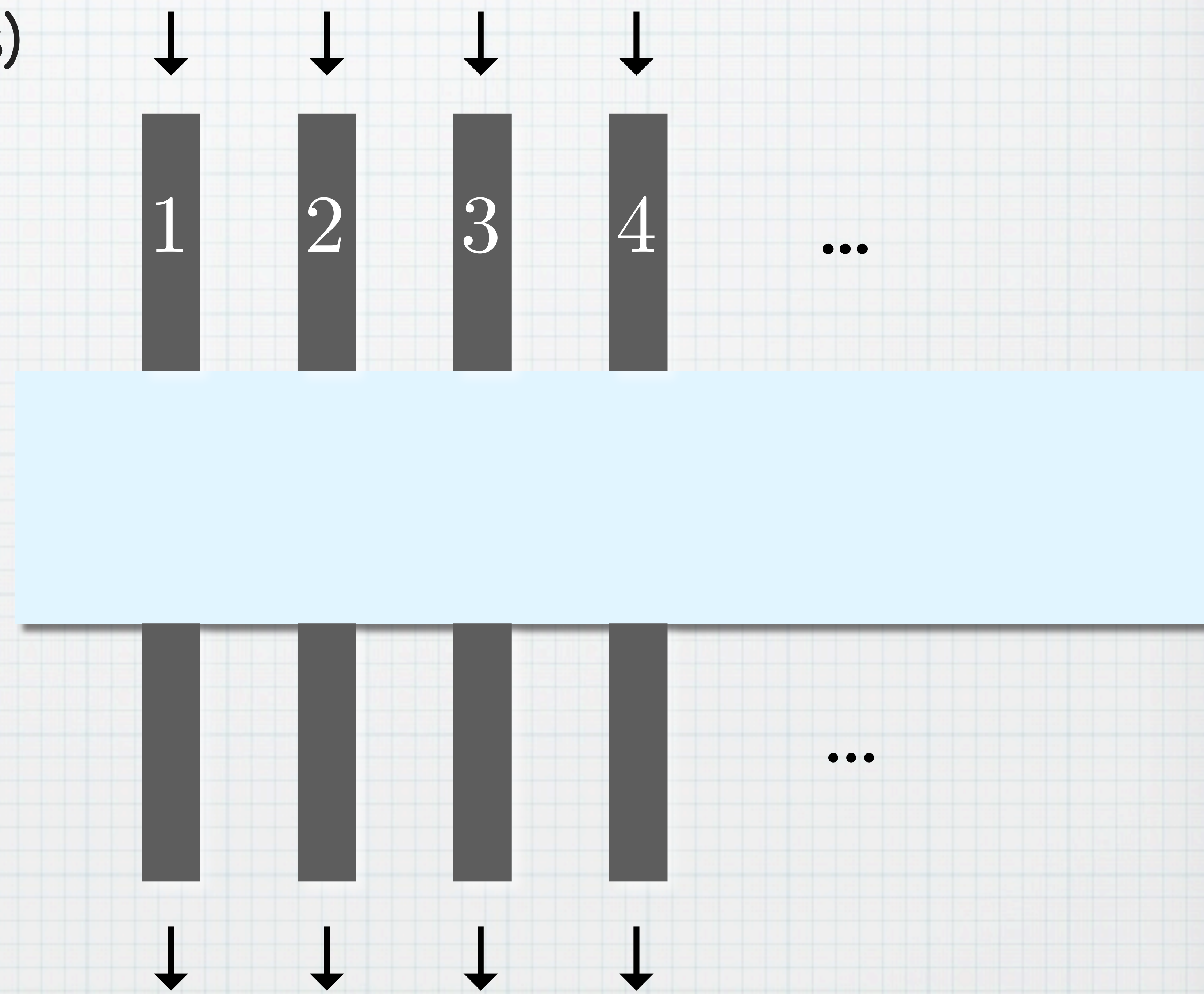

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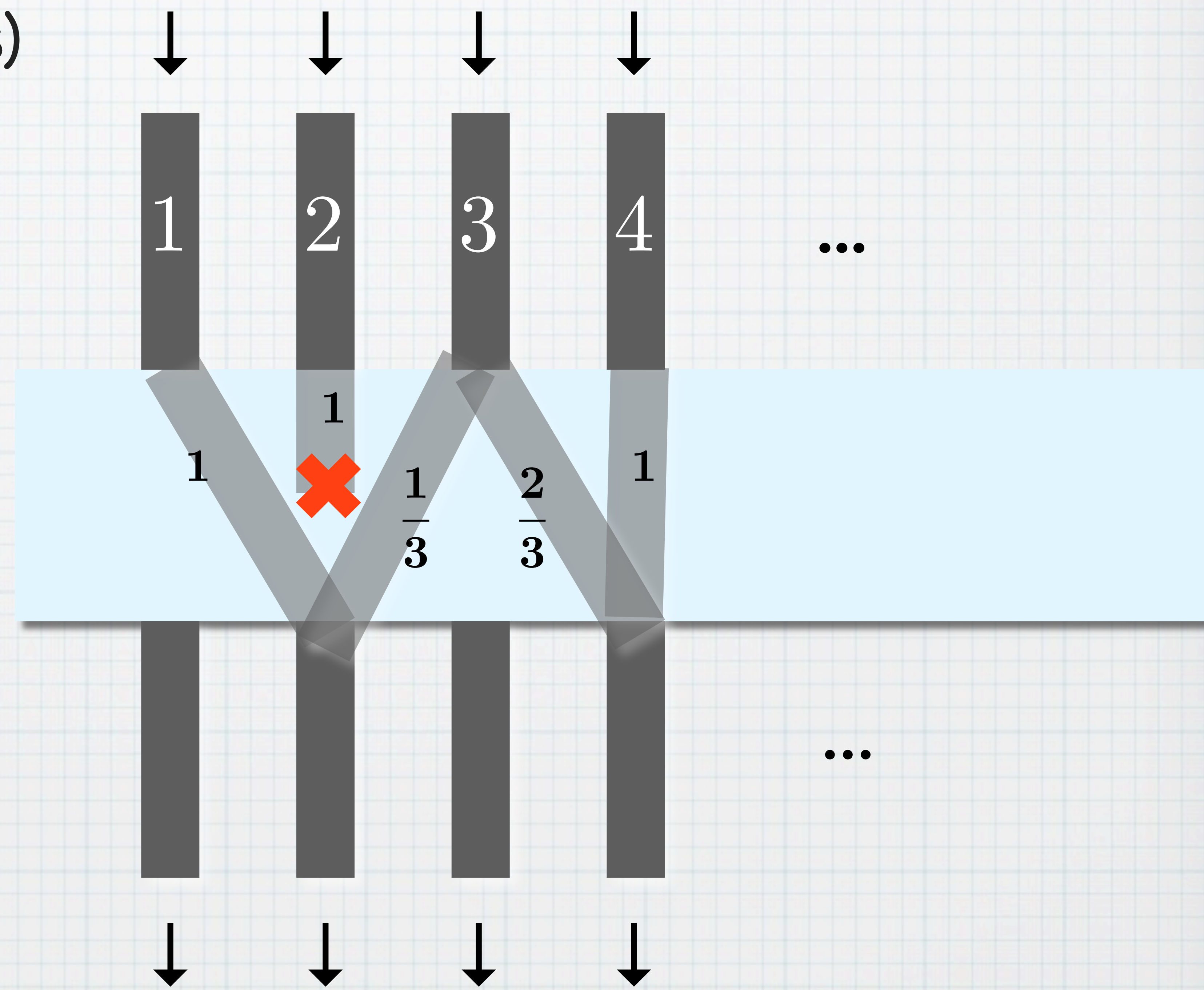

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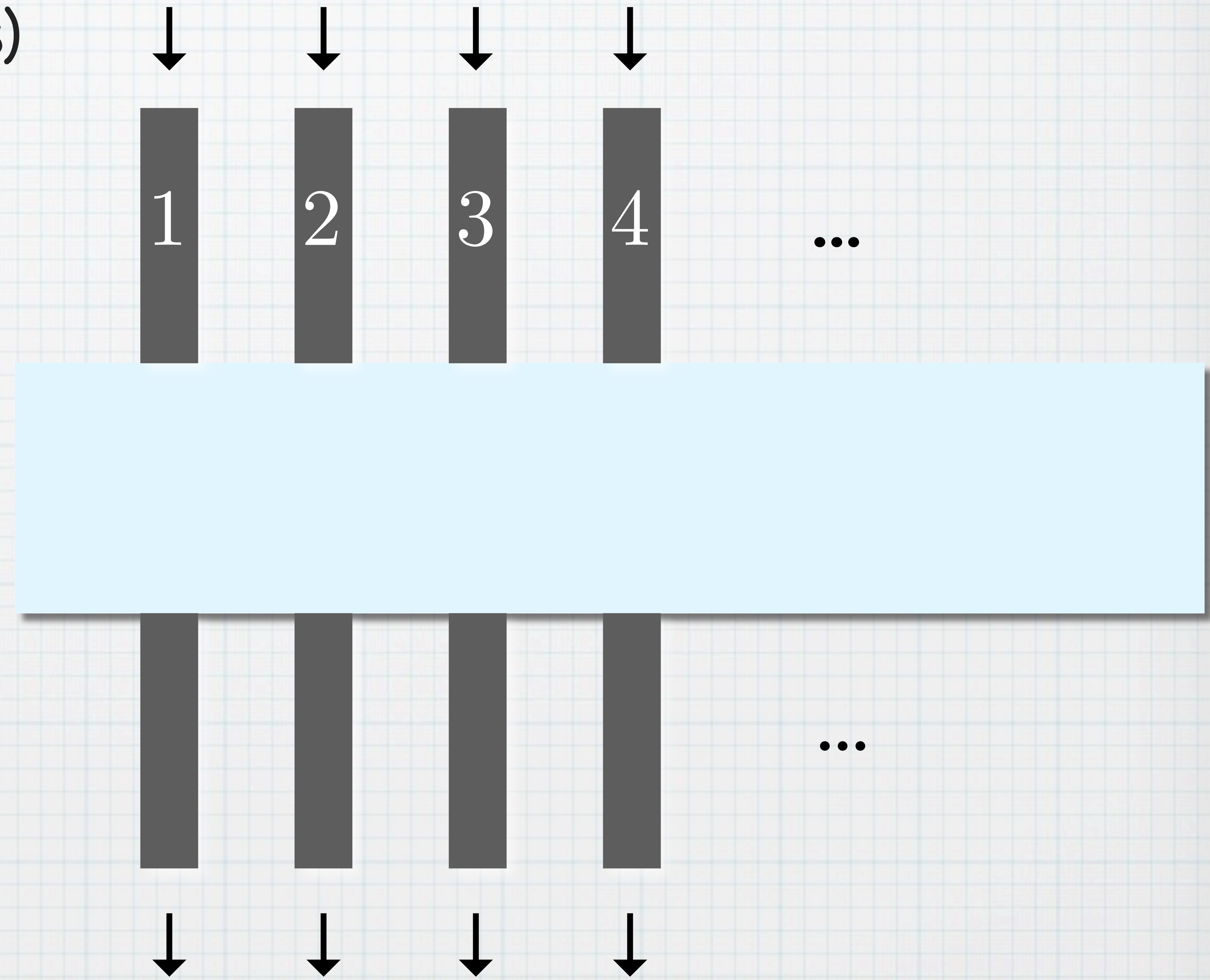

A Coalgebraic View

* Theory of coalgebra = Categorical theory of state-based dynamic systems (LTS, automaton, Markov chain, ...)

* In my thesis (2008):
  * Coalgebras in a Kleisli category $Kl(T)$

$$\frac{X \to Y \text{ in } \mathcal{K}\ell(T)}{X \to TY \text{ in } \mathbf{Sets}}$$

  * ➜ Generic theory of **trace** and **simulations**

Hasuo (Tokyo)

Why Catego… Examples Other Than Pfn

Categories of sets and
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Why Catego… Examples

$Kl(T)$ for different branching monads T

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Probabilistic branching

Branching Monad: Source of Particle-Style GoI Situations

Thm. ([Jacobs,CMCS10])
Given a "branching monad" $\boldsymbol{T}$ on **Sets**, the monoidal category

$$(\mathcal{K}\ell(\boldsymbol{T}), +, \mathbf{0})$$

is

- a *unique decomposition category* [Haghverdi,PhD00], hence is
- a traced symmetric monoidal category.

Cor.
$\big(\,(\mathcal{K}\ell(\boldsymbol{T}), +, \mathbf{0}),\ \mathbb{N}\cdot\_\,,\ \mathbb{N}\,\big)$ is a GoI situation.

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is

- a *unique decomposition category* [Haghverdi,PhD00], hence is
- a traced symmetric monoidal category.

Cor.
$\big(\,(\mathcal{K}\ell(\boldsymbol{T}), +, \mathbf{0}),\ \mathbb{N}\cdot\_\,,\ \mathbb{N}\,\big)$ is a GoI situation.

Monads in [Hasuo,Jacobs&Sokolova07]

- Kl(T) is $\mathrm{Cpo}_\perp$-enriched
- like L, P, D

Hasuo (Tokyo)

Branching Monad: Source of Particle-Style GoI Situations

Thm. ([Jacobs,CMCS10])
Given a "branching monad" T on **Sets**, the monoidal category

$$(\mathcal{K}\ell(T), +, \mathbf{0})$$

is

- a *unique decomposition category* [Haghverdi,PhD00], hence is
- a traced symmetric monoidal category.

Cor.
$\big(\, (\mathcal{K}\ell(T), +, \mathbf{0}),\ \mathbb{N}\cdot\_\ ,\ \mathbb{N}\,\big)$ is a GoI situation.

Monads in [Hasuo,Jacobs&Sokolova07]

- Kl(T) is $\mathrm{Cpo}_{\perp}$-enriched
- like L, P, D

Particle-style: trace via the execution formula

$$\mathbf{tr}(f) = f_{XY} \sqcup \left(\coprod_{n \in \mathbb{N}} f_{ZY} \circ (f_{ZZ})^n \circ f_{XZ} \right)$$

The Categorical GoI Workflow

Traced monoidal category $\mathbb{C}$

+ other constructs ➜ "GoI situation" [AHS02]

Categorical GoI [AHS02]

Linear combinatory algebra

Realizability

Linear category

The Categorical GoI Workflow

Branching monad B

Coalgebraic trace semantics

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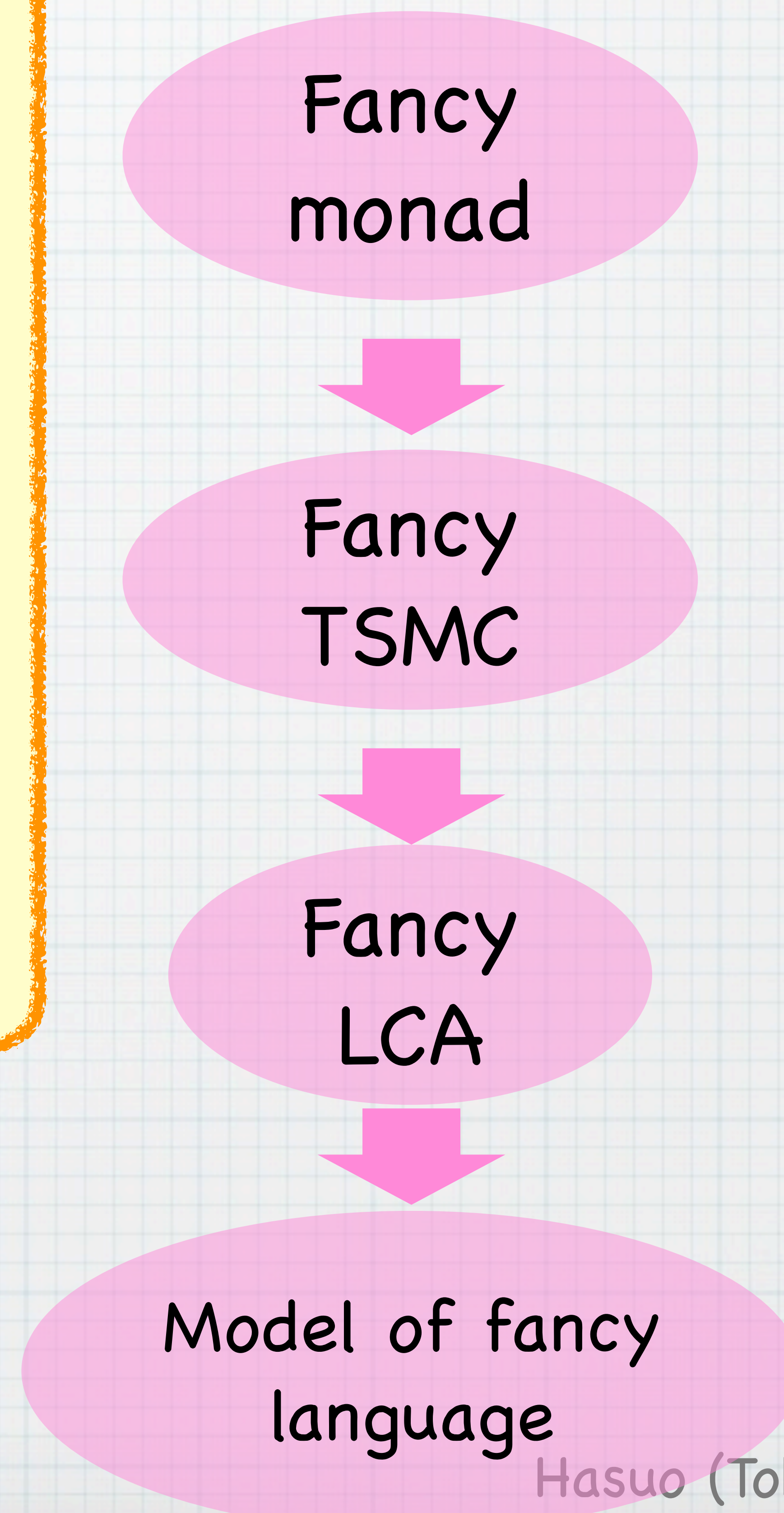
* Model for (a variant of) the Selinger-Valiron **quantum λ-calculus**

  (linear λ + prep./Unitary/meas.)
  [Hasuo & Hoshino, LICS'11 & APAL'16]

* via the **quantum branching monad**



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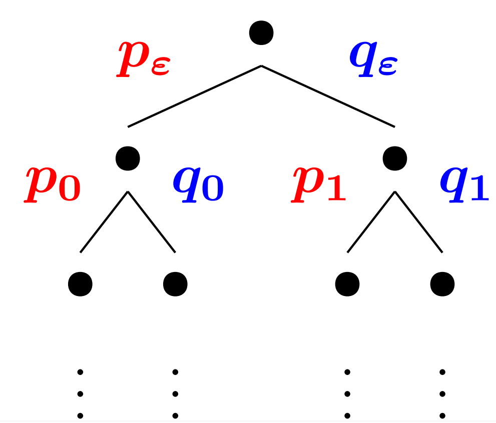
* via the **quantum branching monad**
* ... with considerable complication :(

$$[\![\Gamma \vdash M : \tau]\!] : \ [\![\Gamma]\!] \longrightarrow ([\![\tau]\!] \multimap R) \multimap R$$

where

$R = \Big\{$



$\Big| \ p_\alpha, q_\alpha \in [0,1] \Big\}$

Realizability

Linear category

Workflow

Fancy monad

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Model of fancy language

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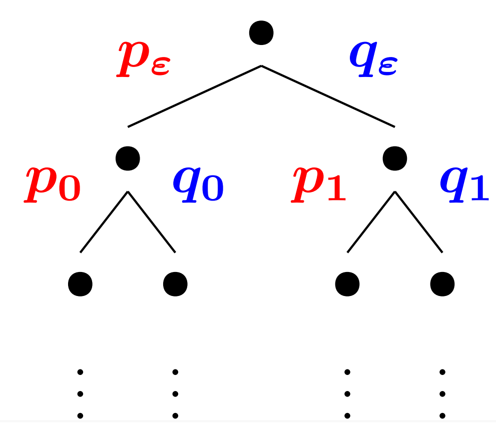


- Records measurement outcomes
- R as a suitable **final coalgebra** in the realizability category

Fancy monad

Fancy LCA

Model of fancy language

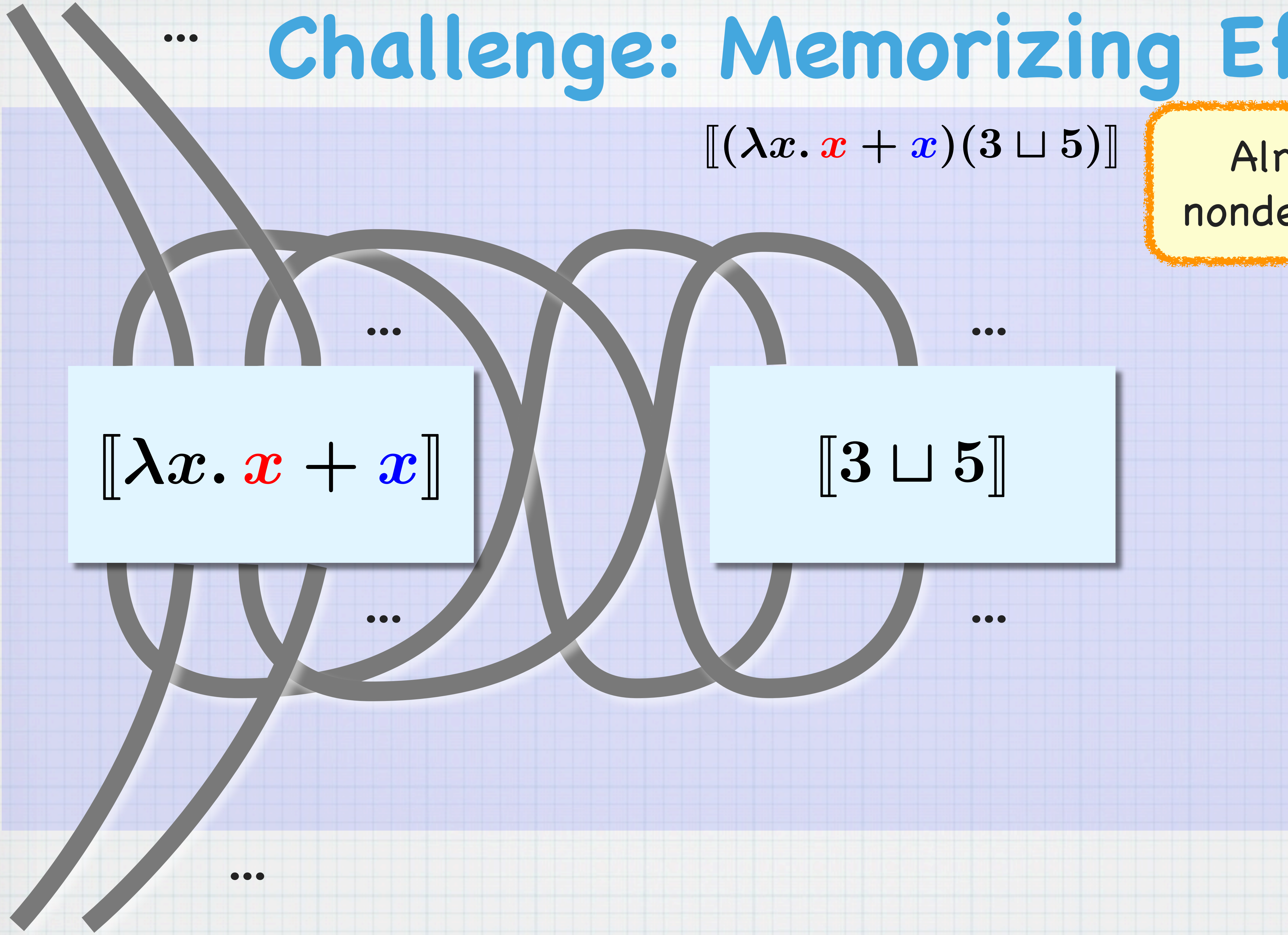
Linear category

Realizability

Challenge: Memorizing Effects

Already w/
nondeterminism!

Challenge: Memorizing Effects



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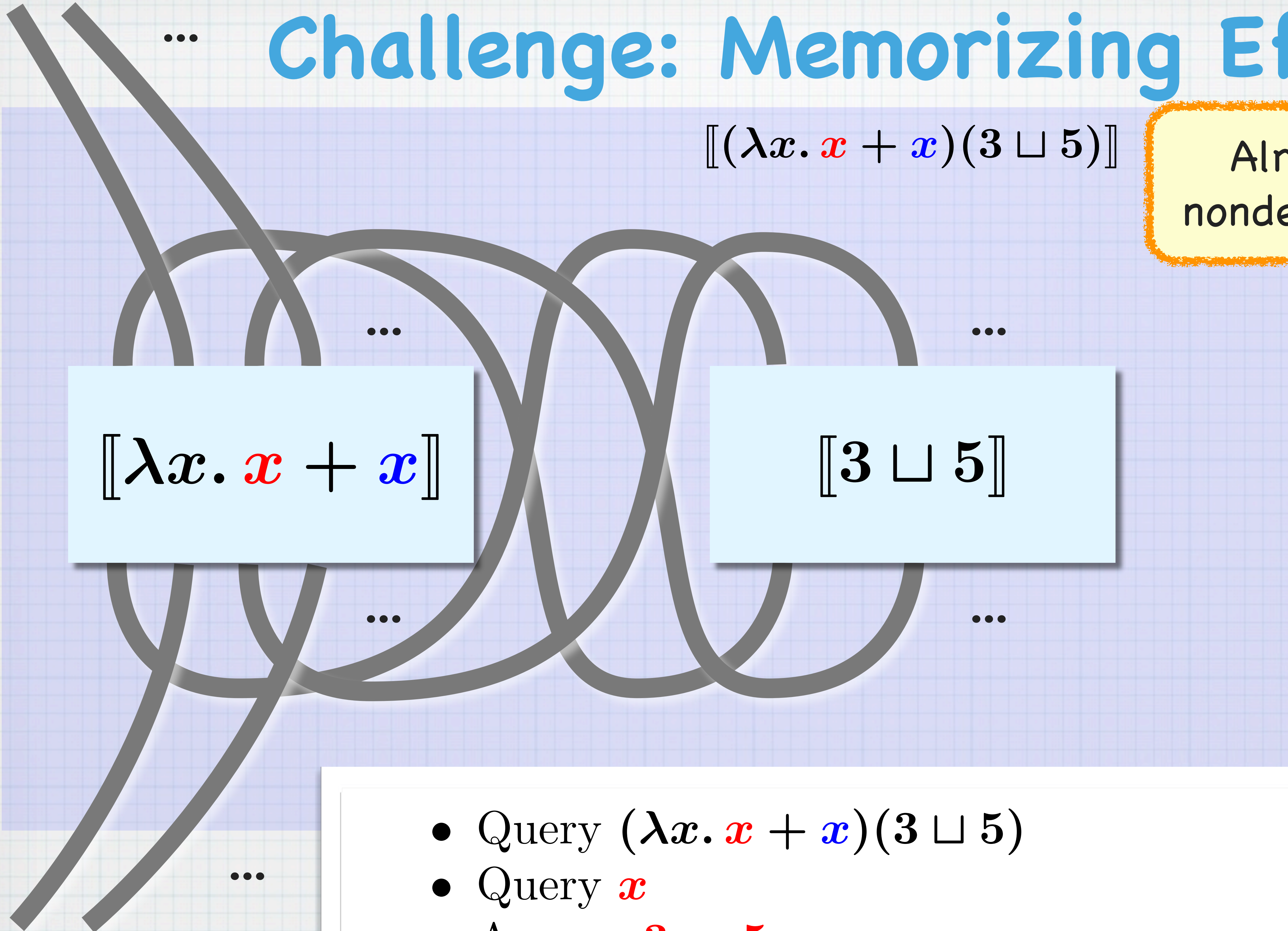
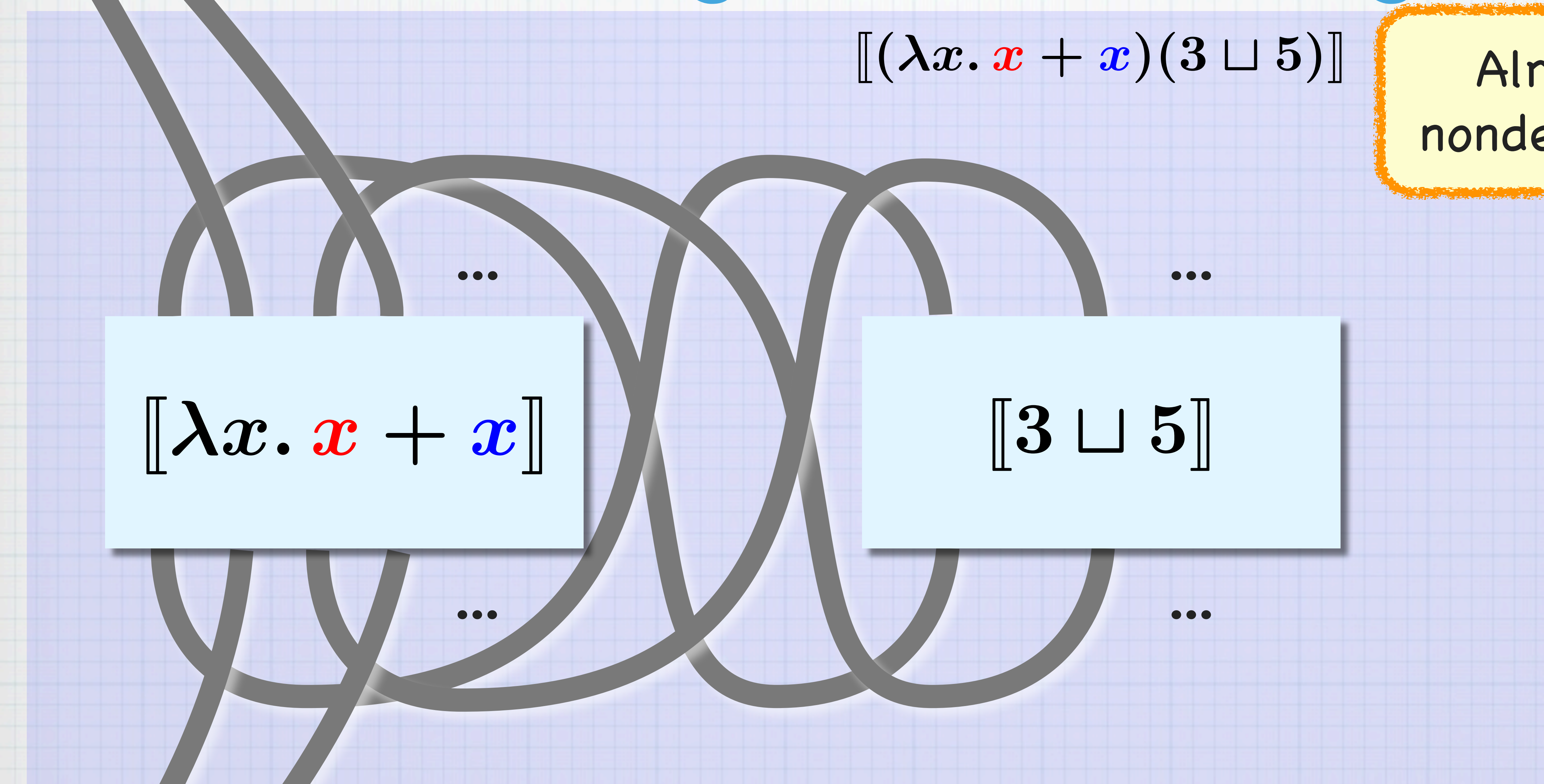


- Query $(\lambda x.\, x + x)(3 \sqcup 5)$
- Query x
- Answer 3 or 5
- Query x
- Answer 3 or 5
- Answer $3 + 3$, $3 + 5$, $5 + 3$ or $5 + 5$

Challenge: Memorizing Effects

Already w/ nondeterminism!



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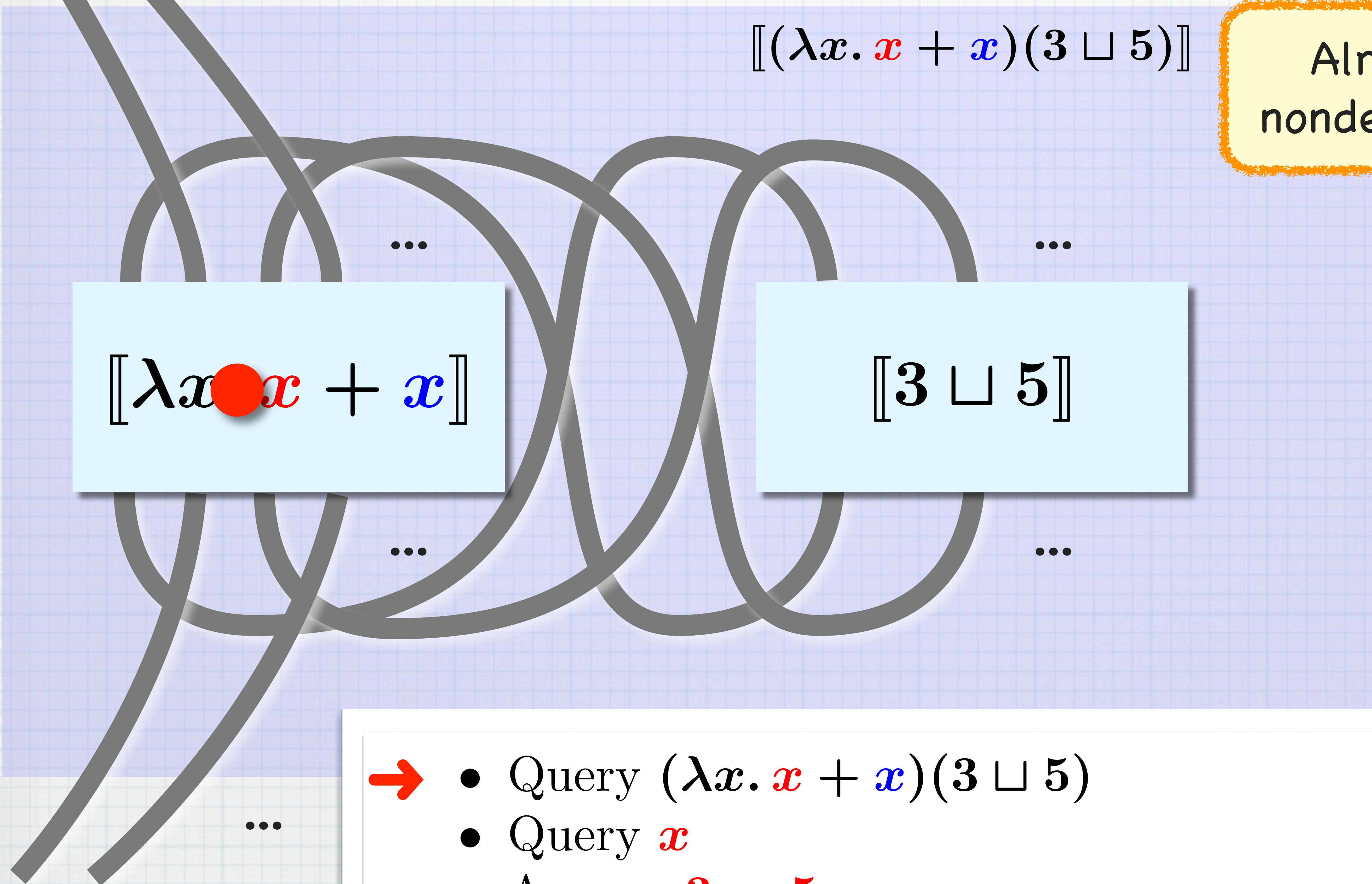

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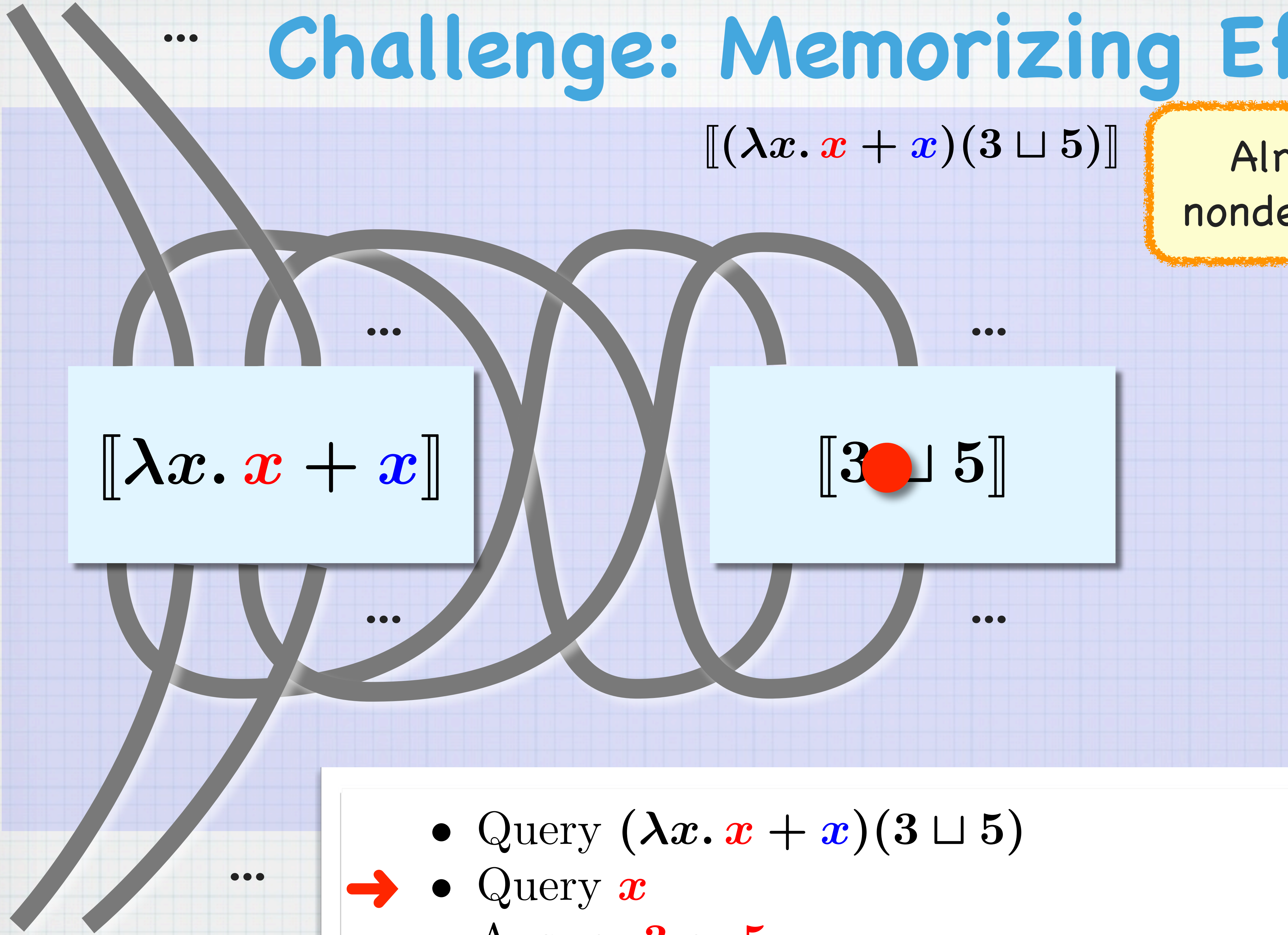

- Query $(\lambda x.\, x + x)(3 \sqcup 5)$
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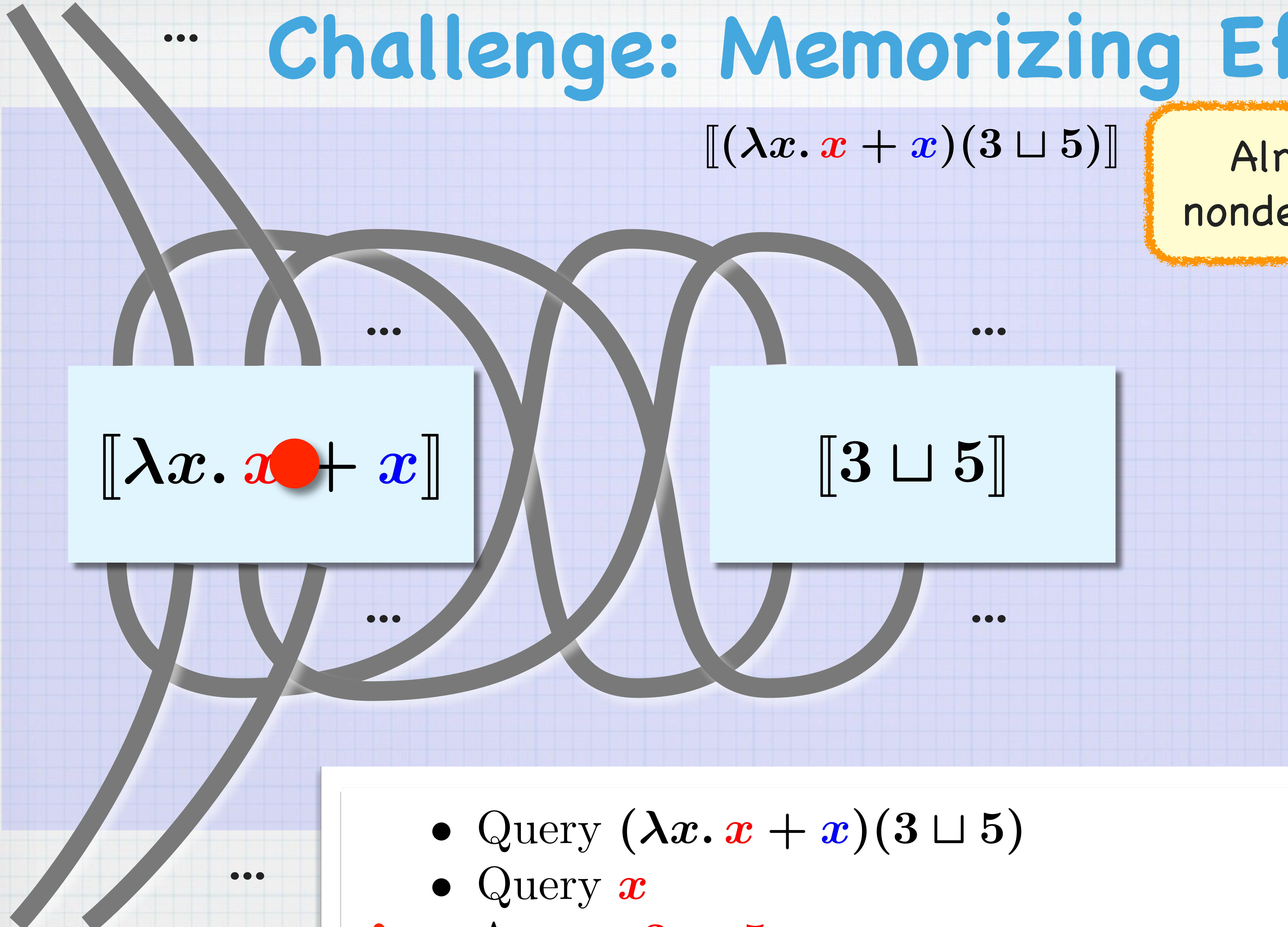
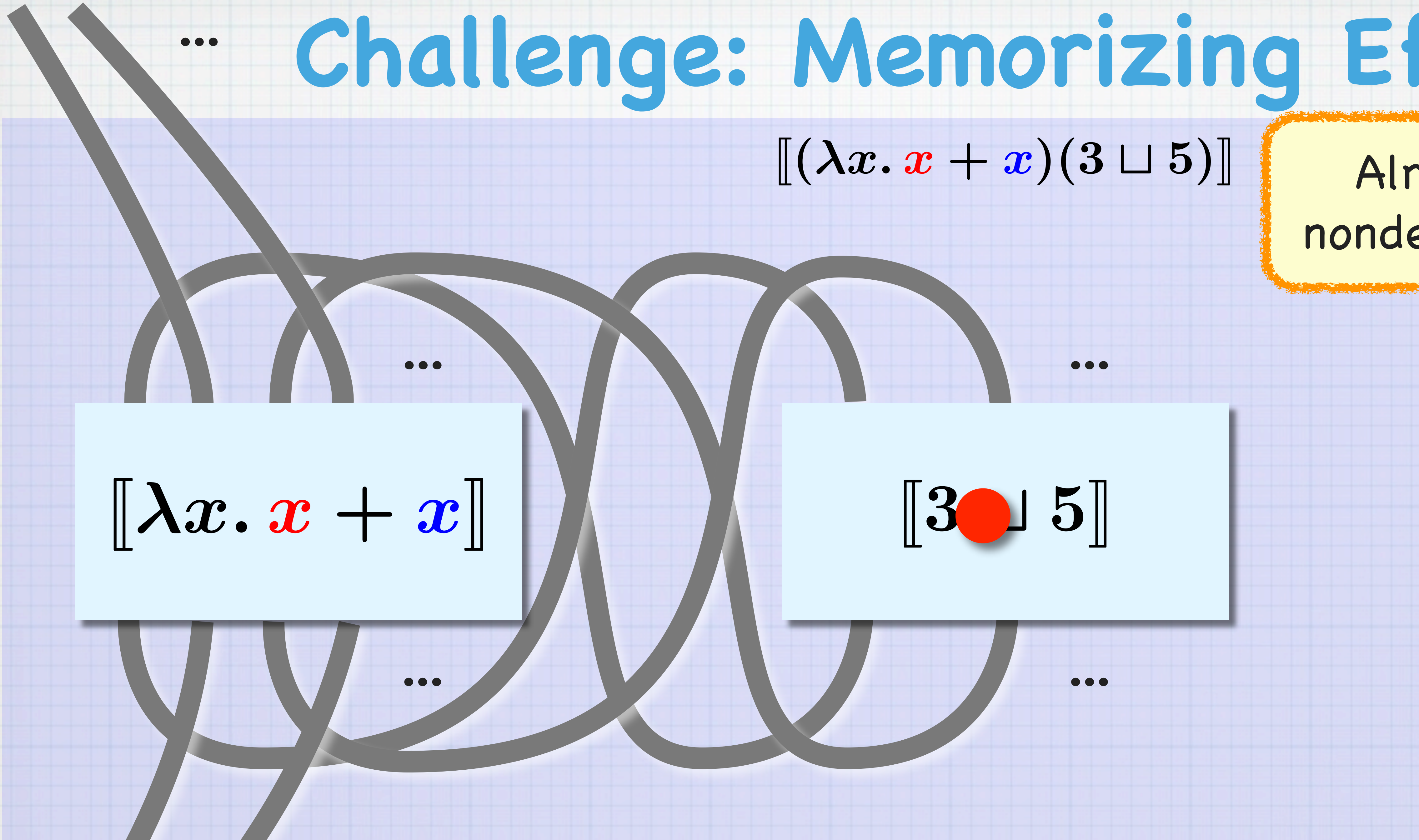


- Query $(\lambda x.\, {\color{red}x} + {\color{blue}x})(3 \sqcup 5)$
- Query ${\color{red}x}$
- → Answer ${\color{red}3}$ or ${\color{red}5}$
- Query ${\color{blue}x}$
- Answer ${\color{blue}3}$ or ${\color{blue}5}$
- Answer ${\color{red}3} + {\color{blue}3}$, ${\color{red}3} + {\color{blue}5}$, ${\color{red}5} + {\color{blue}3}$ or ${\color{red}5} + {\color{blue}5}$

Challenge: Memorizing Effects

Already w/ nondeterminism!



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$[\![(\lambda x.\, {\color{red}x} + {\color{blue}x})(3 \sqcup 5)]\!]$

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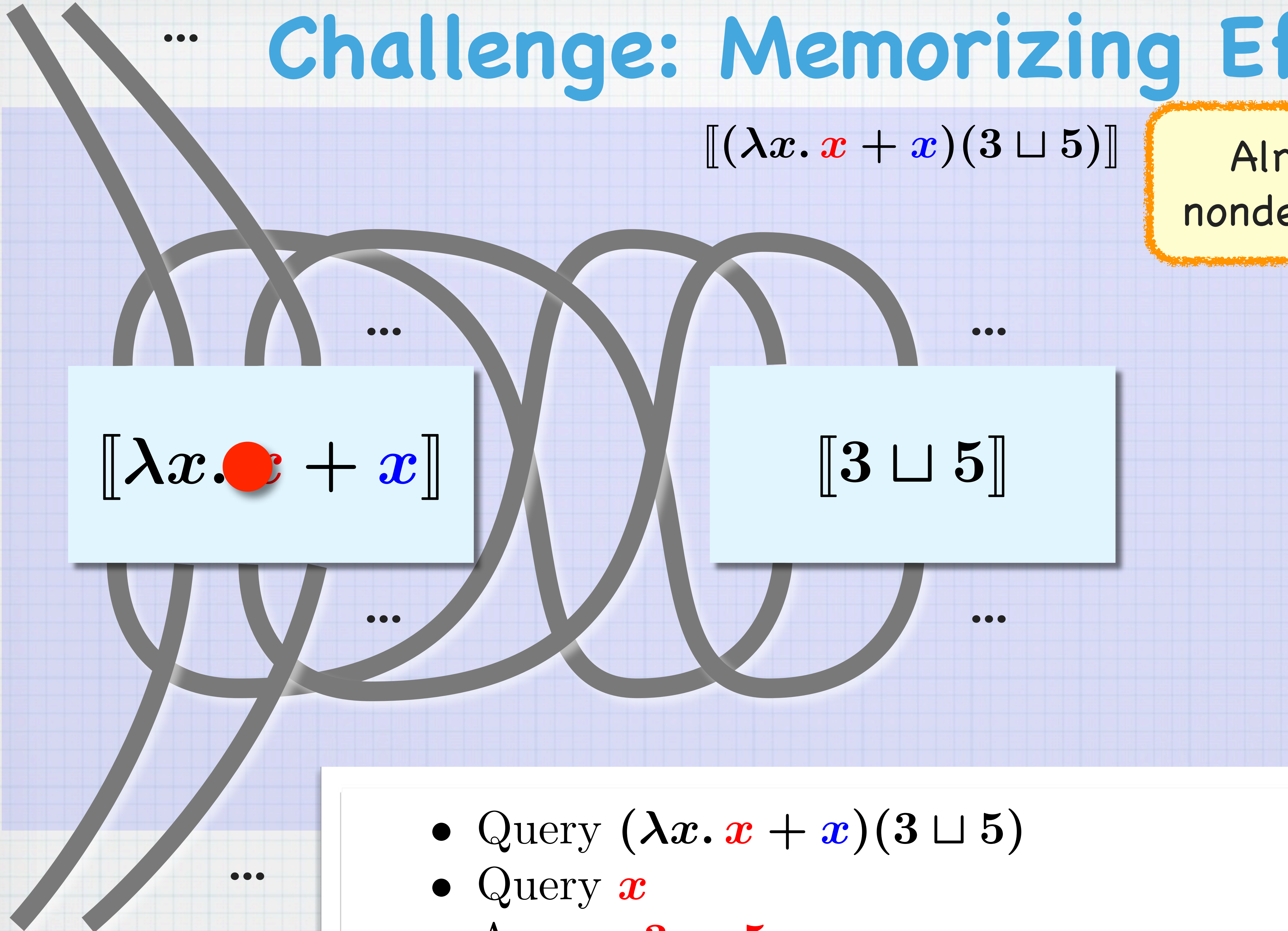

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- Query ${\color{red}x}$
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- Answer ${\color{red}3} + {\color{blue}3}$, ${\color{red}3} + {\color{blue}5}$, ${\color{red}5} + {\color{blue}3}$ or ${\color{red}5} + {\color{blue}5}$

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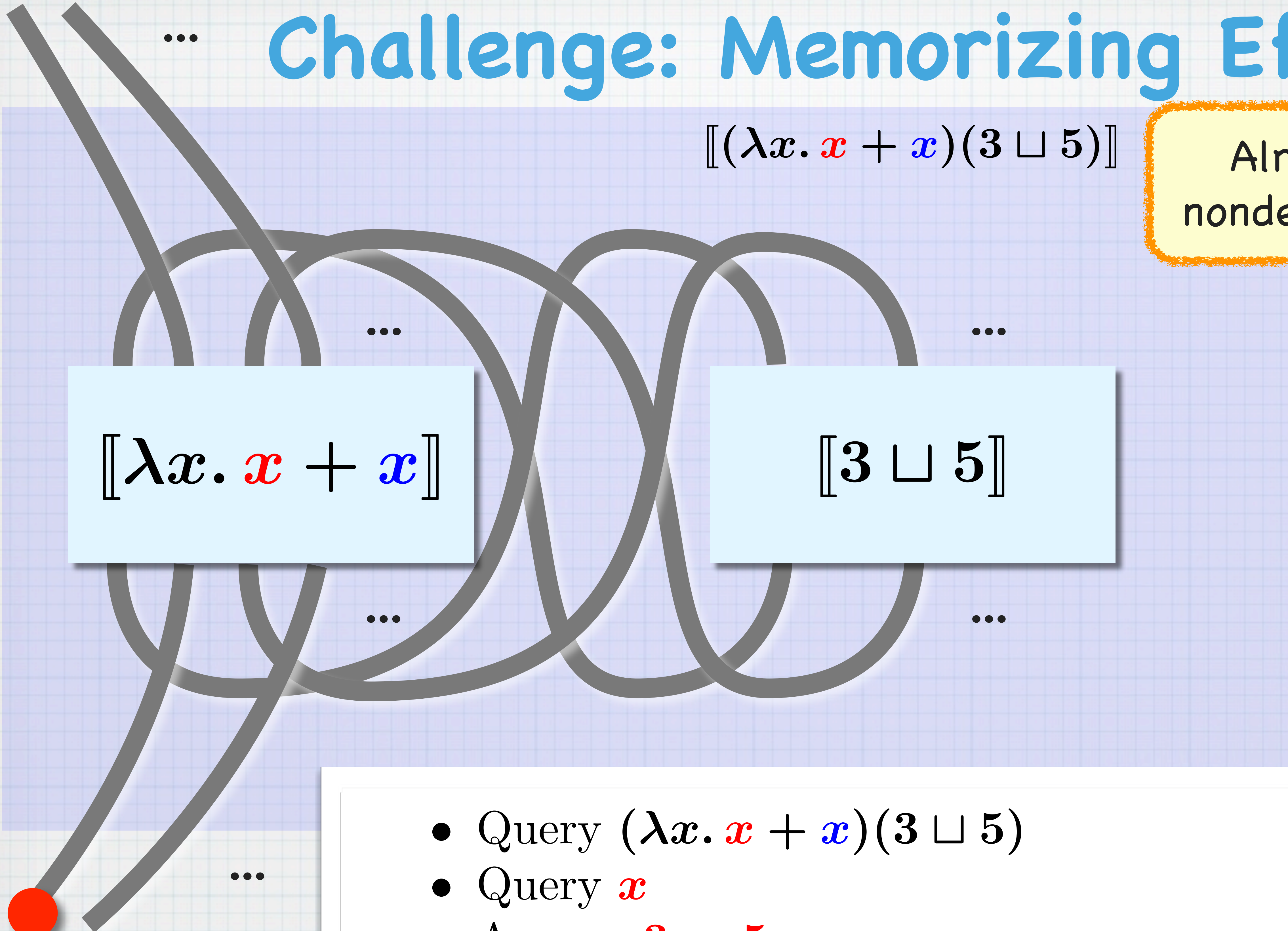

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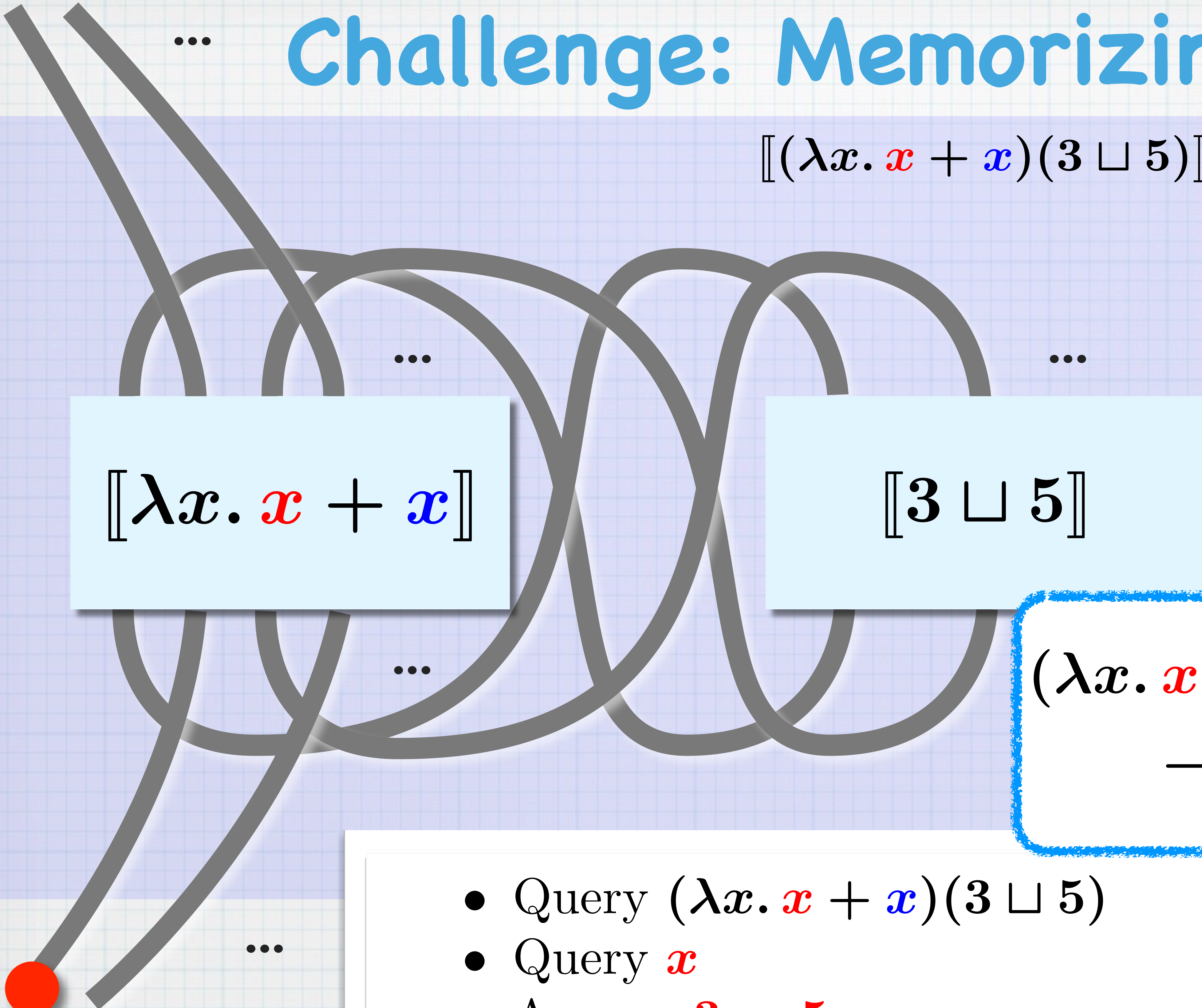
Challenge: Memorizing Effects

$[\![(\lambda x.\, x + x)(3 \sqcup 5)]\!]$



Already w/ nondeterminism!

$(\lambda x.\, x + x)(3 \sqcup 5) \longrightarrow_{\mathrm{CBV}}$ **6** or **10** ??

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Challenge: Memorizing Effects

- Nondeterministic choice is resolved
  ➜ we must **stick to it**!
- Is CBV to blame?
  (GoI is inherently CBN...)

$$(\lambda x.\, x + x)(3 \sqcup 5) \longrightarrow_{\mathrm{CBV}} \mathbf{6} \text{ or } \mathbf{10}$$

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- Not really: it's also hard to get

$$(M \sqcup N)L = ML \sqcup NL$$

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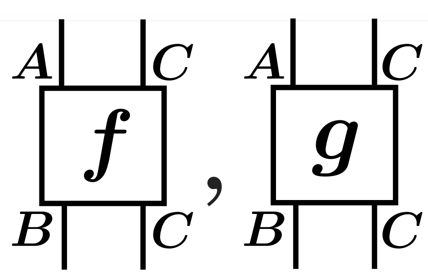
  - Not really: it's also hard to get

$$(M \sqcup N)L = ML \sqcup NL$$

  - Mathematically:

Given  $: A + C \longrightarrow \mathcal{P}(B + C)$,

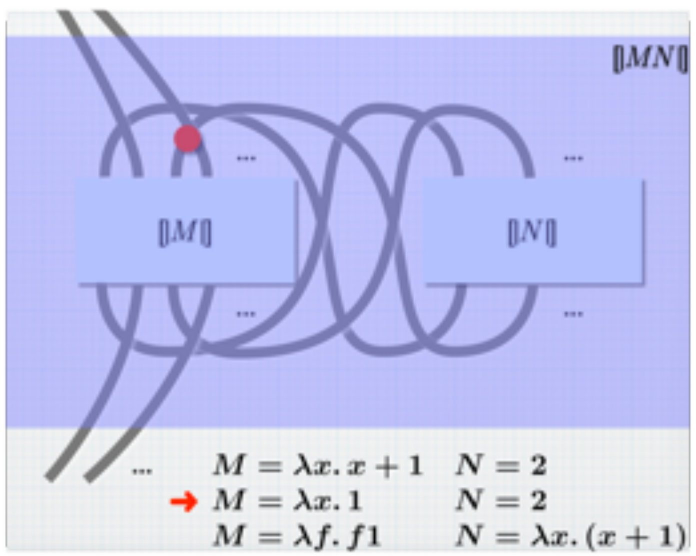
$$\mathbf{tr}(f \cup g) \neq \mathbf{tr}(f) \cup \mathbf{tr}(g)$$

Outline

Coalgebra meets **higher-order computation**

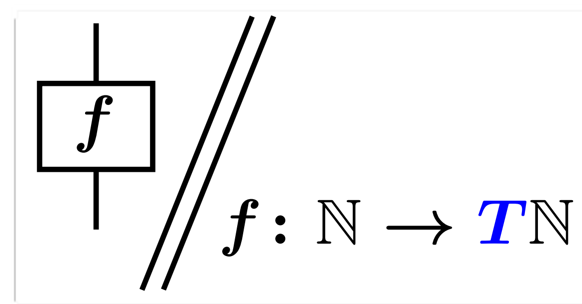
in **Geometry of Interaction** [Girard, LC'88]

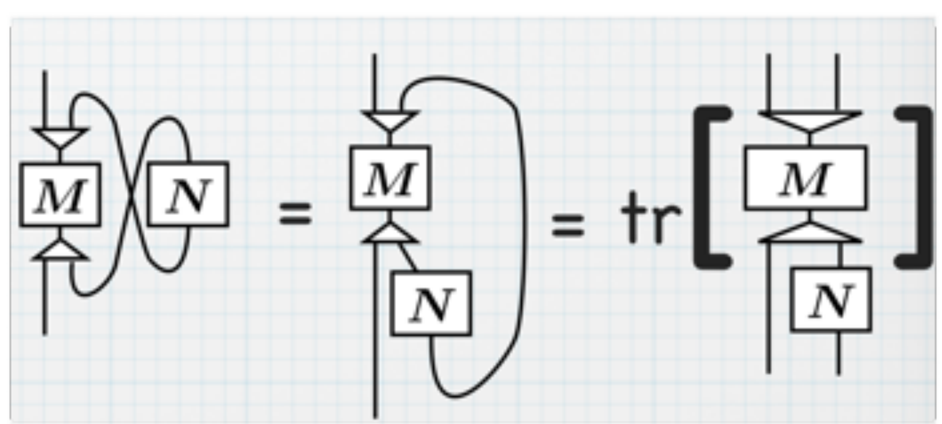
"GoI Animation"



GoI w/ T-branching

[IH & Hoshino, LICS'11]





Categorical GoI

[Abramsky, Haghverdi & Scott, MSCS'02]

Outline

Coalgebra meets **higher-order computation** in **Geometry of Interaction** [Girard, LC'88]

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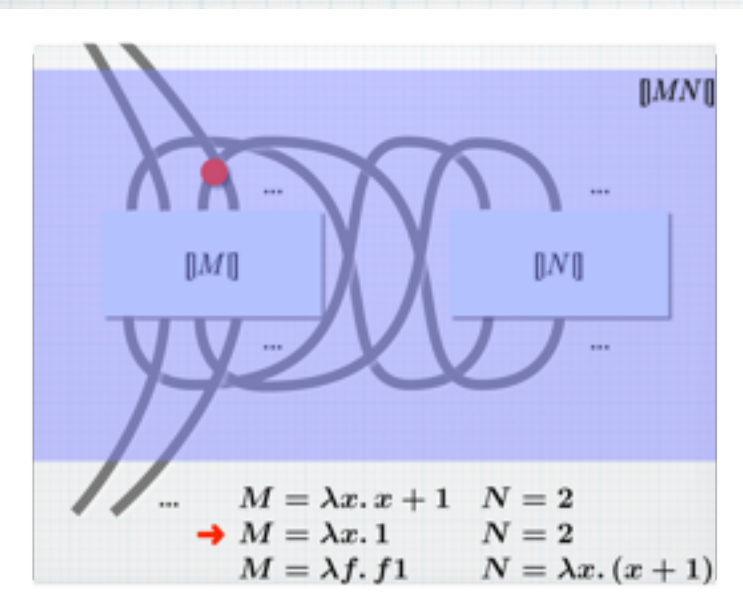

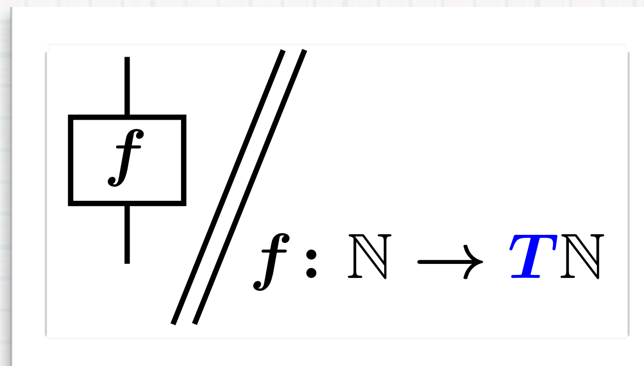

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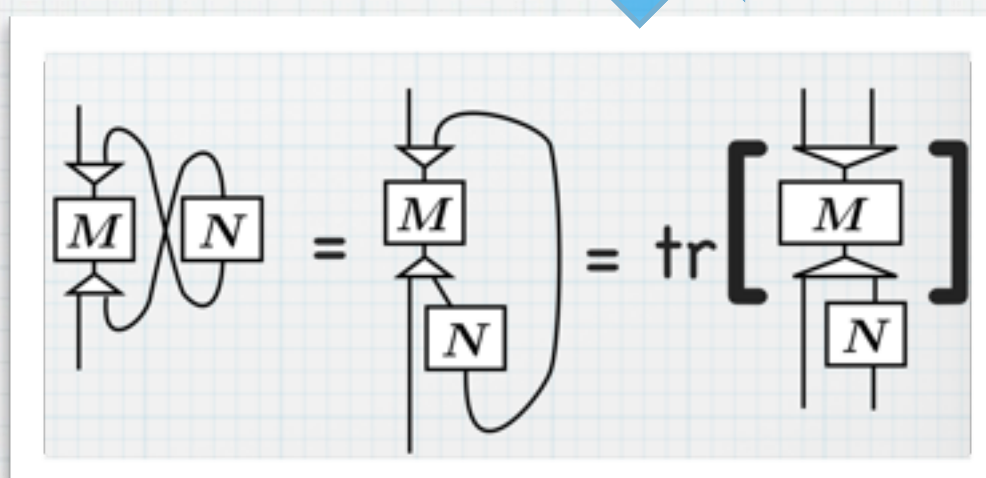

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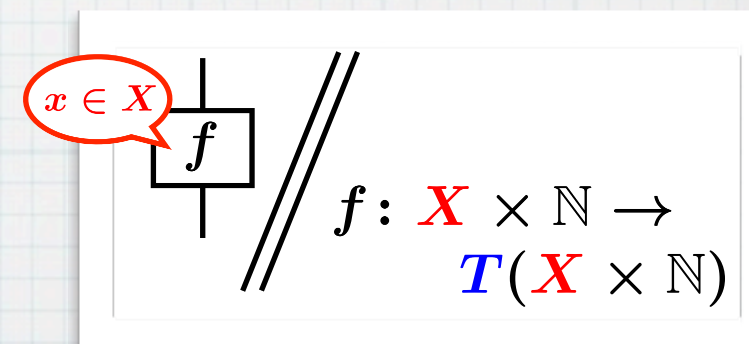
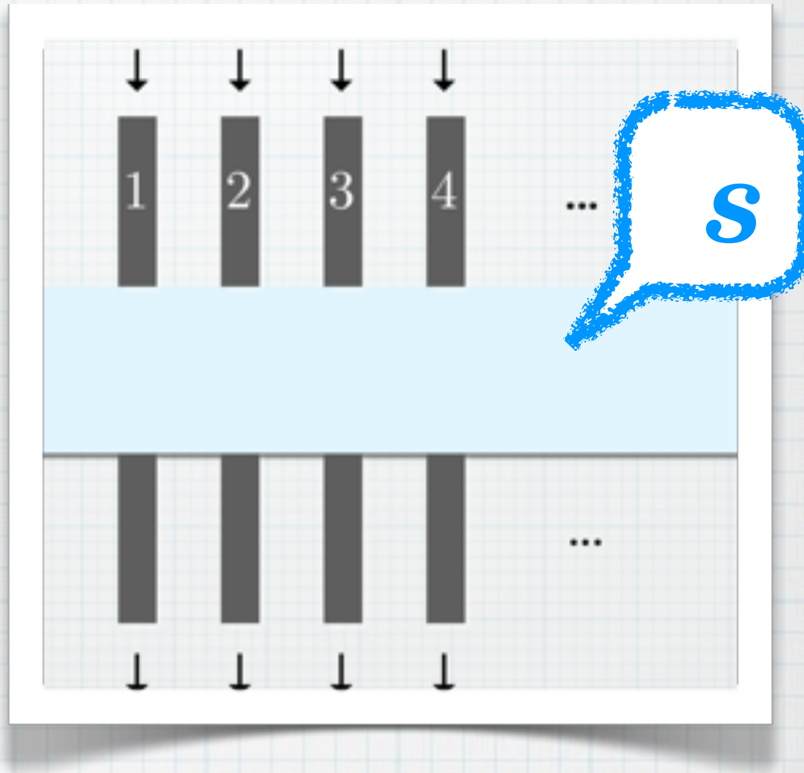


Memoryful GoI

[Hoshino, Muroya & IH, CSL-LICS'14 & POPL'16]

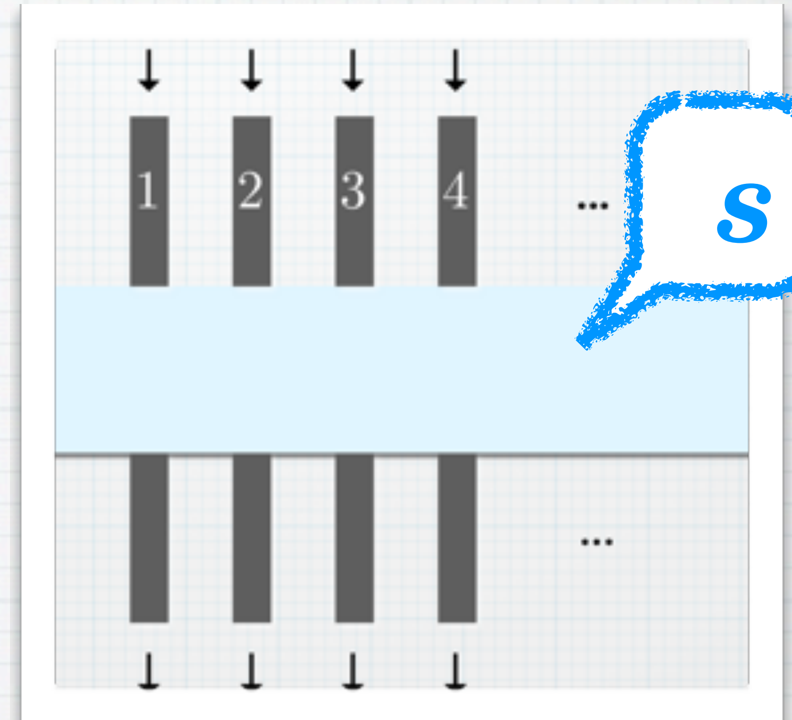
Memoryful GoI

* Equip piping with internal states, or **memory**



Hasuo (Tokyo)

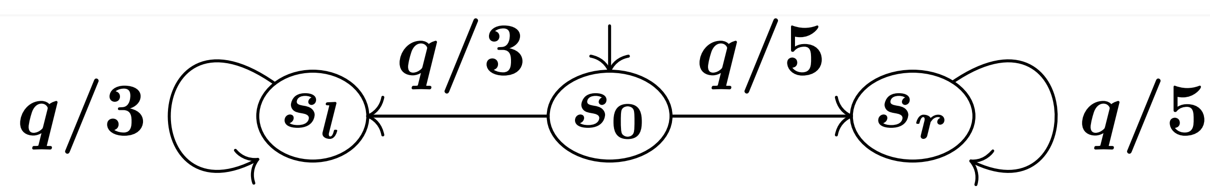
Memoryful GoI



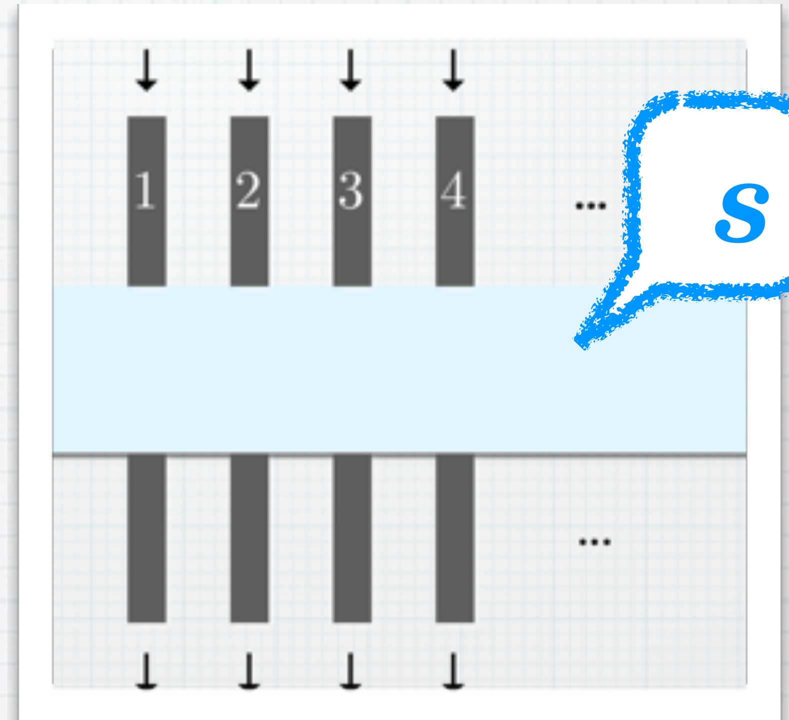
* Equip piping with internal states, or **memory**
* **not** $[\![3 \sqcup 5]\!] : \mathbb{N} \longrightarrow \mathcal{P}\mathbb{N} \ , \quad q \longmapsto \{3,5\}$

but a transducer (Mealy machine)

$[\![3 \sqcup 5]\!] : X \times \mathbb{N} \longrightarrow \mathcal{P}(X \times \mathbb{N}) \ ,$



Memoryful GoI



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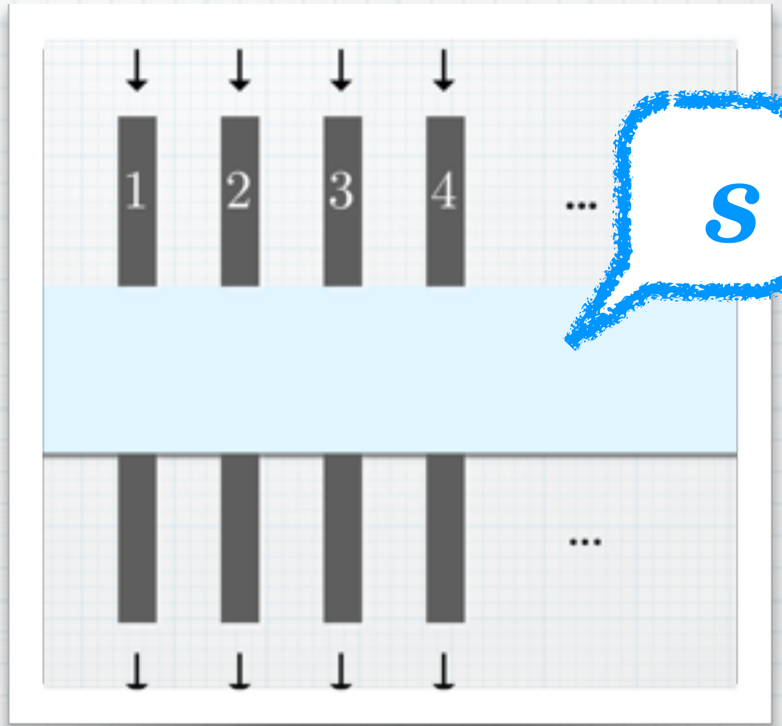
$$[\![3 \sqcup 5]\!] : \boldsymbol{X} \times \mathbb{N} \longrightarrow \mathcal{P}(\boldsymbol{X} \times \mathbb{N}) \ ,$$

$$\begin{tikzcd}
s_l \arrow[loop left, "q/3"] & s_0 \arrow[l, "q/3"'] \arrow[r, "q/5"] & s_r \arrow[loop right, "q/5"]
\end{tikzcd}$$

Memoryful GoI



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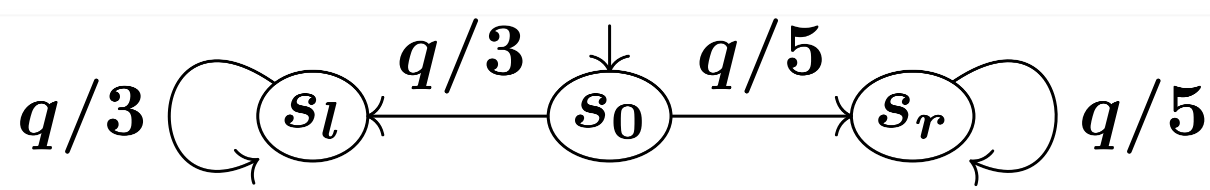
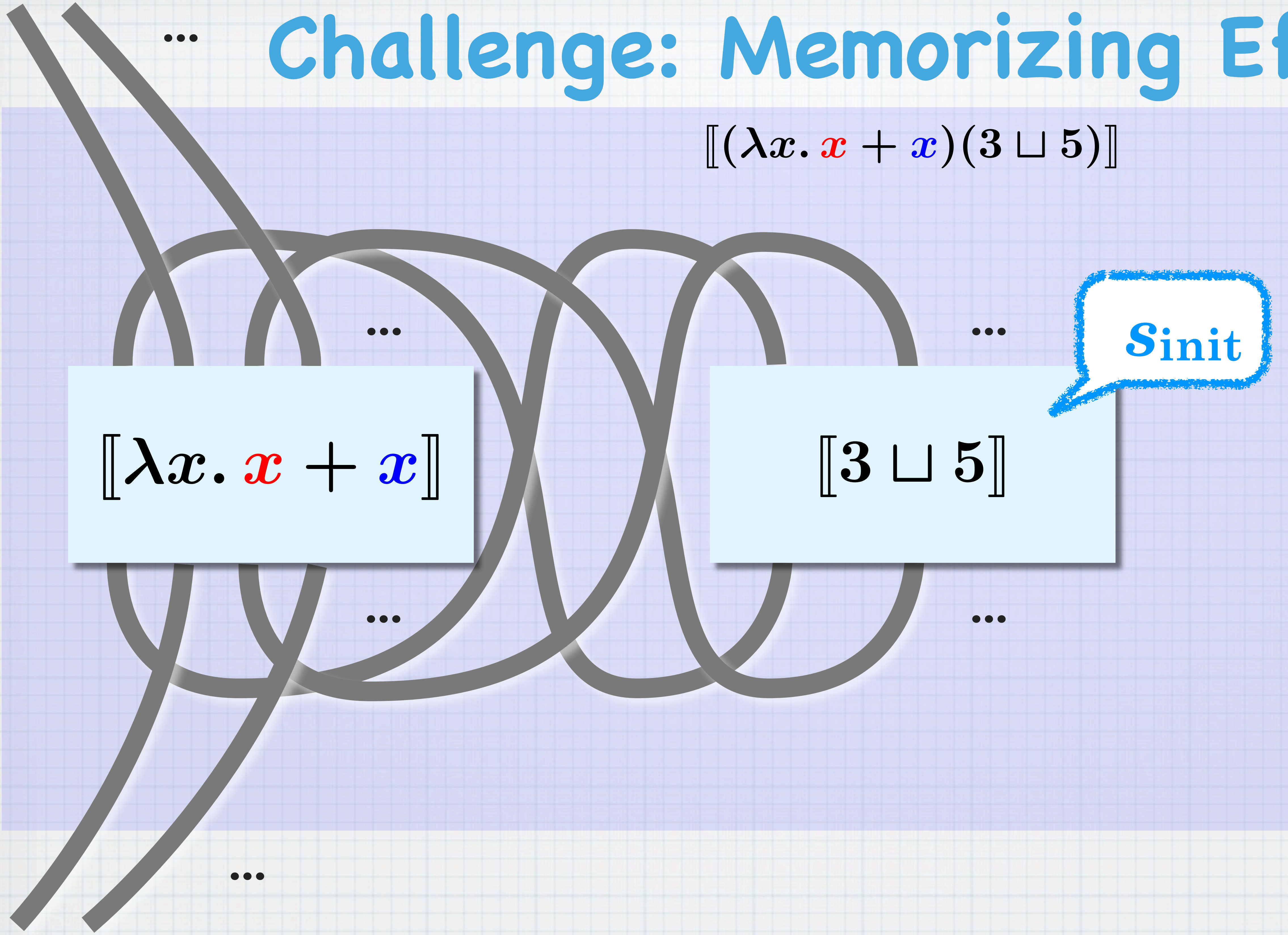


- Not a new idea:
  - Slices in GoI for additives [Laurent, TLCA'01]
  - Resumption GoI [Abramsky, CONCUR'96]

Challenge: Memorizing Effects



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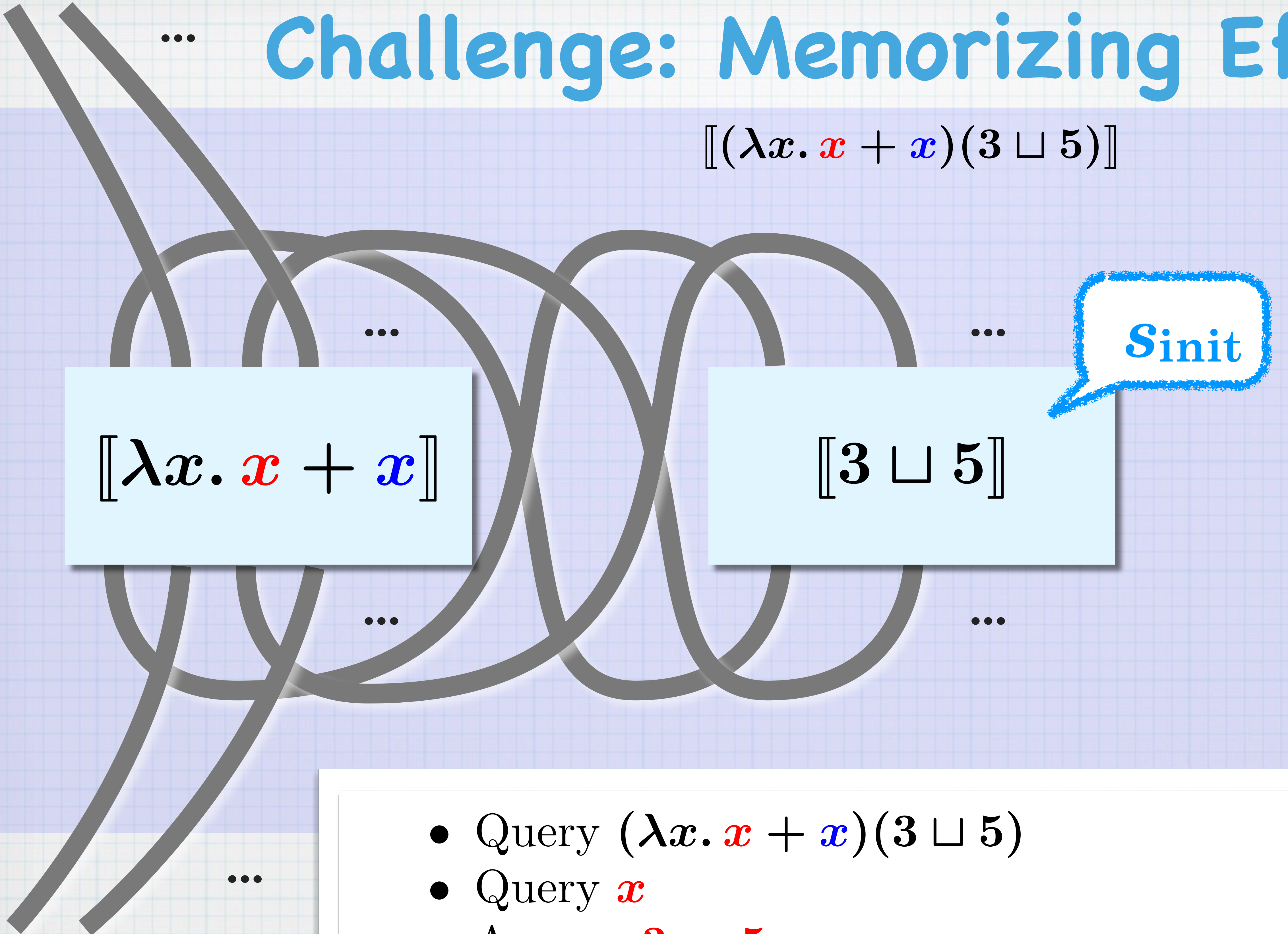

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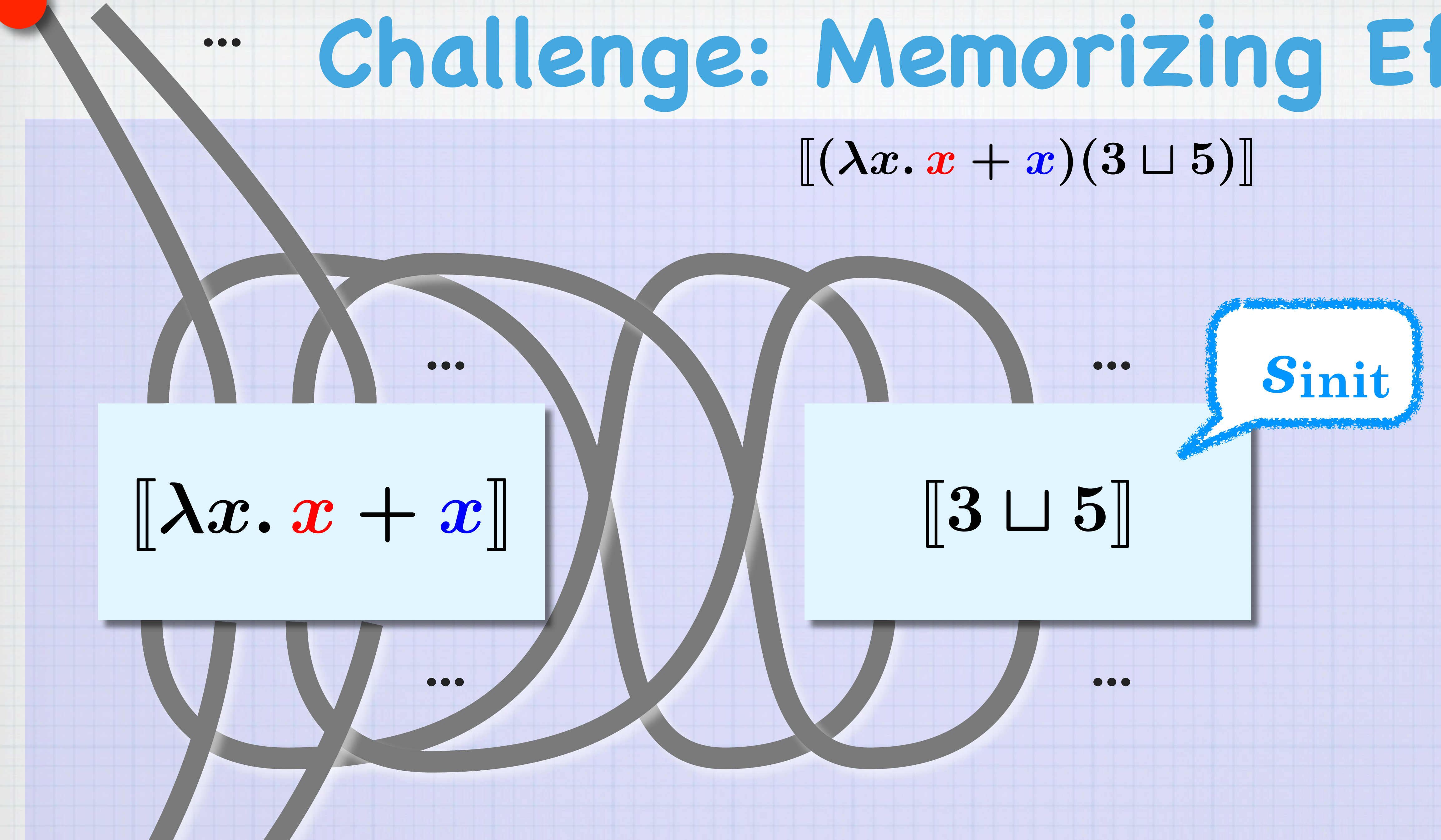

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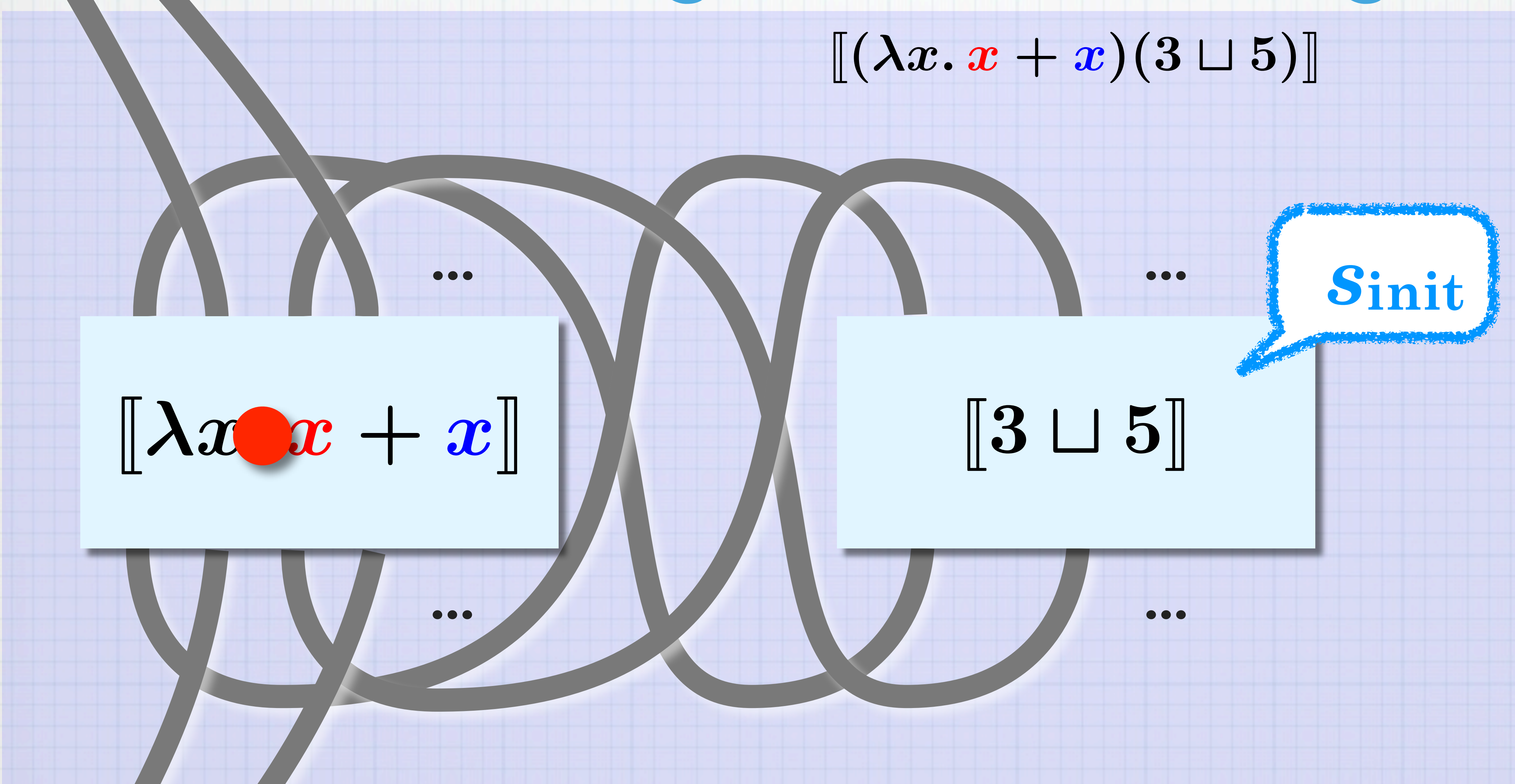

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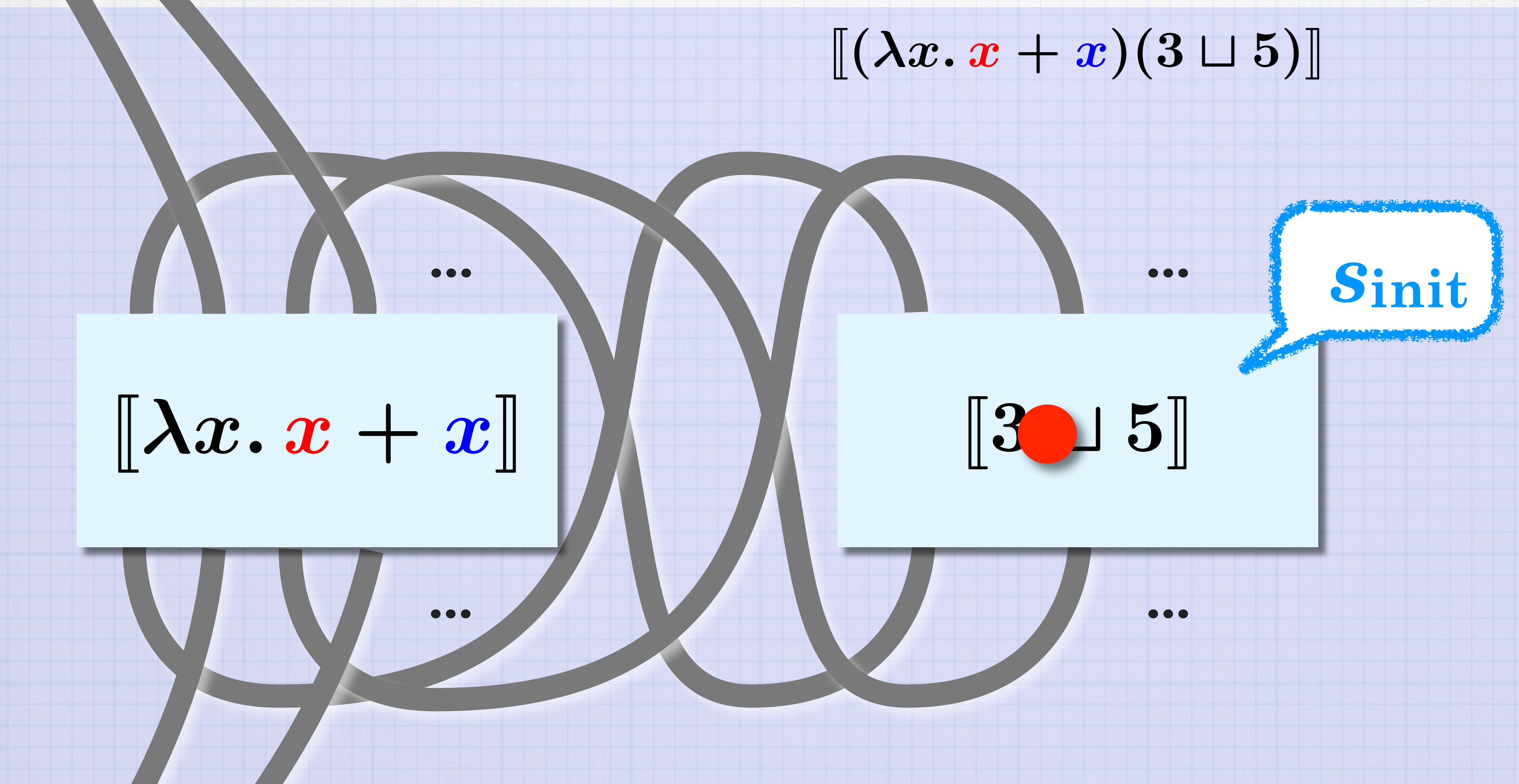

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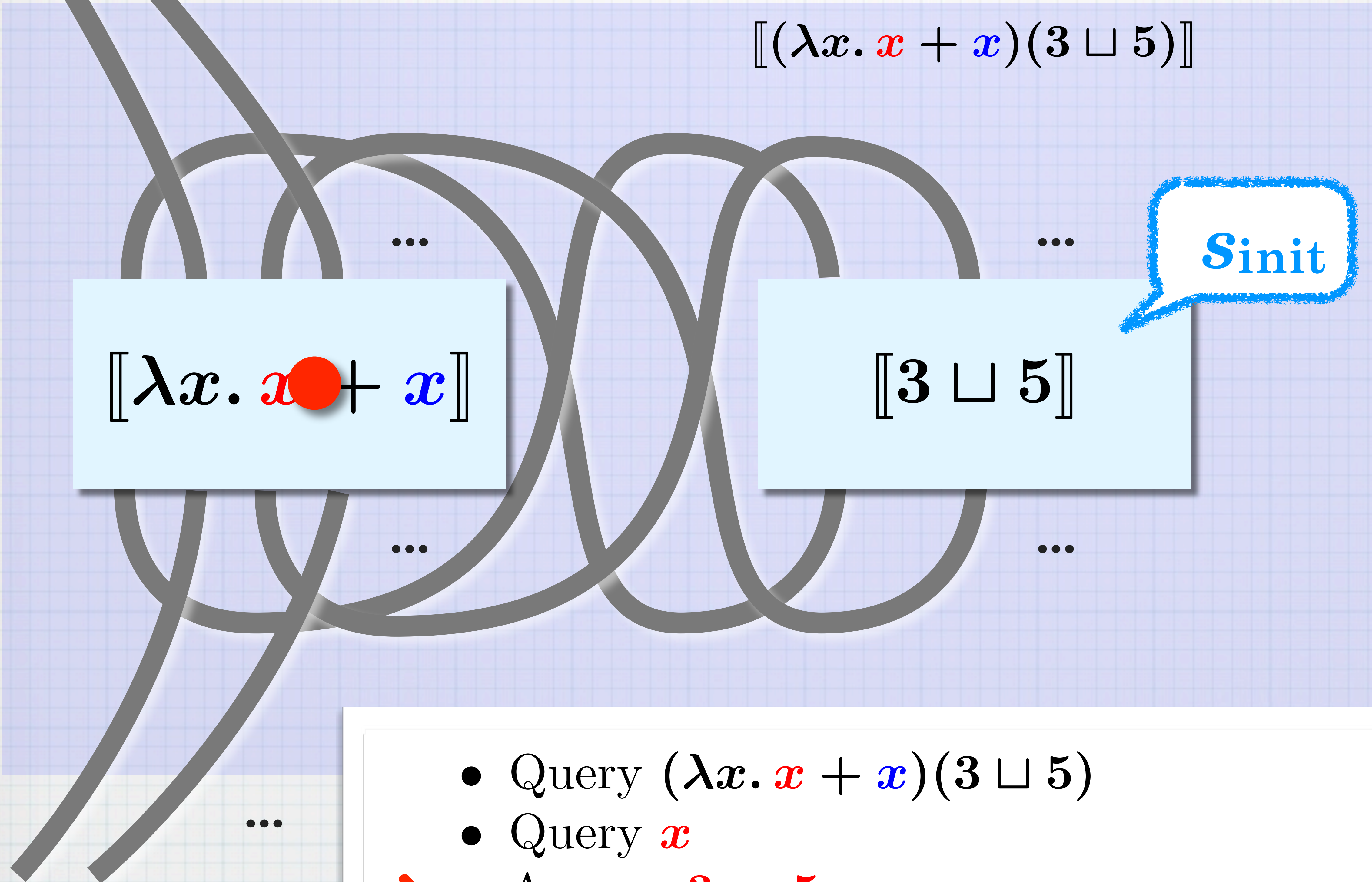
Challenge: Memorizing Effects



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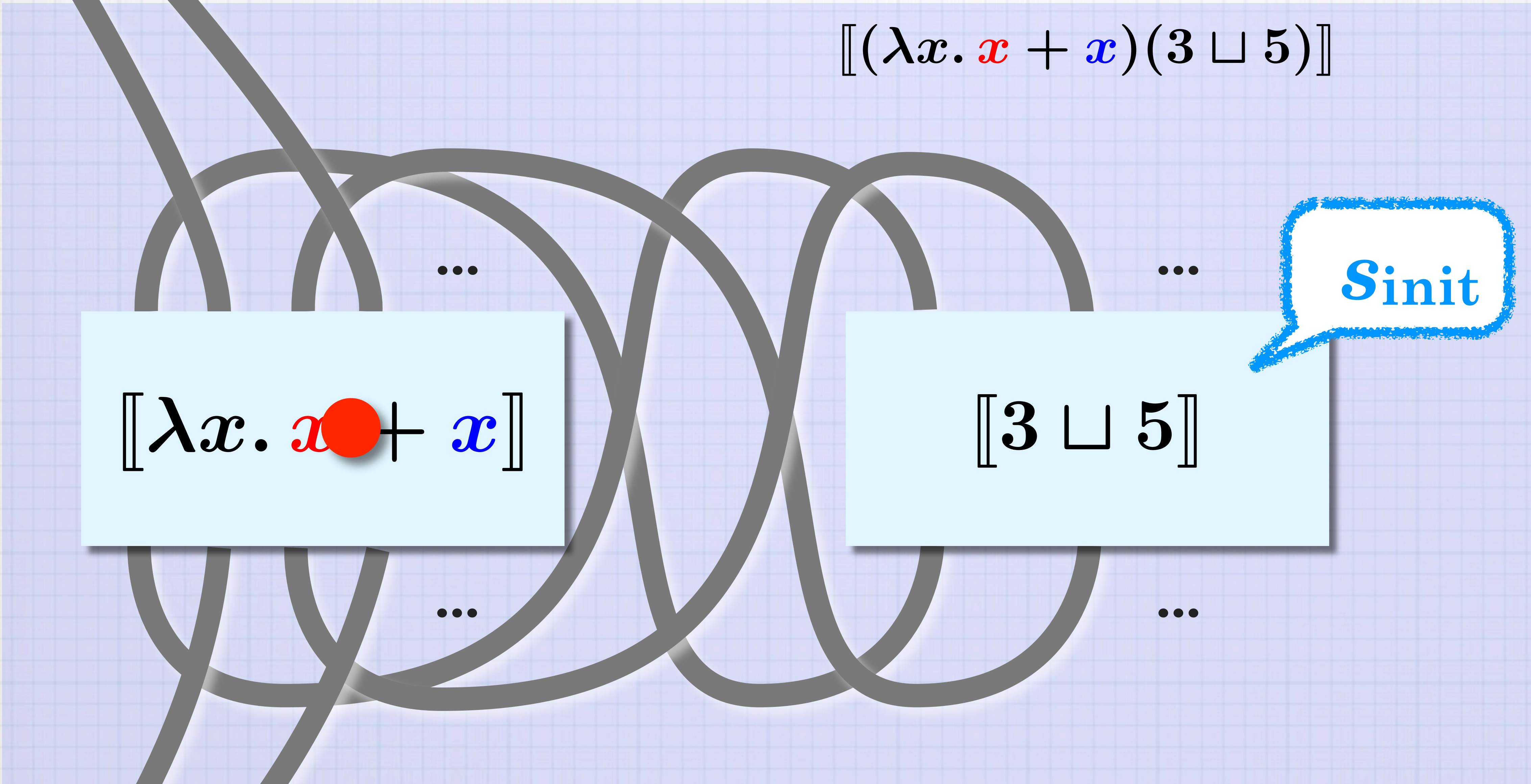

- Query $(\lambda x.\, x + x)(3 \sqcup 5)$
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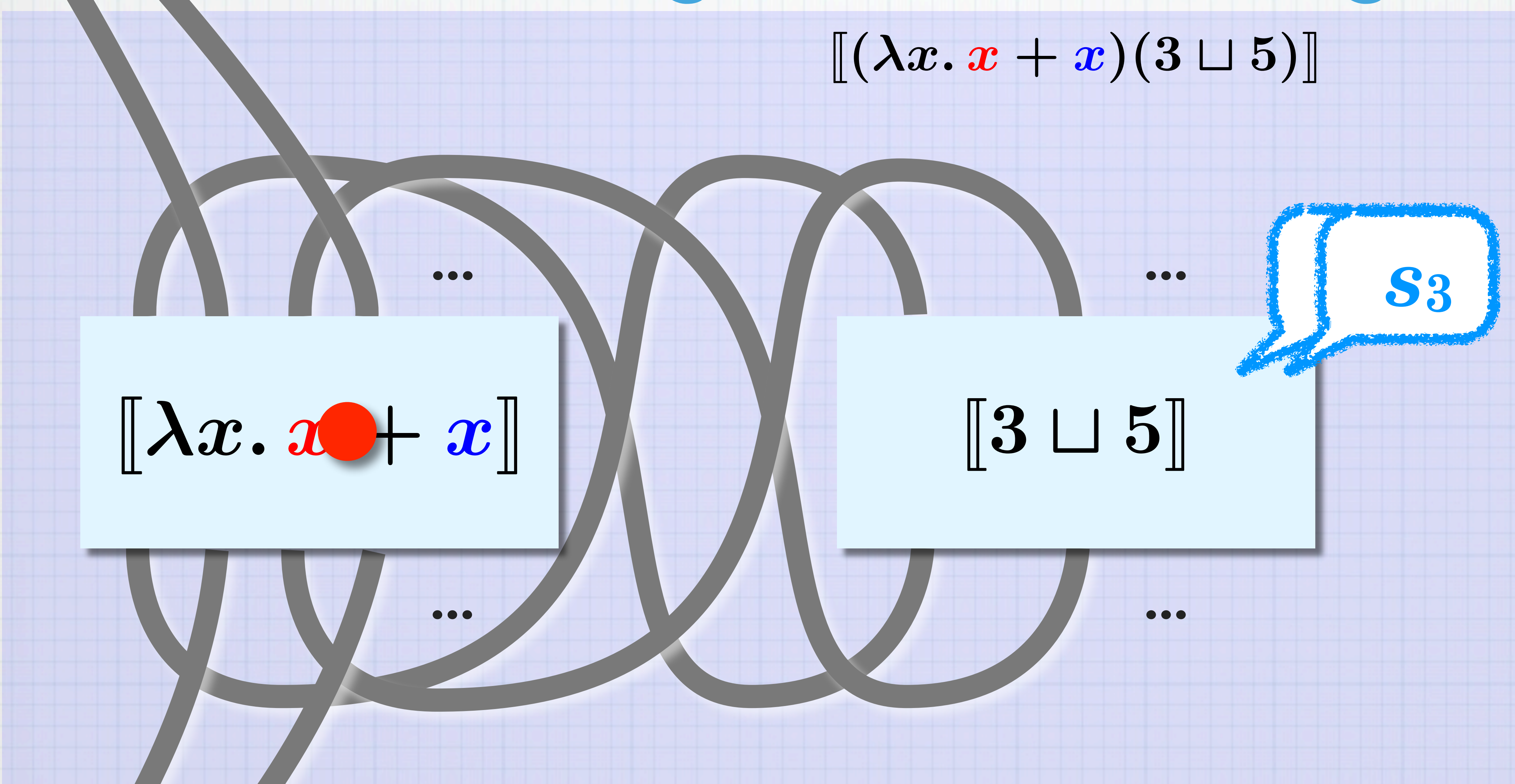
Challenge: Memorizing Effects



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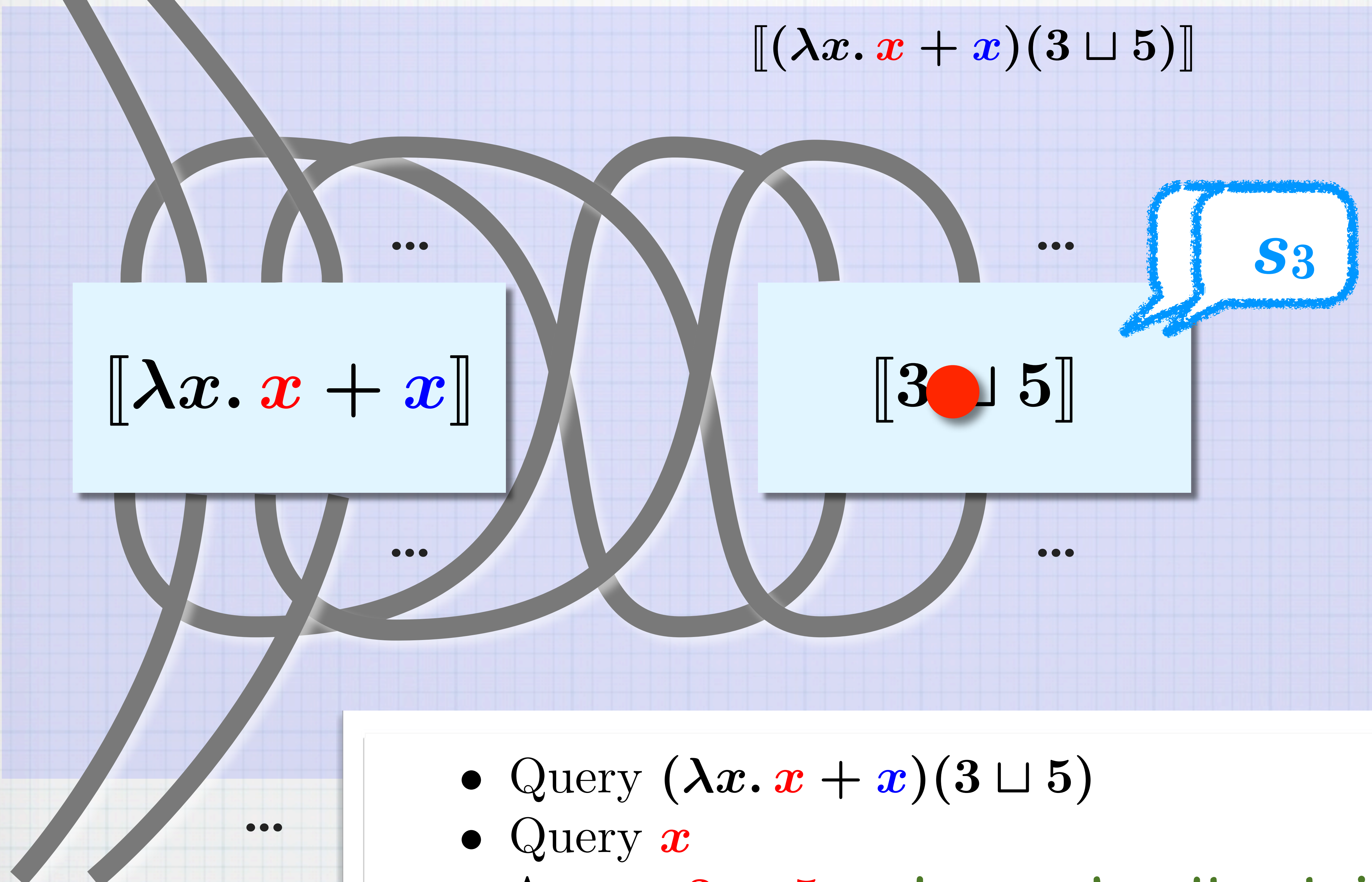
- Query x
- Answer 3 or 5
- Answer $3 + 3$, $3 + 5$, $5 + 3$ or $5 + 5$

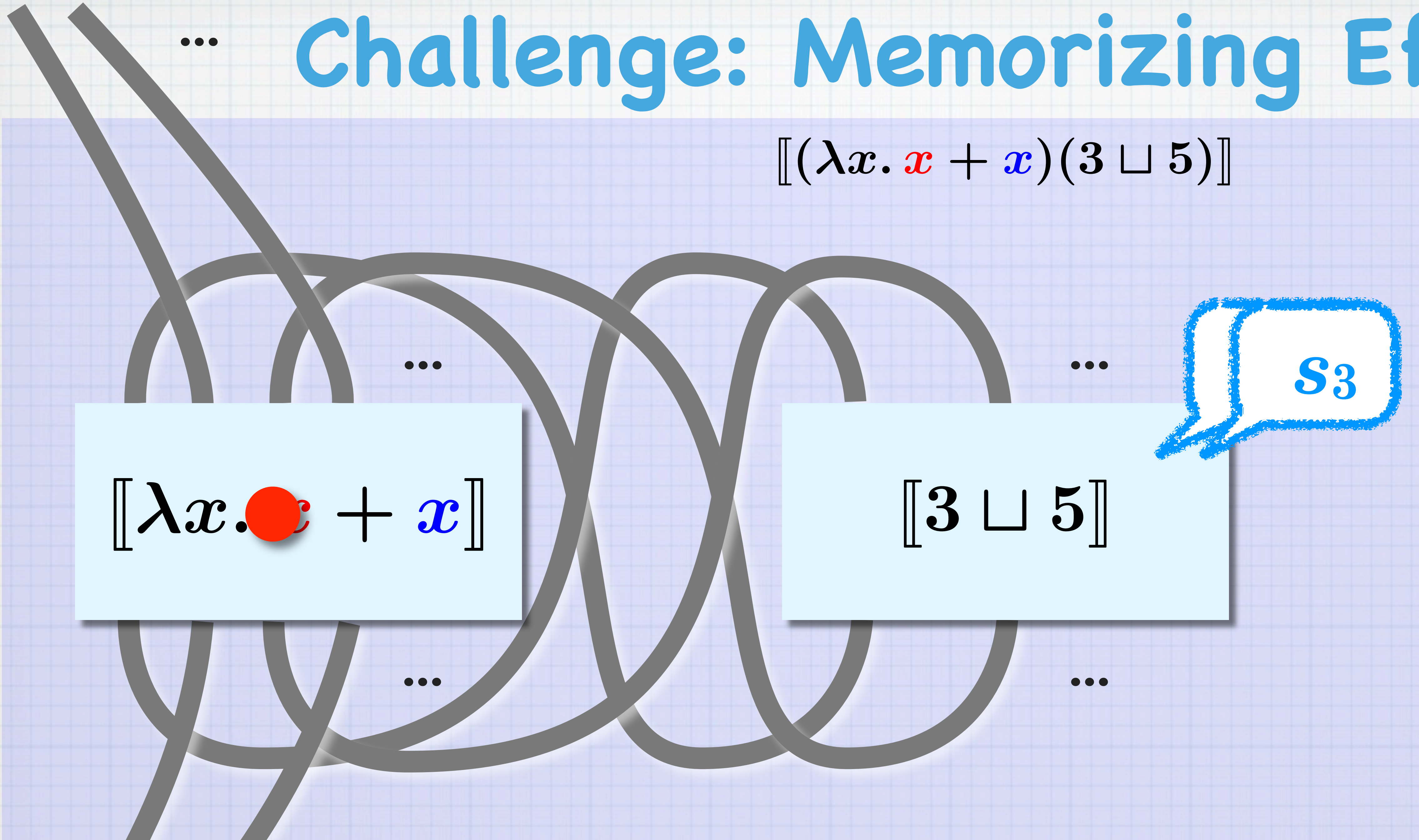
Challenge: Memorizing Effects



- Query $(\lambda x.\, x + x)(3 \sqcup 5)$
- Query x
- Answer $\mathbf{3}$ or $\mathbf{5}$ and remember the choice
- → Query x
- Answer $\mathbf{3}$ or $\mathbf{5}$
- Answer $\mathbf{3} + \mathbf{3}$, $\mathbf{3} + \mathbf{5}$, $\mathbf{5} + \mathbf{3}$ or $\mathbf{5} + \mathbf{5}$

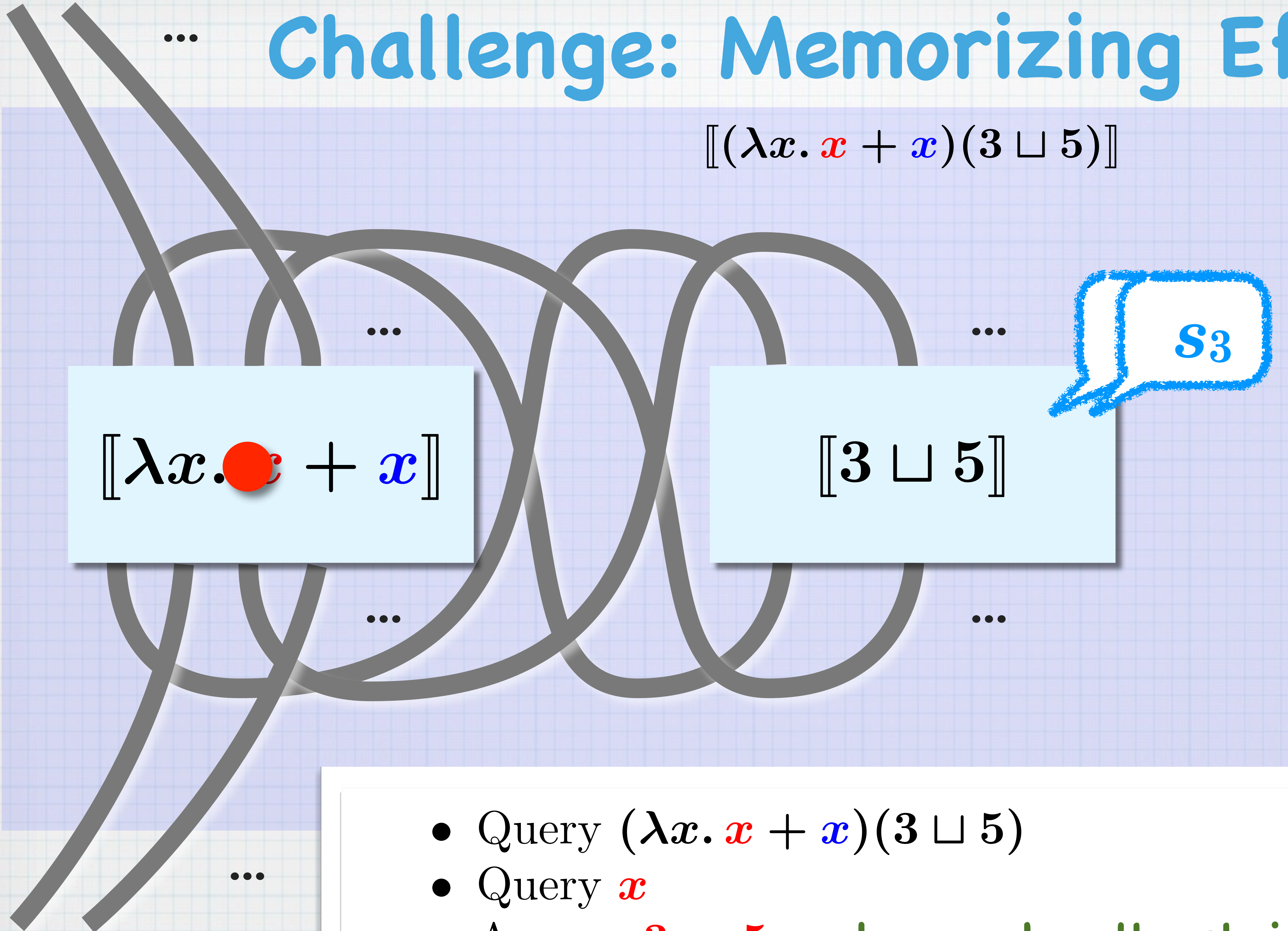
Challenge: Memorizing Effects

$$[\![(\lambda x.\, \textcolor{red}{x} + \textcolor{blue}{x})(3 \sqcup 5)]\!]$$



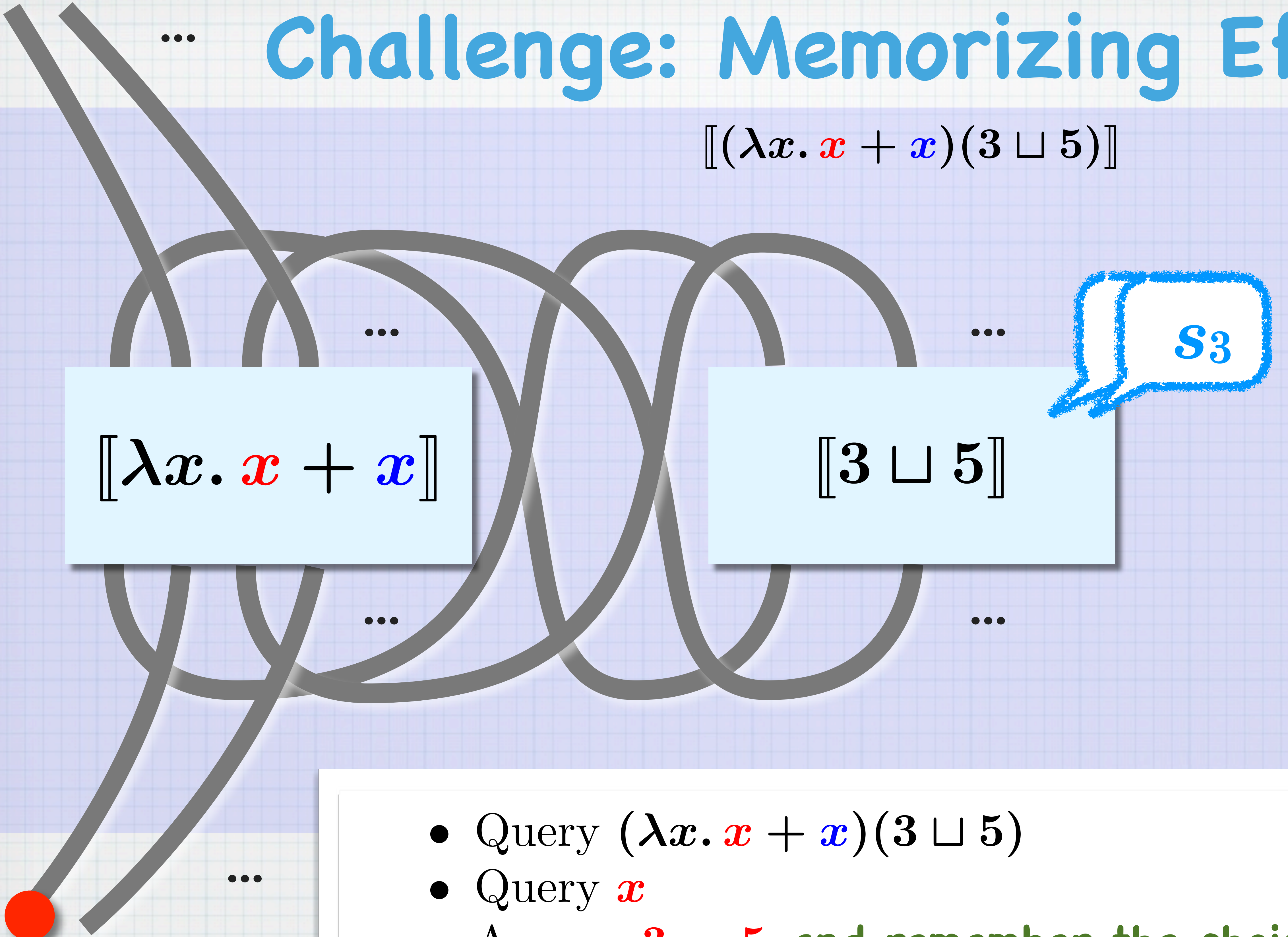
- Query $(\lambda x.\, \textcolor{red}{x} + \textcolor{blue}{x})(3 \sqcup 5)$
- Query $\textcolor{red}{x}$
- Answer $\textcolor{red}{3}$ or $\textcolor{red}{5}$ <u>and remember the choice</u>
- Query $\textcolor{blue}{x}$
- ➜ Answer $\textcolor{blue}{3}$ or $\textcolor{blue}{5}$
- Answer $\textcolor{red}{3} + \textcolor{blue}{3}$, $\textcolor{red}{3} + \textcolor{blue}{5}$, $\textcolor{red}{5} + \textcolor{blue}{3}$ or $\textcolor{red}{5} + \textcolor{blue}{5}$

Challenge: Memorizing Effects



- Query $(\lambda x.\, x + x)(3 \sqcup 5)$
- Query x
- Answer 3 or 5 and remember the choice
- Query x
- → Answer 3 or 5 following the prev. choice
- Answer $3 + 3$, $3 + 5$, $5 + 3$ or $5 + 5$

Challenge: Memorizing Effects



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- Query x
- Answer 3 or 5 <u>and remember the choice</u>
- Query x
- Answer 3 or 5 <u>following the prev. choice</u>
- Answer $3 + 3$, $\cancel{3 + 5}$, $\cancel{5 + 3}$ or $5 + 5$

Memoryful GoI

- That is...
  a traversing token rearranges piping!

Memoryful GoI

* We introduce memory in a structured manner...
  ➜
  the "traced monoidal category" of transducers

$\boxed{\mathbf{Trans}(T)}$	Objects:	sets $A, B, \ldots$
	Arrows:	$$\dfrac{A \longrightarrow B \text{ in } \mathbf{Trans}(T)}{\big(\, X,\ X \times A \xrightarrow{c} T(X \times B),\ x_0 \in X\ \big),\ T\text{-transducer}}$$

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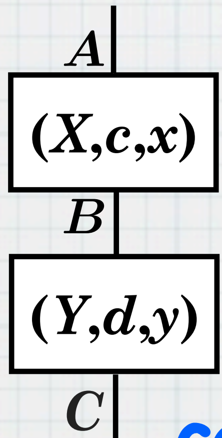
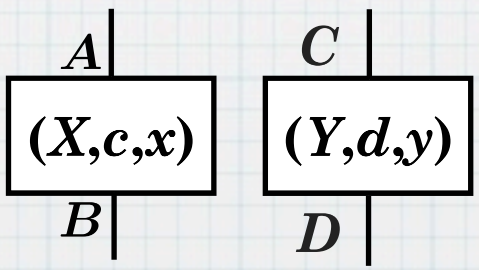
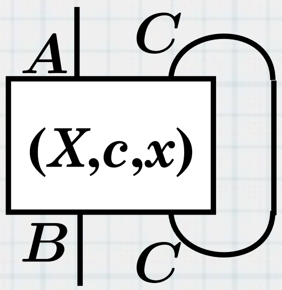
- with operations like



composition $\circ$



tensor $\otimes$



trace

Trans(T) by Coalgebraic Component Calculus

[Barbosa '03][IH & Jacobs '11]

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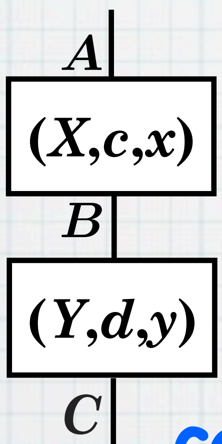

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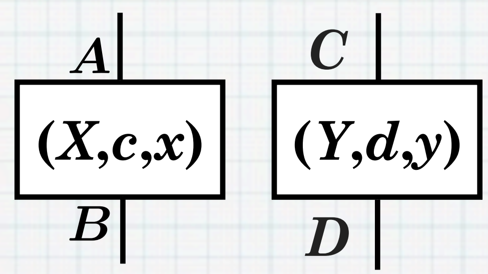

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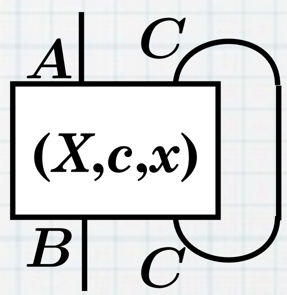

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Hasuo (Tokyo)

Trans(T) by Coalgebraic Component Calculus

[Barbosa '03][IH & Jacobs '11]

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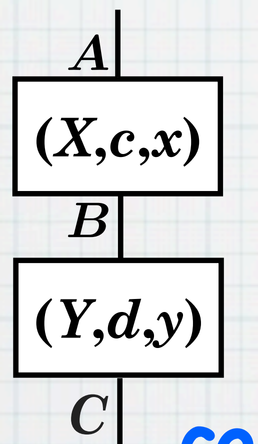

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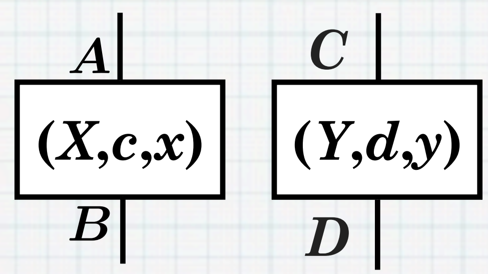
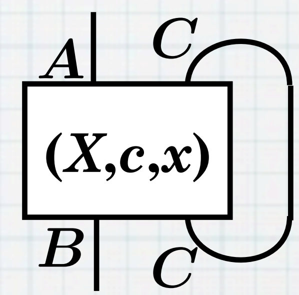


tensor $\otimes$



trace

$$\left(X \times Y,\ \begin{array}{ll} (X \times Y) \times A & \stackrel{\cong}{\longrightarrow} (X \times A) \times Y \\ & \stackrel{c \times Y}{\longrightarrow} T(X \times B) \times Y \\ & \stackrel{\mathrm{str}'}{\longrightarrow} T\big((X \times B) \times Y\big) \\ & \stackrel{\cong}{\longrightarrow} T\big(X \times (Y \times B)\big) \\ & \stackrel{T(X \times d)}{\longrightarrow} T\big(X \times T(Y \times C)\big) \\ & \stackrel{T\mathrm{str}}{\longrightarrow} TT\big(X \times (Y \times C)\big) \\ & \stackrel{\mu^T}{\longrightarrow} T\big(X \times (Y \times C)\big) \\ & \stackrel{\cong}{\longrightarrow} T\big((X \times Y) \times C\big) \end{array},\ (x, y) \right)$$

Hasuo (Tokyo)

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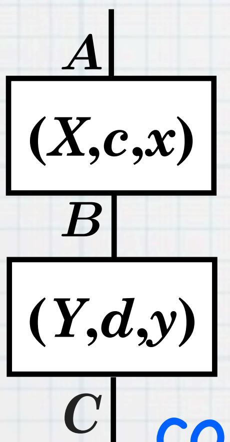

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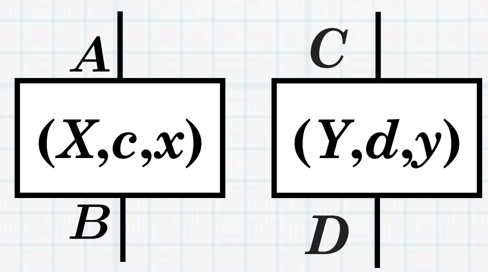

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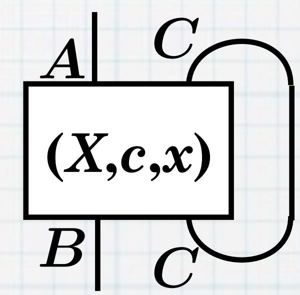
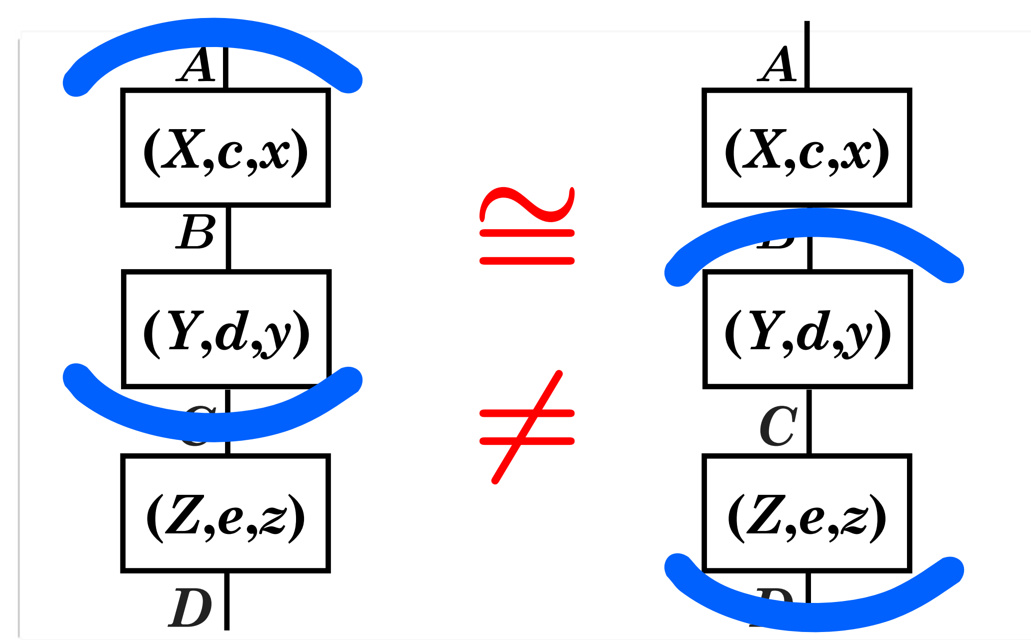


trace

* Trans(T) is a "category"...



Hasuo (Tokyo)

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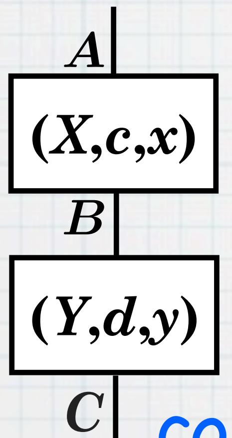

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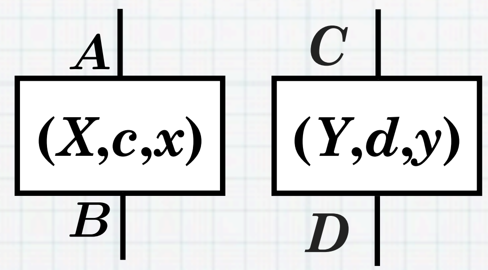

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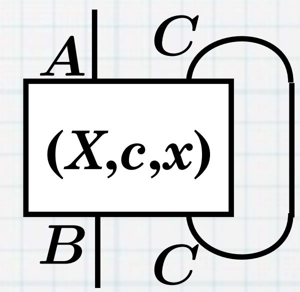
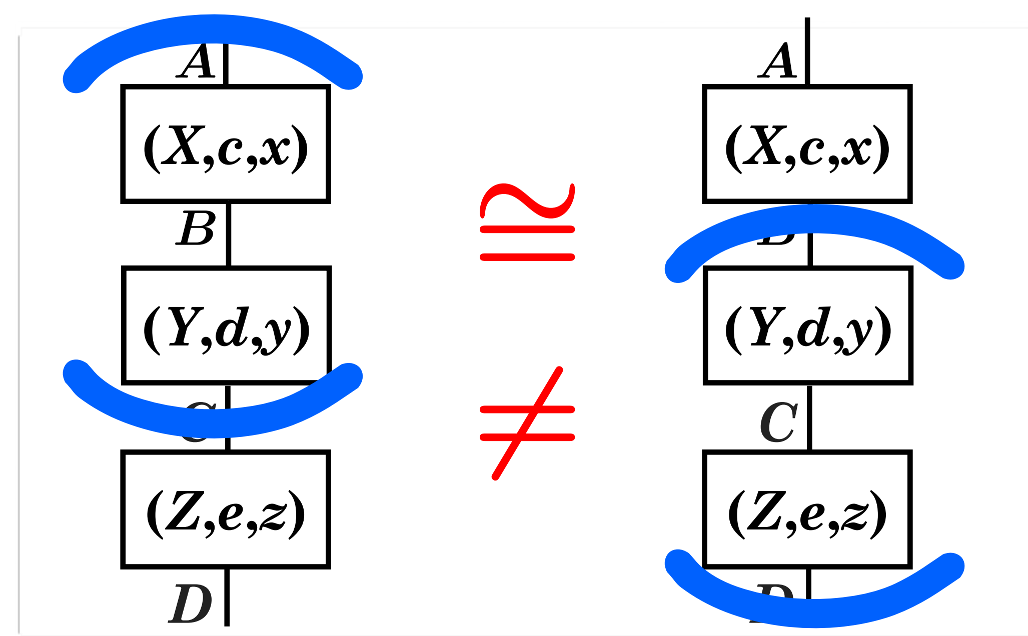


trace

* Trans(T) is a "category"...
* Fix: quotient modulo **behavioral equivalence** (homomorphisms of T-transducers) ➜ **resumptions** [Abramsky]



The Memoryful GoI Framework

* Given:
  * a **monad** T on $\mathbf{Sets}$, s.t. $\mathbf{Kl}(T)$ is $\mathbf{Cppo}$-enriched
  * an alg. signature Σ with **algebraic operations** on T [Plotkin & Power]

$$\left\{ \alpha_{A,B} \colon (A \Rightarrow TB)^{|\alpha|} \longrightarrow (A \Rightarrow TB) \right\}_{A \in \mathbf{Sets}, B \in \mathcal{K}\ell(T)}$$

* For the calculus: $\lambda_{\mathbf{c}}$ + (alg. opr. from Σ) + (co)products + arith.
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  - with $\mathbf{0}$-ary opr. $\mathbf{raise}_e$ ($e \in E$)
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  - with binary opr. $\sqcup$
- *Probability* $\mathcal{D}$, where $\mathcal{D}X = \{d \colon X \to [0,1] \mid \sum_x d(x) \le 1\}$
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  - with $|V|$-ary $\mathbf{lookup}_l$ and unary $\mathbf{update}_{l,v}$

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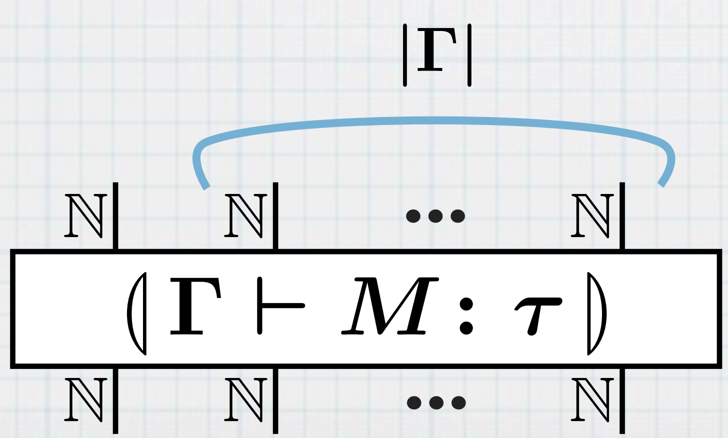

in $\mathbf{Trans}(T)$

Hasuo (Tokyo)

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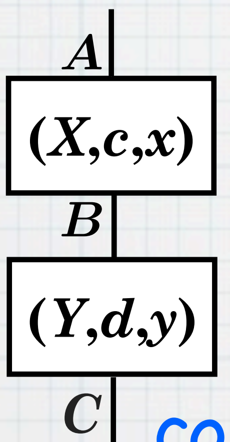
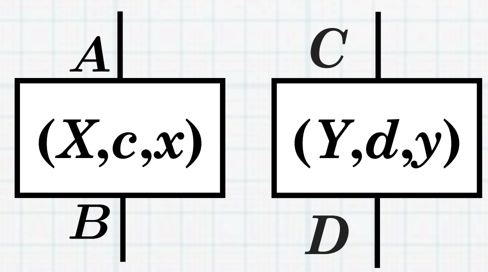
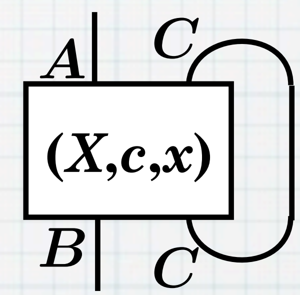


composition



tensor



trace

$$\left(X \times Y,\ \begin{array}{rl} (X \times Y) \times A & \xrightarrow{\cong} (X \times A) \times Y \\ & \xrightarrow{c \times Y} T(X \times B) \times Y \\ & \xrightarrow{\mathrm{str}'} T\big((X \times B) \times Y\big) \\ & \xrightarrow{\cong} T\big(X \times (Y \times B)\big) \\ & \xrightarrow{T(X \times d)} T\big(X \times T(Y \times C)\big) \\ & \xrightarrow{T\mathrm{str}} TT\big(X \times (Y \times C)\big) \\ & \xrightarrow{\mu^T} T\big(X \times (Y \times C)\big) \\ & \xrightarrow{\cong} T\big((X \times Y) \times C\big) \end{array},\ (x, y) \right)$$

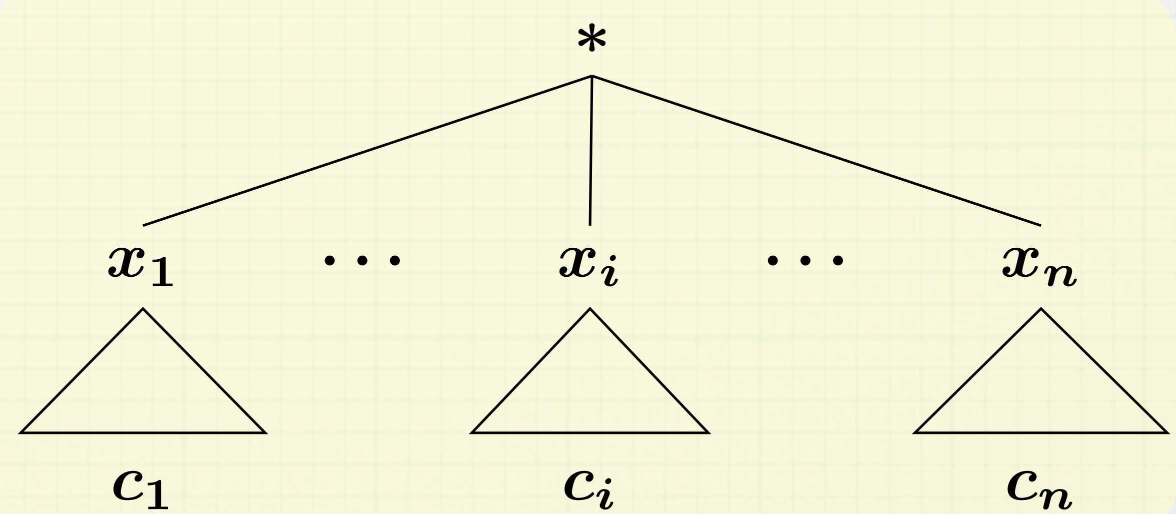
Hasuo (Tokyo)

Missing Ingredient I: Alg. Opr.

* $\alpha \in \Sigma_n$ an alg. operation

$$\alpha\left(\underset{B}{\overset{A}{\boxed{(X_1,c_1,x_1)}}}, \ldots, \underset{B}{\overset{A}{\boxed{(X_n,c_n,x_n)}}}\right)$$

$$= \Bigl(\{*\} + X_1 + \cdots + X_n ,$$



$$, \; * \Bigr)$$

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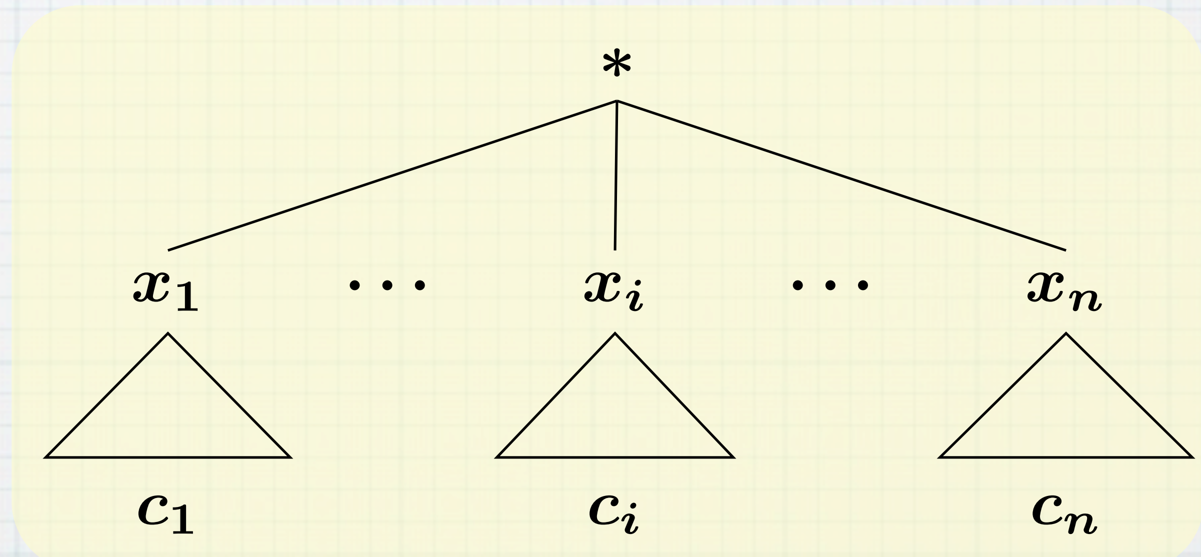

$$,\ * \Big)$$

Fresh initial state

Missing Ingredient I: Alg. Opr.

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T-branching given by
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$$\alpha\left(\overset{A}{\underset{B}{\boxed{(X_1,c_1,x_1)}}} \;,\; \ldots \;,\; \overset{A}{\underset{B}{\boxed{(X_n,c_n,x_n)}}} \right)$$

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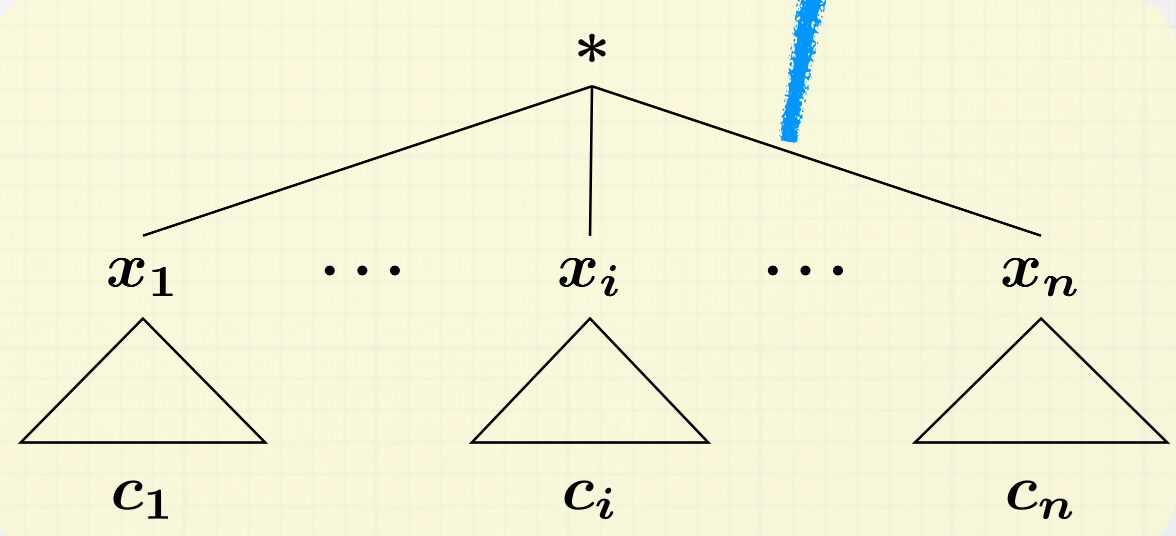
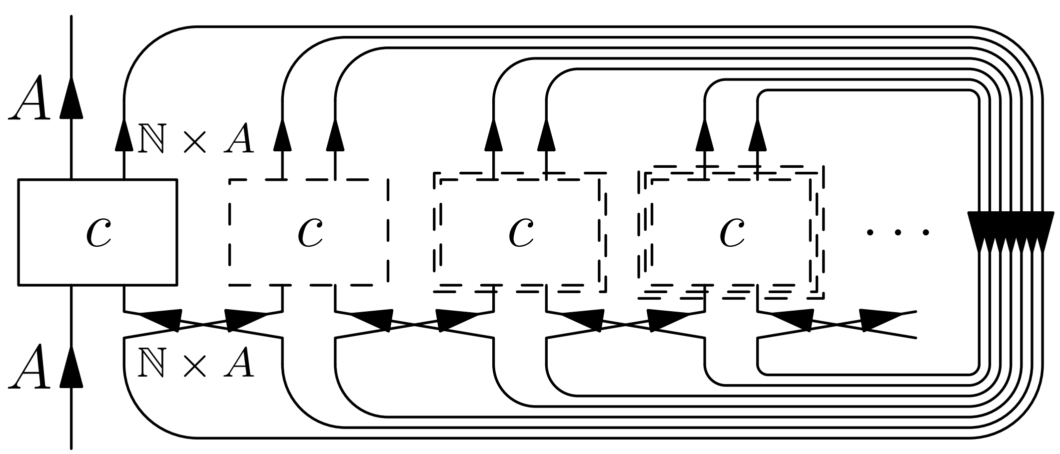


$$\left. ,\; * \right)$$

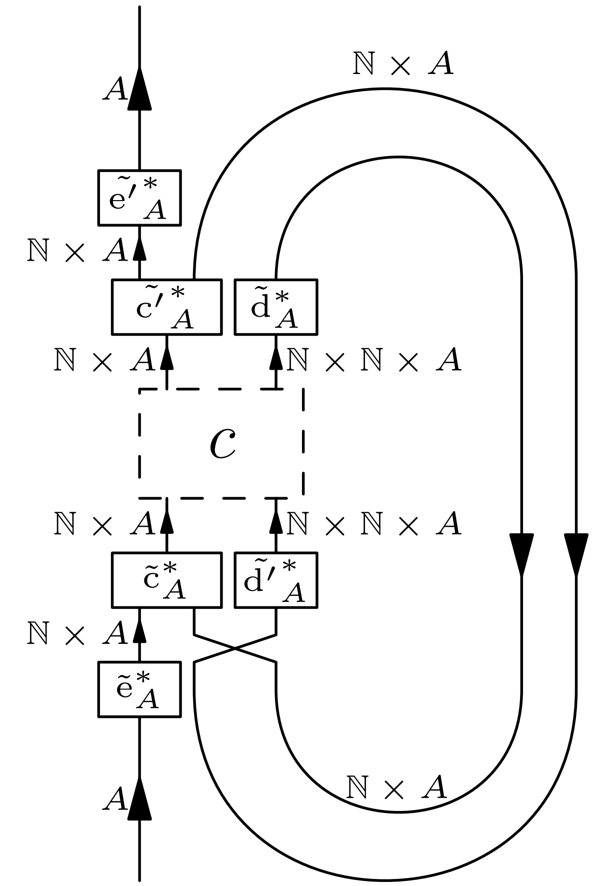
Fresh initial state

Missing Ingredient II: Recursion

Girard style
fixed point operator



Mackie style
fixed point operator



* Obviously a fixed point
* Fixed-point induction
* Finitary string diagram

Missing Ingredient II: Recursion

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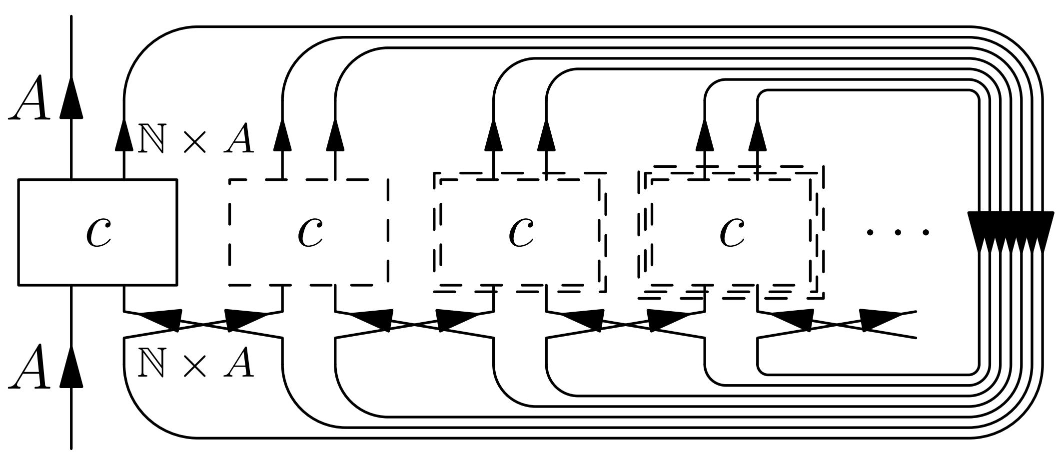

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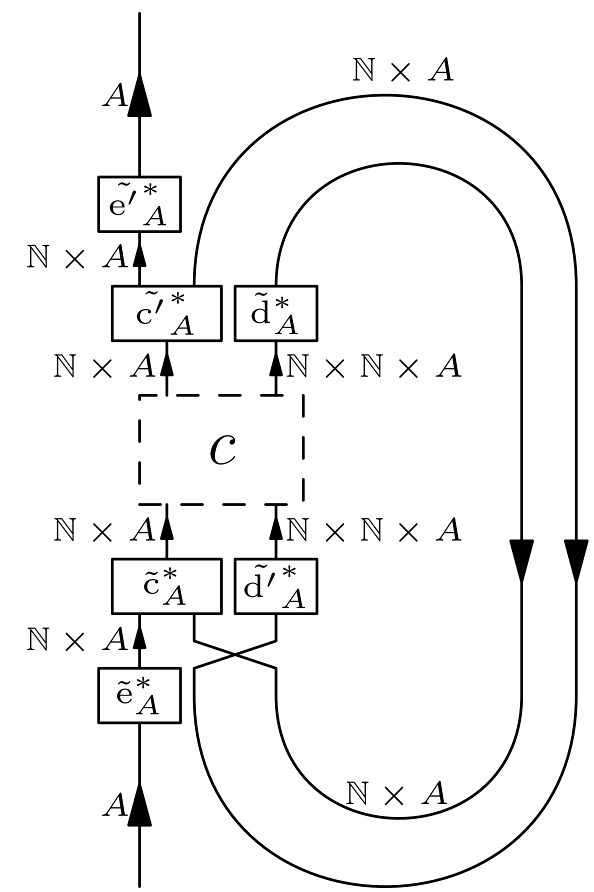

- Obviously a fixed point
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Theorem The two coincide. (for any suitable T!)

Hasuo (Tokyo)

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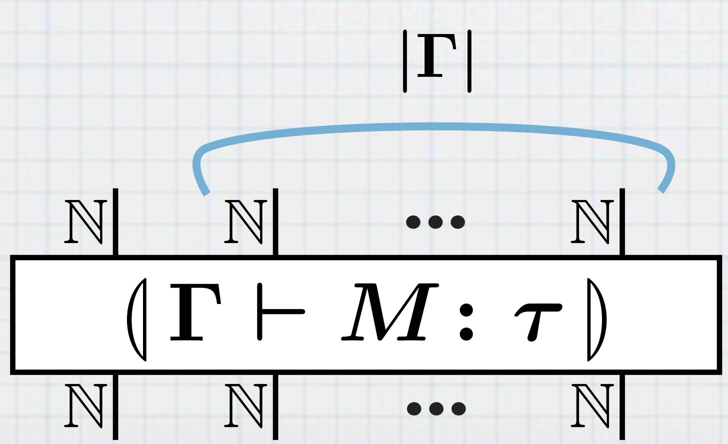
- We give



in $\mathbf{Trans}(T)$

The Memoryful GoI Framework

* Given:
  * a **monad** T on $\mathbf{Sets}$, s.t. $\mathbf{Kl}(T)$ is $\mathbf{Cppo}$-enriched
  * an alg. signature Σ with **algebraic operations** on T
    [Plotkin & Power]

$$\left\{ \alpha_{A,B} \colon (A \Rightarrow TB)^{|\alpha|} \longrightarrow (A \Rightarrow TB) \right\}_{A \in \mathbf{Sets}, B \in \mathcal{K}\ell(T)}$$

- *Exception* $\mathbf{1} + E + (\_)$
  - with $\mathbf{0}$-ary opr. $\mathbf{raise}_e$ $(e \in E)$
- *Nondeterminism* $\mathcal{P}$
  - with binary opr. $\sqcup$
- *Probability* $\mathcal{D}$, where
  $\mathcal{D}X = \{d \colon X \to [0,1] \mid \sum_x d(x) \leq 1\}$
  - with binary opr. $\sqcup_p$ $(p \in [0,1])$
- *Global state* $(\mathbf{1} + S \times \_)^S$
  - with $|V|$-ary $\mathbf{lookup}_l$ and unary $\mathbf{update}_{l,v}$

* For the calculus: λ_c + (alg. opr. from Σ) + (co)products

$$\frac{\Gamma \vdash M_1 : \tau \quad \cdots \quad \Gamma \vdash M_{|\alpha|} : \tau}{\Gamma \vdash \alpha(M_1, \ldots, M_{|\alpha|}) : \tau} \; \alpha \in \Sigma$$

* We give

in $\mathbf{Trans}(T)$

The Memoryful GoI Framework

<u>Theorem</u> (Adequacy)

Let $\vdash \boldsymbol{M} : \mathtt{nat}$. Then, as elem. of $\boldsymbol{T}(\mathbb{N})$,

$$\left(\begin{array}{c} \mathbb{N} \\ \boxed{(\!|\vdash \boldsymbol{M} : \mathtt{nat}\,|\!)} \\ \mathbb{N} \end{array} \right)^{\dagger} = [\![\, |\boldsymbol{M}| \,]\!] \,.$$

* For the calculus: λ_c + (alg. opr. from Σ) + (co)products

$$\frac{\Gamma \vdash M_1 : \tau \quad \cdots \quad \Gamma \vdash M_{|\alpha|} : \tau}{\Gamma \vdash \alpha(M_1, \ldots, M_{|\alpha|}) : \tau}\ \alpha \in \Sigma$$

* We give

$$\begin{array}{c} \overbrace{\mathbb{N} \quad \mathbb{N} \quad \cdots \quad \mathbb{N}}^{|\Gamma|} \\ \boxed{(\!|\Gamma \vdash M : \tau|\!)} \\ \mathbb{N} \quad \mathbb{N} \quad \cdots \quad \mathbb{N} \end{array}$$

in $\mathbf{Trans}(T)$

The Memoryful GoI Framework

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feeding a query
and observing
the outcome

The Memoryful GoI Framework

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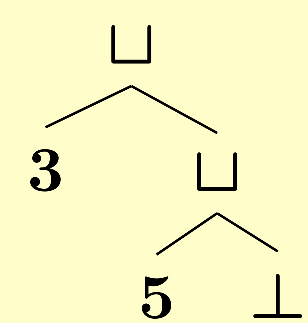
feeding a query and observing the outcome

Opr. sem.: Plotkin-Power **effect-value**. E.g.

$$|\,\mathbf{3} \sqcup (\mathbf{5} \sqcup \mathtt{div})\,| =$$



Interpretation

$$[\![\_]\!] : \mathbf{EffVal}_{\mathbb{N}}^{\Sigma} \longrightarrow T(\mathbb{N})$$

(exploiting free conti. Σ-alg.)

Theorem (Adequacy)

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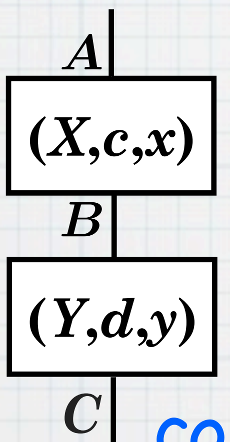
$$|\,3 \sqcup (5 \sqcup \mathtt{div})\,| = \begin{array}{c} \sqcup \\ 3 \quad \sqcup \\ \quad\;\; 5 \quad \bot \end{array}$$

Trans(T) by Coalgebraic Component Calculus

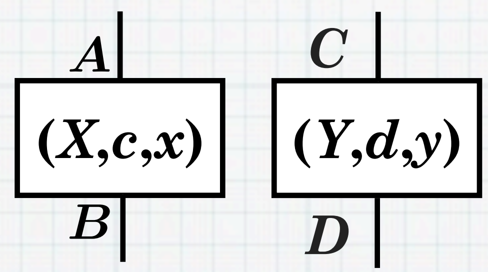
[Barbosa '03][IH & Jacobs '11]

$\boxed{\mathbf{Trans}(T)}$ Objects: sets $A, B, \ldots$

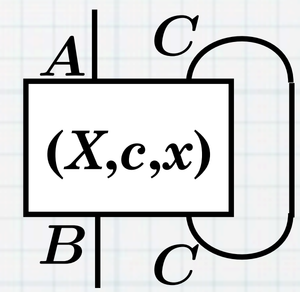
Arrows:
$$\frac{{\color{red}A} \longrightarrow {\color{blue}B} \text{ in } \mathbf{Trans}(T)}{\big(\, X,\ X \times {\color{red}A} \xrightarrow{c} T(X \times {\color{blue}B}),\ x_0 \in X\ \big),\ T\text{-transducer}}$$



composition



tensor

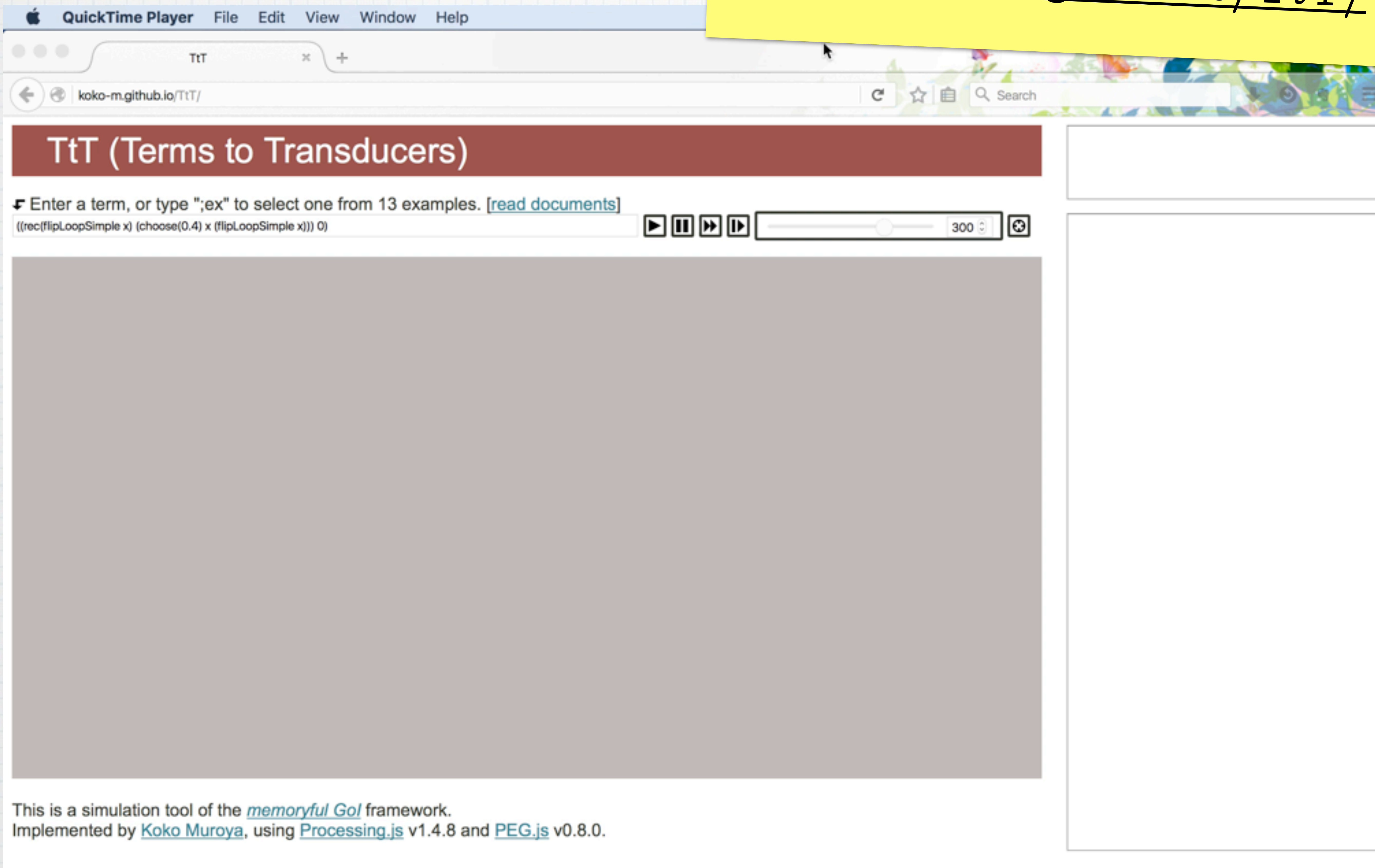


trace

$$\left(\ X \times Y,\quad \begin{aligned} (X \times Y) \times A &\xrightarrow{\cong} (X \times A) \times Y \\ &\xrightarrow{c \times Y} T(X \times B) \times Y \\ &\xrightarrow{\mathrm{str}'} T\big((X \times B) \times Y\big) \\ &\xrightarrow{\cong} T\big(X \times (Y \times B)\big) \\ &\xrightarrow{T(X \times d)} T\big(X \times T(Y \times C)\big) \\ &\xrightarrow{T\mathrm{str}} TT\big(X \times (Y \times C)\big) \\ &\xrightarrow{\mu^T} T\big(X \times (Y \times C)\big) \\ &\xrightarrow{\cong} T\big((X \times Y) \times C\big) \end{aligned},\quad (x, y)\ \right)$$

Our Tool TtT

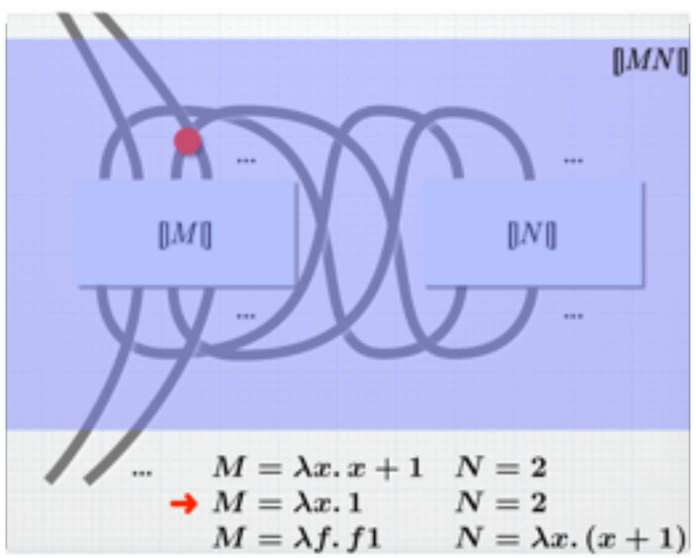
Developed by Koko Muroya
http://koko-m.github.io/TtT/

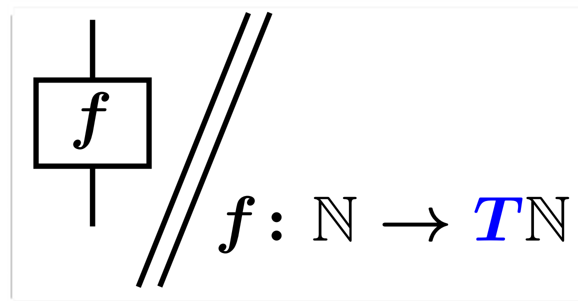


Summary

Coalgebra meets **higher-order computation** in **Geometry of Interaction** [Girard, LC'88]

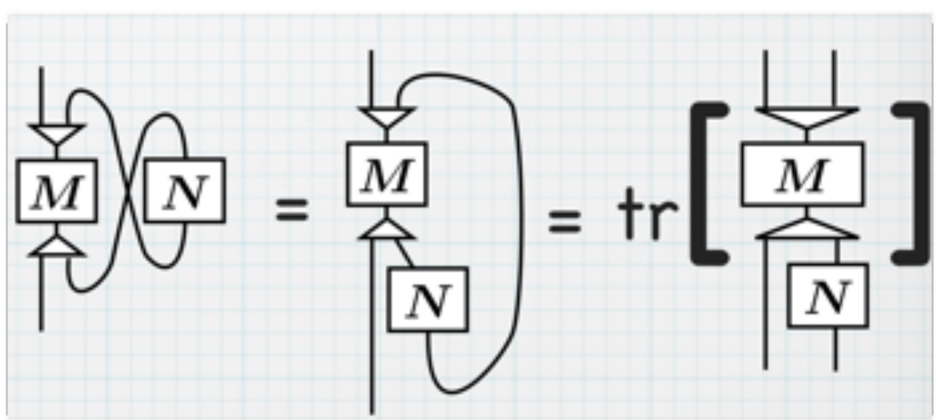
"GoI Animation"





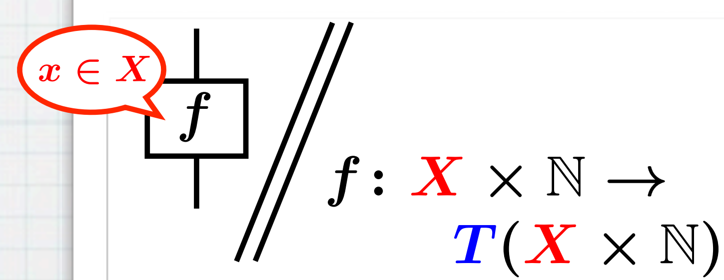
GoI w/ T-branching

[IH & Hoshino, LICS'11]



Categorical GoI

[Abramsky, Haghverdi & Scott, MSCS'02]



Memoryful GoI

[Hoshino, Muroya & IH, CSL-LICS'14 & POPL'16]

Hasuo (Tokyo)

Friday, June 24, 16

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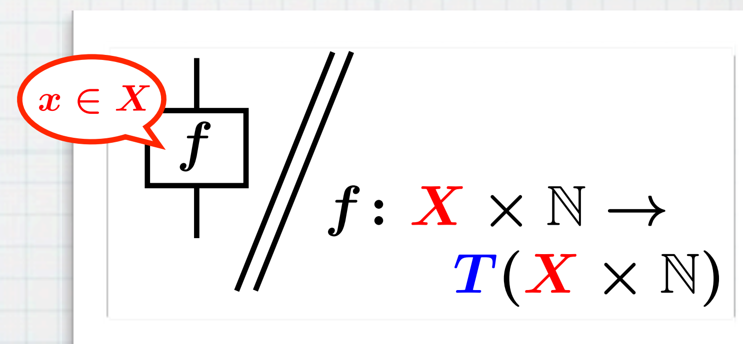

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Hasuo (Tokyo)

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on [Girard, LC'88]

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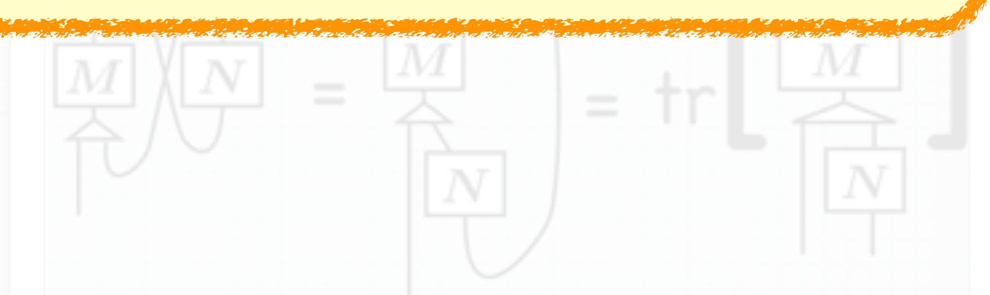

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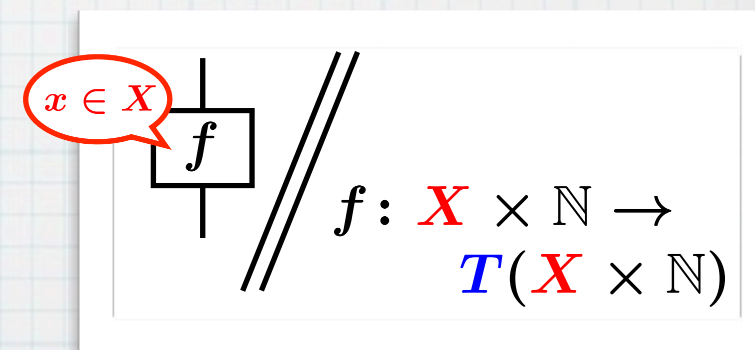

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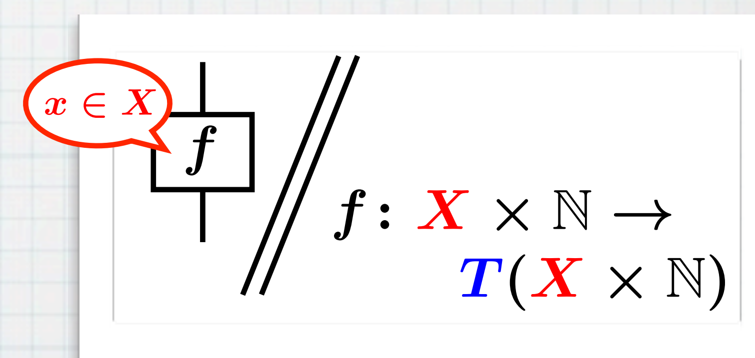

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$$(\!|\, \Gamma \vdash M : \tau \,|\!)$$

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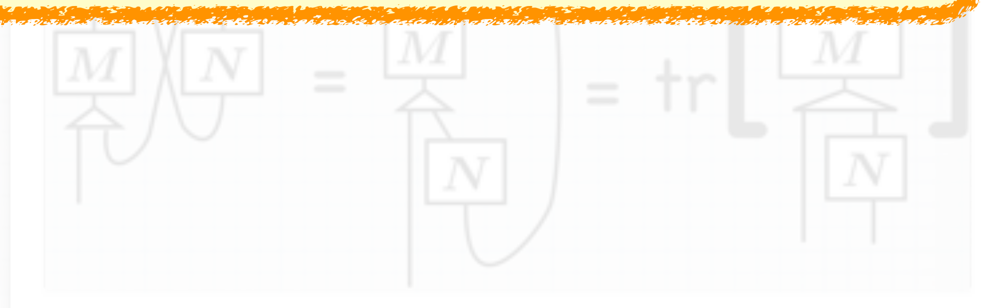

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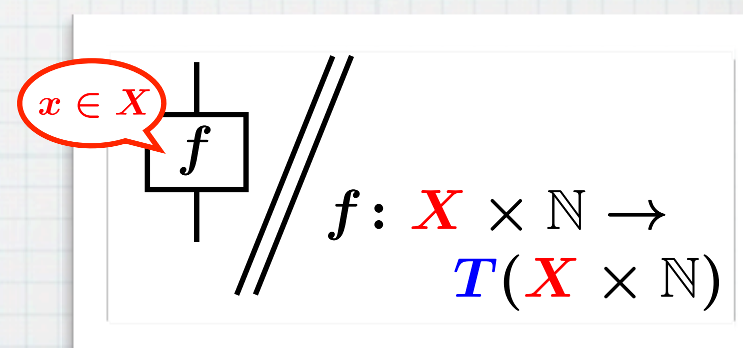

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- GoI + algebraic effects [Plotkin & Power]
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- Compositional translation

$$(\!|\, \Gamma \vdash M : \tau \,|\!)$$

==> **implementation TtT**

- (Categorical GoI + realizability)

==> **categorical model**

  - <u>Thm.</u> Adequacy
  - "Correct-by-construction" compilation!

[Girard, LC'88]

GoI w/ T-branching

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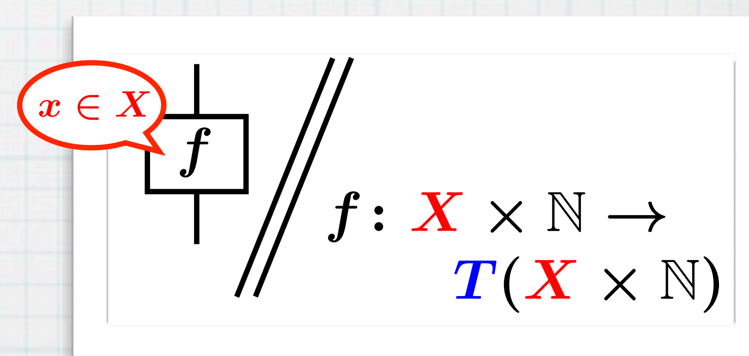

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Hasuo (Tokyo)

Retracing some paths in Process Algebra

Samson Abramsky
Laboratory for the Foundations of Computer Science
University of Edinburgh

1 Introduction

The very existence of the CONCUR conference bears witness to the fact that "concurrency theory" has developed into a subject unto itself, with substantially different emphases and techniques to those prominent elsewhere in the semantics of computation.

Whatever the past merits of this separate development, it seems timely to look for some convergence and unification. In addressing these issues, I have found it instructive to trace some of the received ideas in concurrency back to their origins in the early 1970's. In particular, I want to focus on a seminal paper by Robin Milner [Mil75][1], which led in a fairly direct line to his enormously influential work on CCS [Mil80, Mil89]. I will take (to the extreme) the liberty of of applying hindsight, and show how some different paths could have been taken, which, it can be argued, lead to a more unified approach to the semantics of computation, and moreover one which may be better suited to modelling today's concurrent, object-oriented languages, and the type systems and logics required to support such languages.

2 The semantic universe: transducers

Milner's starting point was the classical automata-theoretic notion of *transducers*, *i.e.* structures

$$(Q, X, Y, q_0, \delta)$$

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Thank you for your attention!
Ichiro Hasuo (Dept. CS, U Tokyo)
http://www-mmm.is.s.u-tokyo.ac.jp/~ichiro/

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