Talk based on:
I. Hasuo & N. Hoshino,
Semantics of Higher-Order Quantum Computation via Geometry of Interaction,
In Proc. Logic in Computer Science (LICS), June 2011.

Quantum Geometry of Interaction

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東京大学
THE UNIVERSITY OF TOKYO

京都大学
KYOTO UNIVERSITY

What's Done

* The Categorical GoI workflow
  * GoI = "Geometry of Interaction"
  * General, standard construction of denotational models
* Applied to quantum computation
  * Quantum λ-calculus = linear λ-cal. + quantum constructs
  * with insights from theory of coalgebra
  * Outcome: first adequate denotational semantics for a full quantum language (with ! and recursion)

Plan

* The categorical GoI workflow
  [Abramsky, Haghverdi, Scott, Jacobs, Longley, Lenisa, Hoshino, ...]
    * GoI + realizability
    * Generic —— still concrete and dynamic
    * Coalgebraic view ➜ let's do something fancy
* Elements of quantum computation
    * Not much, really!
* The calculus $q\lambda_\ell$ Based on [Selinger-Valiron'09]
* The denotational model

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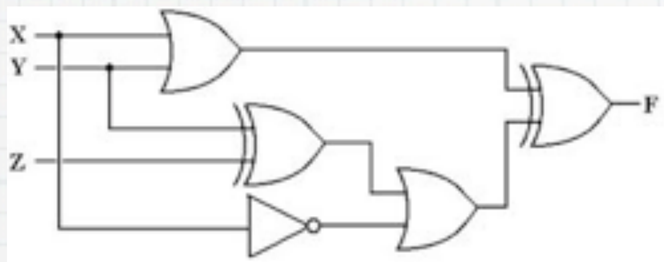
Quantum λ-calculus

Classical

(Boolean) circuit



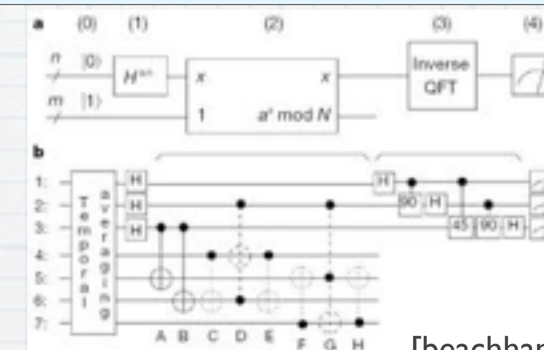
[Null-Lobur]

Programming language

```
int i,j;
int factorial(int k)
{
    j=1;
    for (i=1; i<=k; i++)
        j=j*i;
    return j;
}
```

Quantum

Quantum circuit



[beachhandball.es]

Quantum λ-calculus

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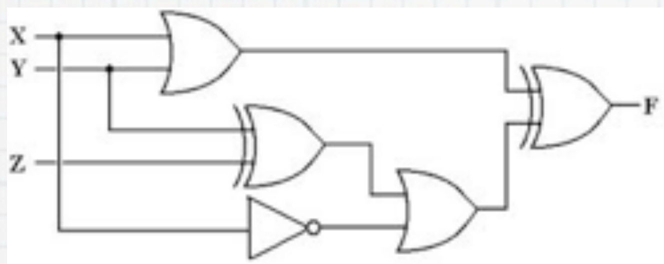

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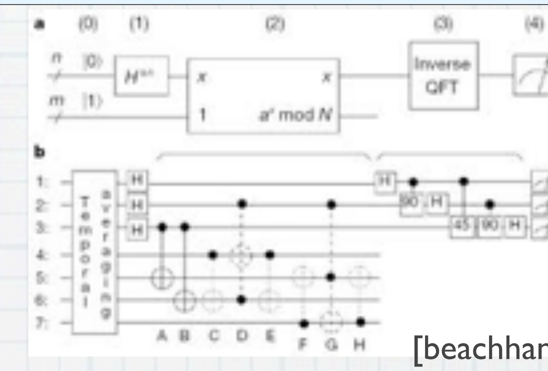

[beachhandball.es]

Quantum programming language

$$\begin{aligned}\textbf{telep} = \quad & \mathit{let}\ \langle x, y\rangle = \textbf{EPR} * \mathit{in} \\ & \quad \mathit{let}\ f = \textbf{BellMeasure}\ x\ \mathit{in} \\ & \quad \mathit{let}\ g = \textbf{U}\ y \\ & \quad \mathit{in}\ \langle f, g\rangle.\end{aligned}$$

[Selinger-Valiron]

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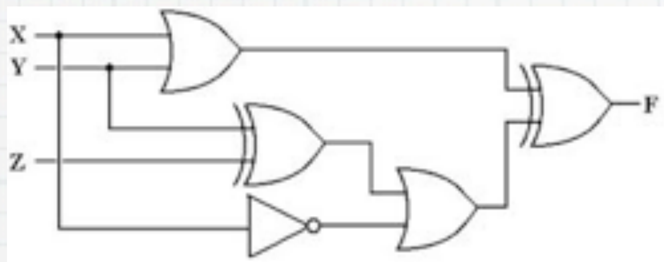
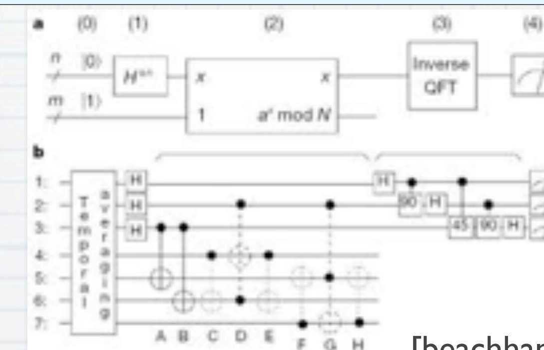
Quantum λ-calculus

Classical		Quantum	
(Boolean) circuit	 [Null-Lobur]	Quantum circuit	 [beachhandball.es]
Programming language	(code below)	Quantum programming language	(term below) [Selinger-Valiron]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- Quantum λ:
  prototype of quantum functional language

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Quantum λ-Calculus: Prototype of Quantum Functional Languages

* **Why** (high-level) language?
  ➜ structured programming
    * Discovery of new algorithms
    * Program verification

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Quantum λ-Calculus: Prototype of Quantum Functional Languages

* **Why** (high-level) language?
  ➜ structured programming
  * Discovery of new algorithms
  * Program verification

* **Why** functional language?
  ➜ Mathematically nice and clean
  * Aids (denotational) semantics
  * Transfer from classical to quantum

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Quantum λ-Calculus: Prototype of Quantum Functional Languages

* **Why** denotational semantics?
  → For quantum communication as well as for quantum computation

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  * "Absolute security" via e.g. quantum key distr.

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Quantum λ-Calculus: Prototype of Quantum Functional Languages

* **Why** denotational semantics?
  ➜ For quantum communication as well as for quantum computation
  * "Absolute security" via e.g. quantum key distr.
  * Being tested for real-world usege
  * Comm. protocols are notoriously error-prone; quantum primitives make it worse

Quantum λ-Calculus: Prototype of Quantum Functional Languages

* Linear λ-calculus
  * "No cloning" by linearity:
  * Classical data (duplicable) via !
* + Quantum primitives
  * State preparation
  * Unitary transformation
  * Measurement

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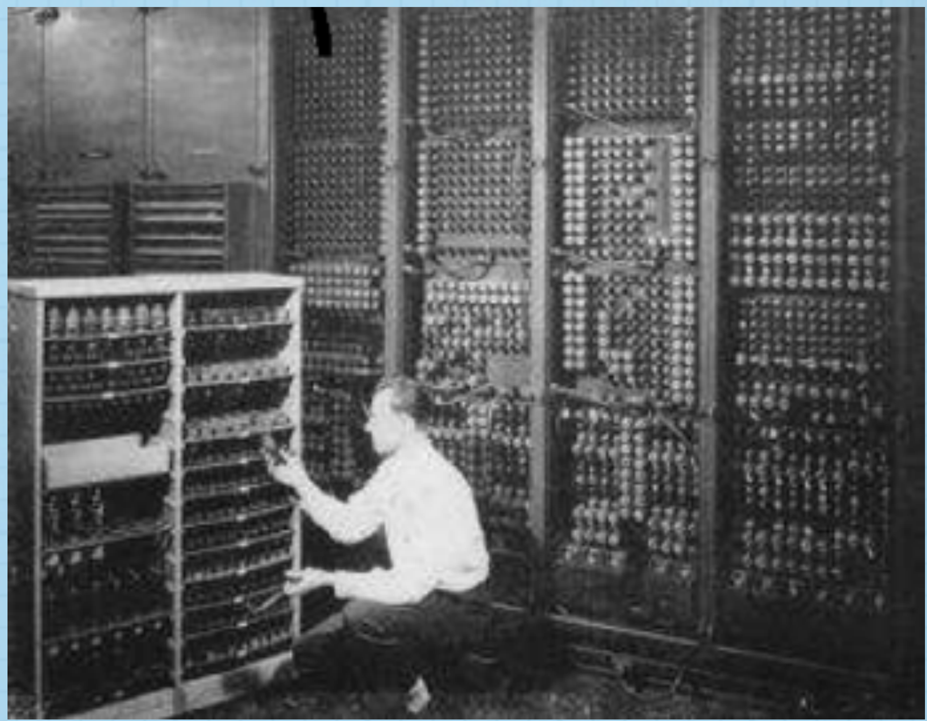
"Quantum Data, Classical Control"

Illustration by N. Hoshino

Quantum data

Classical control



Hasuo (Tokyo)

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Quantum data

$$\frac{1}{\sqrt{2}}$$

Classical control

Hasuo (Tokyo)

“Quantum Data, Classical Control”

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Quantum data

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$$+\frac{1}{\sqrt{2}}$$

Classical control

Hasuo (Tokyo)

Denotational Semantics for Quantum λ

* In **Hilb** ?
  * Not that easy. Classical data?
* [Selinger&Valiron'08] Den. sem. for the !-free fragment
* [Selinger&Valiron'09] Operational semantics (nice!)
* [Current Work]
  * The first model for the full fragment (with ! and recursion)
  * Categorical GoI: useful for "Quantum Data, Classical Control"

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Critical Acclaim (?) for:

I. Hasuo & N. Hoshino, Semantics of Higher-Order Quantum Computation via Geometry of Interaction

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* **Now their pain is yours!!**

Part 1

Categorical GoI (Geometry of Interaction)

GoI:
Geometry of Interaction

* J.-Y. Girard, at Logic Colloquium '88

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GoI:
Geometry of Interaction

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* Disclaimer (and sincere apologies):
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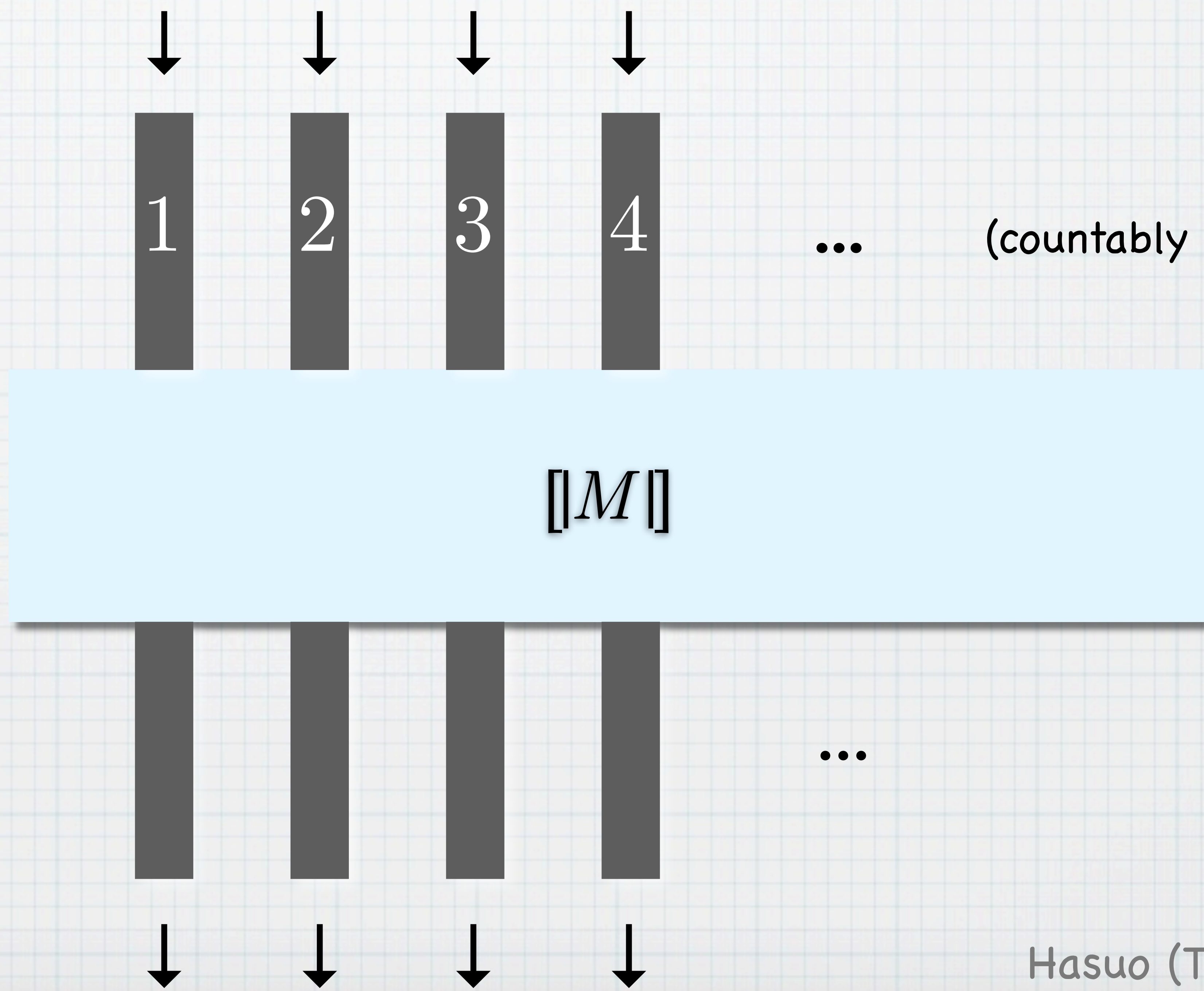
* In this talk:
  * Its categorical formulation [Abramsky,Haghverdi&Scott'02]
  * "The GoI Animation"

Hasuo (Tokyo)

The GoI Animation

$$[\![M]\!] \;=\; (\,\mathbb{N} \rightharpoonup \mathbb{N},\;\text{a partial function}\,)$$

$$=\;\text{“piping”}$$



(countably many)

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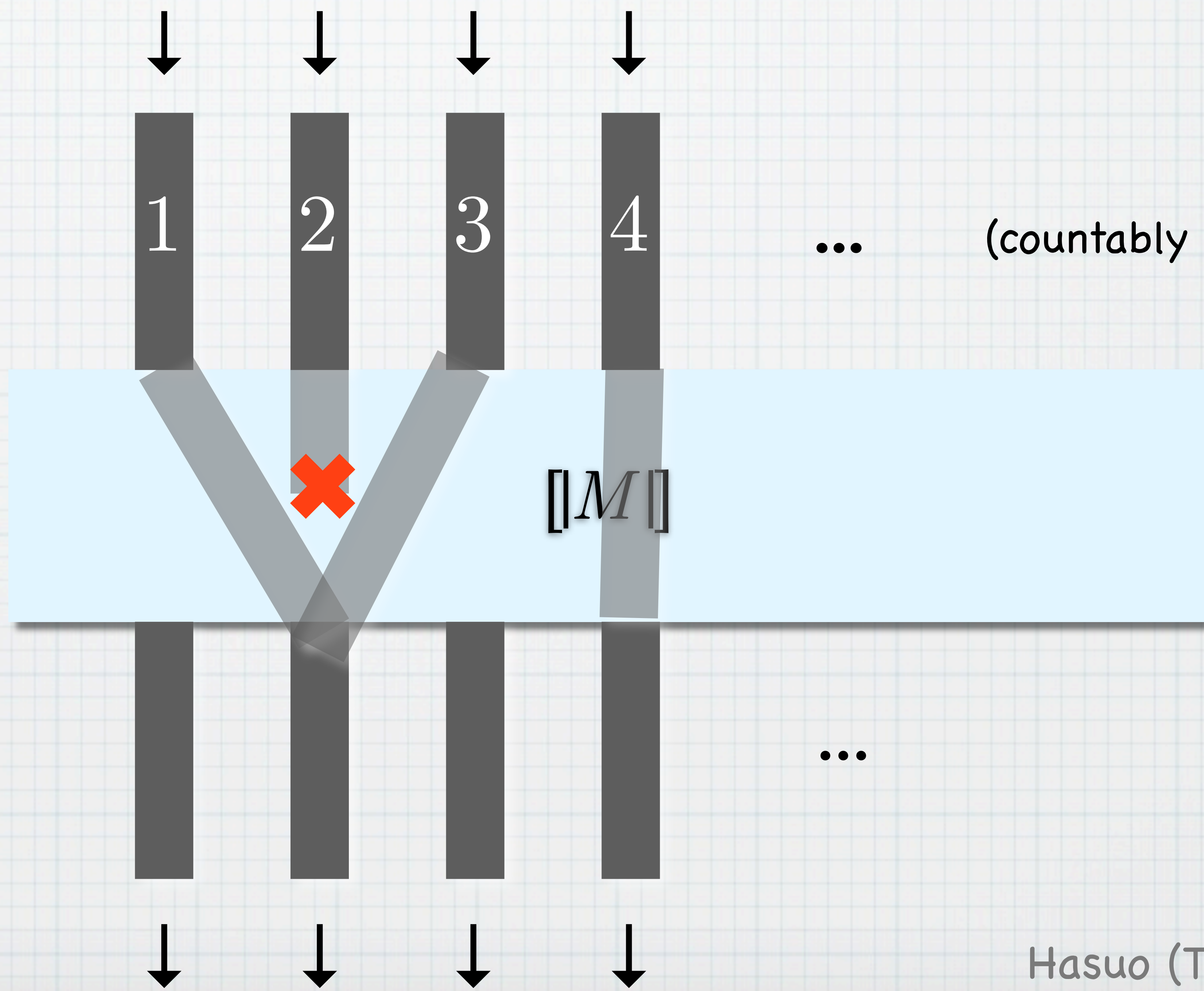

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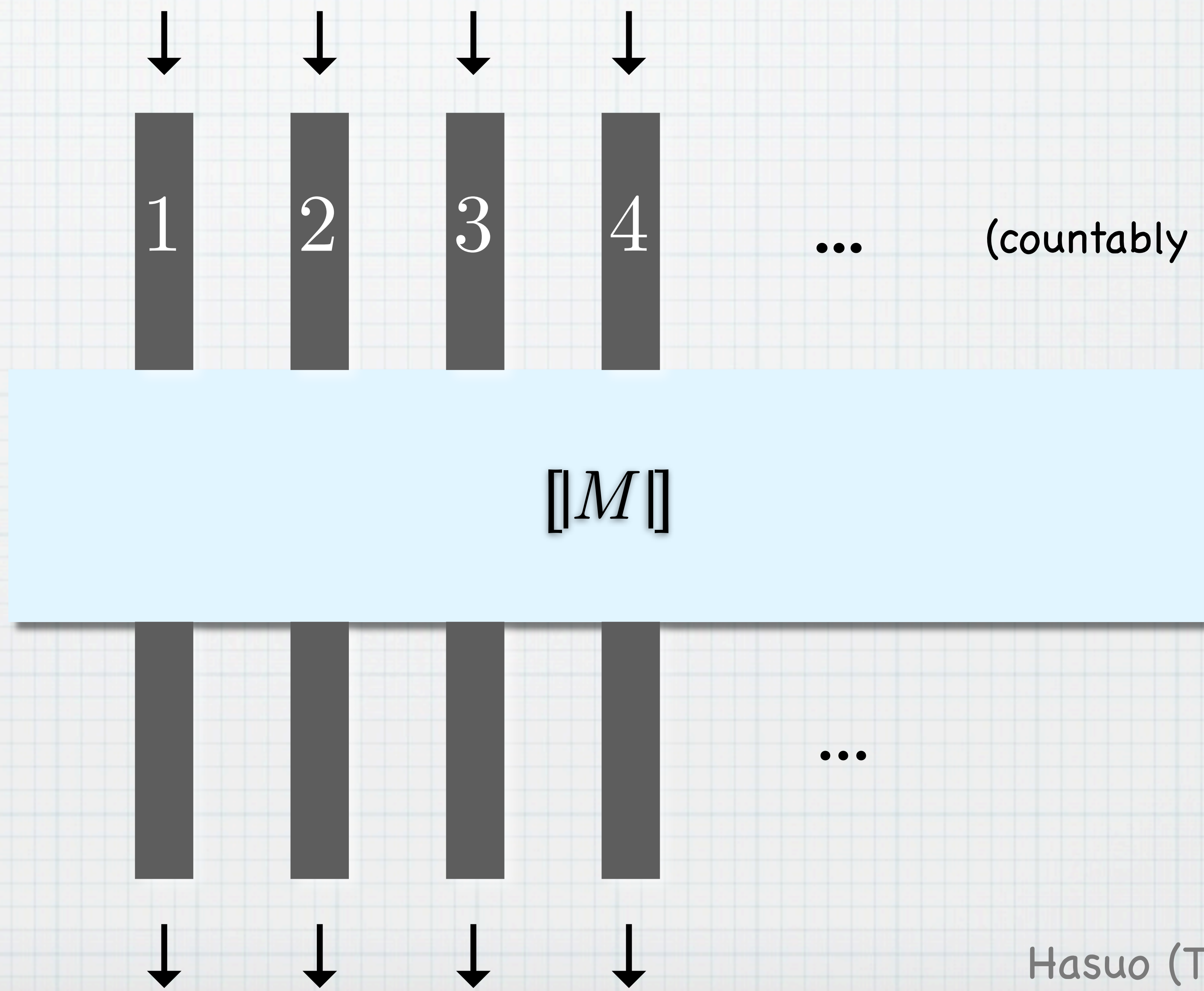

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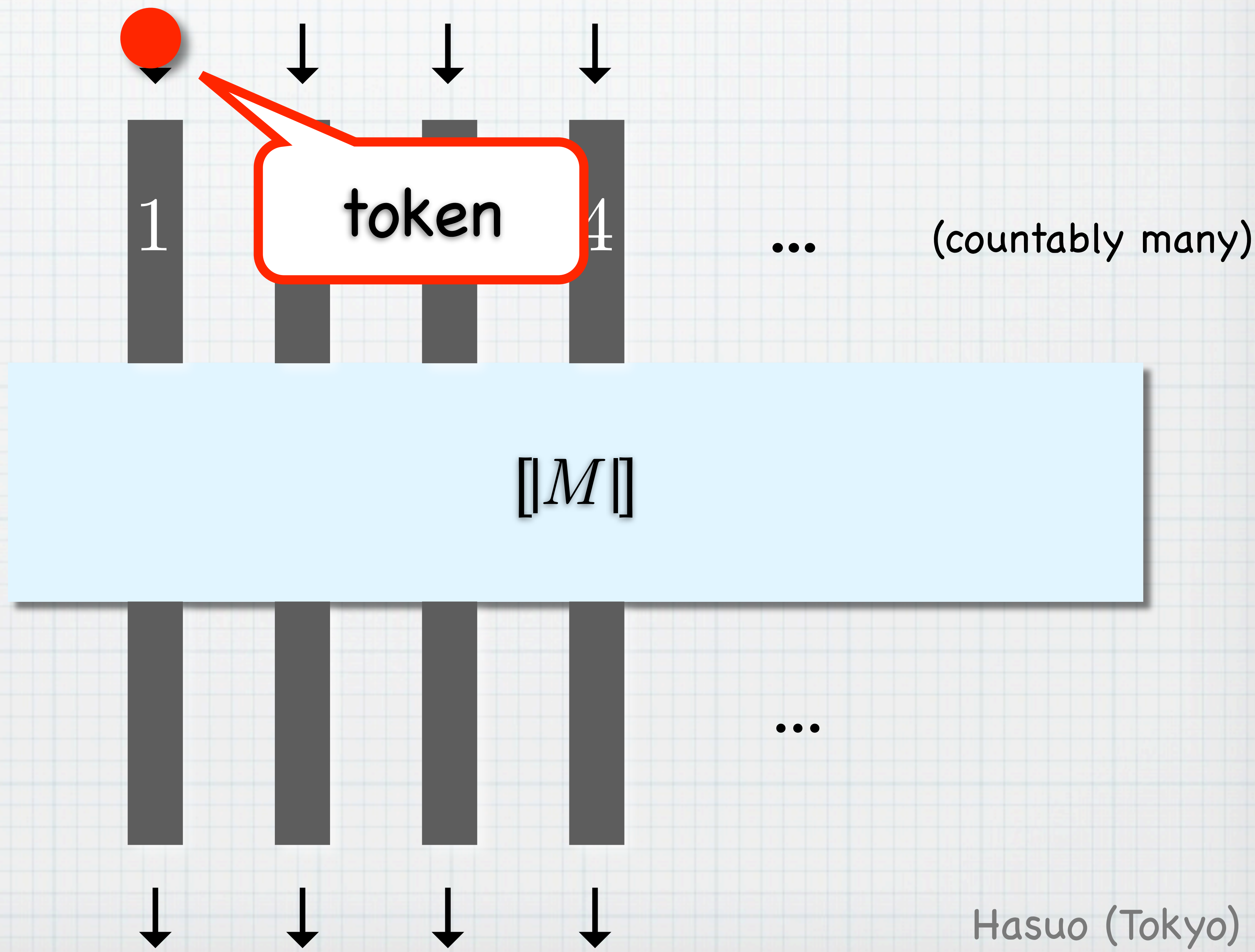

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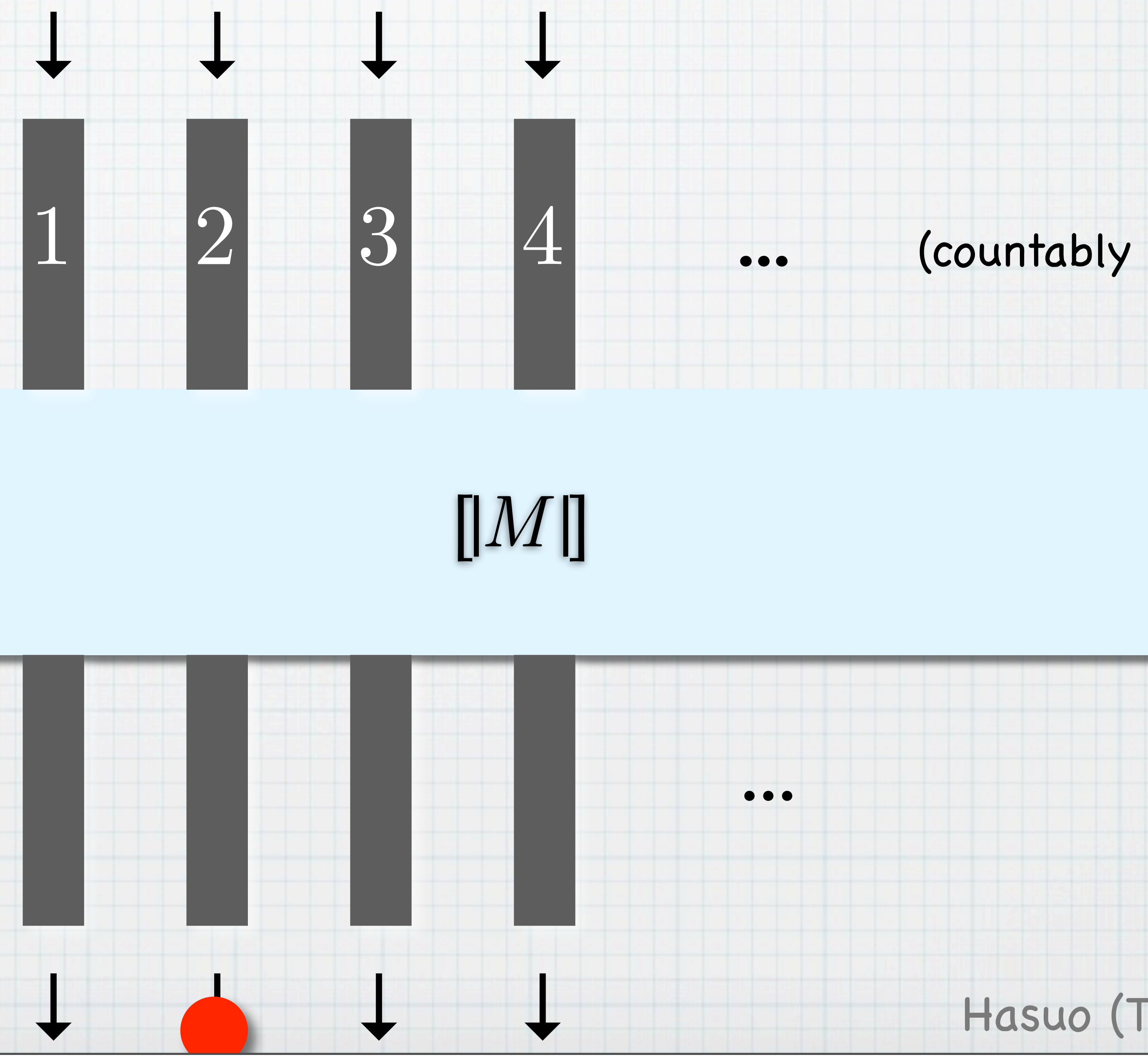

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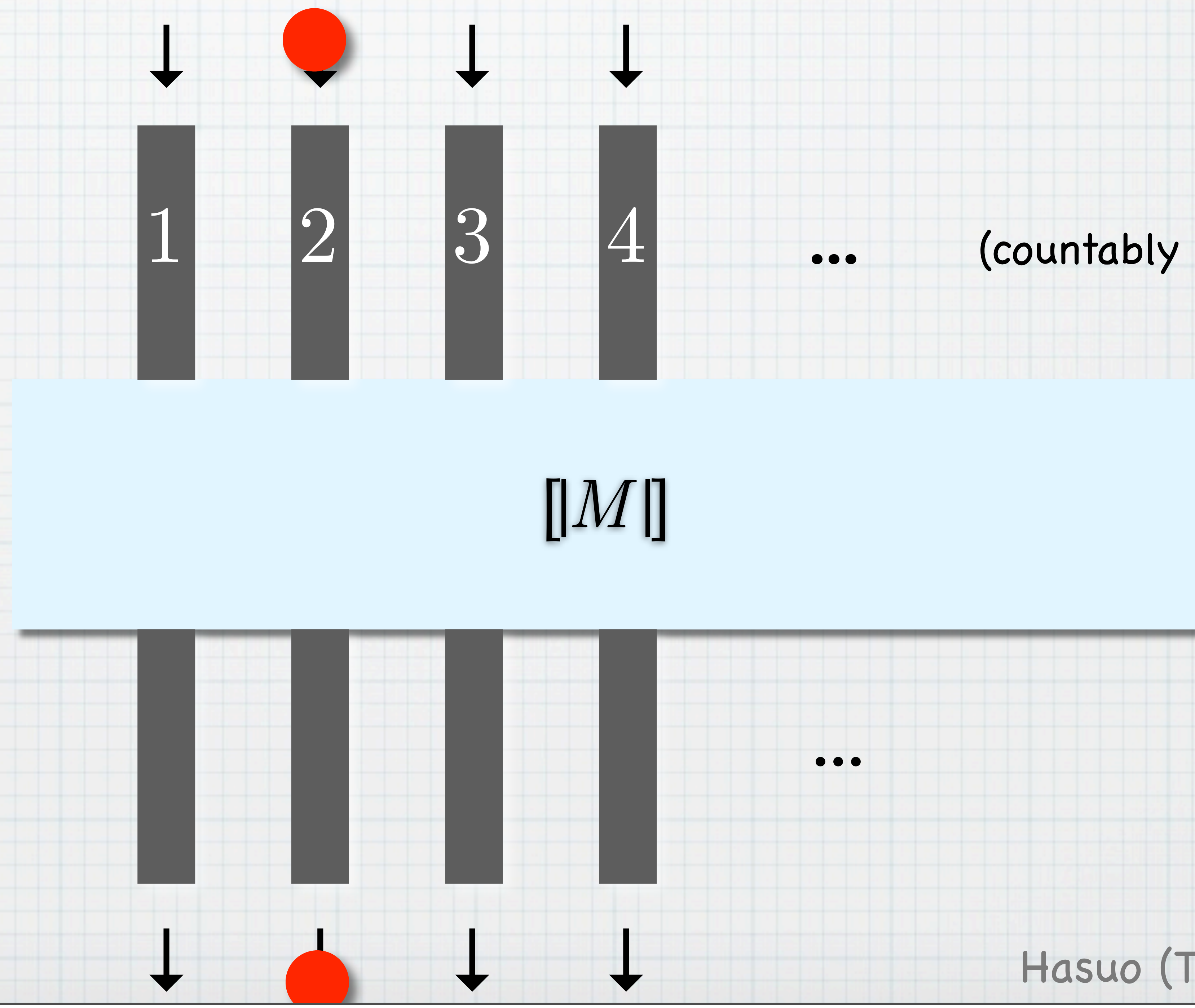

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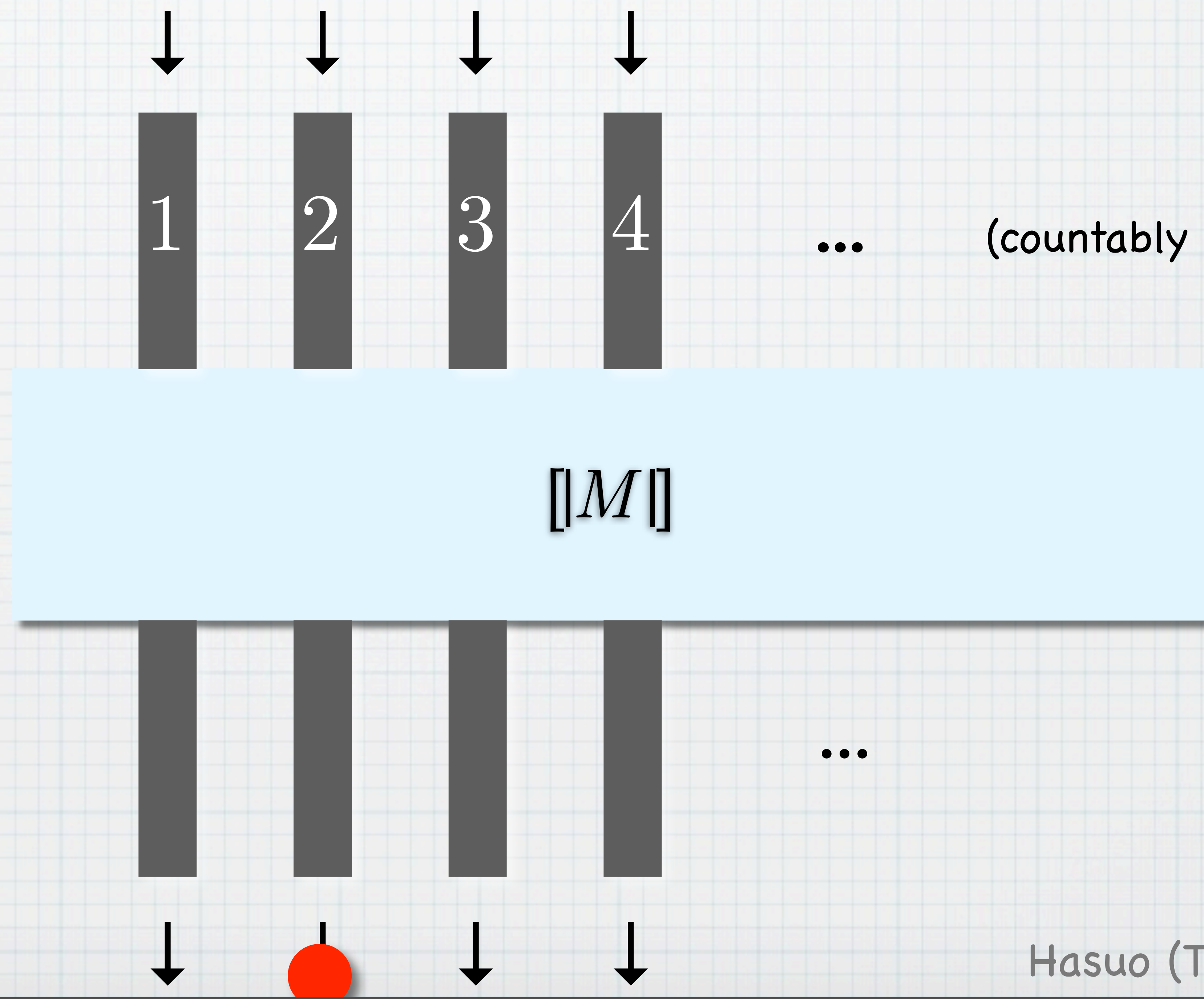

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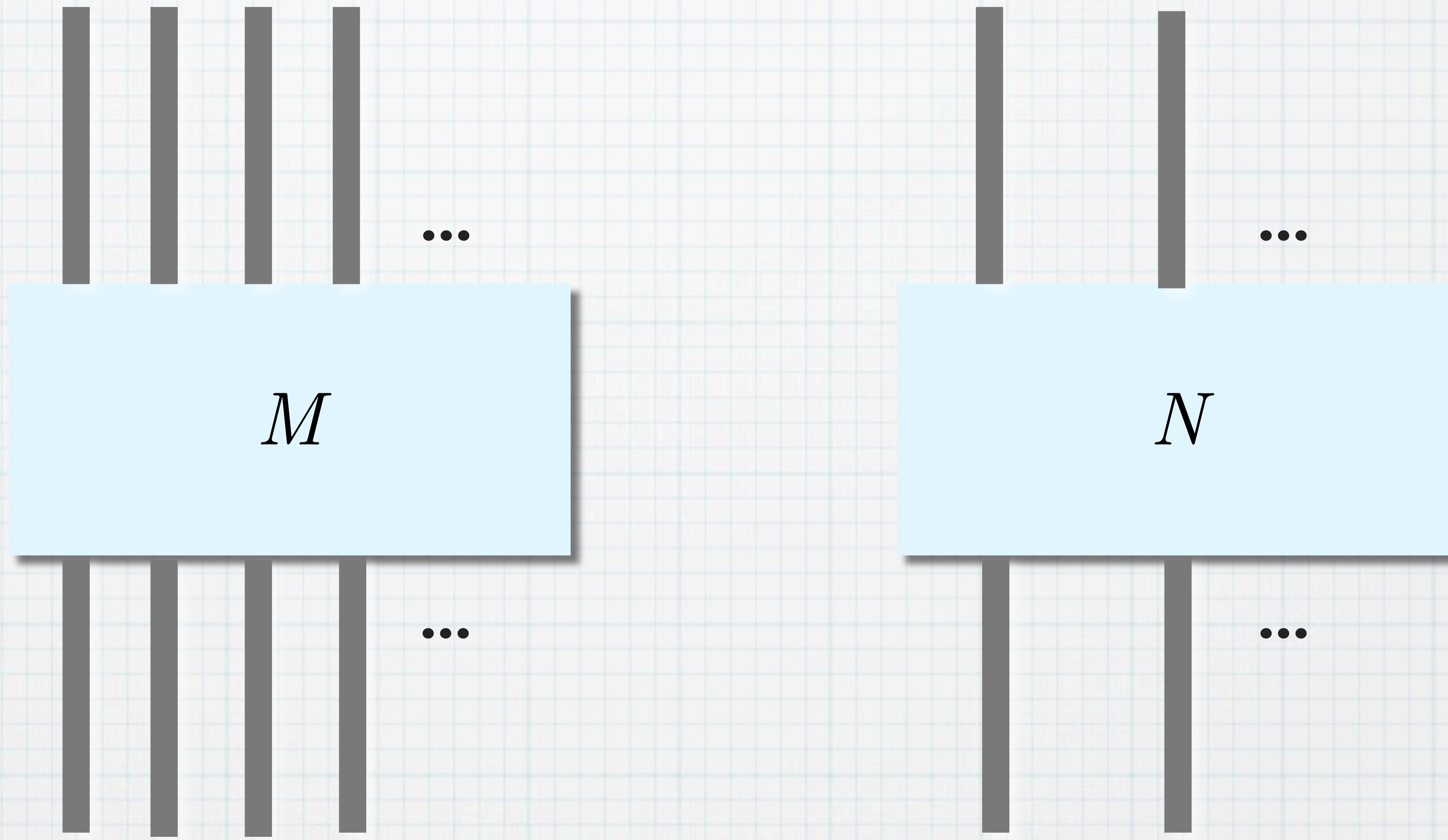
The GoI Animation

* Function application $[\![MN]\!]$
  * by “parallel composition + hiding”

$$[\![\boldsymbol{M} \boldsymbol{N}]\!] =$$



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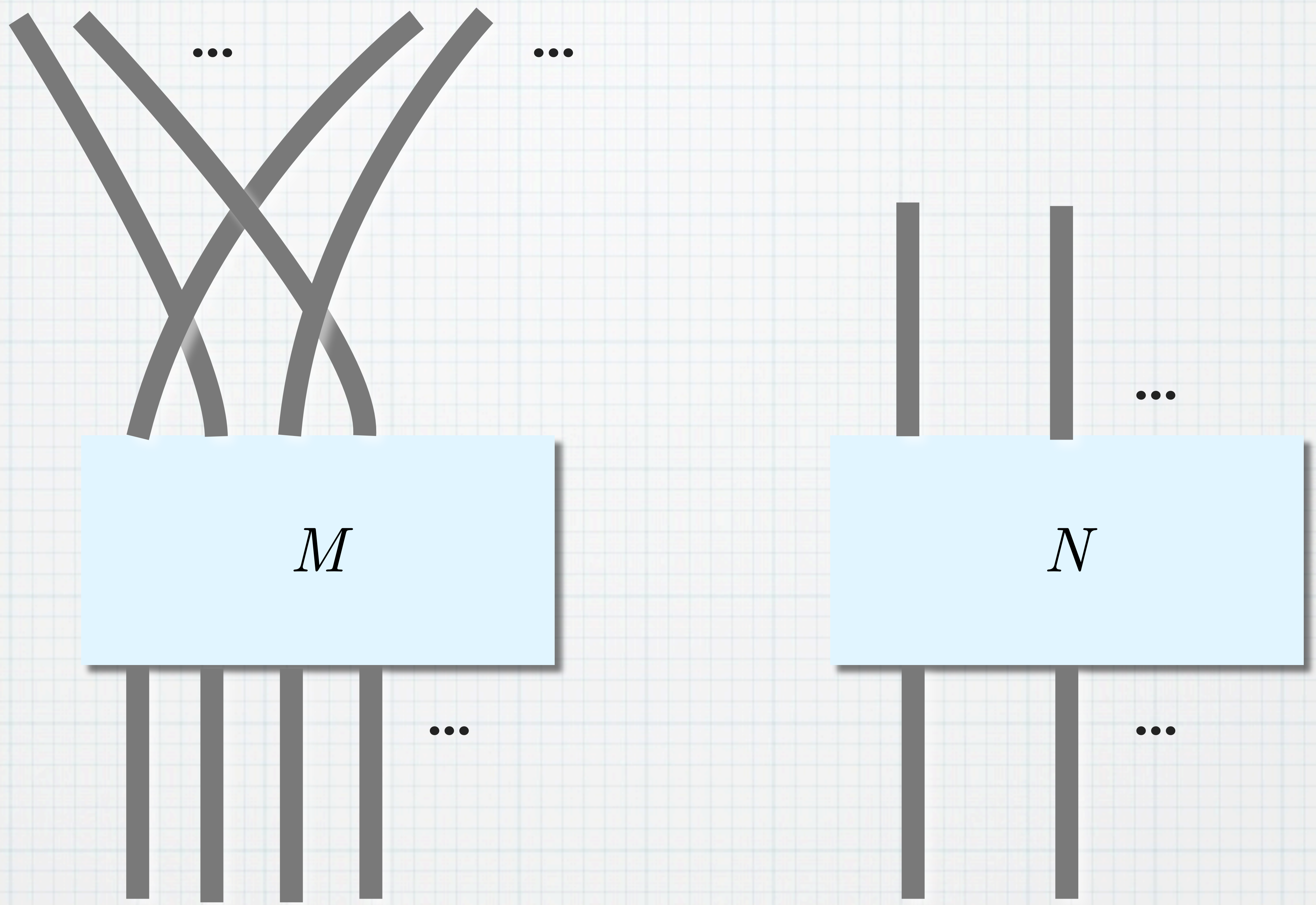

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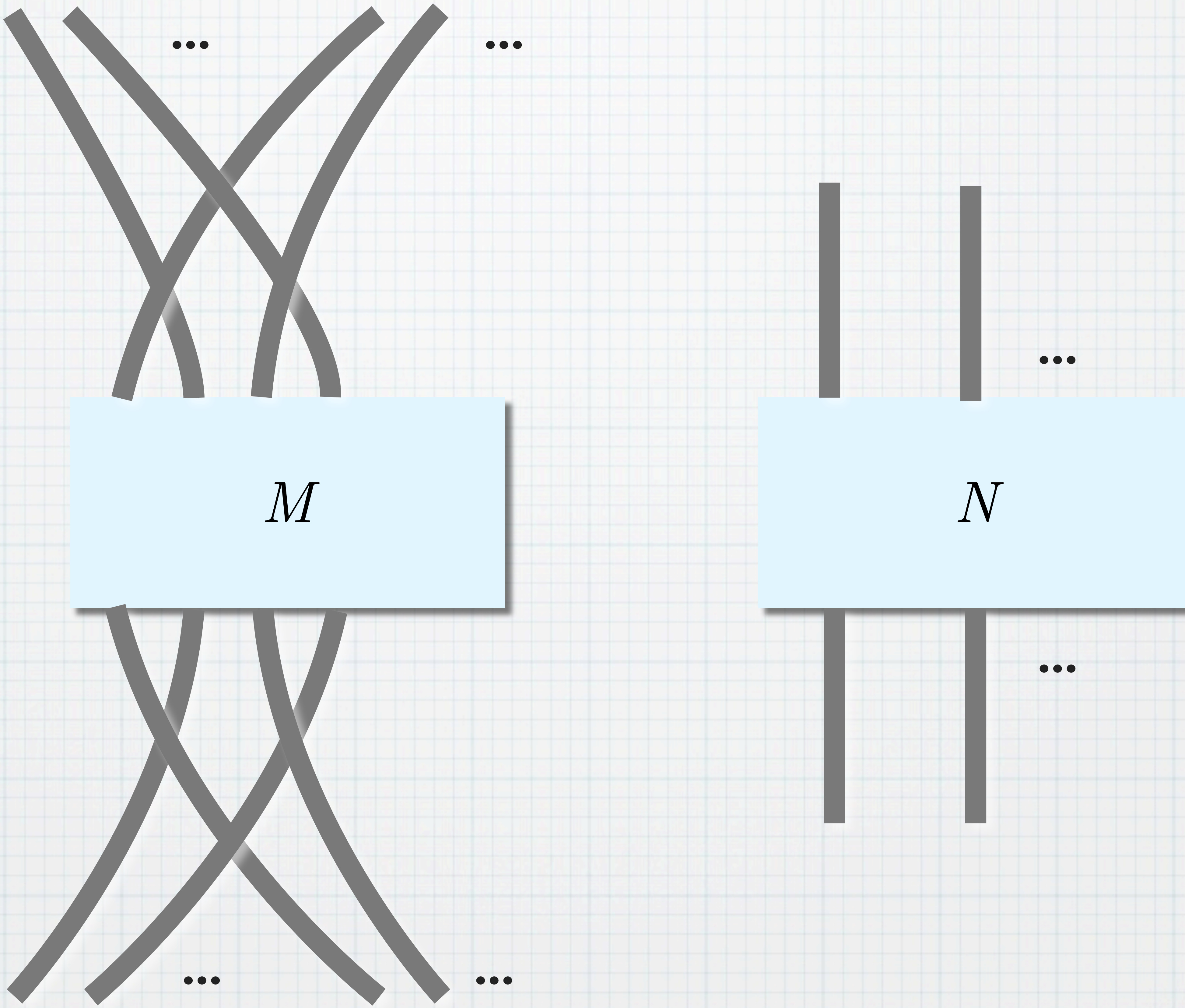

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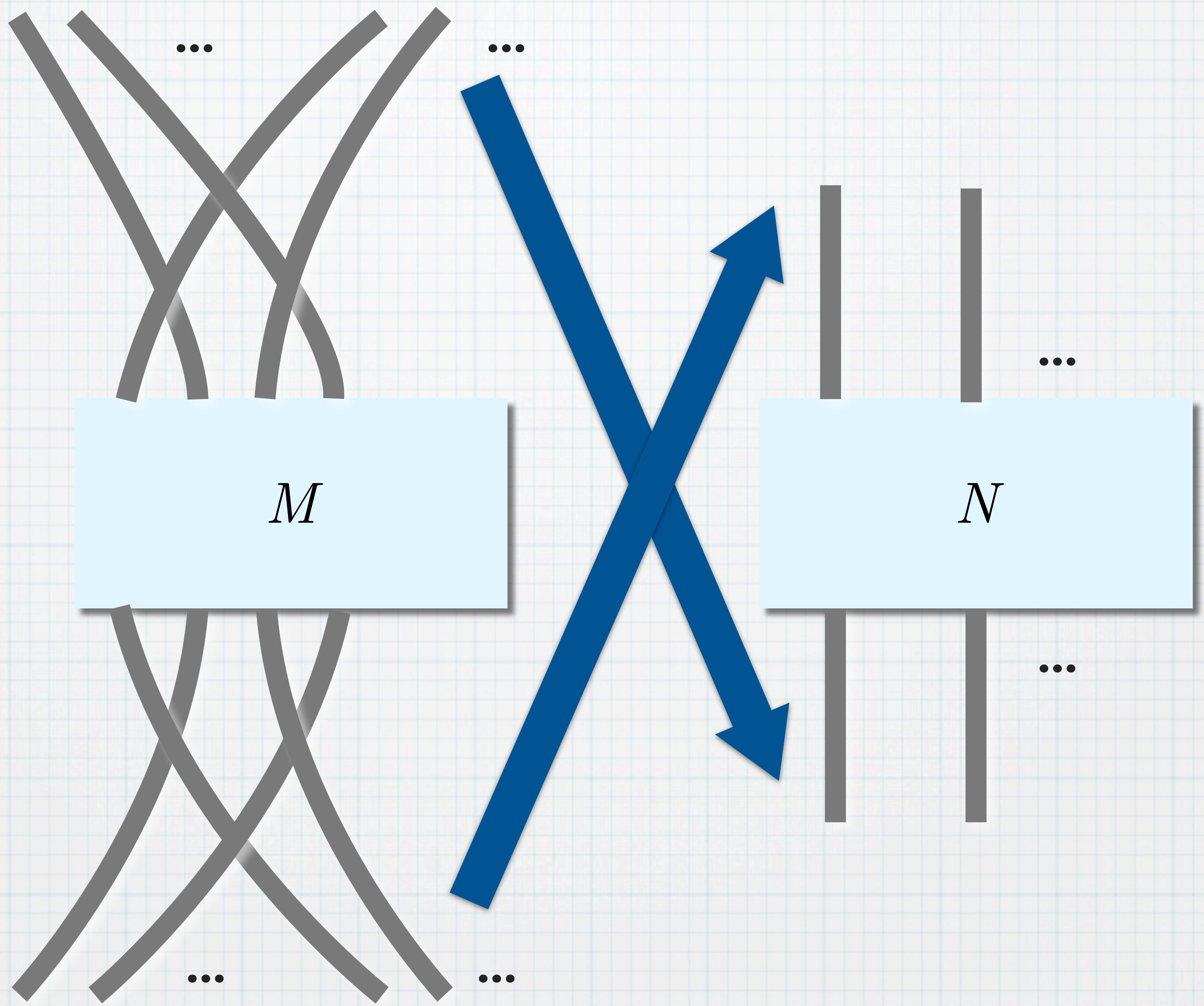

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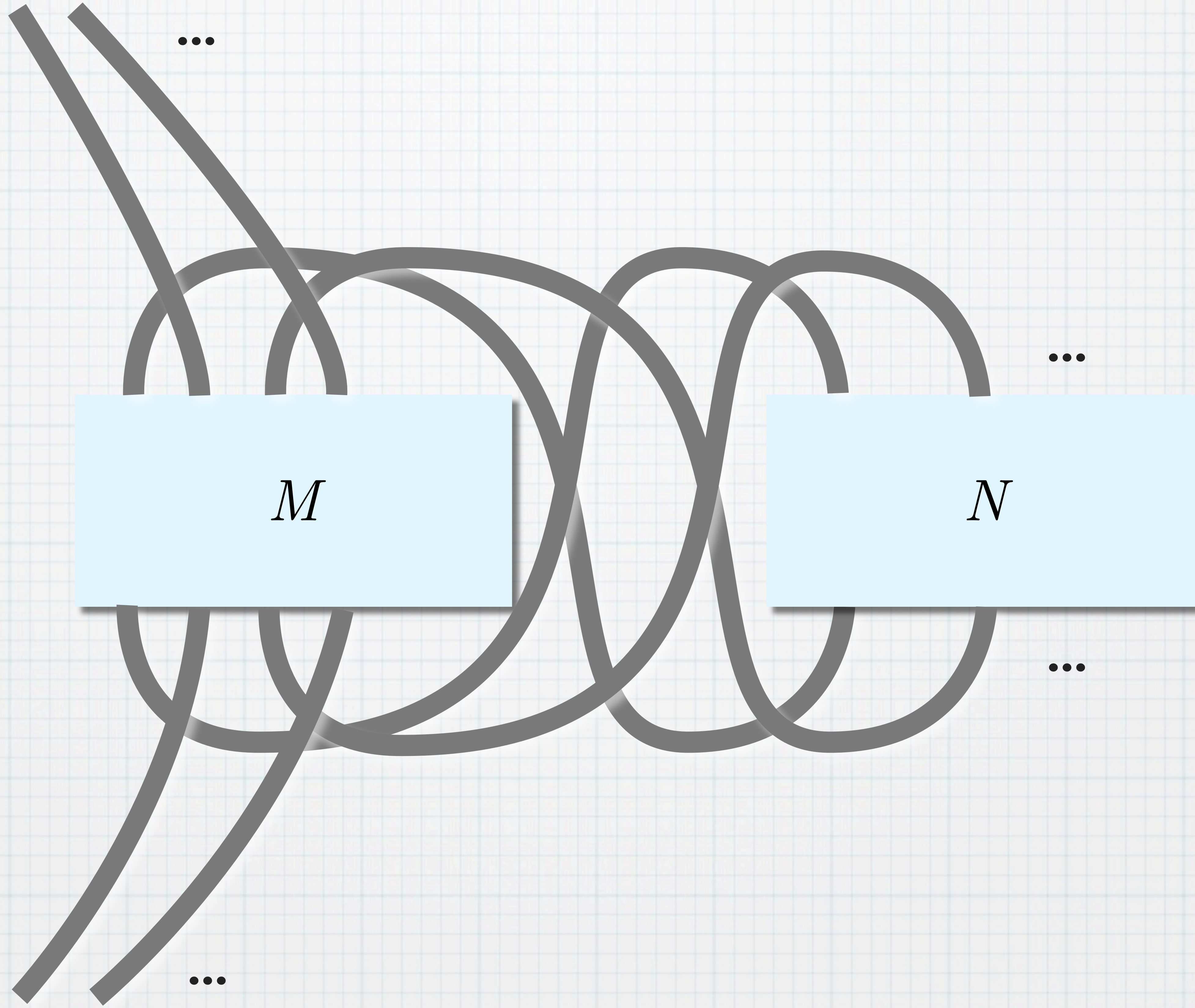

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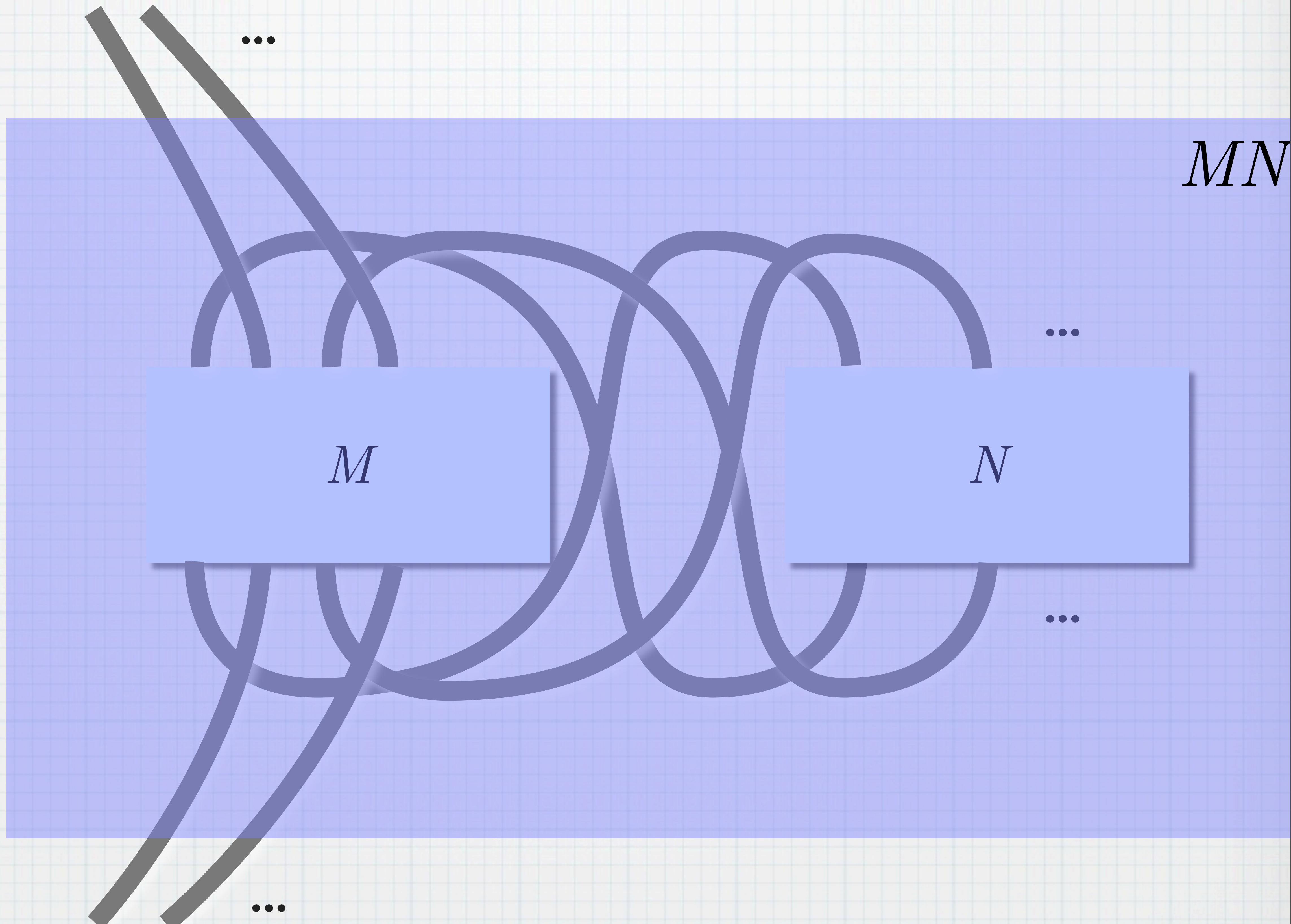

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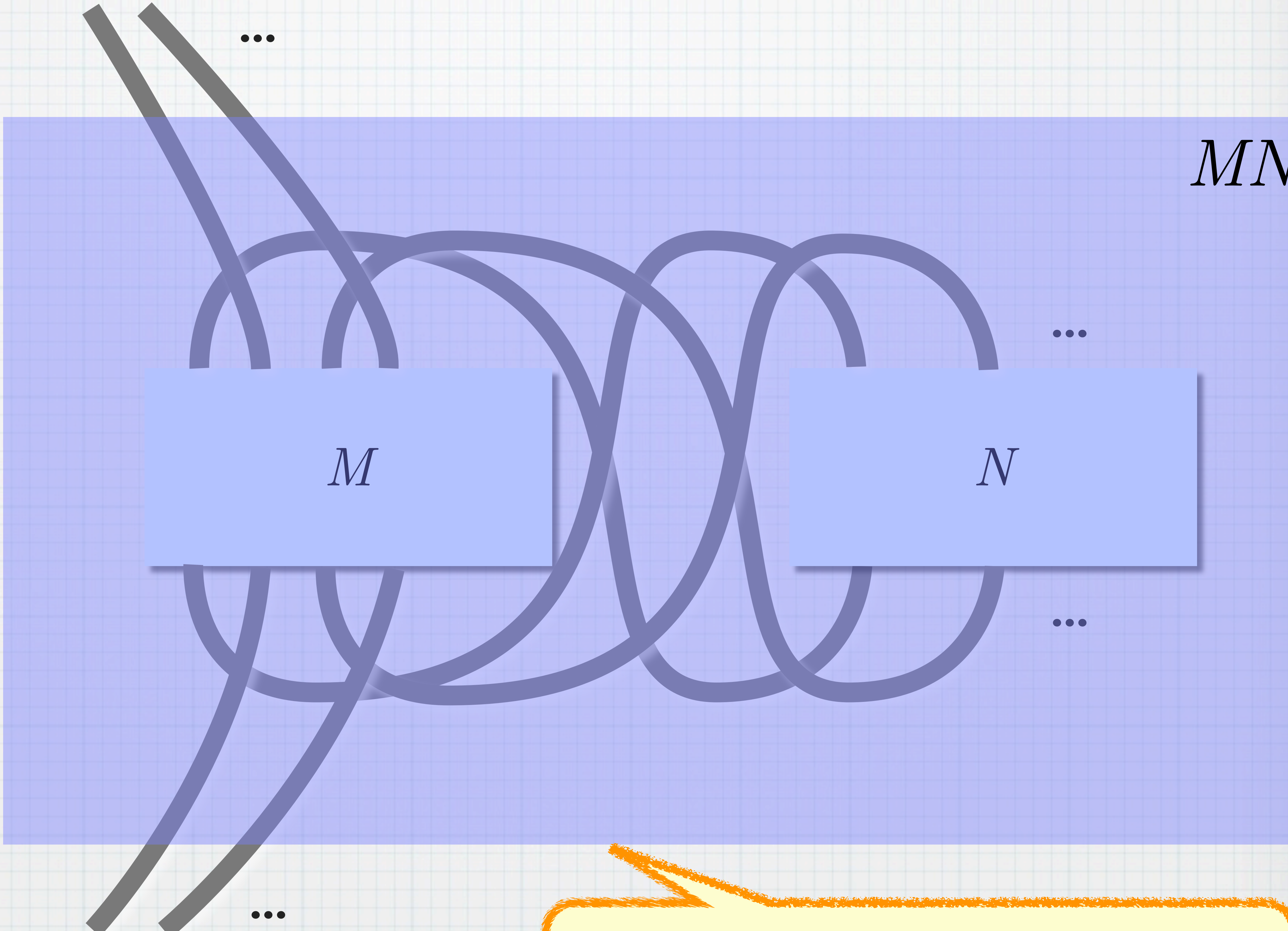
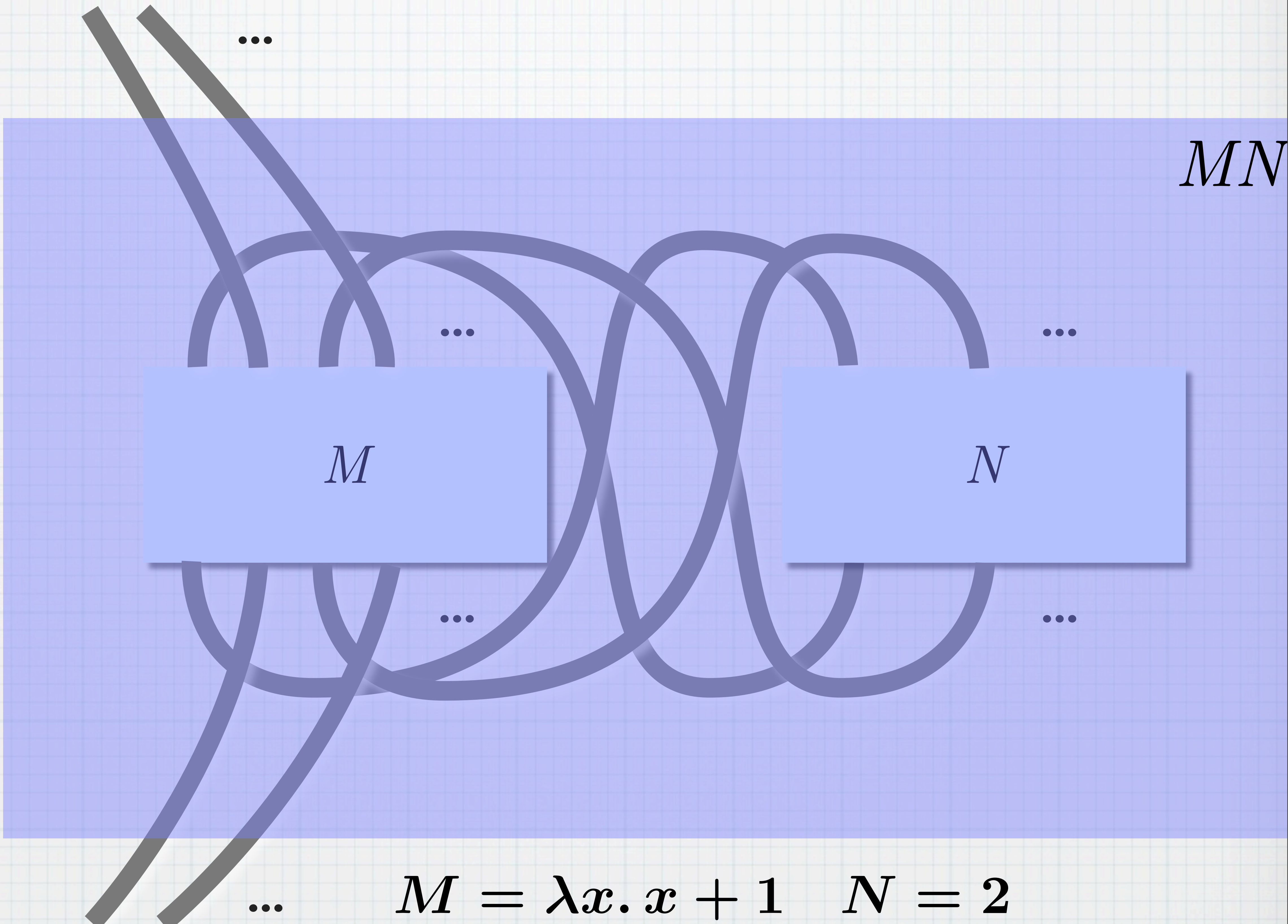


“parallel composition + hiding”
(cf. games)

$[\![MN]\!]$

$=$



$M = \lambda x.\, x + 1 \quad N = 2$

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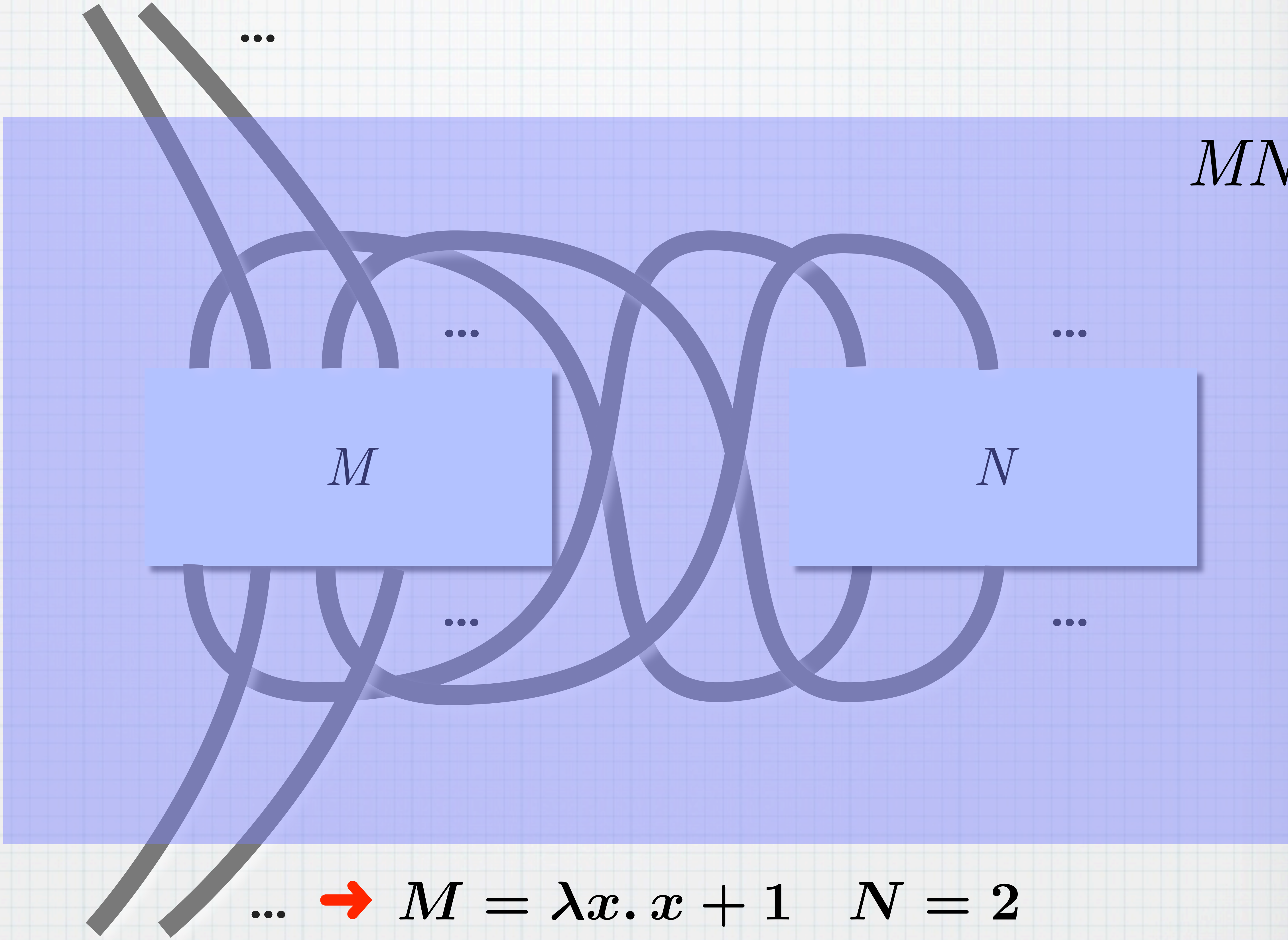

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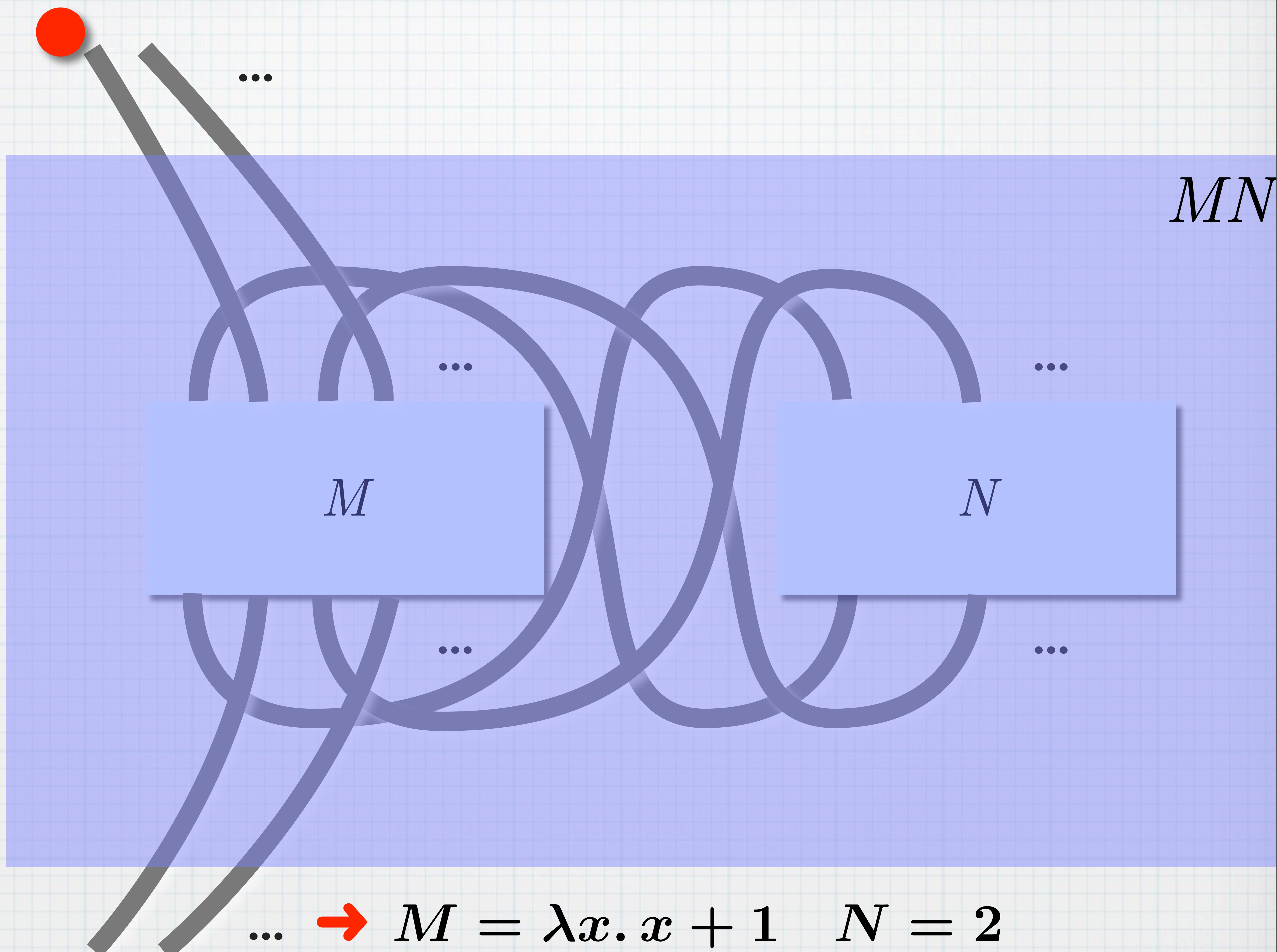

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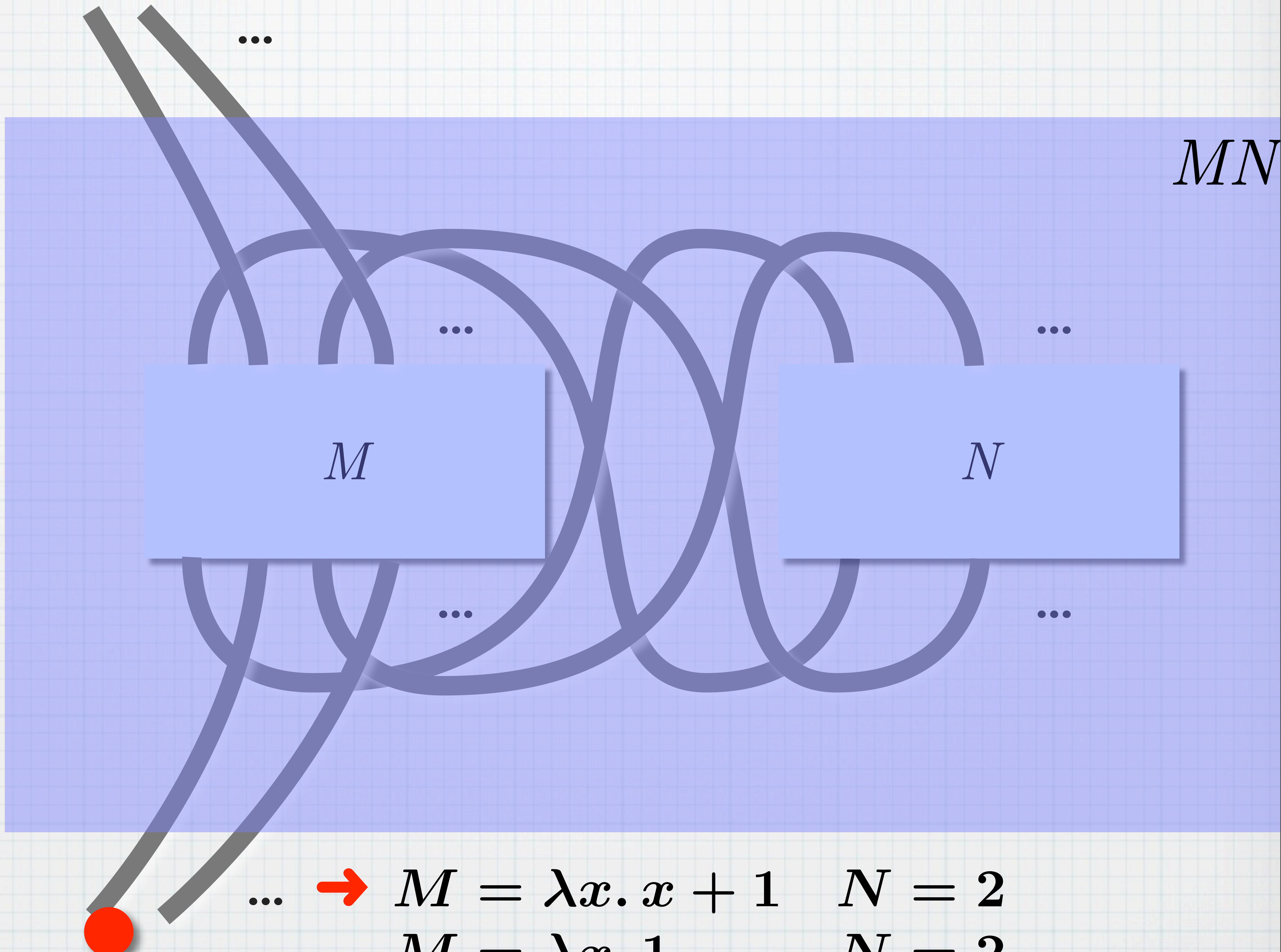

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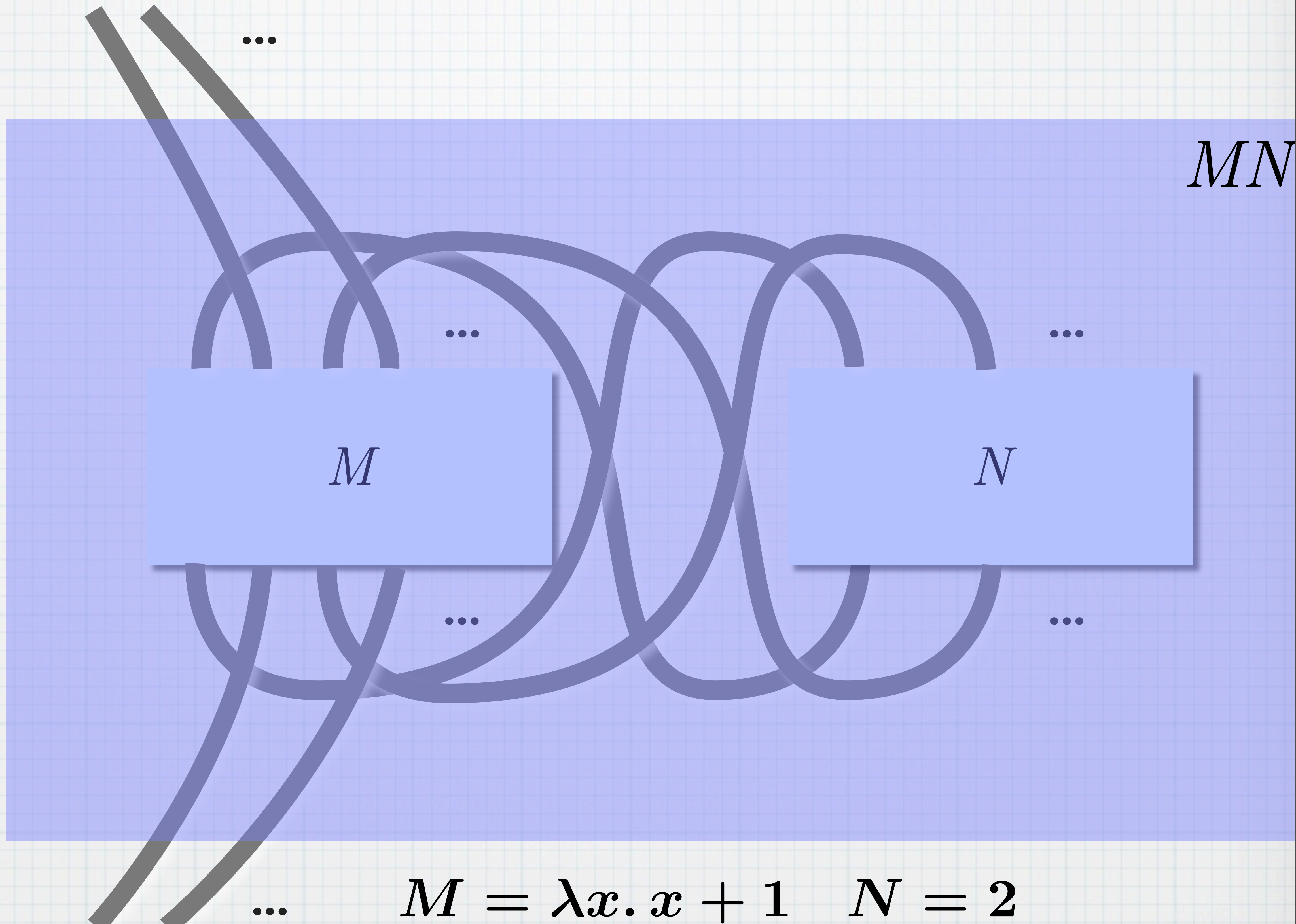

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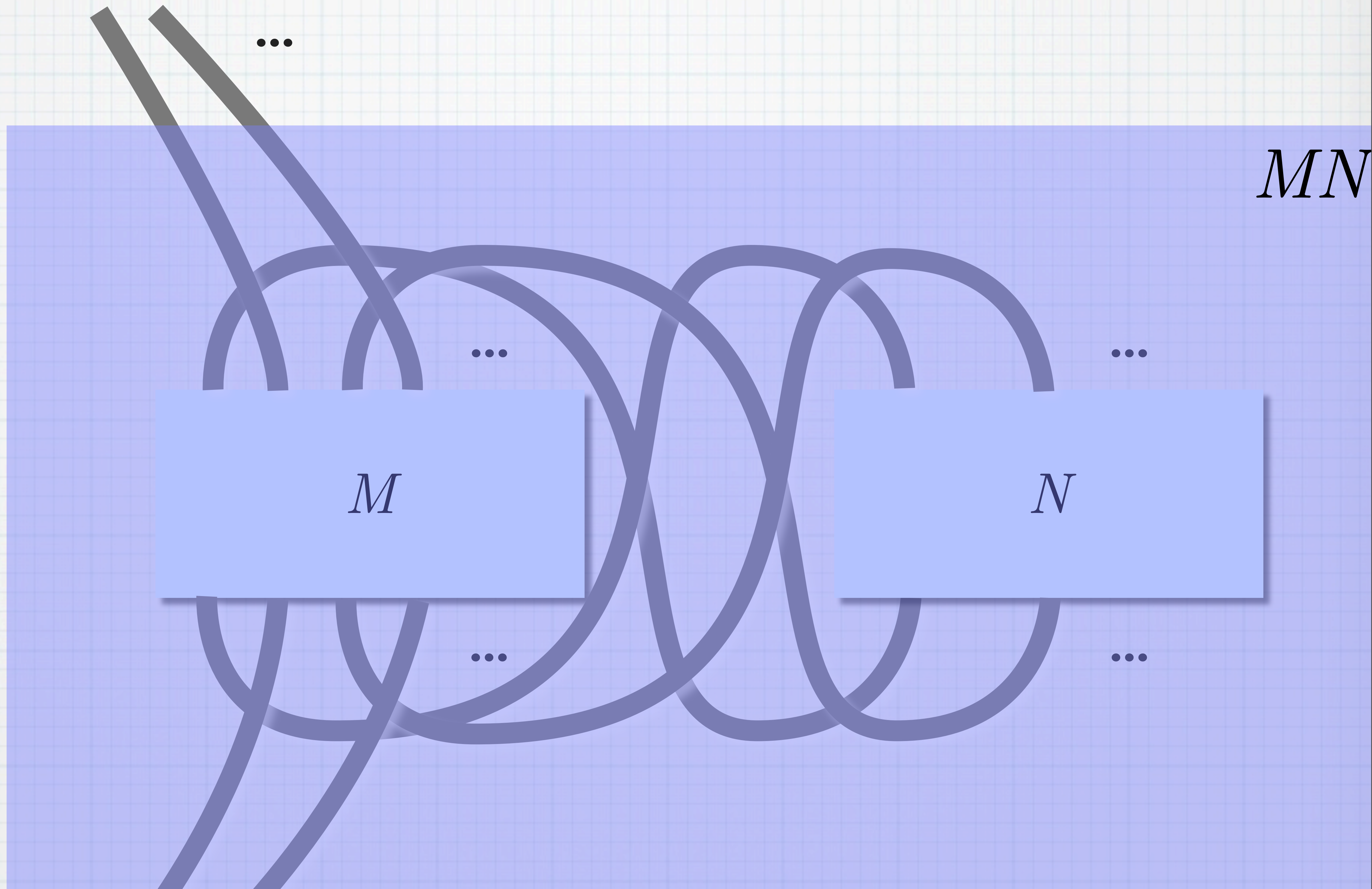

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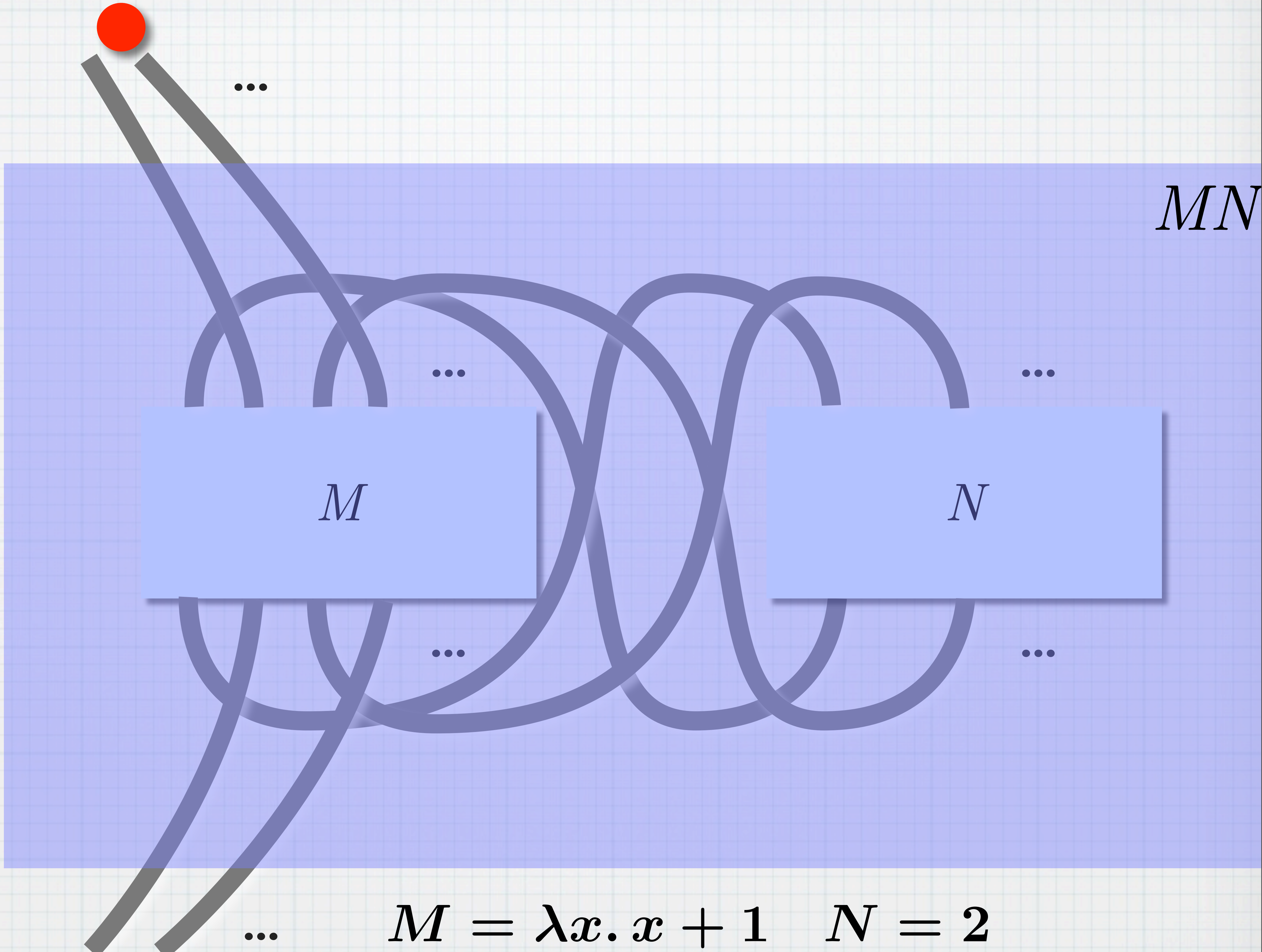

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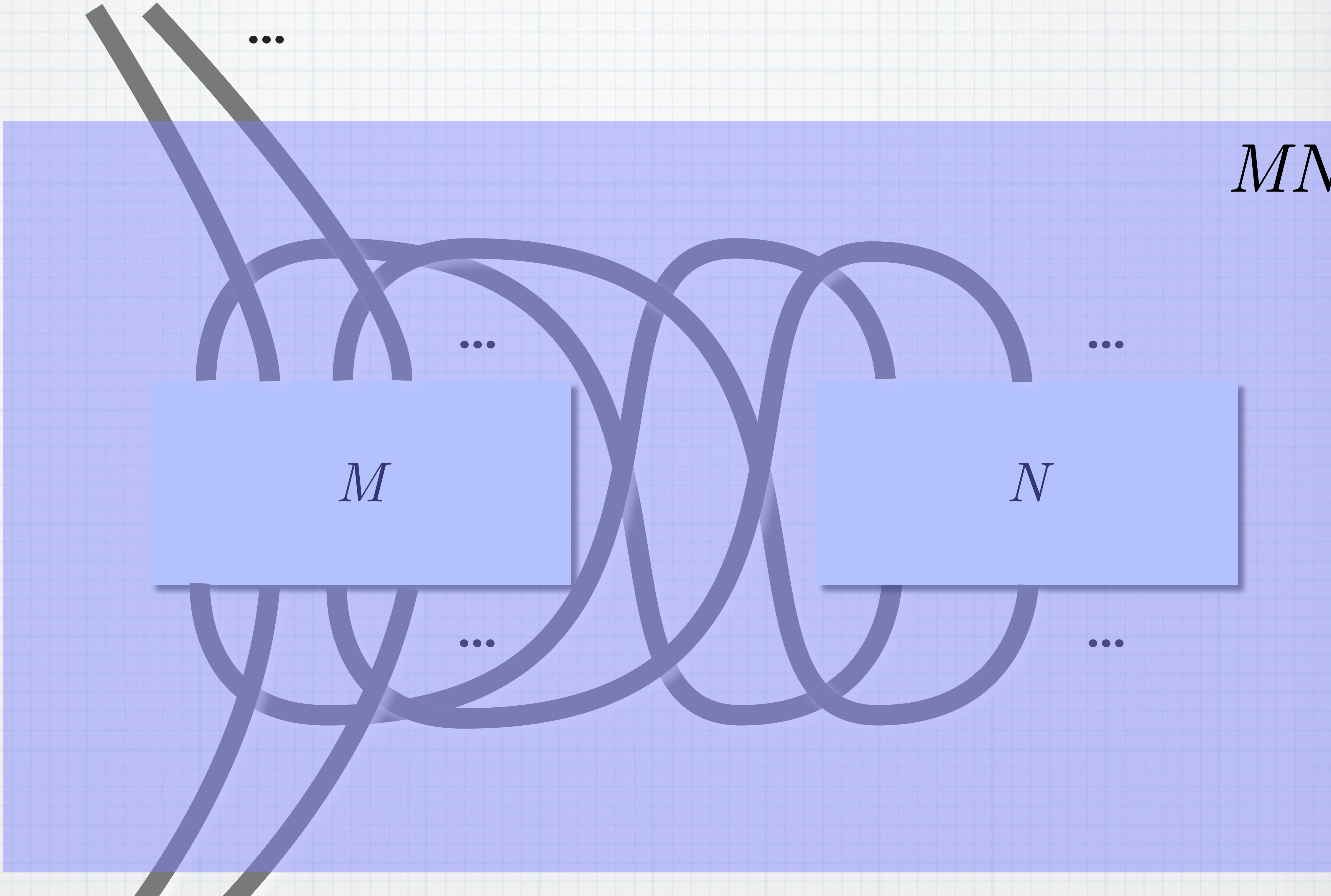

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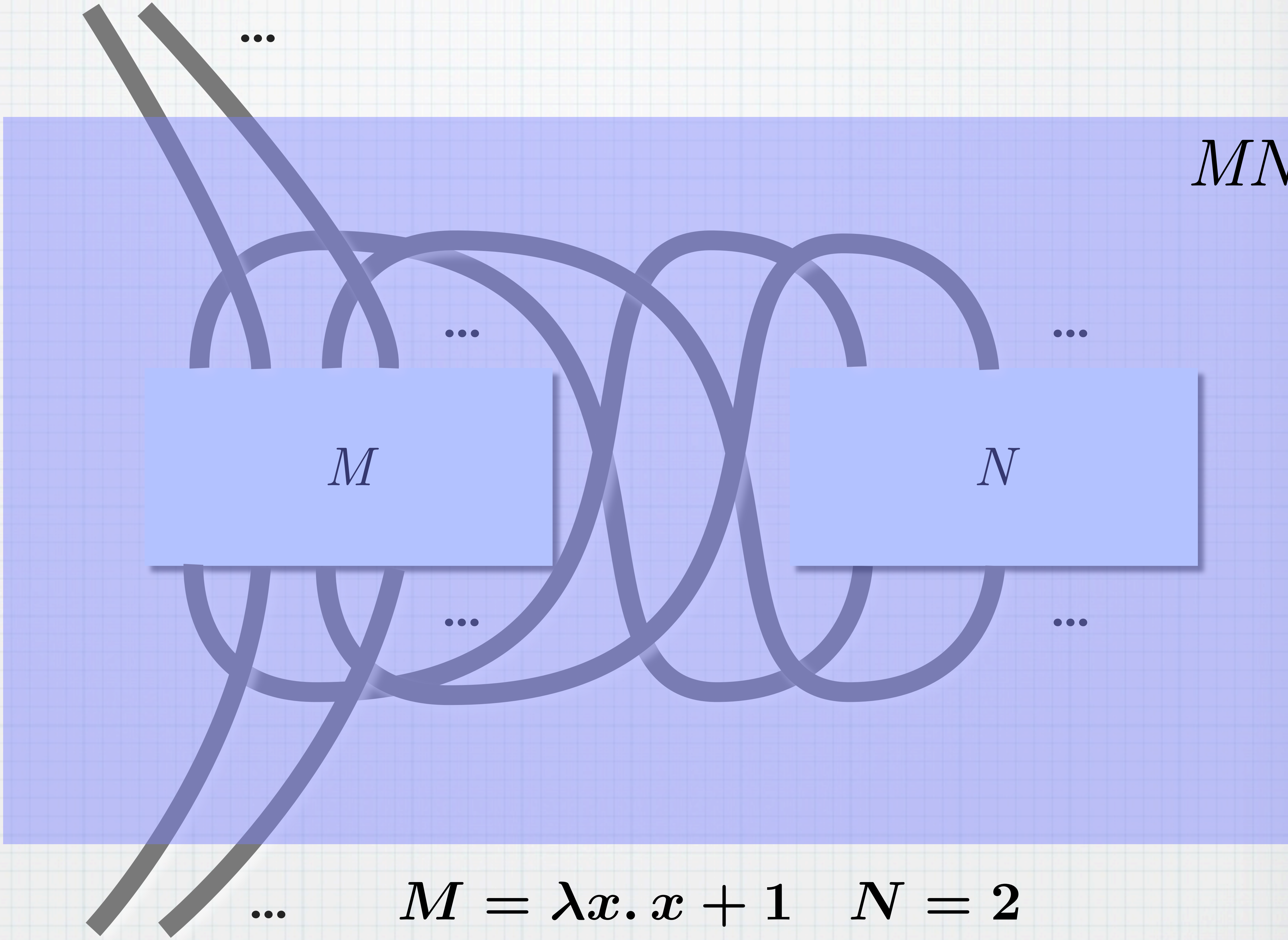

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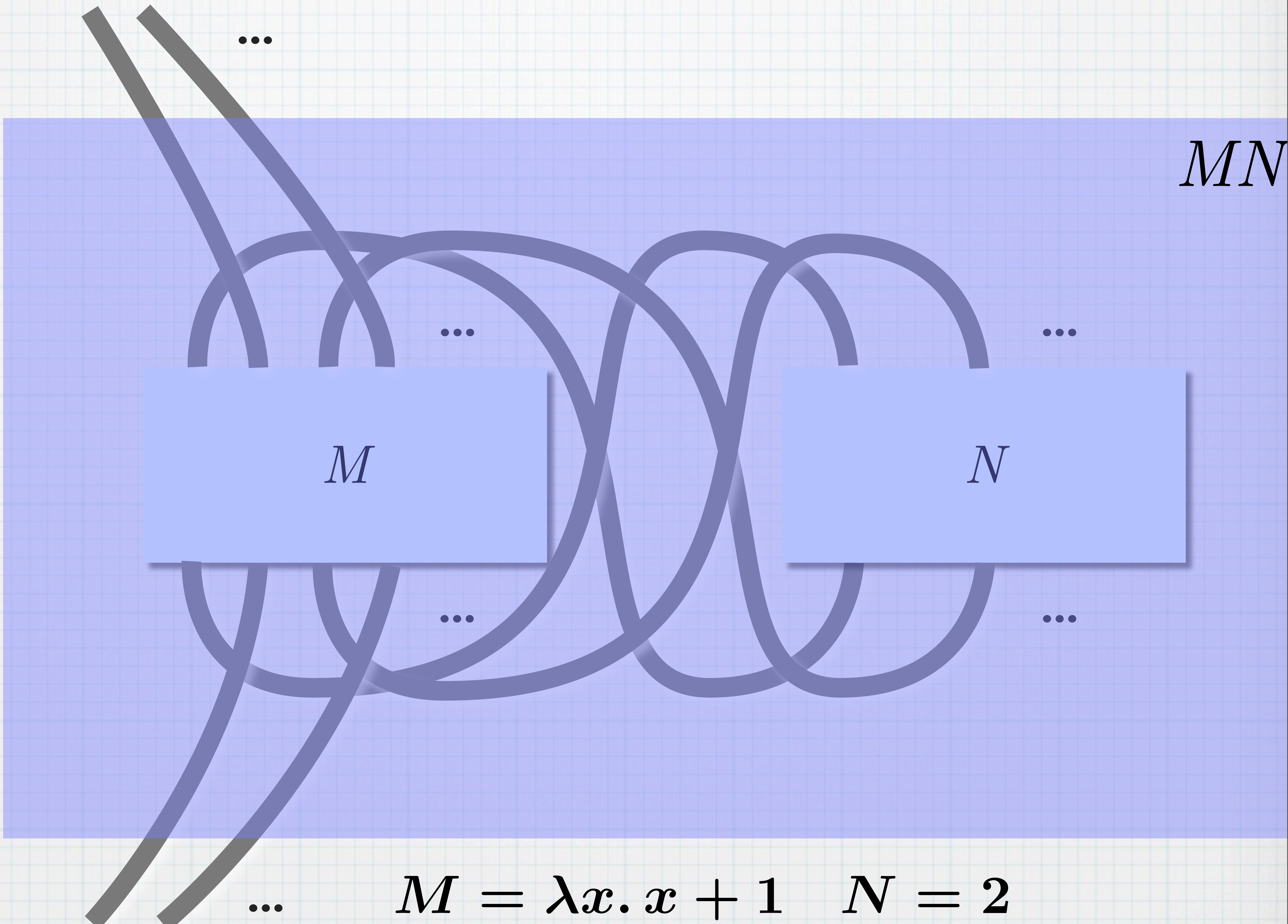

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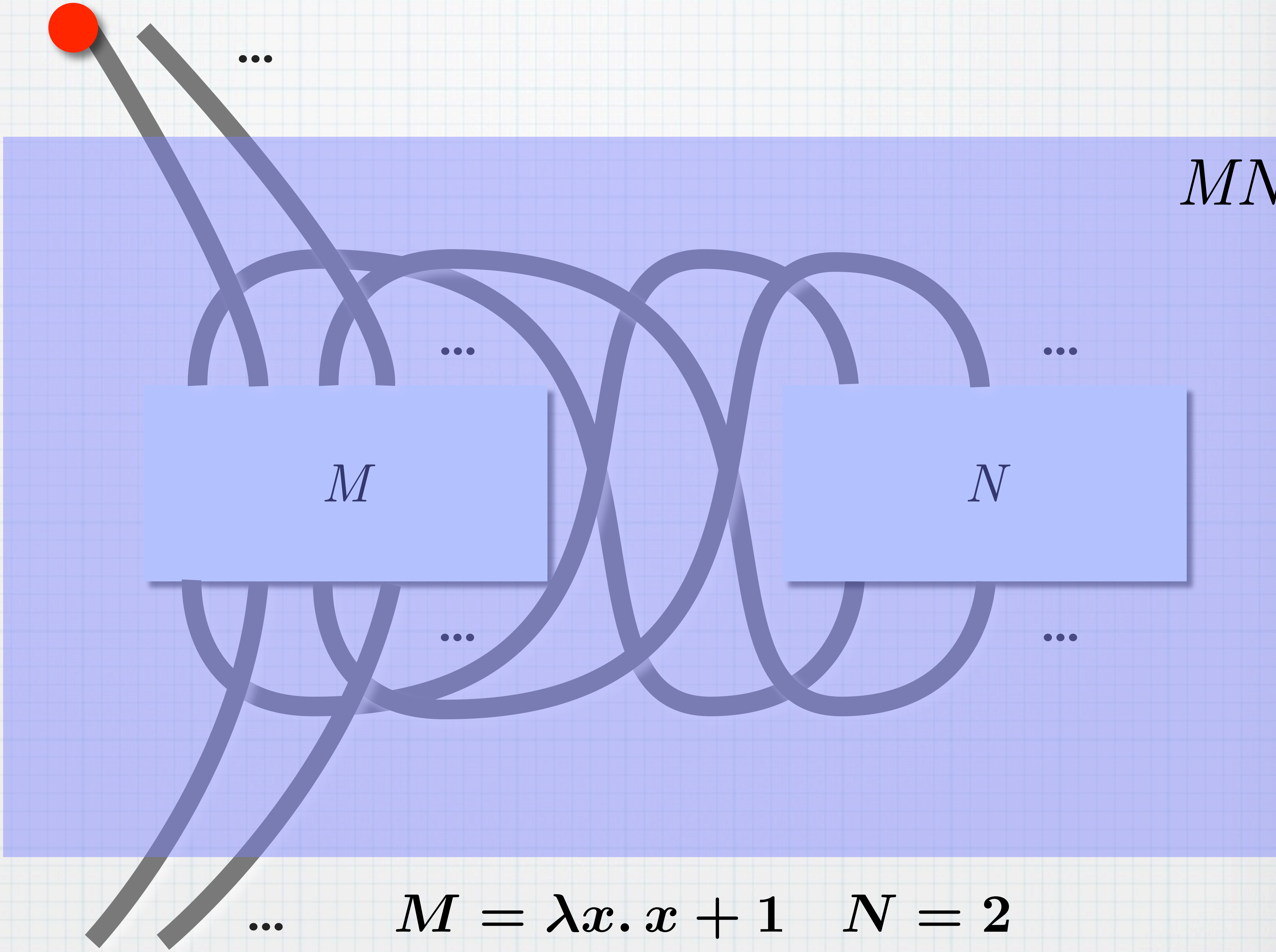

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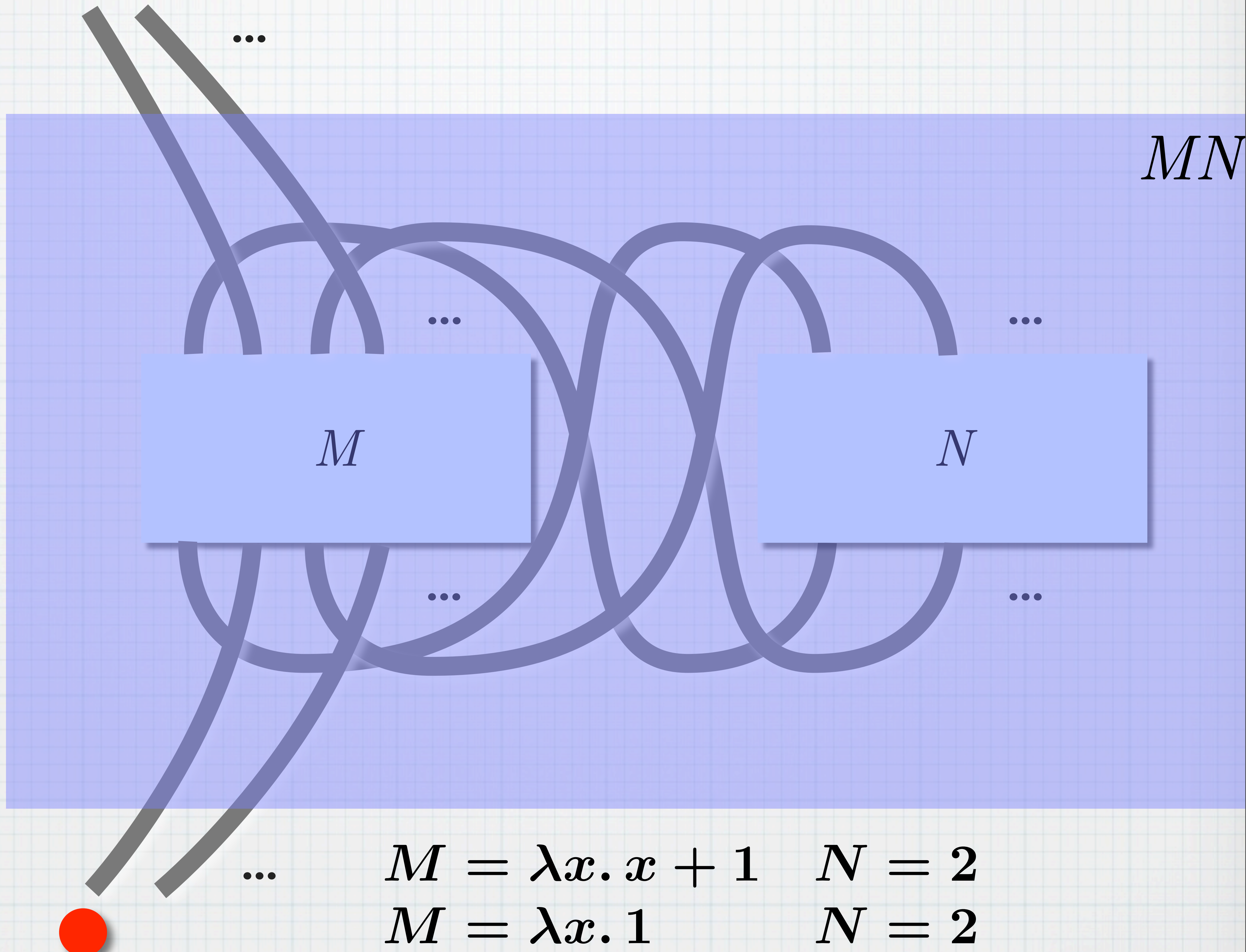

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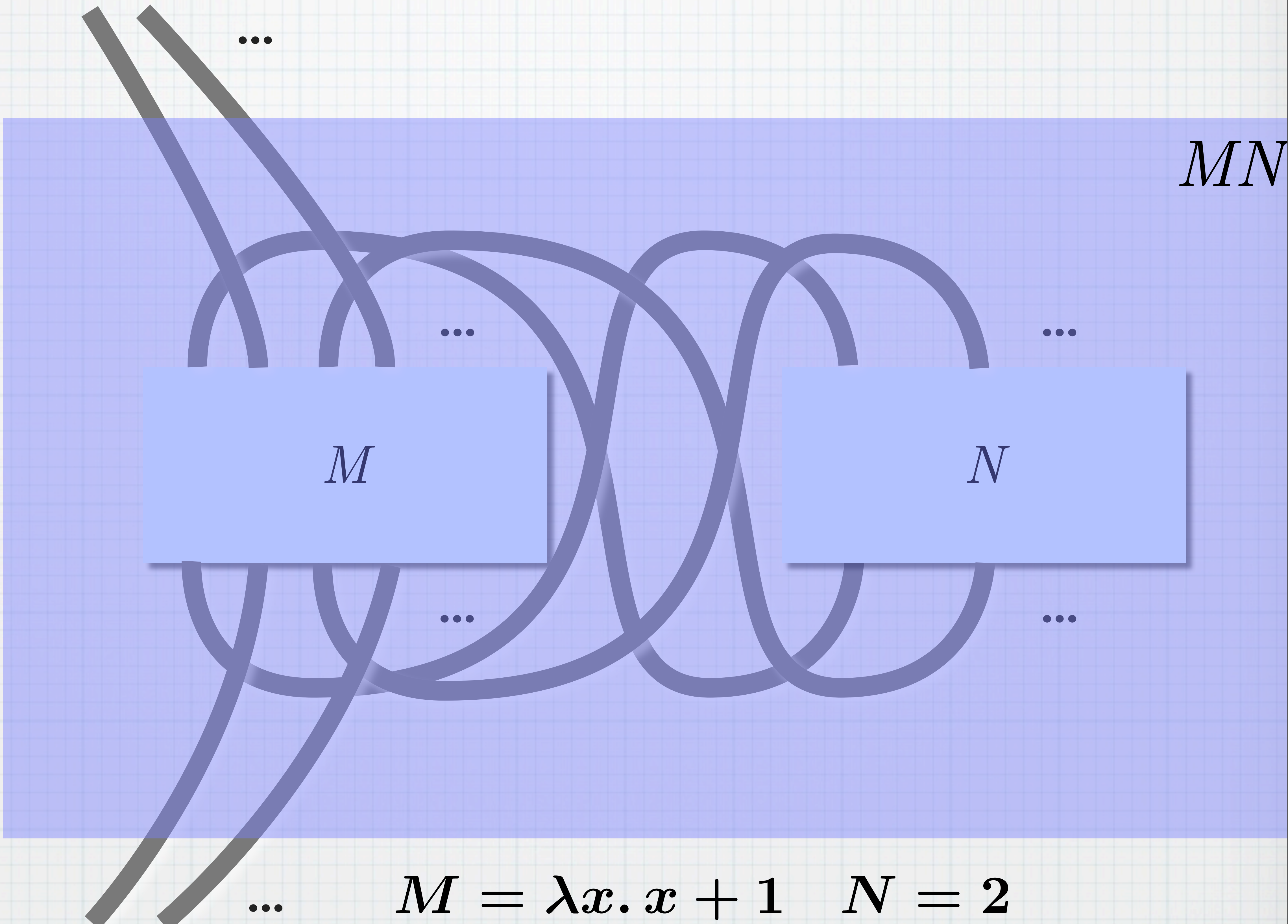

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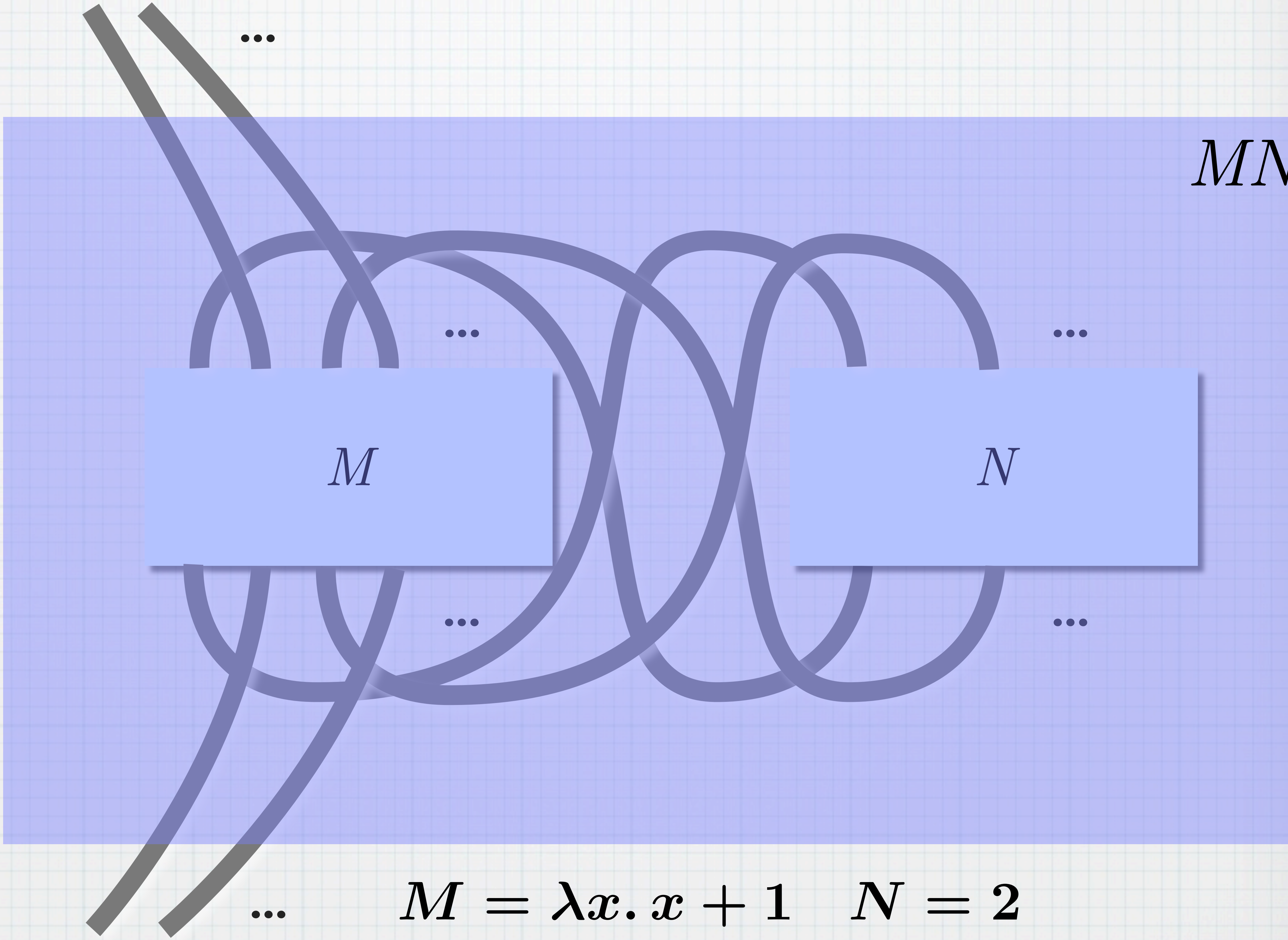

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Categorical GoI

* Axiomatics of GoI in the categorical language
* Abstraction & genericity, which we exploit

* Our main reference (recommended!):
  * [AHS02] S. Abramsky, E. Haghverdi, and P. Scott, "Geometry of interaction and linear combinatory algebras," MSCS 2002
  * Especially its technical report version (Oxford CL), since it's more detailed

The Categorical GoI Workflow

Traced monoidal category $\mathbb{C}$

+ other constructs ➜ "GoI situation" [AHS02]

⬇ Categorical GoI [AHS02]

Linear combinatory algebra

⬇ Realizability

Linear category

Hasuo (Tokyo)

The Categorical GoI Workflow

Traced monoidal category $\mathbb{C}$

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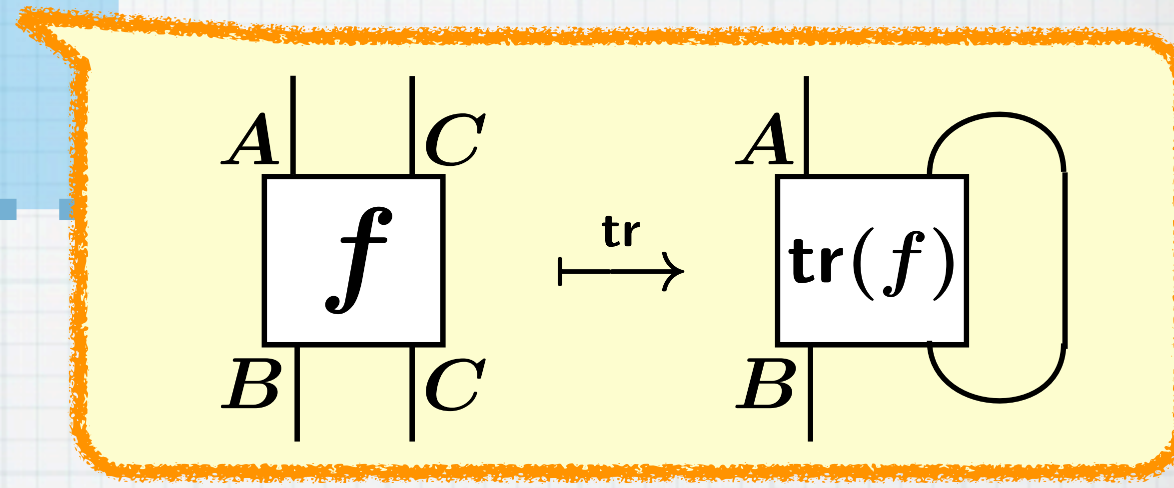

Categorical GoI [AHS02]

Linear combinatory algebra

Realizability

Linear category

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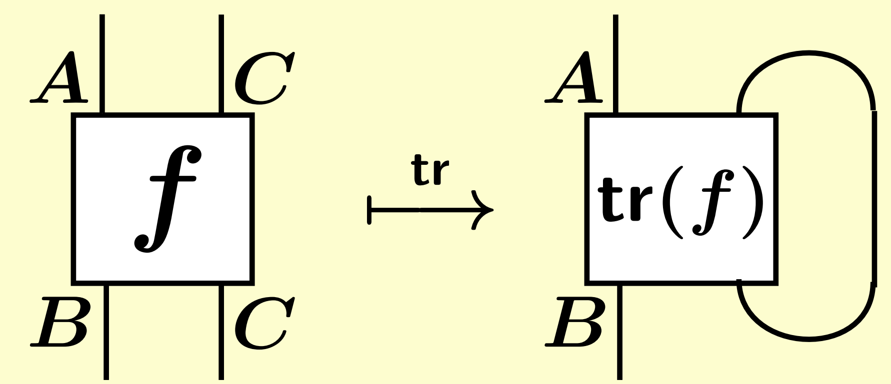


Categorical GoI [AHS02]

Linear combinatory algebra

- Applicative str. + combinators
- Model of untyped calculus

Realizability

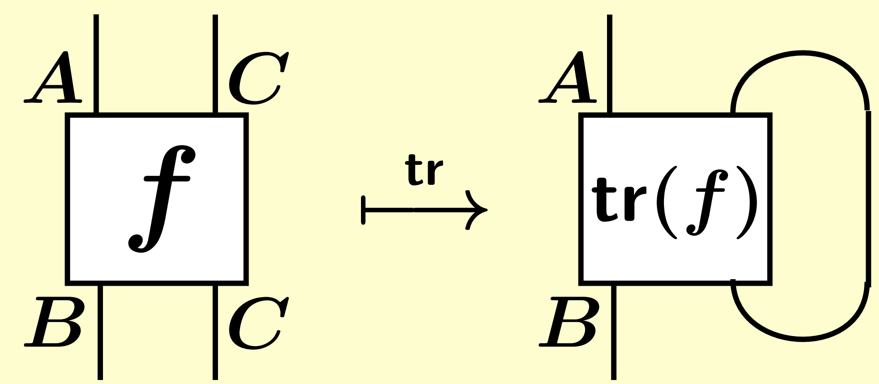
Linear category

Hasuo (Tokyo)

The Categorical GoI Workflow

Traced monoidal category $\mathbb{C}$

+ other constructs ➜ "GoI situation" [AHS02]



⬇ Categorical GoI [AHS02]

Linear combinatory algebra

- Applicative str. + combinators
- Model of untyped calculus

⬇ Realizability

Linear category

Model of typed calculus

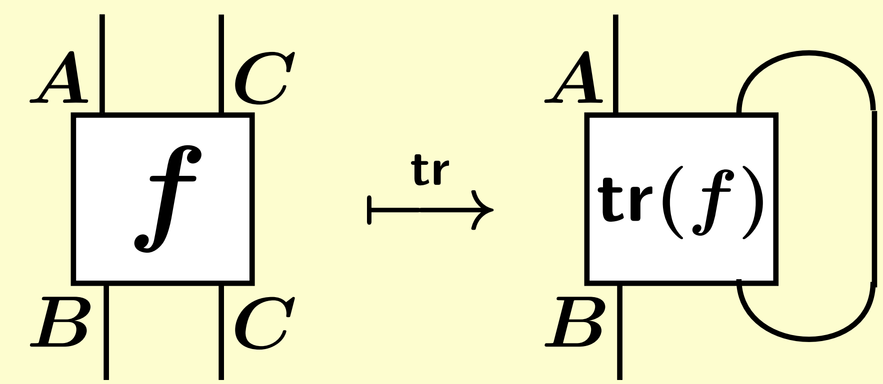
The Categorical GoI Workflow

Weak linear category Int($\mathbb{C}$)

Traced monoidal category $\mathbb{C}$

+ other constructs → "GoI situation" [AHS02]

Int-constr. [Joyal,Street&Verity96]



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Hasuo (Tokyo)

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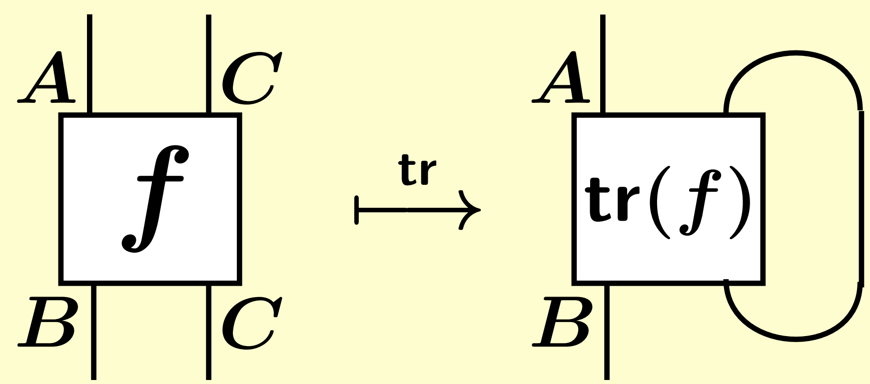

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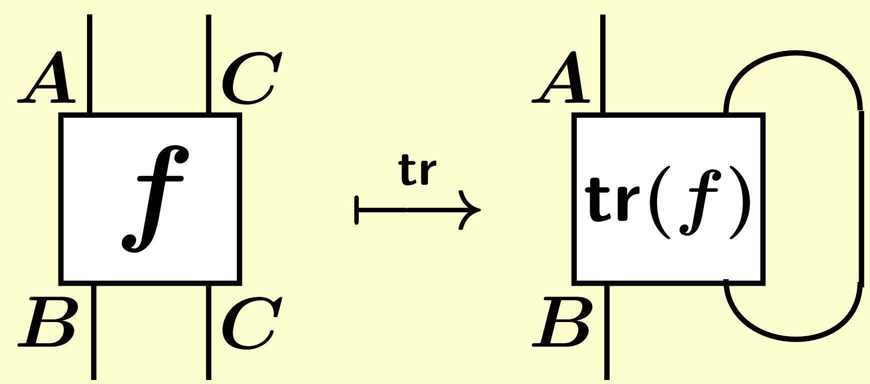

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Hasuo (Tokyo)

Linear Combinatory Algebra (LCA)

Defn. (LCA)
A *linear combinatory algebra (LCA)* is a set $\boldsymbol{A}$ equipped with

- a binary operator (called an *applicative structure*)
$$\cdot \ : \ \boldsymbol{A}^2 \longrightarrow \boldsymbol{A}$$
- a unary operator
$$! \ : \ \boldsymbol{A} \longrightarrow \boldsymbol{A}$$
- (*combinators*) distinguished elements $\mathsf{B}, \mathsf{C}, \mathsf{I}, \mathsf{K}, \mathsf{W}, \mathsf{D}, \delta, \mathsf{F}$ satisfying

$$\begin{array}{rcll}
\mathsf{B}xyz &=& x(yz) & \text{Composition, Cut} \\
\mathsf{C}xyz &=& (xz)y & \text{Exchange} \\
\mathsf{I}x &=& x & \text{Identity} \\
\mathsf{K}x\,!\,y &=& x & \text{Weakening} \\
\mathsf{W}x\,!\,y &=& x\,!\,y\,!\,y & \text{Contraction} \\
\mathsf{D}\,!\,x &=& x & \text{Dereliction} \\
\delta\,!\,x &=& !\,!\,x & \text{Comultiplication} \\
\mathsf{F}\,!\,x\,!\,y &=& !(xy) & \text{Monoidal functoriality}
\end{array}$$

Here: $\cdot$ associates to the left; $\cdot$ is suppressed; and $!$ binds stronger than $\cdot$ does.

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What
we want (outcome)

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- Model of untyped linear λ
- a ∈ A ≈ closed linear λ-term

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- Model of untyped linear λ
- a ∈ A ≈ closed linear λ-term
- No **S** or **K** (linear!)
- Combinatory completeness: e.g.

$$\lambda xyz.\, zxy$$

designates elem. of A

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What we use (ingredient)

GoI situation

Defn. (GoI situation [AHS02])
A *GoI situation* is a triple $(\mathbb{C}, \boldsymbol{F}, \boldsymbol{U})$ where

- $\mathbb{C} = (\mathbb{C}, \otimes, \boldsymbol{I})$ is a traced symmetric monoidal category (TSMC);
- $\boldsymbol{F} : \mathbb{C} \to \mathbb{C}$ is a traced symmetric monoidal functor, equipped with the following retractions (which are monoidal natural transformations).

$$\begin{array}{rcrcl} \boldsymbol{e} & : & \boldsymbol{FF} \lhd \boldsymbol{F} & : & \boldsymbol{e}' \\ \boldsymbol{d} & : & \mathbf{id} \lhd \boldsymbol{F} & : & \boldsymbol{d}' \\ \boldsymbol{c} & : & \boldsymbol{F} \otimes \boldsymbol{F} \lhd \boldsymbol{F} & : & \boldsymbol{c}' \\ \boldsymbol{w} & : & \boldsymbol{K_I} \lhd \boldsymbol{F} & : & \boldsymbol{w}' \end{array} \qquad \begin{array}{l} \text{Comultiplication} \\ \text{Dereliction} \\ \text{Contraction} \\ \text{Weakening} \end{array}$$

  Here $\boldsymbol{K_I}$ is the constant functor into the monoidal unit $\boldsymbol{I}$;
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$$\begin{array}{rcrcl} \boldsymbol{j} & : & \boldsymbol{U} \otimes \boldsymbol{U} \lhd \boldsymbol{U} & : & \boldsymbol{k} \\ & & \boldsymbol{I} \lhd \boldsymbol{U} & & \\ \boldsymbol{u} & : & \boldsymbol{FU} \lhd \boldsymbol{U} & : & \boldsymbol{v} \end{array}$$

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- **Monoidal category** $(\mathbb{C}, \otimes, \boldsymbol{I})$
  - **String diagrams**

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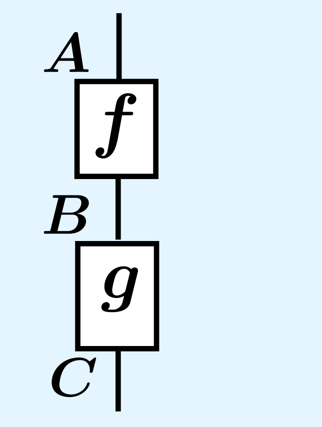
$$\frac{A \xrightarrow{f} B \qquad B \xrightarrow{g} C}{A \xrightarrow{g \circ f} C}$$



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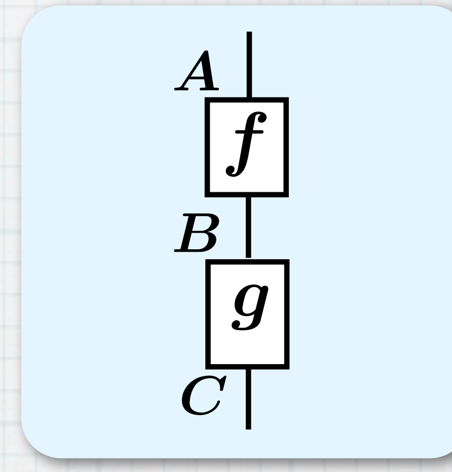
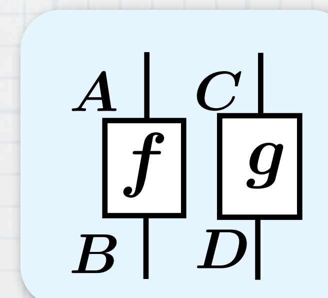


$$\frac{A \xrightarrow{f} B \quad C \xrightarrow{g} D}{A \otimes C \xrightarrow{f \otimes g} B \otimes D}$$



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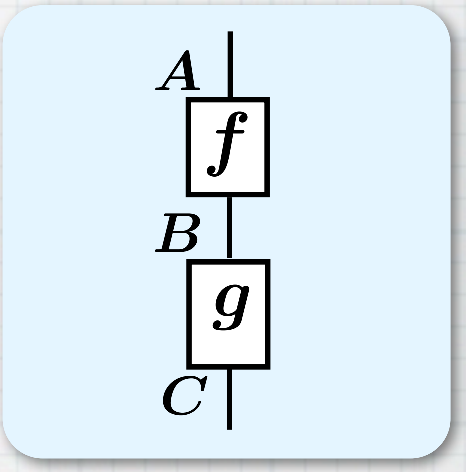

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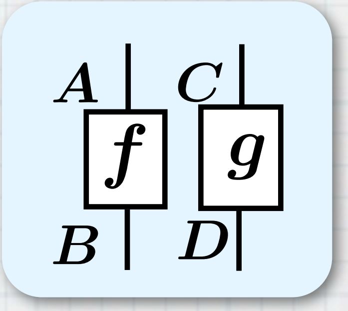
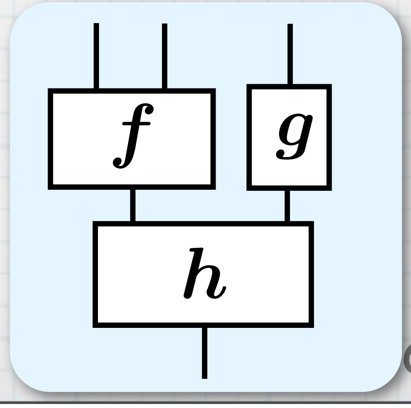


$$h \circ (f \otimes g)$$



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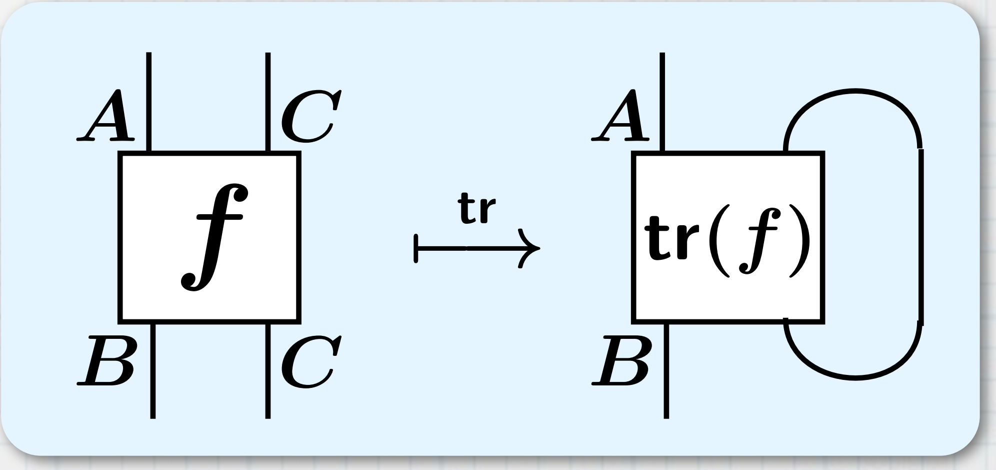
* **Traced** monoidal category

  * "feedback"

$$\frac{A \otimes C \xrightarrow{f} B \otimes C}{A \xrightarrow{\mathsf{tr}(f)} B}$$

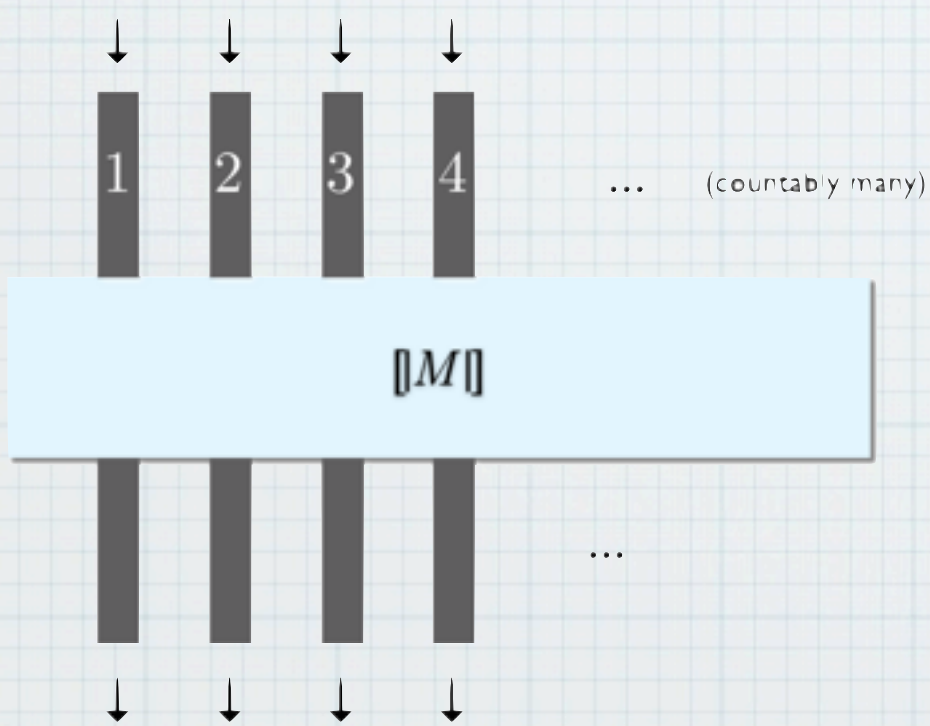
that is



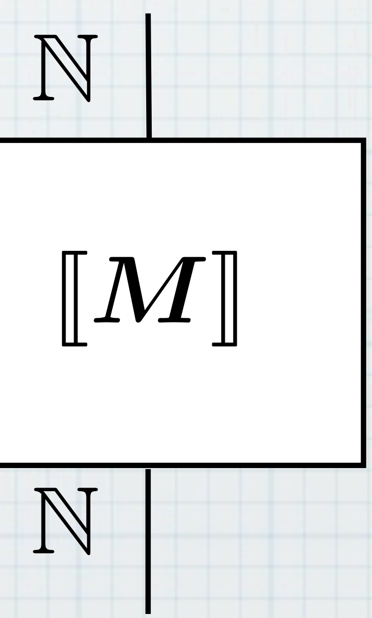
Hasuo (Tokyo)

String Diagram vs. "Pipe Diagram"

* In this talk, I use two ways of depicting partial functions $\mathbb{N} \rightharpoonup \mathbb{N}$



Pipe diagram



String diagram

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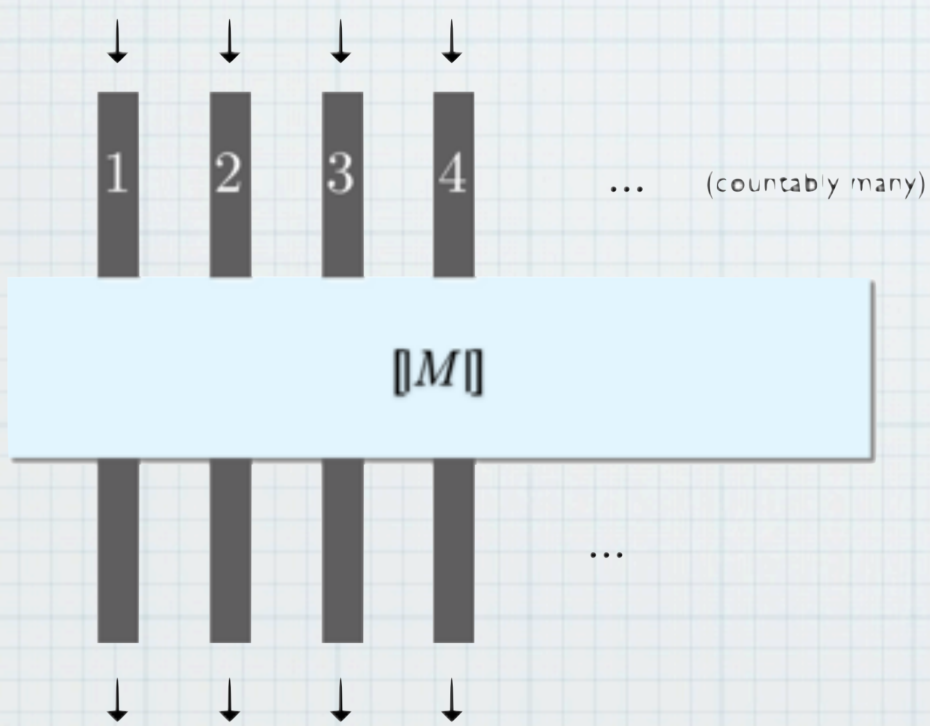
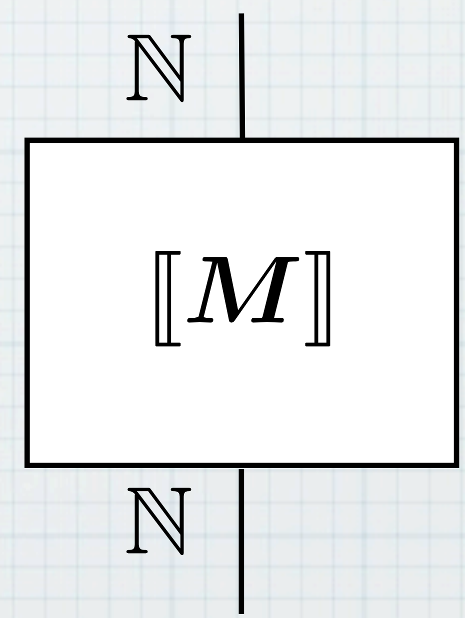
In the monoidal category $(\mathbf{Pfn}, +, 0)$



Pipe diagram



String diagram

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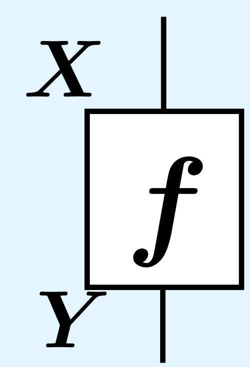
Traced Sym. Monoidal Category
$(\mathbf{Pfn}, +, 0)$

- Category $\mathbf{Pfn}$ of **partial functions**
  - <u>**Obj.**</u> A set X
  - <u>**Arr.**</u> A partial function

$$\frac{X \to Y \ \text{ in } \mathbf{Pfn}}{X \rightharpoonup Y, \ \text{partial function}}$$



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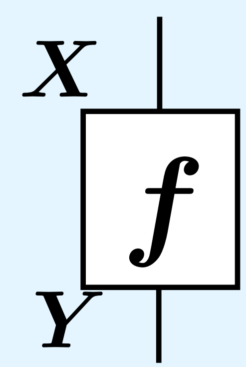


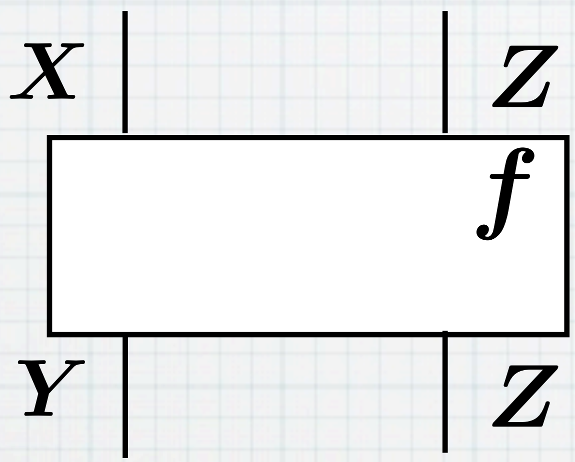
* is traced symmetric monoidal

Traced Sym. Monoidal Category $(\mathbf{Pfn}, +, 0)$

* Given $\quad X + Z \xrightarrow{f} Y + Z$ in $\mathbf{Pfn}$
*

Hasuo (Tokyo)

Traced Sym. Monoidal Category $(\mathbf{Pfn}, +, 0)$

- Given $\quad X + Z \xrightarrow{f} Y + Z$ in $\mathbf{Pfn}$
- 

Traced Sym. Monoidal Category $(\mathbf{Pfn}, +, 0)$

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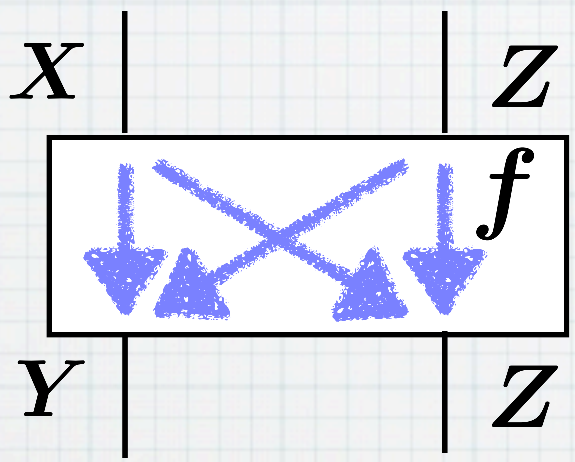
- 

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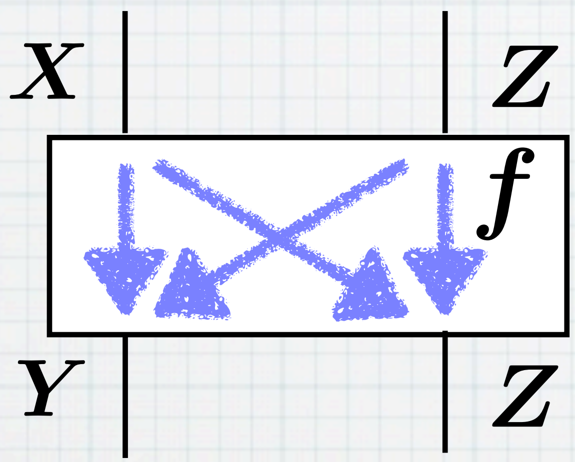
* 

$$f_{XY}(x) := \begin{cases} f(x) & \text{if } f(x) \in Y \\ \perp & \text{o.w.} \end{cases}$$

Similar for f_{XZ}, f_{ZY}, f_{ZZ}

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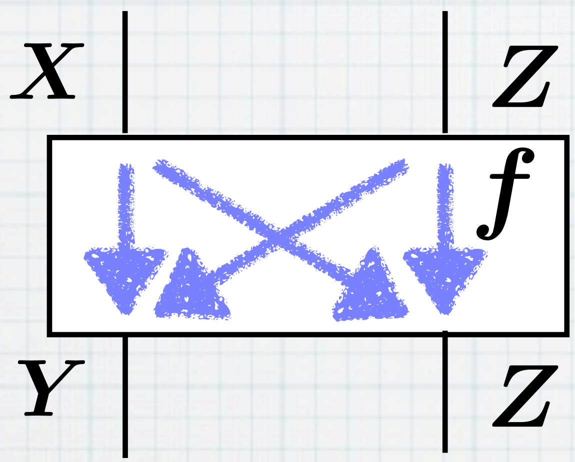
* 

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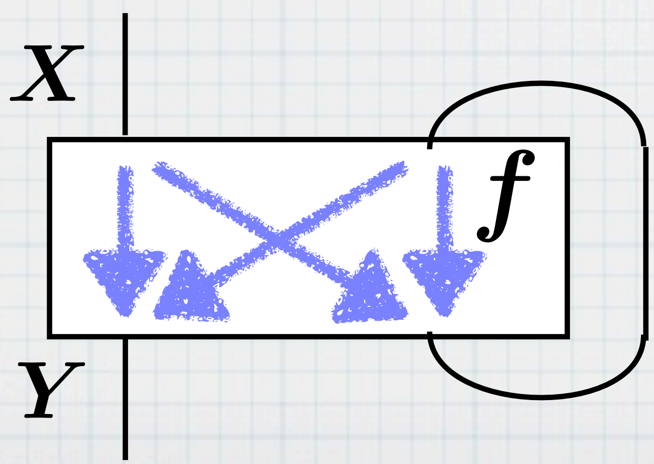

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Traced Sym. Monoidal Category
$(\mathbf{Pfn}, +, 0)$

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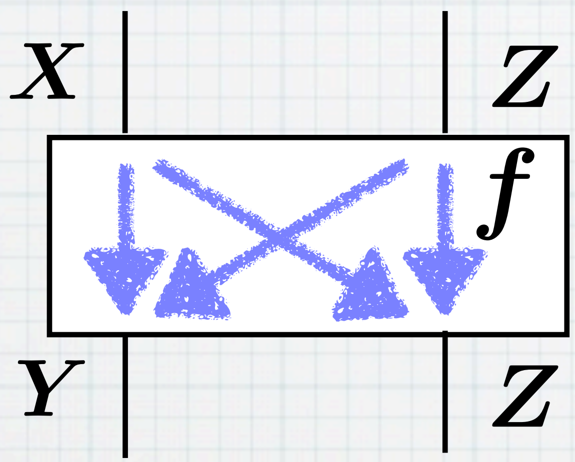
* 

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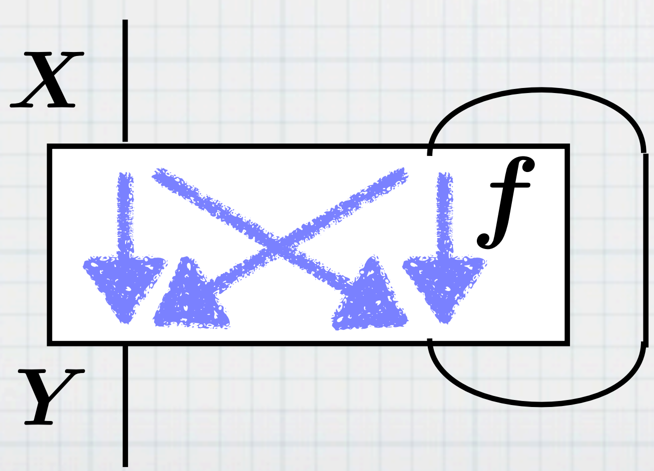

$$\mathbf{tr}(f) = f_{XY} \sqcup \left(\coprod_{n \in \mathbb{N}} f_{ZY} \circ (f_{ZZ})^n \circ f_{XZ} \right)$$

(Tokyo)

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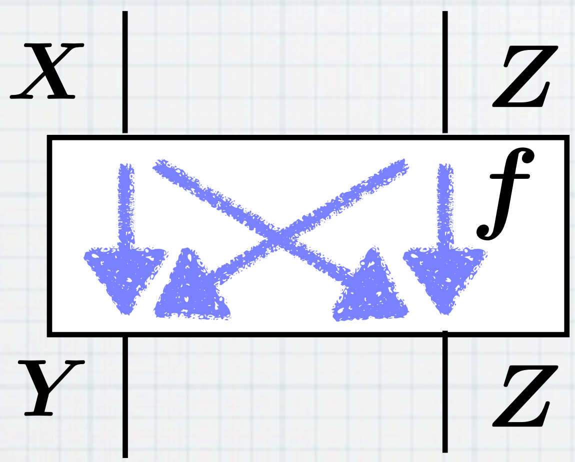

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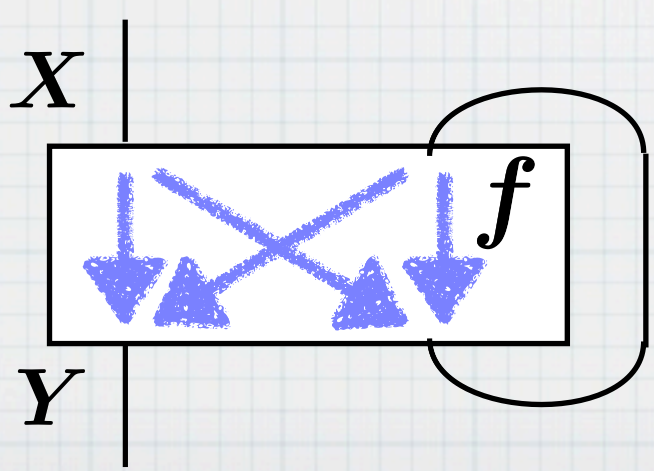

* **Execution formula**
* Partiality is essential (infinite loop)

$$\mathbf{tr}(f) =$$

$$f_{XY} \sqcup \left(\coprod_{n \in \mathbb{N}} f_{ZY} \circ (f_{ZZ})^n \circ f_{XZ} \right)$$

(Tokyo)

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Defn. (GoI situation [AHS02])
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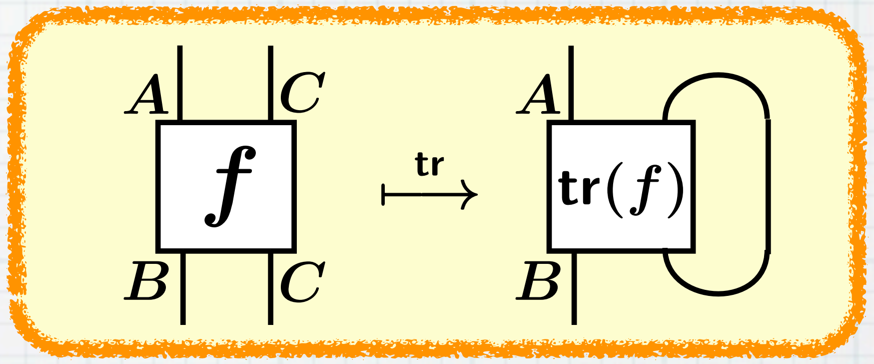
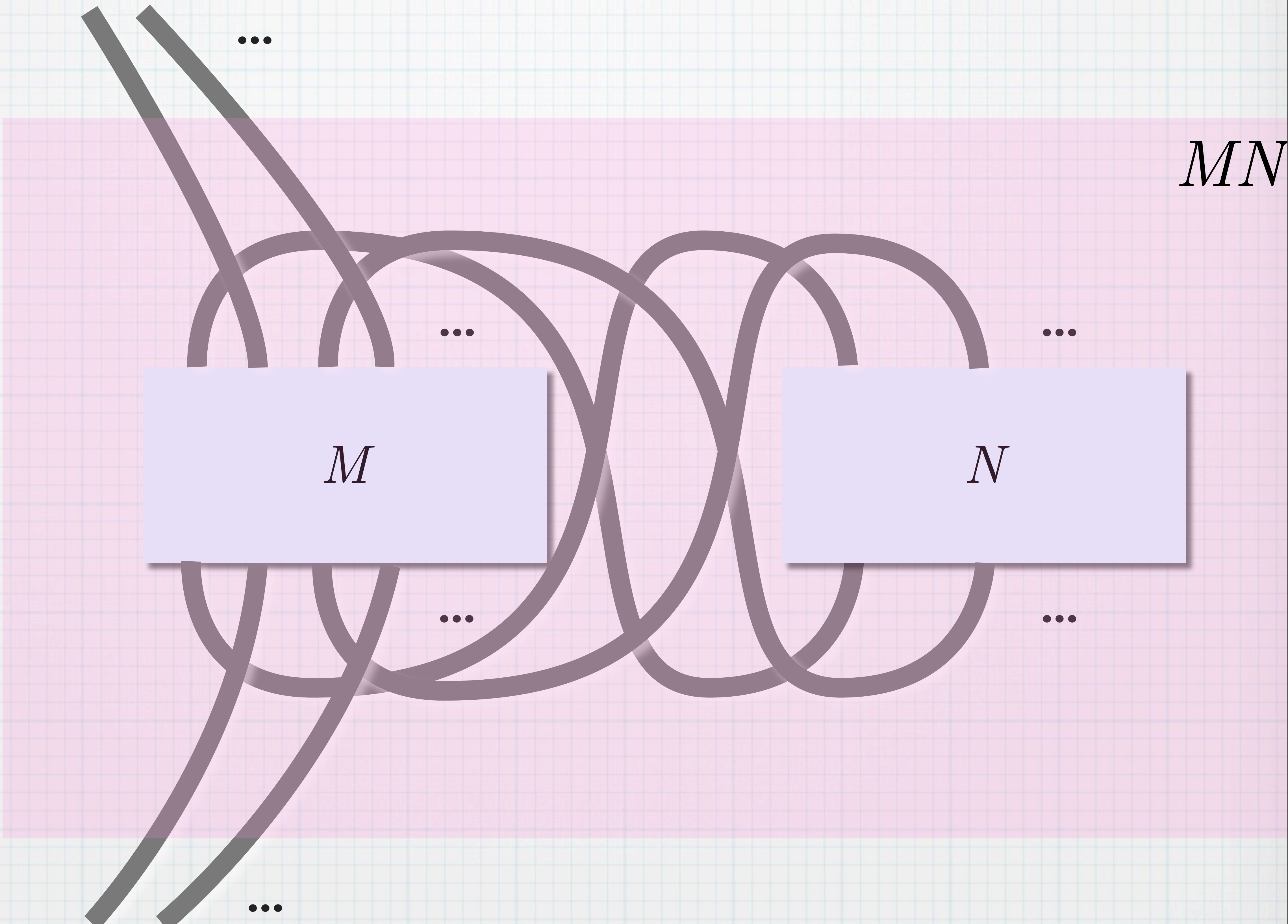
* Traced sym. monoidal cat.
  * Where one can “feedback”



  * Why for GoI?

$$[\![\boldsymbol{MN}]\!] =$$



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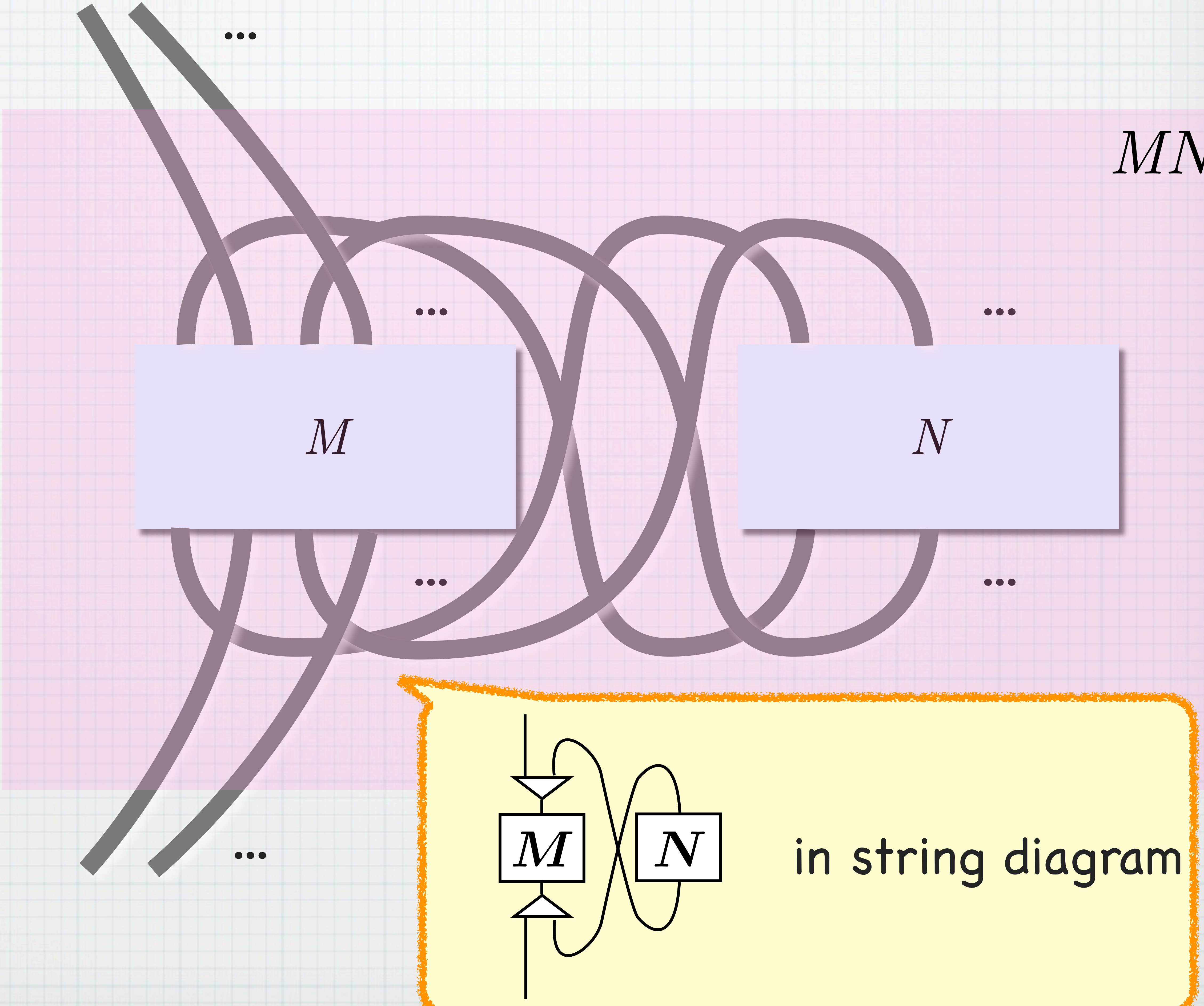

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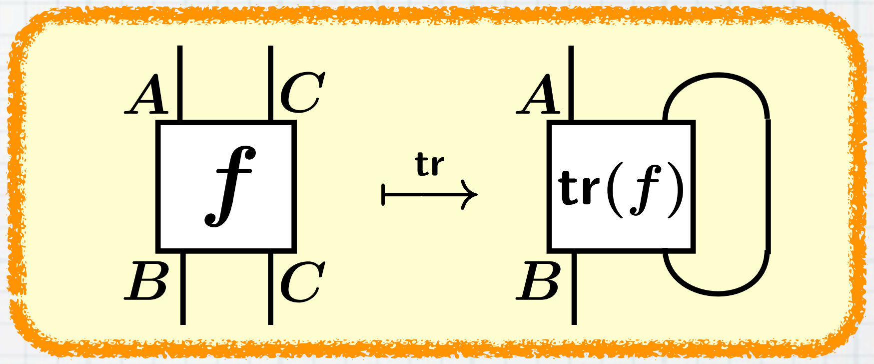

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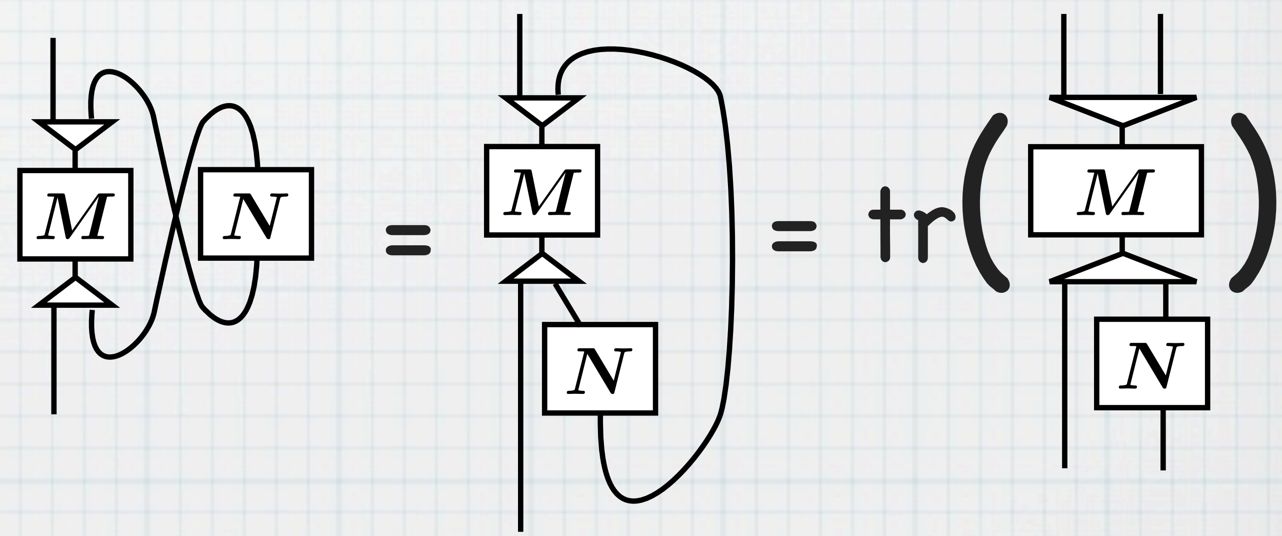

  - Leading example: **Pfn**

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Defn. (Retraction)
A *retraction* from X to Y,

$$f : X \lhd Y : g \;,$$

is a pair of arrows

"embedding"

$$\begin{tikzcd} X \arrow[loop left, "\mathrm{id}"] \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd}$$

"projection"

such that $g \circ f = \mathrm{id}_X$.

- Functor F
  - For obtaining $! : A \to A$

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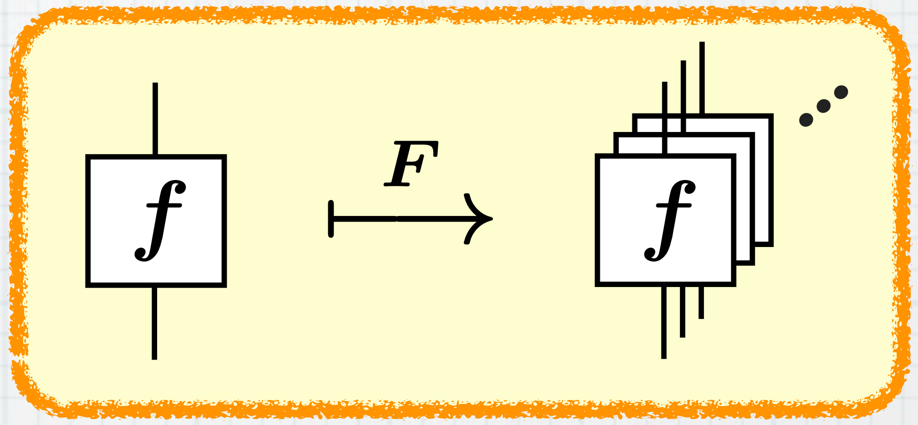
- Functor F
  - For obtaining $! : A \to A$
  - Pictorially:



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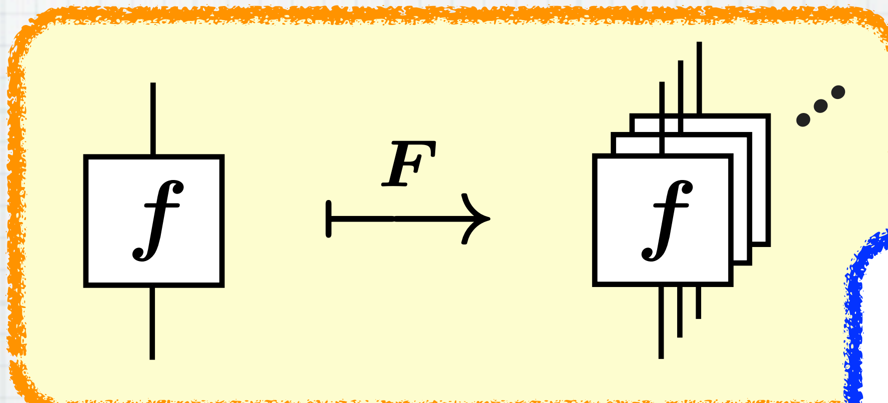

  * Example in **Pfn**:

$$(n, x) \mapsto (n, f(x))$$

$$\mathbb{N} \cdot \_ : \quad \mathbf{Pfn} \longrightarrow \mathbf{Pfn}$$

$$\begin{array}{ccc} X & & \mathbb{N} \cdot X \\ \downarrow f & \longmapsto & \downarrow \mathbb{N} \cdot f \\ Y & & \mathbb{N} \cdot Y \end{array}$$

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* The reflexive object U

* Retr. $\begin{tikzcd} U \otimes U \arrow[r, bend left, "j"] & U \arrow[l, bend left, "k"] \end{tikzcd}$

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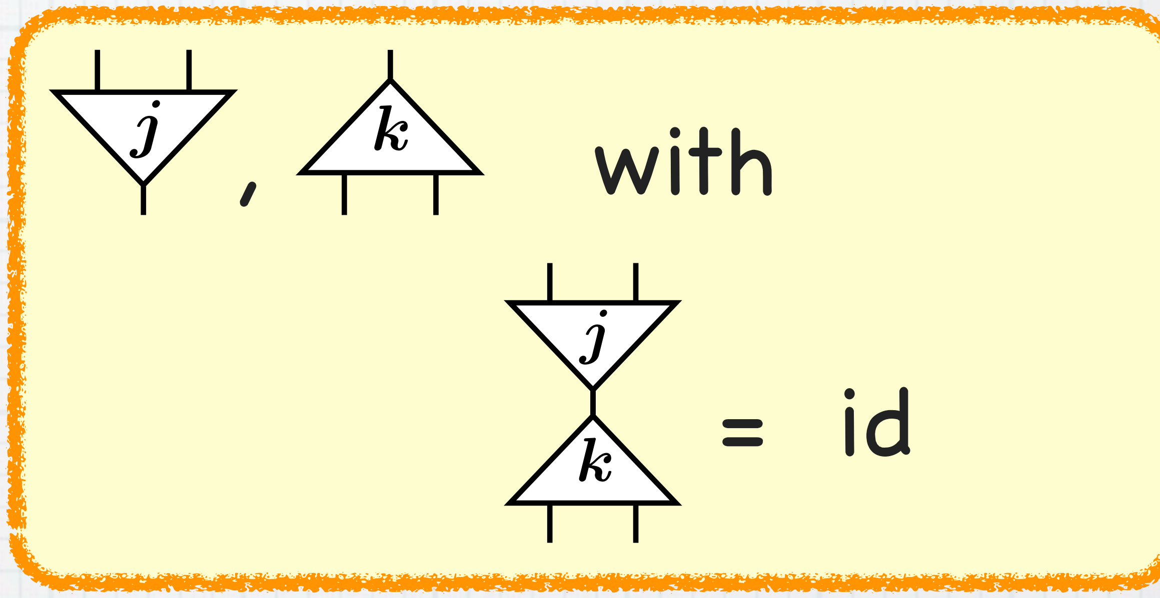

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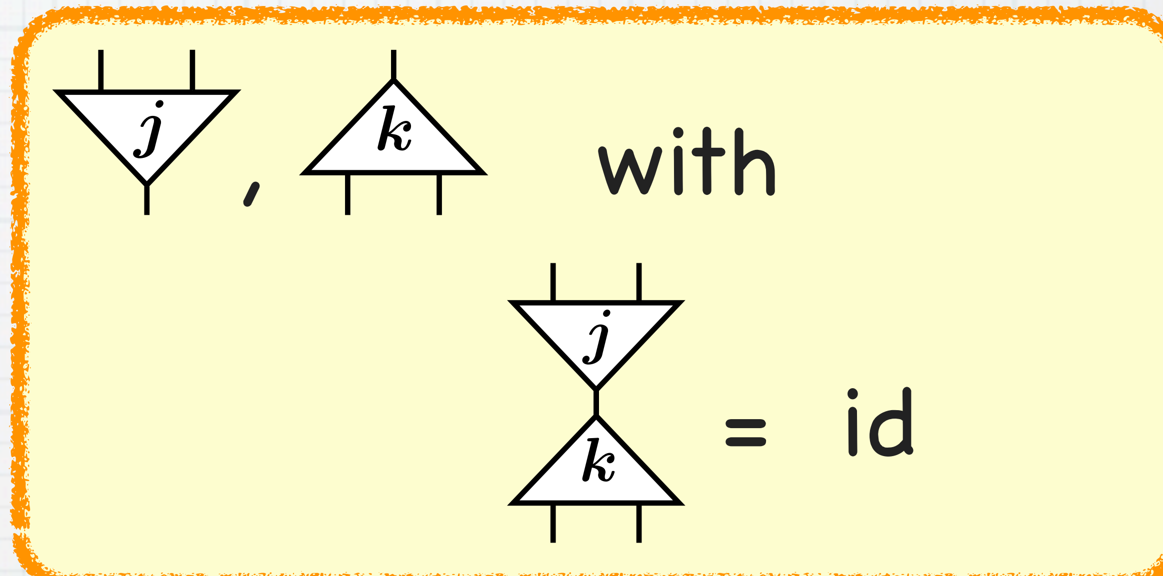

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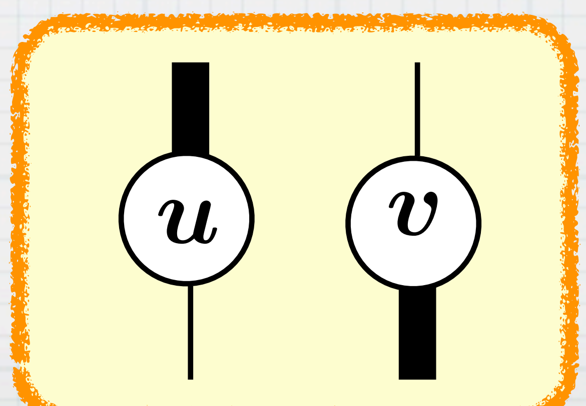

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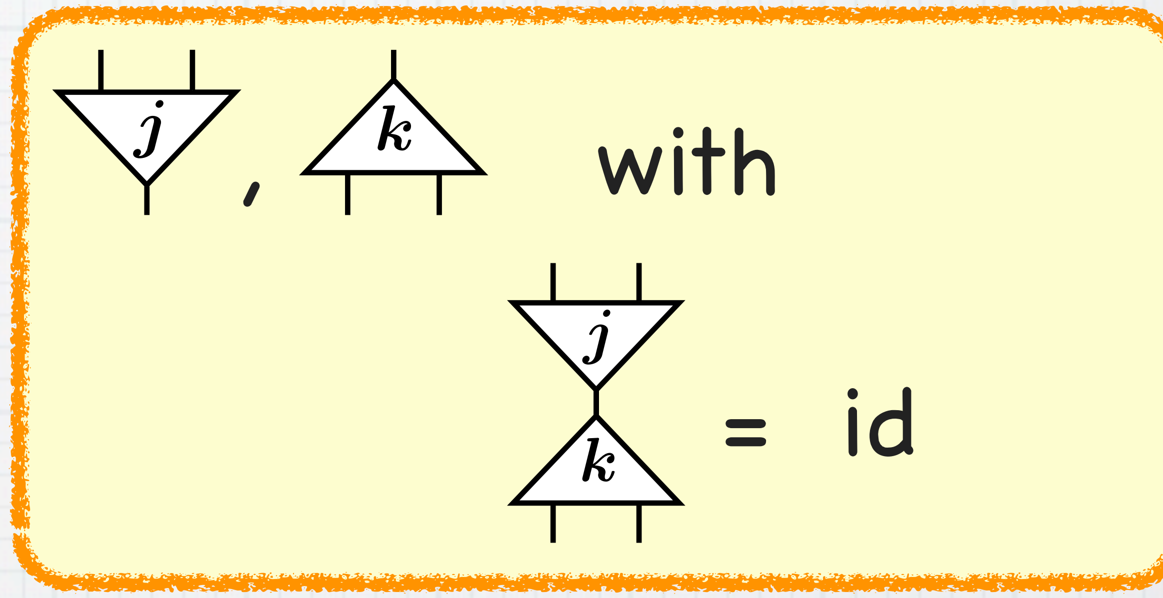

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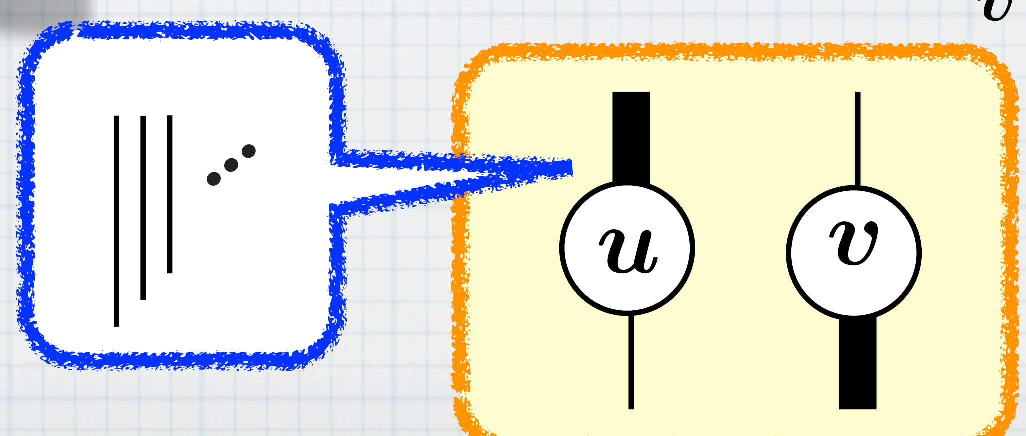

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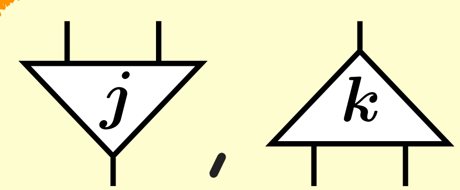

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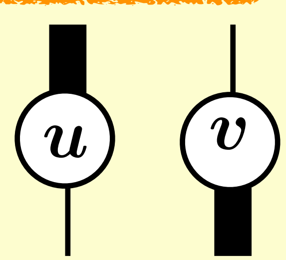

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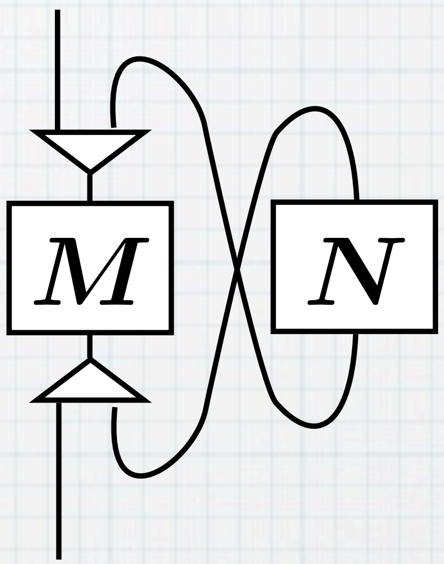

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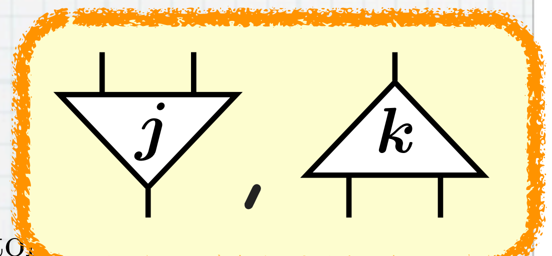

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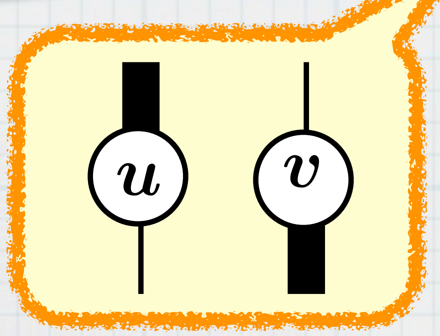
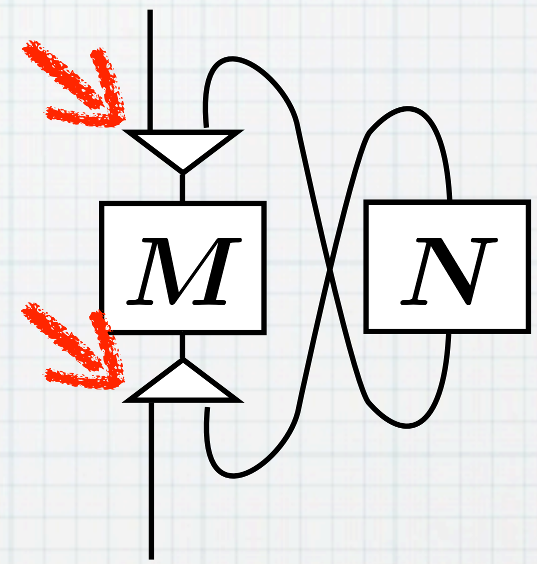


* The reflexive object U

* Why for GoI?



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Hasuo (Tokyo)

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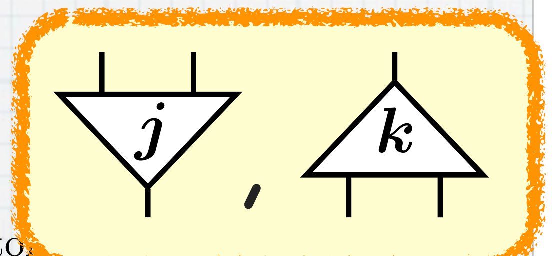

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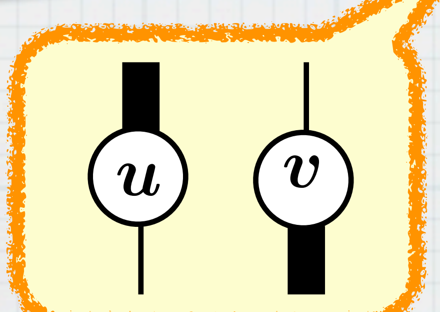

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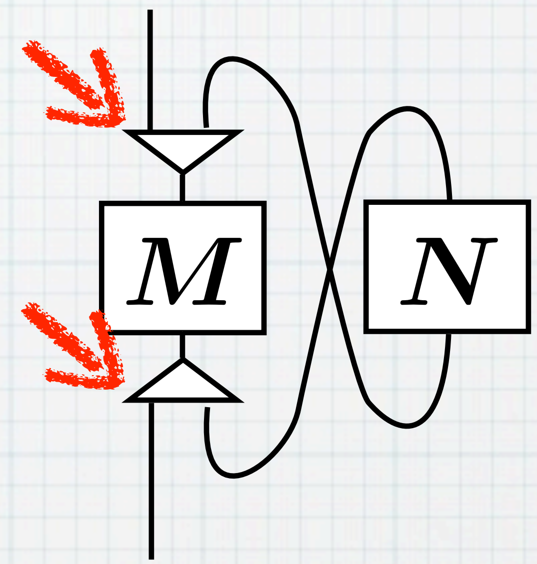

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$$\mathbb{N} + \mathbb{N} \cong \mathbb{N},$$
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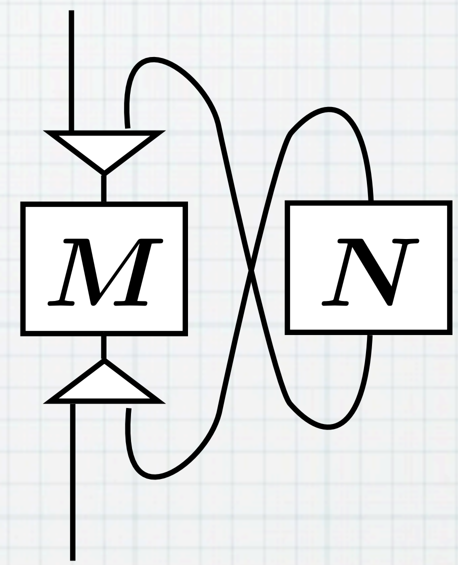

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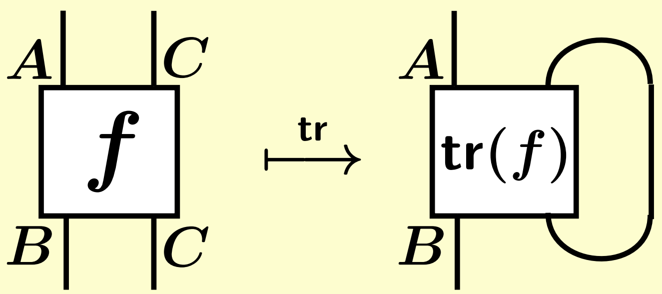
Hasuo (Tokyo)



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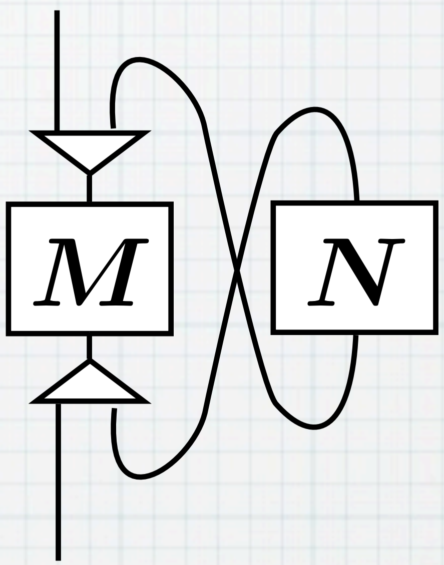

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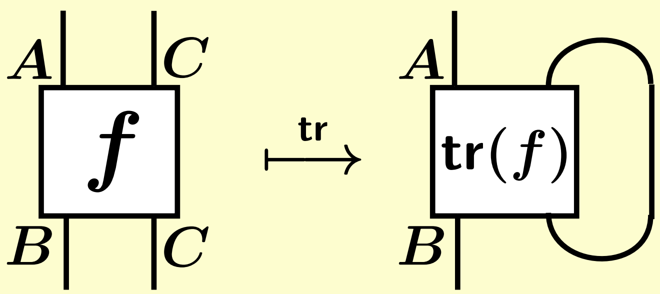

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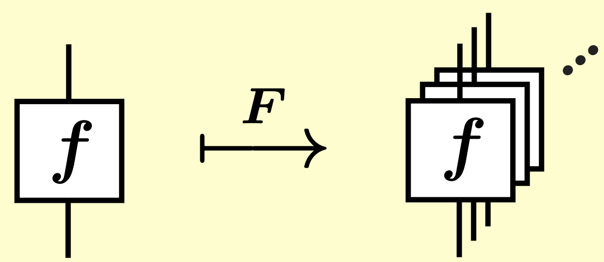

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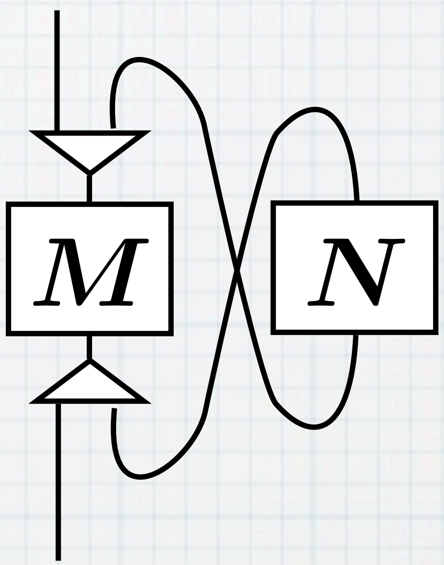

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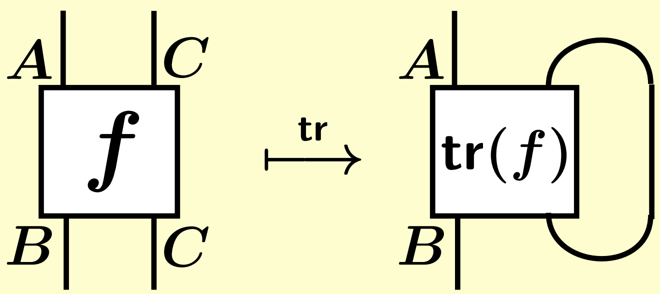

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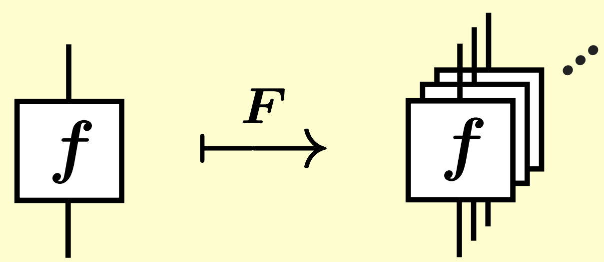

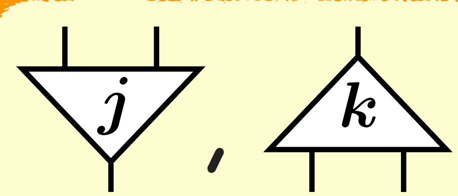

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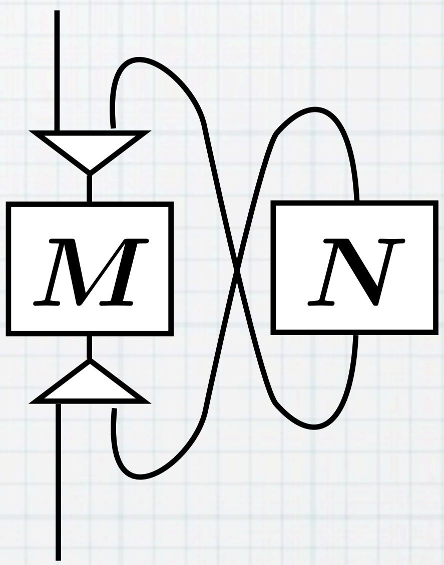

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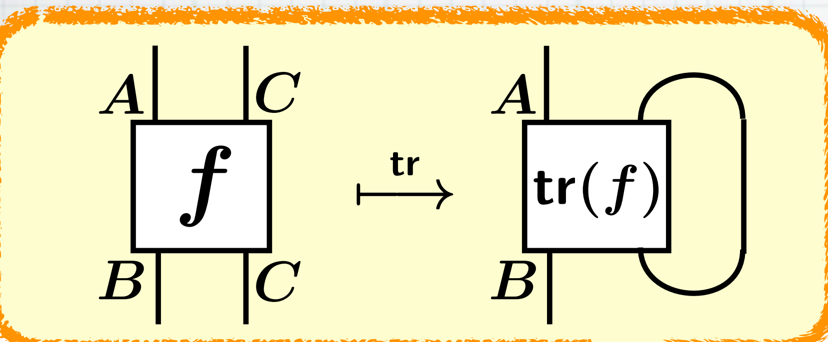

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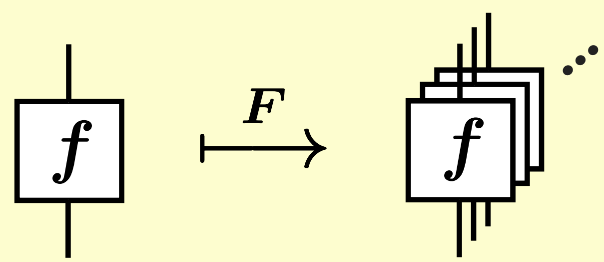

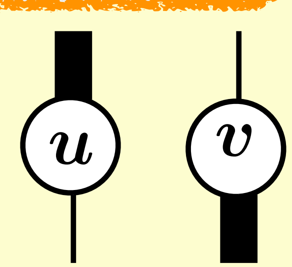

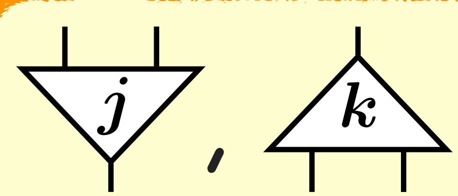

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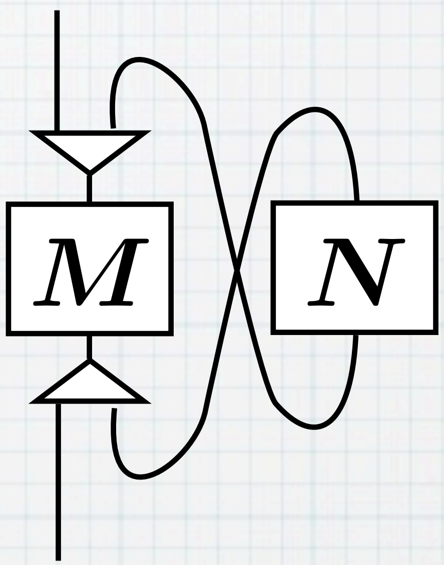

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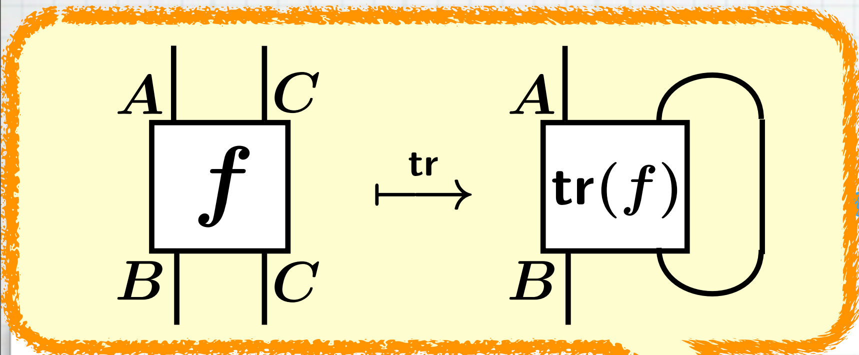

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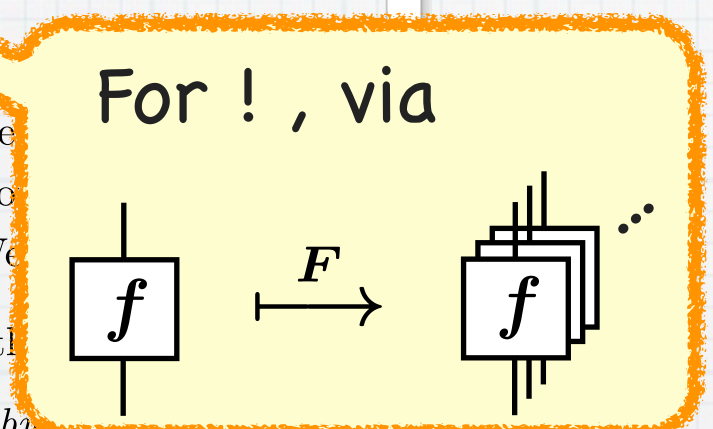

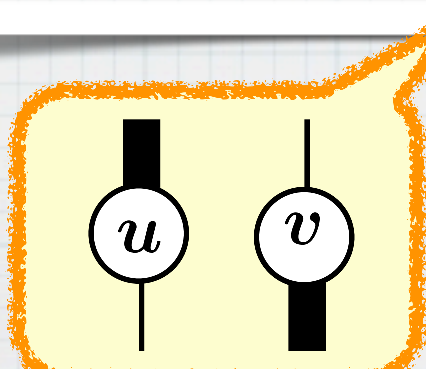

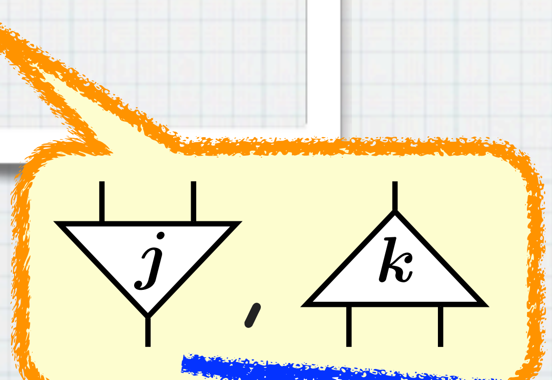

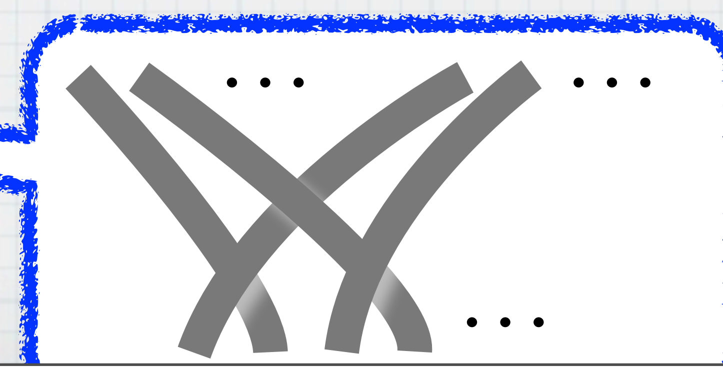

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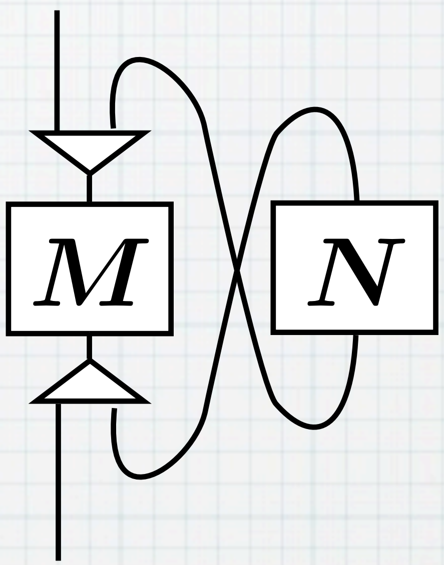

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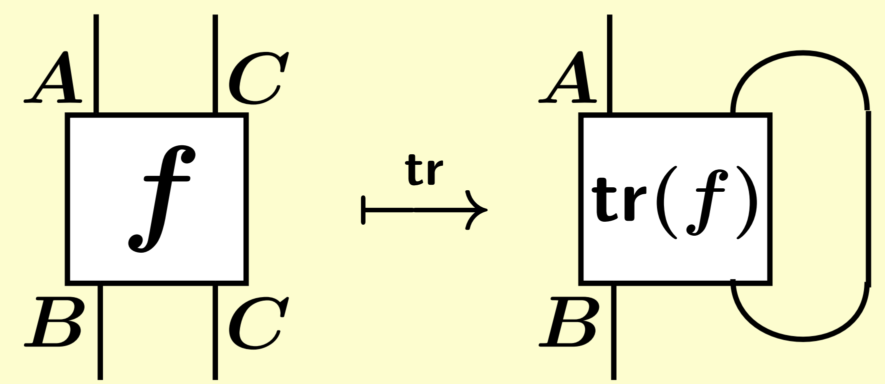
The Categorical GoI Workflow

Traced monoidal category $\mathbb{C}$

+ other constructs ➜ "GoI situation" [AHS02]



Categorical GoI [AHS02]

Linear combinatory algebra

* Applicative str. + combinators
* Model of untyped calculus

Realizability

Linear category

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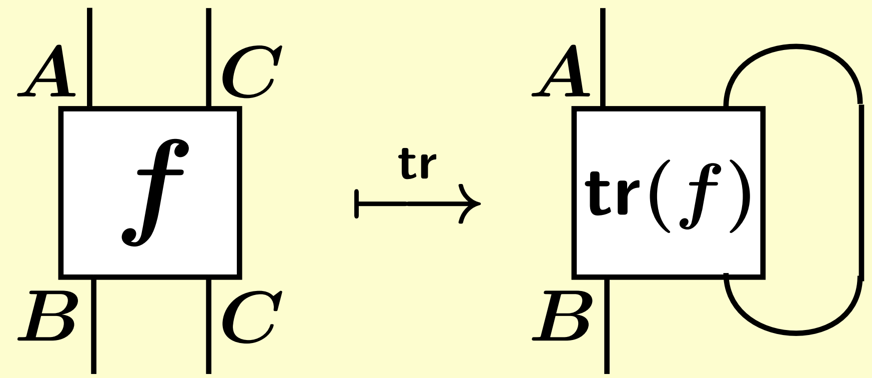

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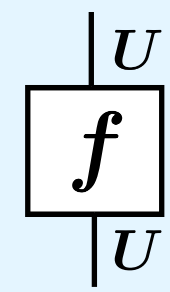

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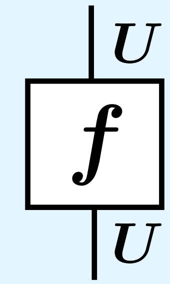
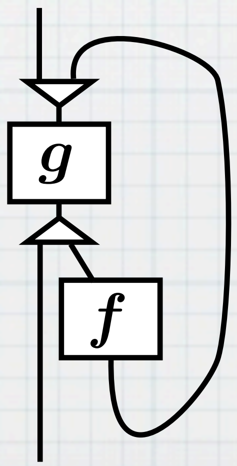
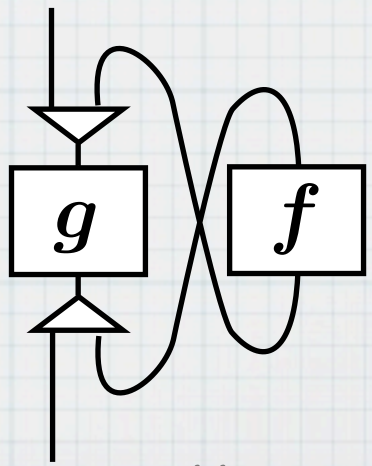


$\in \mathbb{C}(\boldsymbol{U}, \boldsymbol{U})$

- $\boldsymbol{g} \cdot \boldsymbol{f}$

$$:= \mathbf{tr}\Big((\boldsymbol{U} \otimes \boldsymbol{f}) \circ \boldsymbol{k} \circ \boldsymbol{g} \circ \boldsymbol{j} \Big)$$

$=$ 

$=$ 

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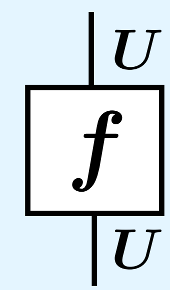
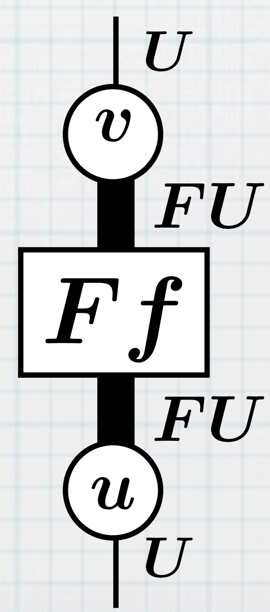


$\in \mathbb{C}(U, U)$

- $!f \;:=\; u \circ Ff \circ v$

$=$ 

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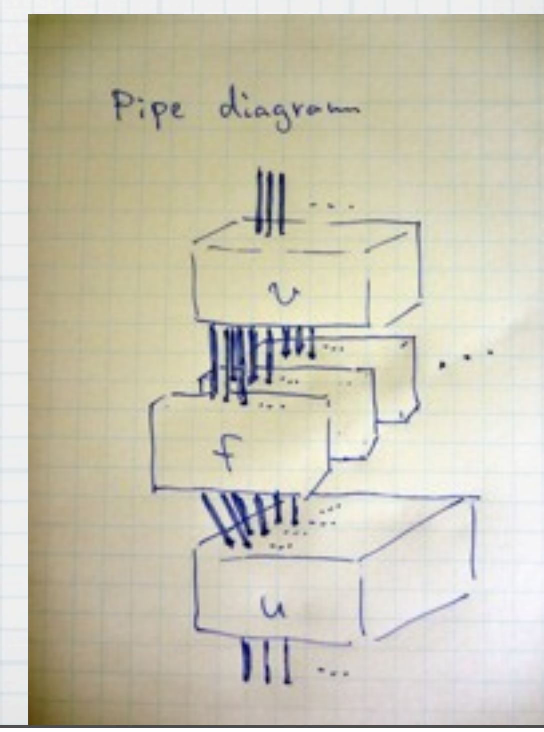

Categorical GoI: Constr. of an LCA

* Combinator $Bxyz = x(yz)$

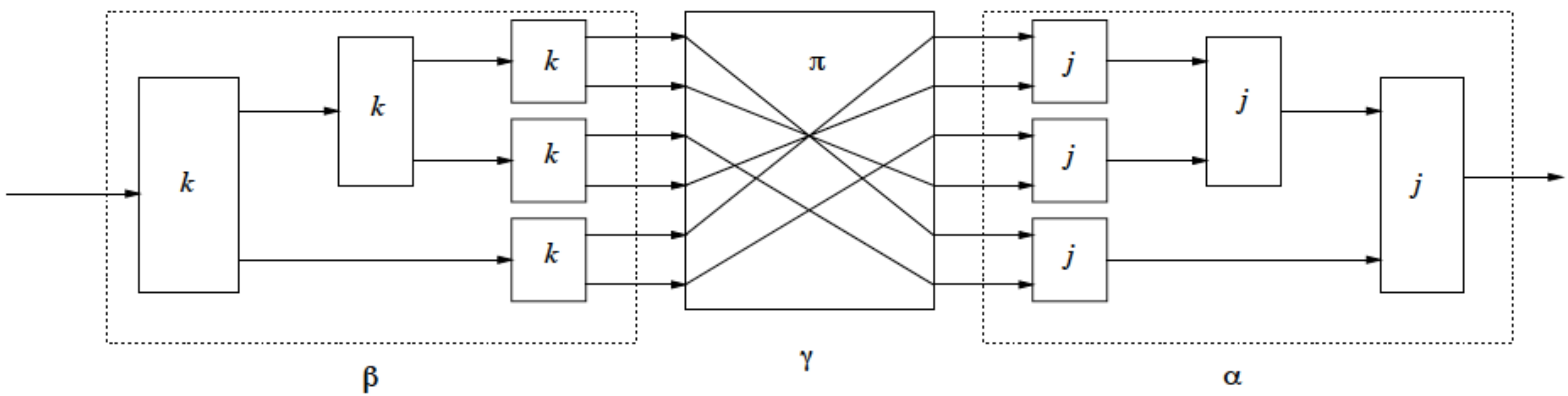

Figure 7: Composition Combinator B

from [AHS02]

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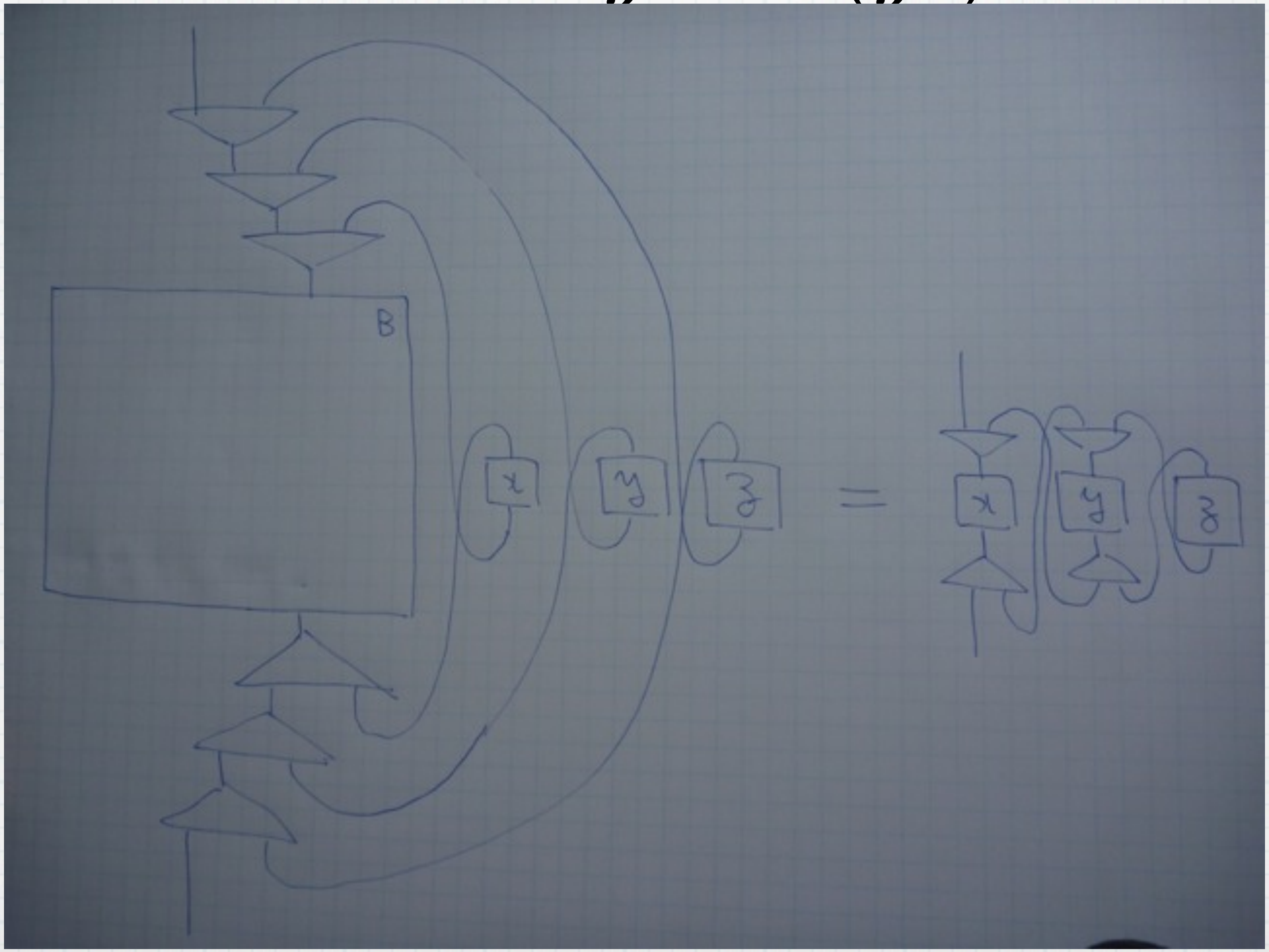

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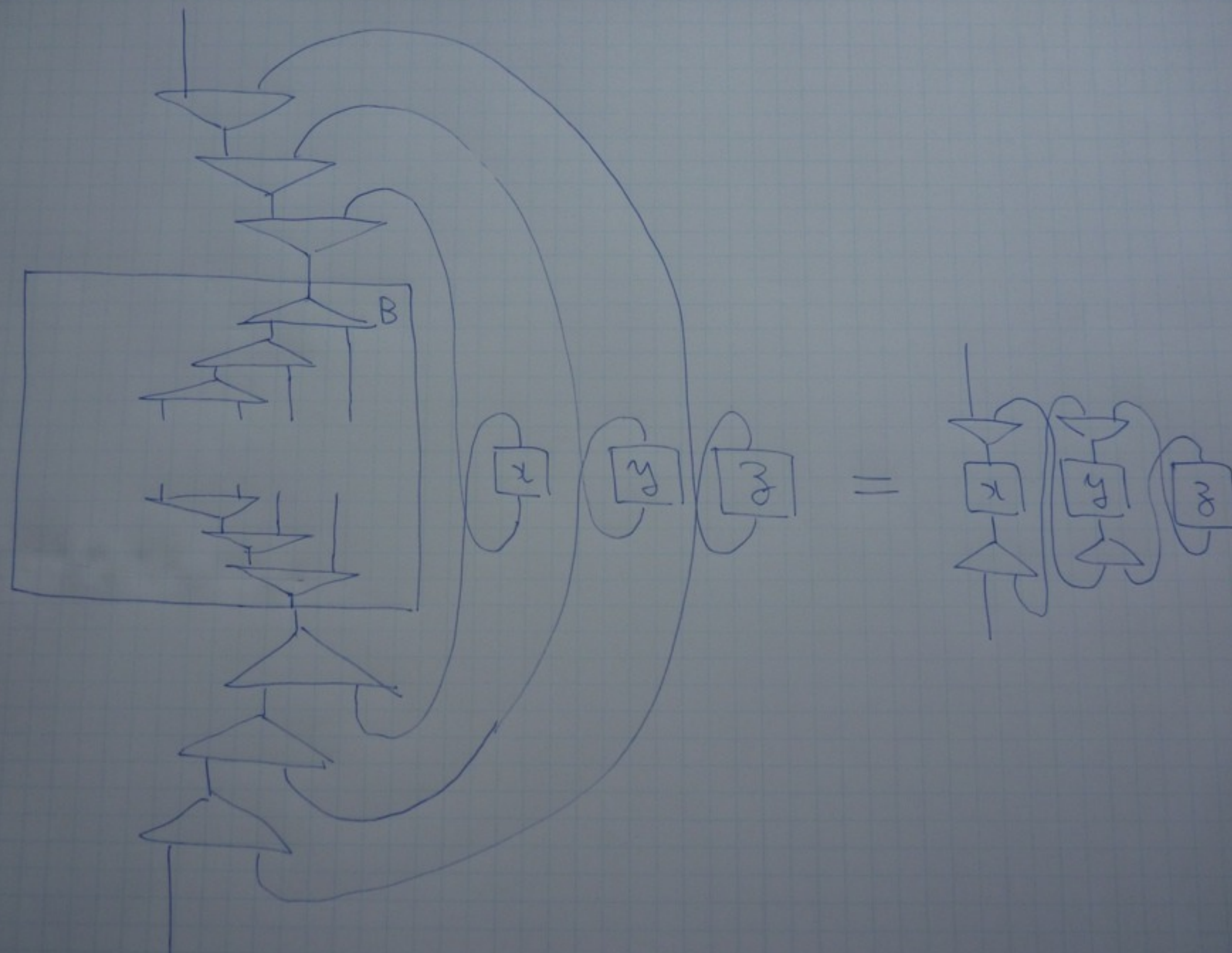
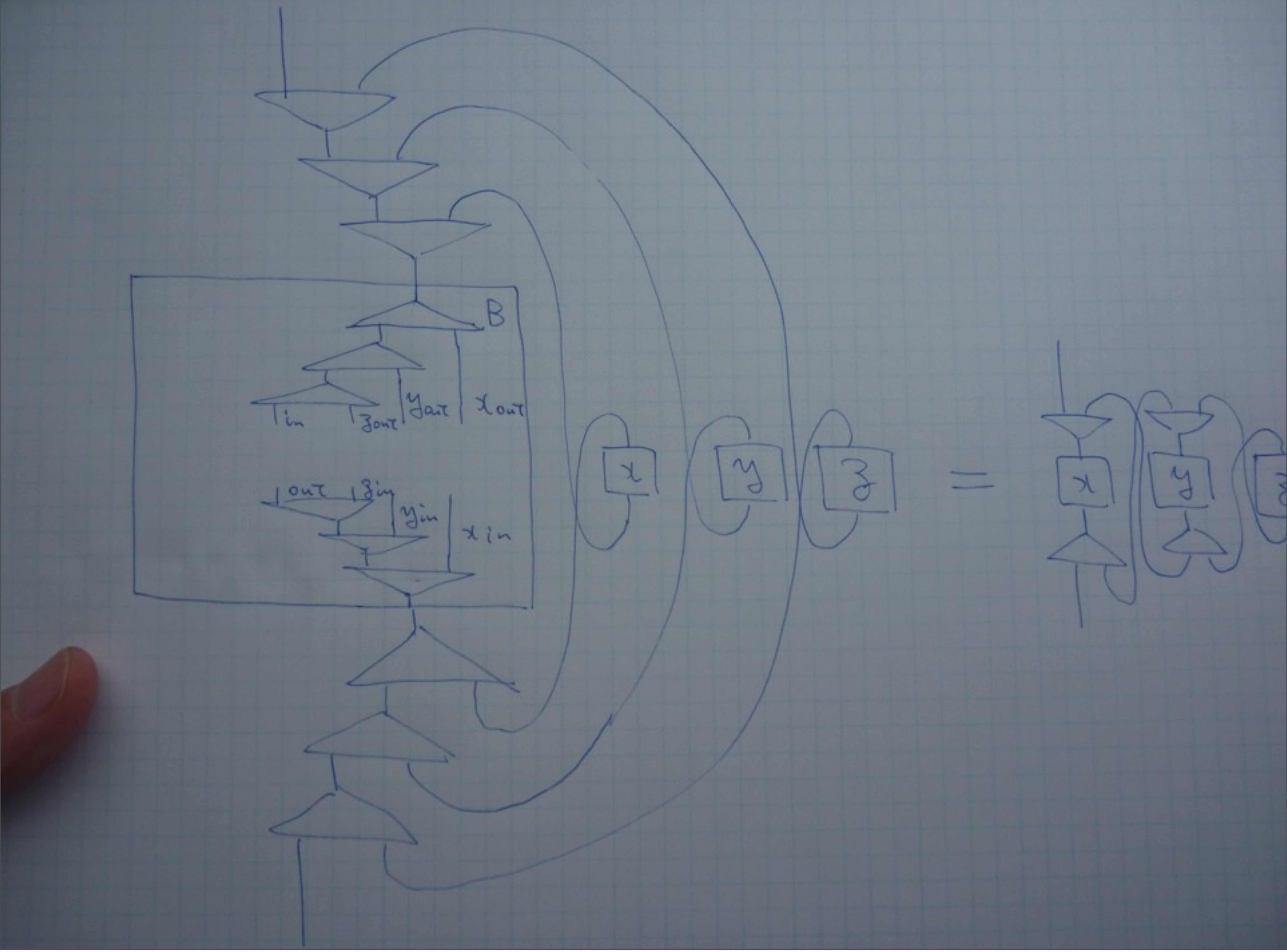
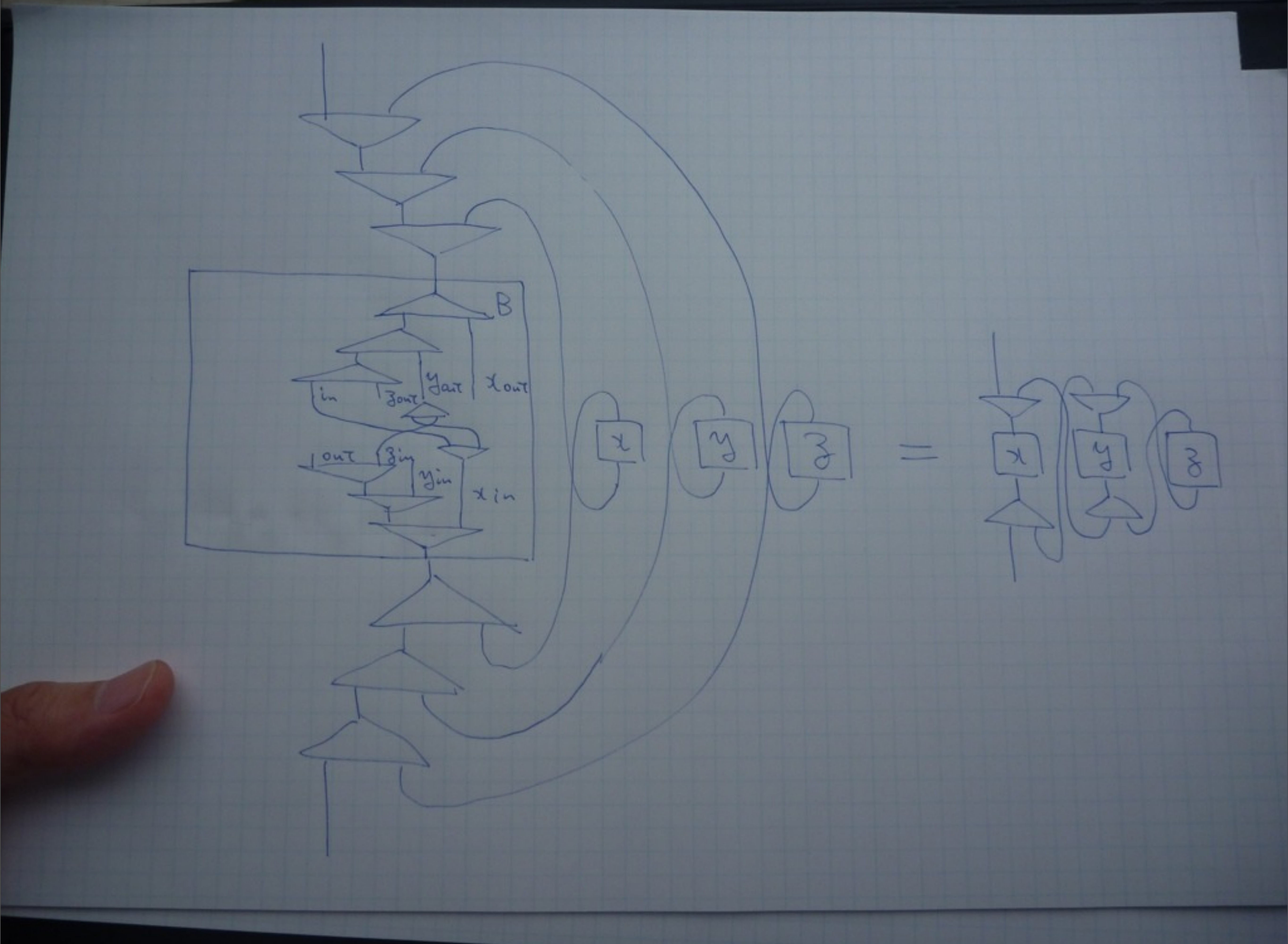
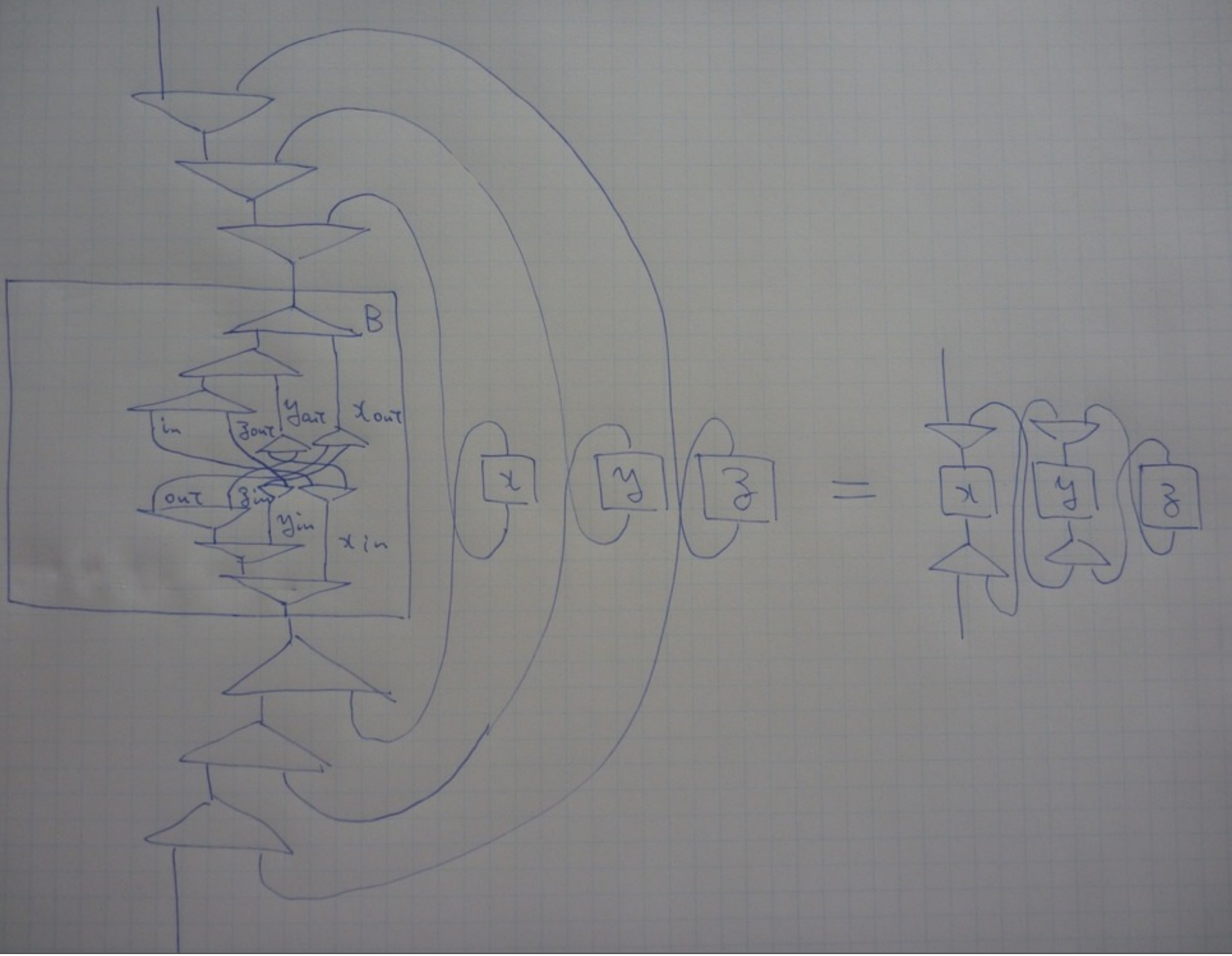

B
x
y
z
=
x
y
z



B
in
z out
y out
x out
out
z in
y in
x in
x
y
z
=
x
y


B
in
zout
yout
xout
out
zin
yin
xin
x
y
z
=
x
y
z



B
in
z out
y out
x out
out
z in
y in
x in
x
y
z
=
x
y
z

Categorical GoI: Constr. of an LCA

* Combinator $Bxyz = x(yz)$

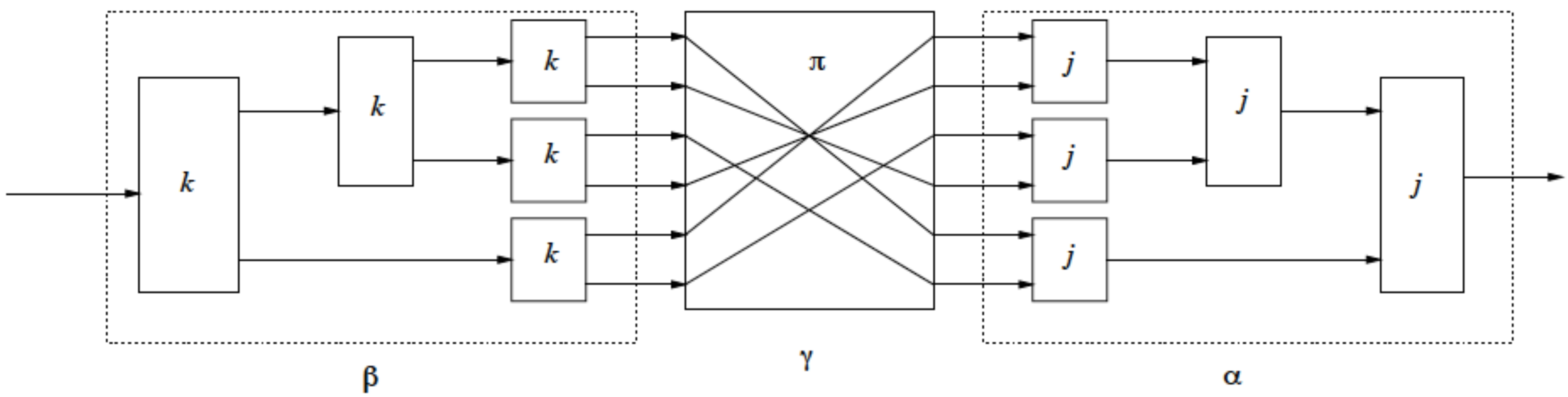


Figure 7: Composition Combinator B

from [AHS02]

Hasuo (Tokyo)

Categorical GoI: Constr. of an LCA

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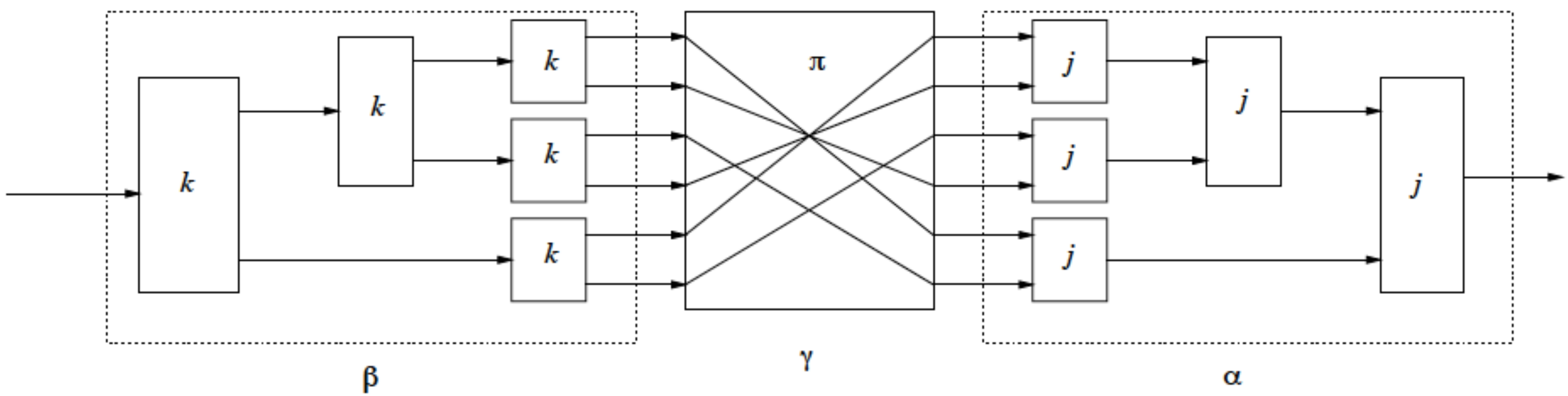

Figure 7: Composition Combinator B

from [AHS02]

Nice dynamic interpretation of (linear) computation!!

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Summary: Categorical GoI

Defn. (GoI situation [AHS02])
A *GoI situation* is a triple $(\mathbb{C}, \boldsymbol{F}, \boldsymbol{U})$ where

- $\mathbb{C} = (\mathbb{C}, \otimes, \boldsymbol{I})$ is a traced symmetric monoidal category (TSMC);
- $\boldsymbol{F} : \mathbb{C} \to \mathbb{C}$ is a traced symmetric monoidal functor, equipped with the following retractions (which are monoidal natural transformations).

$$
\begin{array}{rcl}
\boldsymbol{e} : \boldsymbol{FF} \lhd \boldsymbol{F} : \boldsymbol{e'} & \quad & \text{Comultiplication} \\
\boldsymbol{d} : \mathbf{id} \lhd \boldsymbol{F} : \boldsymbol{d'} & & \text{Dereliction} \\
\boldsymbol{c} : \boldsymbol{F} \otimes \boldsymbol{F} \lhd \boldsymbol{F} : \boldsymbol{c'} & & \text{Contraction} \\
\boldsymbol{w} : \boldsymbol{K_I} \lhd \boldsymbol{F} : \boldsymbol{w'} & & \text{Weakening}
\end{array}
$$

  Here $\boldsymbol{K_I}$ is the constant functor into the monoidal unit $\boldsymbol{I}$;
- $\boldsymbol{U} \in \mathbb{C}$ is an object (called *reflexive object*), equipped with the following retractions.

$$
\begin{array}{c}
\boldsymbol{j} : \boldsymbol{U} \otimes \boldsymbol{U} \lhd \boldsymbol{U} : \boldsymbol{k} \\
\boldsymbol{I} \lhd \boldsymbol{U} \\
\boldsymbol{u} : \boldsymbol{FU} \lhd \boldsymbol{U} : \boldsymbol{v}
\end{array}
$$

Thm. ([AHS02])
Given a GoI situation $(\mathbb{C}, \boldsymbol{F}, \boldsymbol{U})$, the homset

$$\mathbb{C}(\boldsymbol{U}, \boldsymbol{U})$$

carries a canonical LCA structure.

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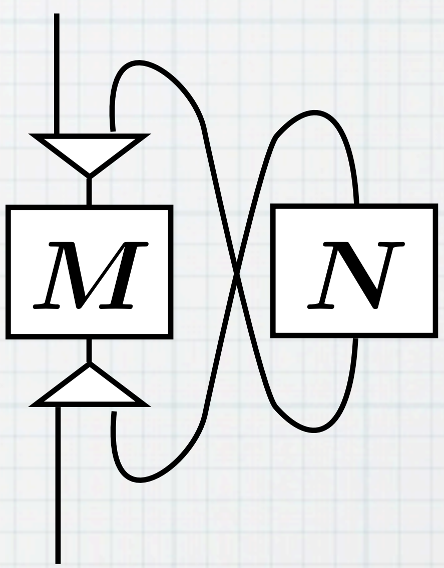
Why Categorical Generalization?: Examples Other Than Pfn



- Strategy: find a TSMC!
- "Wave-style" examples
  - $\otimes$ is Cartesian product(-like)
    - in which case,

      trace $\approx$ fixed point operator [Hasegawa/Hyland]
  - An example: $\big(\,(\omega\text{-}\mathbf{Cpo}, \times, 1),\ (\_)^{\mathbb{N}},\ A^{\mathbb{N}}\,\big)$
  - (... less of a dynamic flavor)

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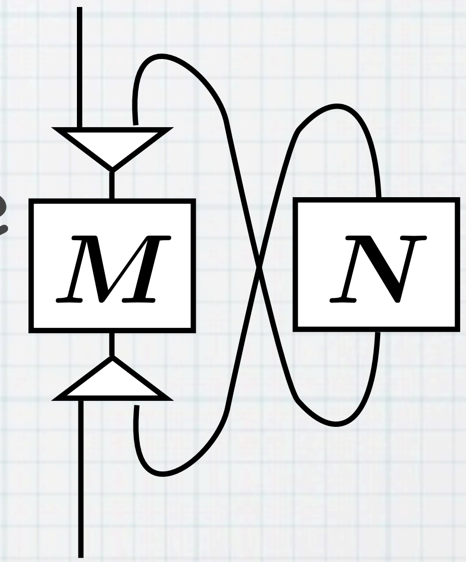
Why Categorical Generalization?: Examples Other Than $\mathbf{Pfn}$



- "Particle-style" examples
  - Obj. $X \in C$ is set-like; $\otimes$ is coproduct-like
    - The GoI animation is valid
  - Examples:
    - Partial functions $\big(\, (\mathbf{Pfn}, +, \mathbf{0}),\ \mathbb{N} \cdot \_\,,\ \mathbb{N} \,\big)$
    - Non-det. functions (i.e. relations) $\big(\, (\mathbf{Rel}, +, \mathbf{0}),\ \mathbb{N} \cdot \_\,,\ \mathbb{N} \,\big)$
    - Probabilistic functions ("discrete stochastic relations") $\big(\, (\mathbf{DSRel}, +, \mathbf{0}),\ \mathbb{N} \cdot \_\,,\ \mathbb{N} \,\big)$

Why Categorical Generalization?: Examples Other Than Pfn

- **Pfn** (partial functions)

$$\dfrac{\dfrac{X \to Y \ \text{ in } \mathbf{Pfn}}{X \rightharpoonup Y, \text{ partial function}}}{X \to \mathcal{L}Y \ \text{ in } \mathbf{Sets}} \qquad \text{where } \mathcal{L}Y = \{\perp\} + Y$$

- **Rel** (relations)

$$\dfrac{\dfrac{X \to Y \ \text{ in } \mathbf{Rel}}{R \subseteq X \times Y, \text{ relation}}}{X \to \mathcal{P}Y \ \text{ in } \mathbf{Sets}} \qquad \text{where } \mathcal{P} \text{ is the powerset monad}$$

- **DSRel**

$$\dfrac{X \to Y \ \text{ in } \mathbf{DSRel}}{X \to \mathcal{D}Y \ \text{ in } \mathbf{Sets}}$$

$$\text{where } \mathcal{D}Y = \{d : Y \to [0,1] \mid \sum_{y} d(y) \leq 1\}$$

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Why Catego... Examples Other Than Pfn

Categories of sets and
(functions with different branching/partiality)

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Why Catego...
Examples

Categories of sets and
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* **Pfn** (partial functions)

(Potential) non-termination

$$\frac{\dfrac{X \to Y \text{ in } \mathbf{Pfn}}{X \rightharpoonup Y, \text{ partial function}}}{X \to \mathcal{L}Y \text{ in } \mathbf{Sets}} \quad \text{where } \mathcal{L}Y = \{\perp\} + Y$$

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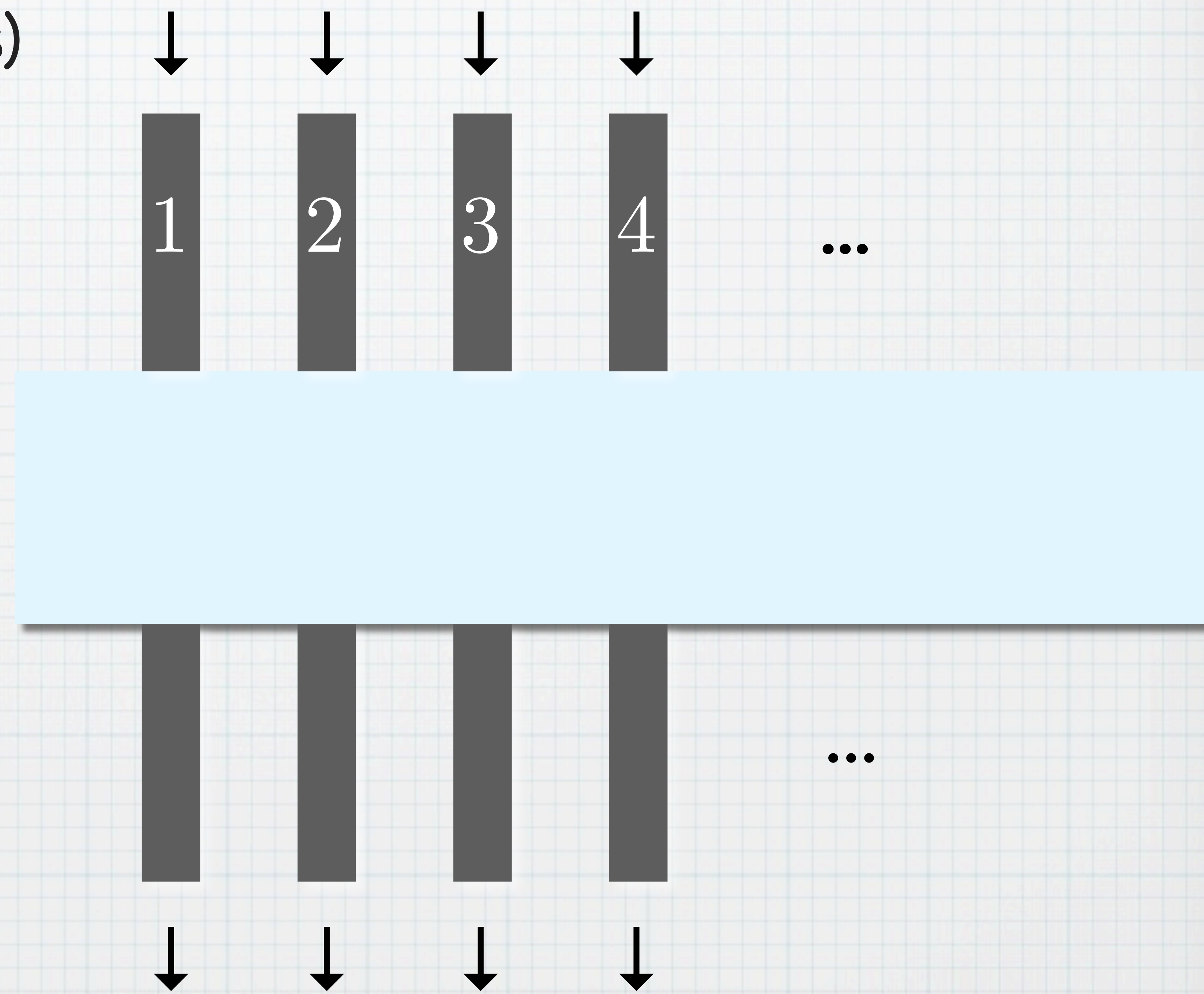
Different Branching in The GoI Animation

- **Pfn** (partial functions)
  - Pipe can be stuck
- **Rel** (relations)
  - Pipe can branch
- **DSRel**
  - Pipe can branch probabilistically



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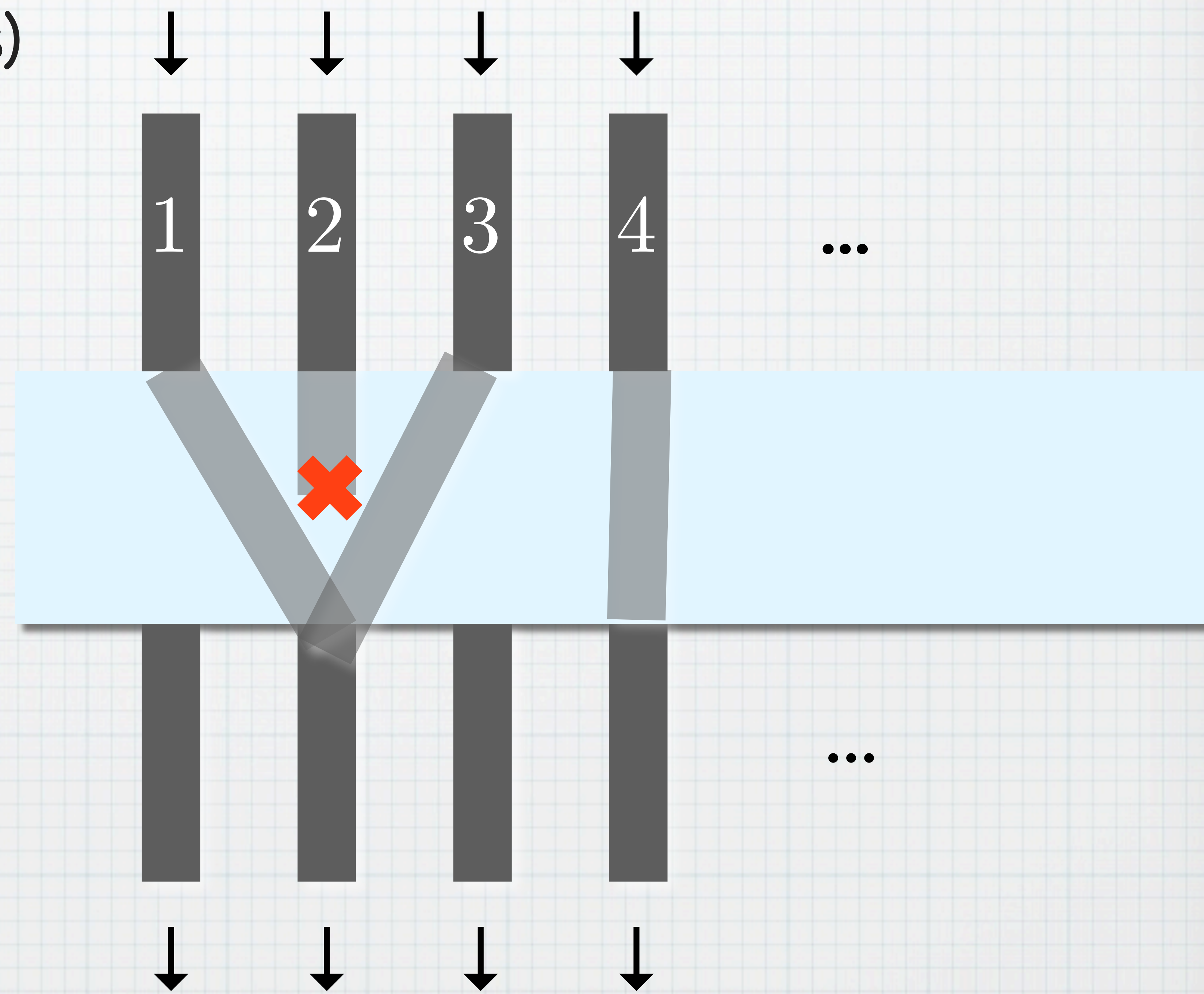

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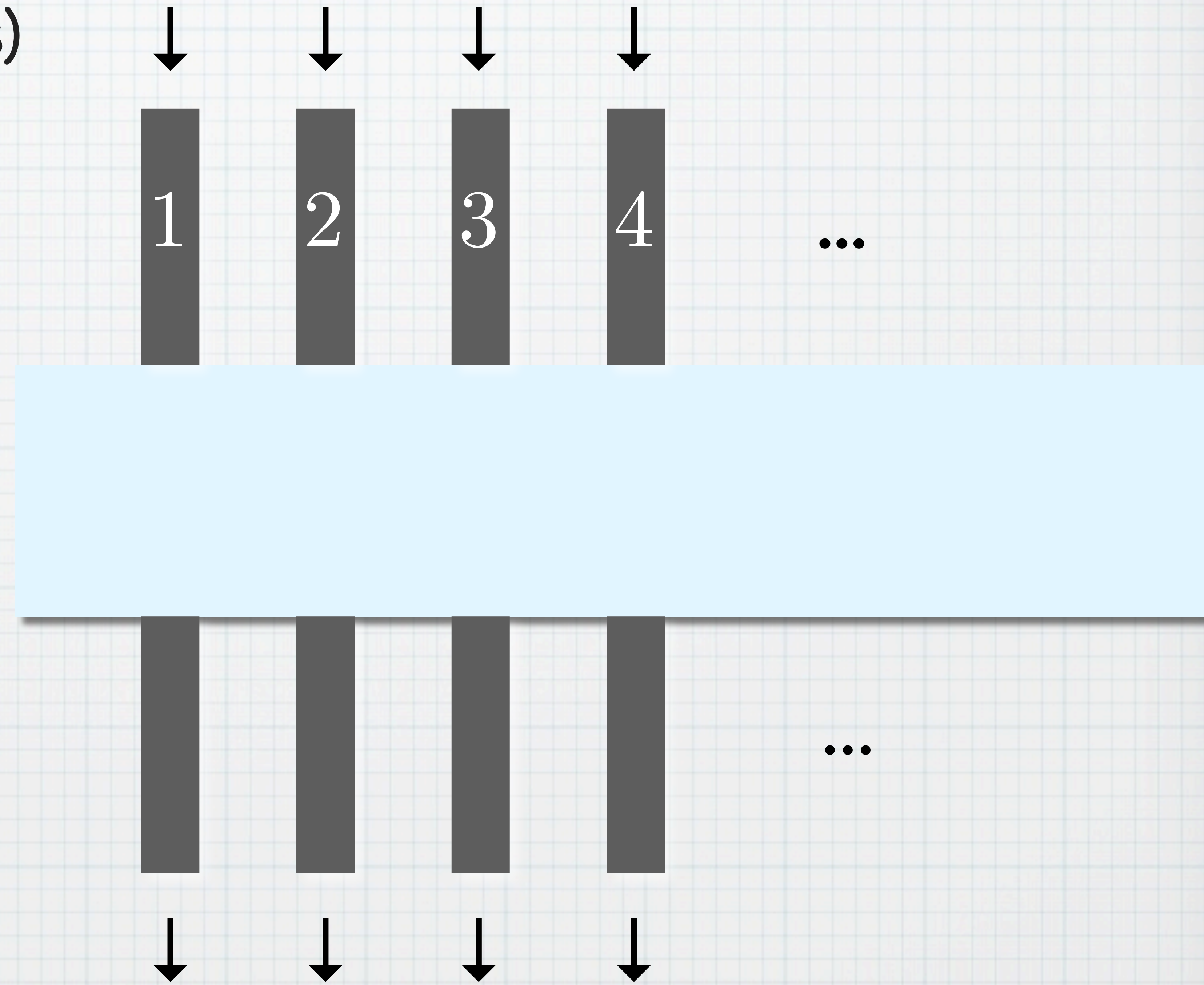

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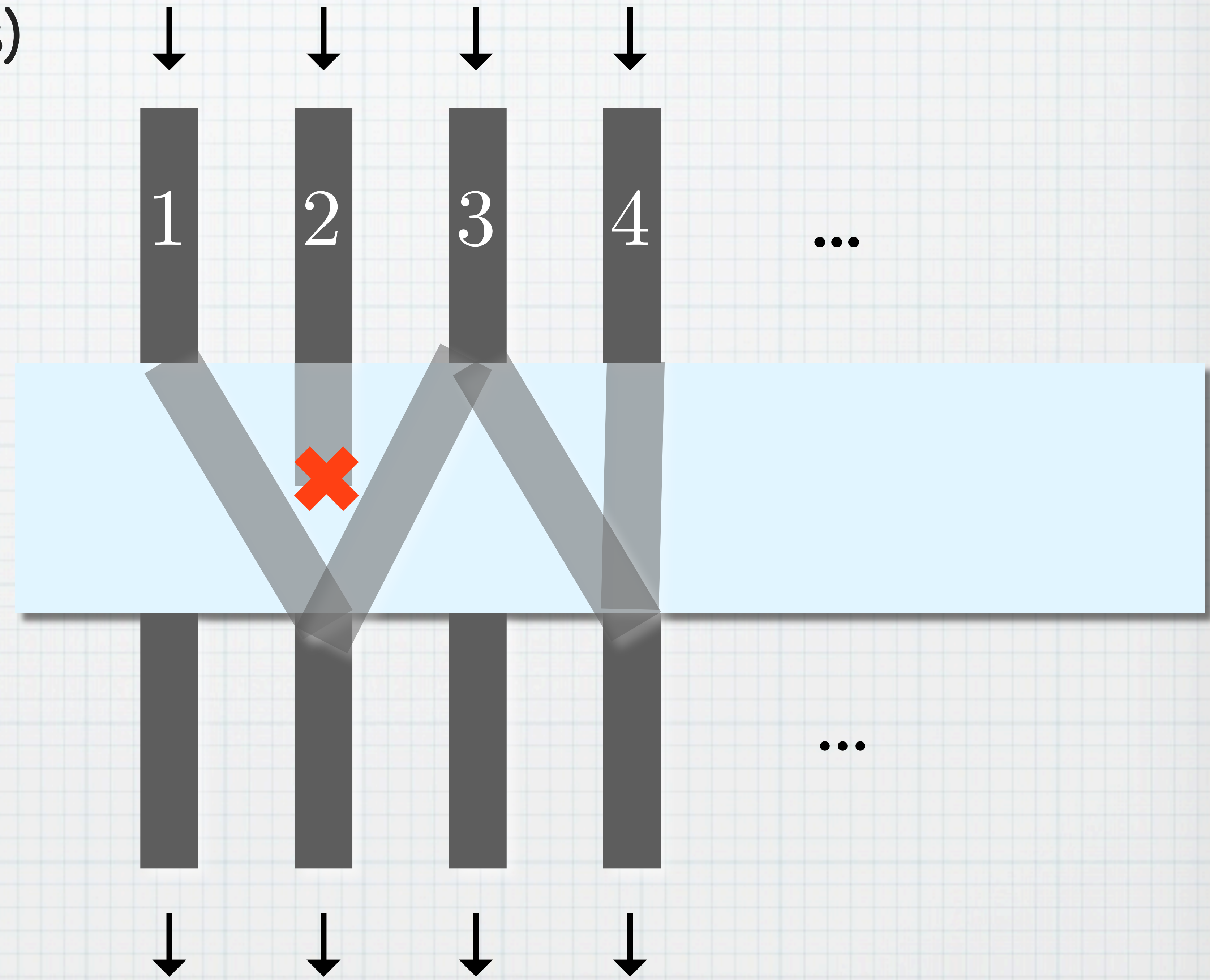

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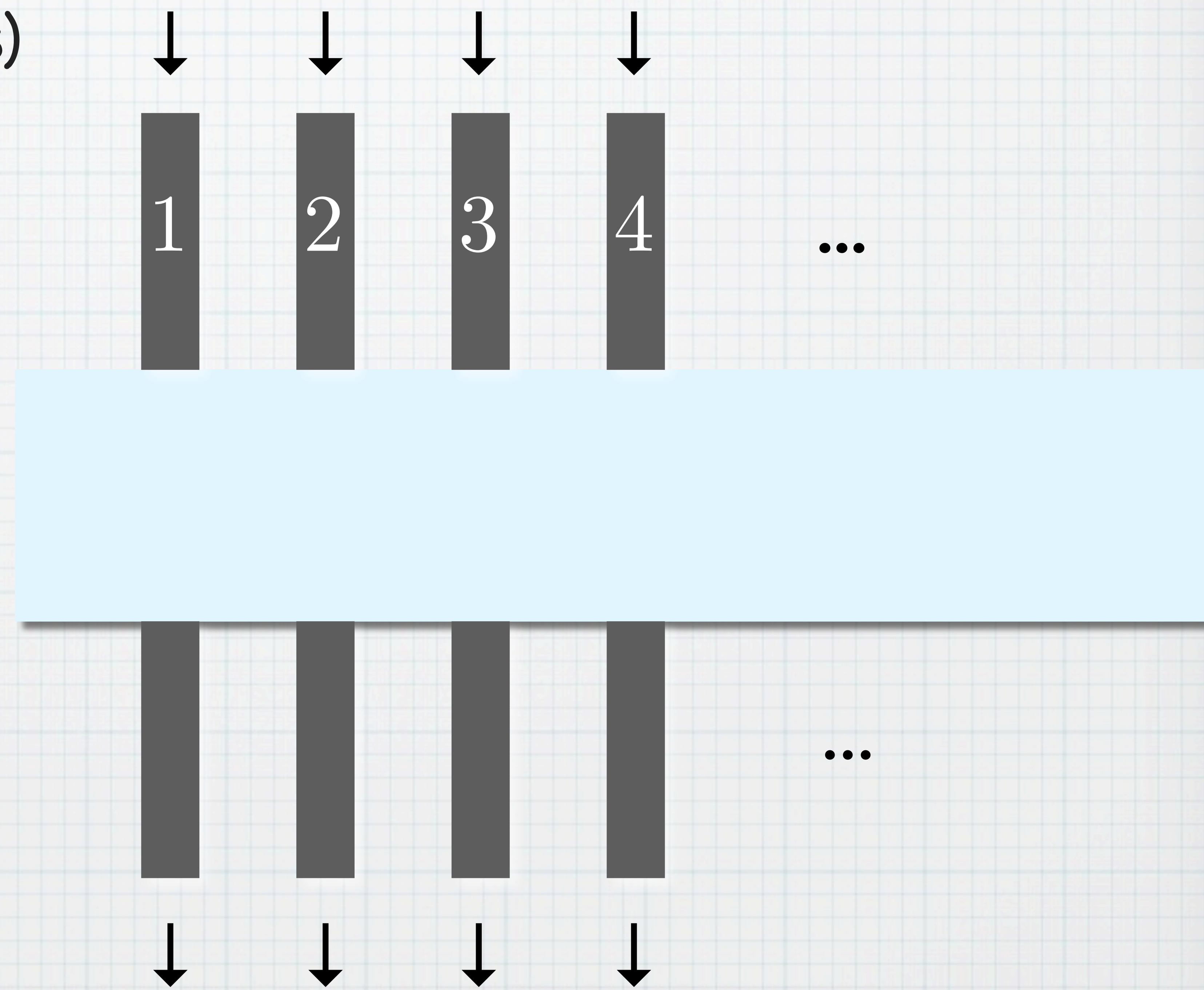

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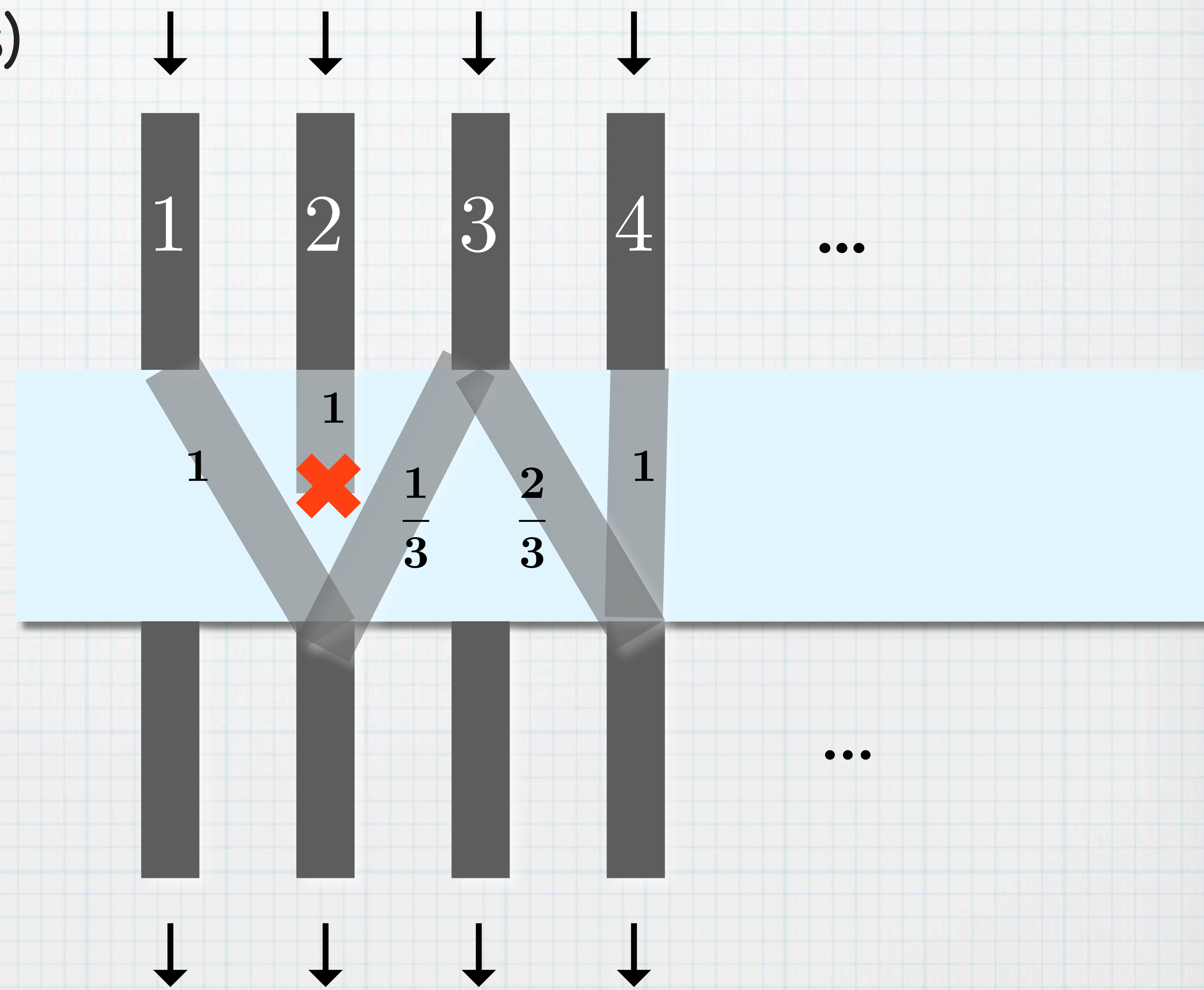

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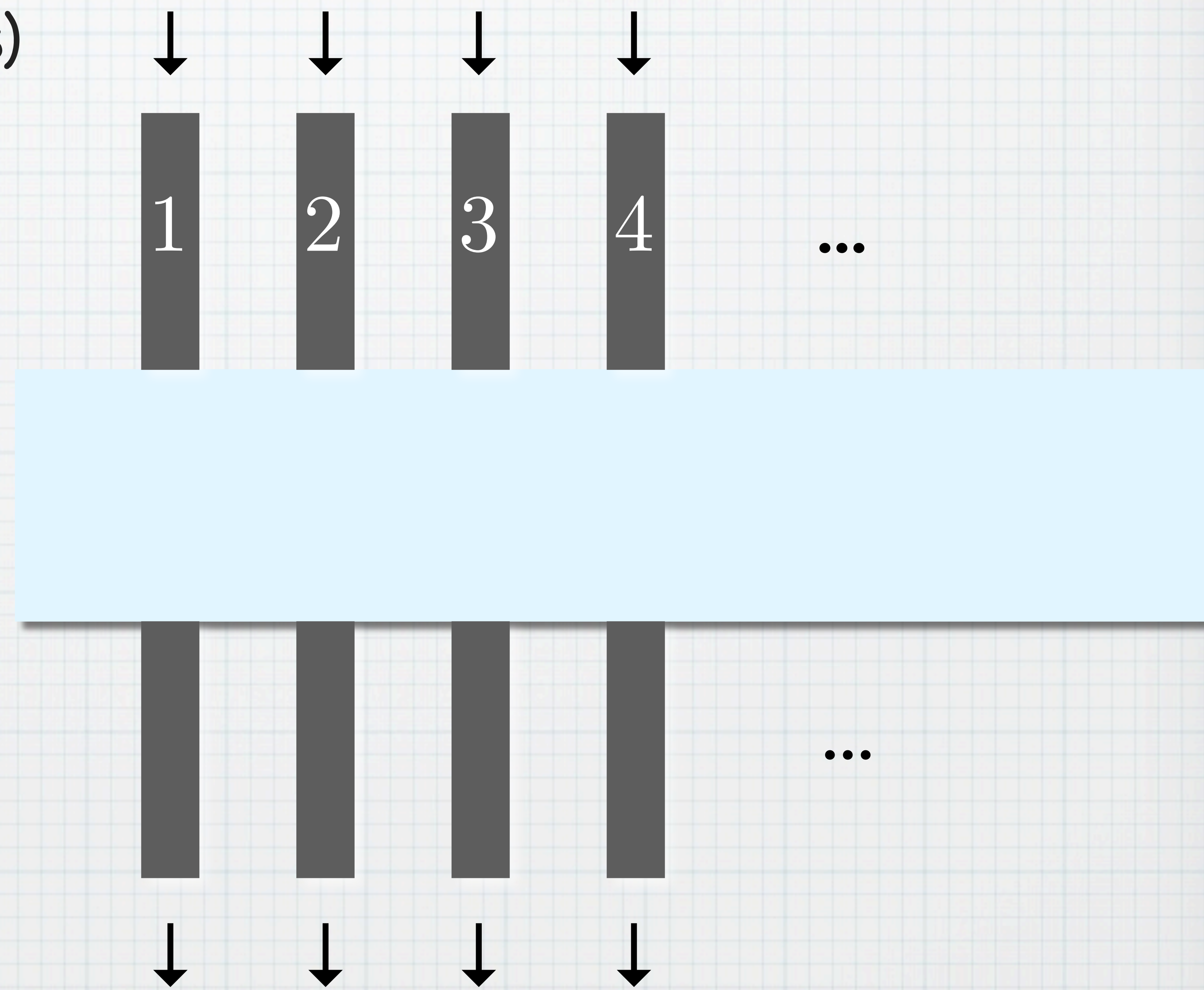

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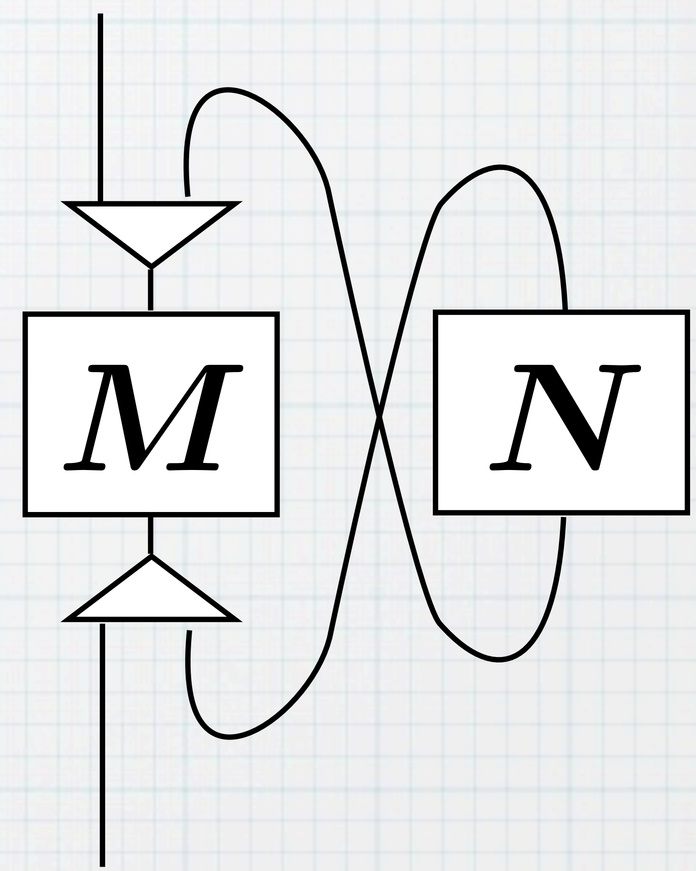

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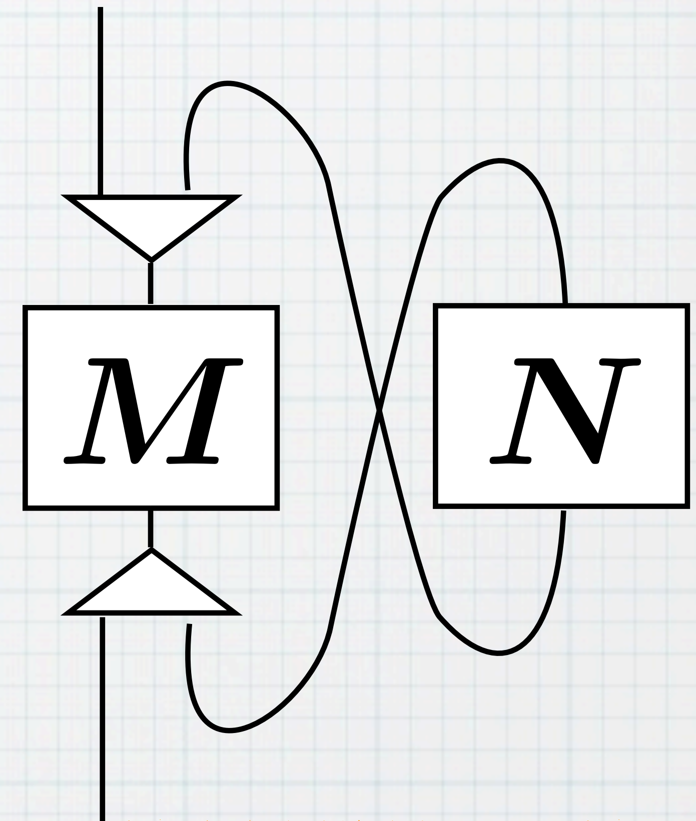

Essential to have **sub**distribution, for infinite loops

The Coauthor

* Naohiko Hoshino
  * DSc
    * Kyoto U. (JP), 2011
    * Supervisor: Masahito "Hassei" Hasegawa
  * Assist. Prof., RIMS, Kyoto U. (2011-)

A Coalgebraic View

* Theory of coalgebra =
  Categorical theory of state-based dynamic systems (LTS, automaton, Markov chain, ...)

* In my thesis (2008):
  * Coalgebras in a Kleisli category $Kl(B)$

$$\frac{X \to Y \ \text{ in } \mathcal{K}\ell(B)}{X \to BY \ \text{ in } \mathbf{Sets}}$$

  * ➜ Generic theory of "trace semantics"

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Why Catego
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Why Catego
Examples

$Kl(B)$ for different branching monads B

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Probabilistic branching

Branching Monad: Source of Particle-Style GoI Situations

Thm. ([Jacobs,CMCS10])
Given a "branching monad" B on **Sets**, the monoidal category

$$(\mathcal{K}\ell(B), +, \mathbf{0})$$

is

- a *unique decomposition category* [Haghverdi,PhD00], hence is
- a traced symmetric monoidal category.

Cor.
$\big((\mathcal{K}\ell(B), +, \mathbf{0}),\ \mathbb{N}\cdot\_\,,\ \mathbb{N} \big)$ is a GoI situation.

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Monads in [Hasuo,Jacobs&Sokolova07]

* Kl(B) is $\mathbf{Cpo}_\perp$-enriched
* like $\mathcal{L}$, $\mathcal{P}$, $\mathcal{D}$

Hasuo (Tokyo)

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- Kl(B) is $\mathbf{Cpo}_{\perp}$-enriched
- like $\mathcal{L}$, $\mathcal{P}$, $\mathcal{D}$

Particle-style: trace via the execution formula

$$\mathbf{tr}(f) = f_{XY} \sqcup \left(\coprod_{n\in\mathbb{N}} f_{ZY} \circ (f_{ZZ})^n \circ f_{XZ} \right)$$

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The Categorical GoI Workflow

Traced monoidal category $\mathbb{C}$

+ other constructs ➜ "GoI situation" [AHS02]

Categorical GoI [AHS02]

Linear combinatory algebra

Realizability

Linear category

Hasuo (Tokyo)

The Categorical GoI Workflow

Branching monad B

Coalgebraic trace semantics

Traced monoidal category $\mathbb{C}$

+ other constructs ➔ "GoI situation" [AHS02]

Categorical GoI [AHS02]

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Model of fancy language

Hasuo (Tokyo)

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↓ Categorical GoI [AHS02]

Linear combinatory algebra

↓ Realizability

Linear category

Fancy monad

↓

Fancy TSMC

↓

Fancy LCA

↓

Model of fancy language

What is Fancy, Nowadays?

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What is Fancy, Nowadays?

* Biology?

Hasuo (Tokyo)

What is Fancy, Nowadays?

* Biology?

What is Fancy, Nowadays?

- Biology?
- Hybrid systems?
  - Both discrete and continuous data, typically in **cyber-physical systems** (CPS)
  - ➜ Our approach via **non-standard analysis** [Suenaga&Hasuo,ICALP11]

What is Fancy, Nowadays?

* Biology?
* Hybrid systems?
  * Both discrete and continuous data, typically in **cyber-physical systems** (CPS)
  * ➜ Our approach via **non-standard analysis** [Suenaga&Hasuo,ICALP11]
* Quantum?
  * Yes this worked!

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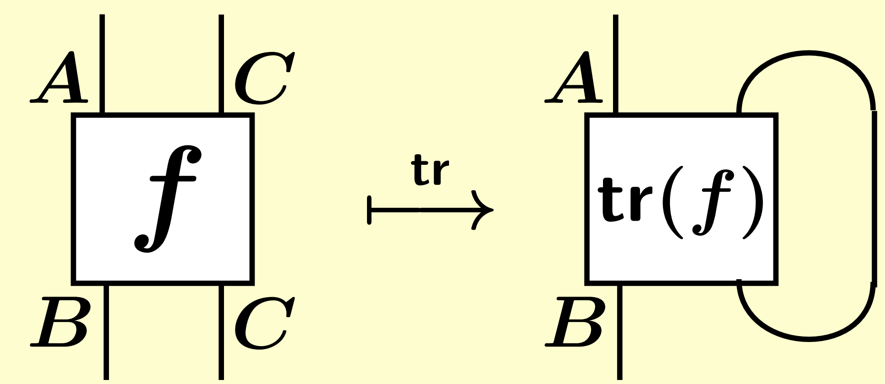
The Categorical GoI Workflow

Branching monad B

Coalgebraic trace semantics

Traced monoidal category $\mathbb{C}$

+ other constructs ➔ "GoI situation" [AHS02]



Categorical GoI [AHS02]

Linear combinatory algebra

* Applicative str. + combinators
* Model of untyped calculus

Realizability

Linear category

Model of typed calculus

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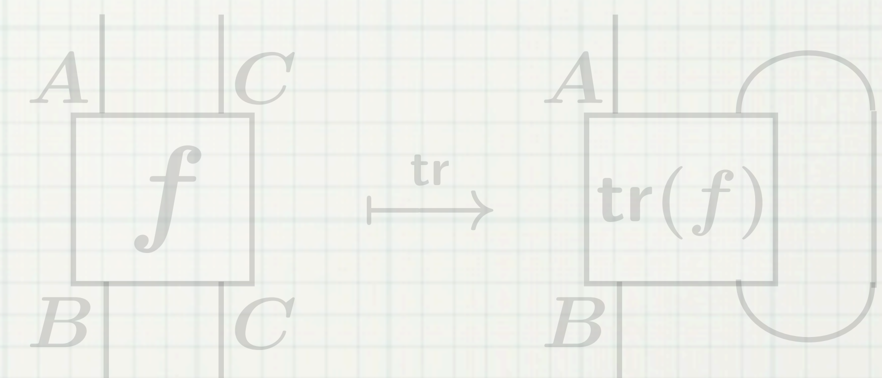

Categorical GoI [AHS02]

Linear combinatory algebra

- Applicative str. + combinators
- Model of untyped calculus

Realizability

Linear category

Model of typed calculus

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Part 2

Realizability:

from Untyped to Typed

Realizability

* Dates back to Kleene
* Cf. the Brouwer-Heyting-Kolmogorov (BHK) interpretation
  * A p'f of $A \wedge B$ is a pair: (p'f of A, p'f of B)
  * A p'f of $A \to B$ is a function carrying (p'f of A) to (p'f of B)
  * Proof = "realizer"

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Realizability

* Our technical view on realizability: a construction
  * from a combinatory algebra,
  * of a categorical model of a typed calculus
* Here: construct a linear category from an LCA

* References:
  * [AL05] S. Abramsky and M. Lenisa, "Linear realizability and full completeness for typed lambda-calculi," APAA 2005.
  * [Hos07] N. Hoshino, "Linear realizability," CSL 2007.

Realizability

* Either by **ω-sets** (intuitive) or by **PERs** (tech. convenient)

Defn.
Given an LCA $\boldsymbol{A}$, an $\boldsymbol{\omega}$-*set* is a pair

$$\left(\boldsymbol{S}, \quad \boldsymbol{r} : \boldsymbol{S} \to \mathcal{P}_{+}(\boldsymbol{A}) \right)$$

where

- $\boldsymbol{S}$ is a set;
- for each $\boldsymbol{x} \in \boldsymbol{S}$, the nonempty subset $\boldsymbol{r}(\boldsymbol{x}) \subseteq \boldsymbol{A}$ is the set of *realizers*.

Hasuo (Tokyo)

Realizability

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Could as well be a **partial combinatory algebra**. Its examples:

* $\mathbb{N}$ with n·m = comp(n,m)
* { closed λ-terms }

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* { closed λ-terms }

a ∈ r(x) :

* "realizes" x, or
* "witnesses existence of" x

Realizability

Defn.
A *partial equivalence relation (PER)* $\boldsymbol{X}$ is a transitive and symmetric relation on $\boldsymbol{A}$.

$$
\begin{aligned}
|\boldsymbol{X}| &:= \{\boldsymbol{a} \mid (\boldsymbol{a}, \boldsymbol{a}) \in \boldsymbol{X}\} \\
&= \{\boldsymbol{a} \mid \exists \boldsymbol{b}.\, (\boldsymbol{a}, \boldsymbol{b}) \in \boldsymbol{X}\} \\
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Hasuo (Tokyo)

Realizability

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- Also: ω-set to PER

Hasuo (Tokyo)

$\mathbf{PER}_A$: The Category of PERs

* **Obj.** A PER X on A
* **Arr.** The homset is

$$\mathbf{PER}_A(X,Y) = \left\{ c \in A \;\middle|\; (x,x') \in X \Longrightarrow (cx, cx') \in Y \right\}$$

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Type Constructors in $\mathbf{PER}_A$

Thm. ([AL05])
If A is an affine LCA, then $\mathbf{PER}_A$ is a linear category.
Furthermore, $\mathbf{PER}_A$ has finite products and coproducts.

* Linear category [Benton&Wadler,LICS'96][Bierman,TLCA'95]
  * Categorical model of linear logic/linear λ, with
    * Monoidal closed with $\boxtimes, \mathbf{I}, \multimap$
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$\underline{n} := \lambda fx.f(f \cdots (fx) \cdots)$	Church numeral
$\overline{\mathsf{K}} := \mathsf{KI}$	
$\mathsf{P} := \lambda xyz.zxy$	Paring
$\mathsf{P}_{\mathsf{l}} := \lambda w.w\mathsf{K}$	Left projection
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  - Cf. Combinaroty completeness

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$$\frac{X \in \mathbf{PER}_A}{\overline{\overline{X \subseteq A \times A, \text{ sym., trans.}}}}$$

Hasuo (Tokyo)

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$$X \boxtimes Y \ := \ \left\{ (\mathsf{P}xy, \mathsf{P}x'y') \;\middle|\; (x,x') \in X \wedge (y,y') \in Y \right\}$$

$$X \times Y \ := \ \left\{ \big(\mathsf{P}k_1(\mathsf{P}k_2 u),\ \mathsf{P}k_1'(\mathsf{P}k_2' u')\big) \;\middle|\; (k_1 u, k_1' u') \in X \wedge (k_2 u, k_2' u') \in Y \right\}$$

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CPS-style. k_1, k_2: "access methods"

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Summary: Realizability

Affine LCA A

$a \cdot b, \quad !\,a, \quad \mathsf{B}, \mathsf{C}, \mathsf{I}, \ldots$

$\longmapsto$

Linear category $\mathbf{PER}_A$

* $$\begin{array}{ccc} X & \xrightarrow{[c]} & Y \\ [a] & \longmapsto & [c \cdot a] \end{array}$$ $(a, c \in A)$

* Type constructors via "programming in untyped λ"
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for distinction

suo (Tokyo)

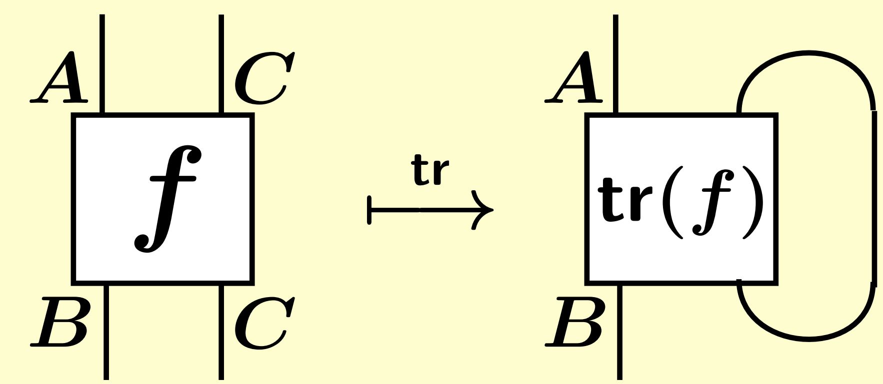
The Categorical GoI Workflow

Branching monad B

Coalgebraic trace semantics

Traced monoidal category $\mathbb{C}$

+ other constructs ➜ "GoI situation" [AHS02]



Categorical GoI [AHS02]

Linear combinatory algebra

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Hasuo (Tokyo)

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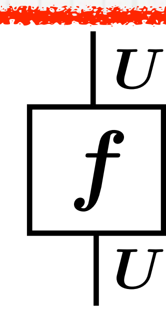
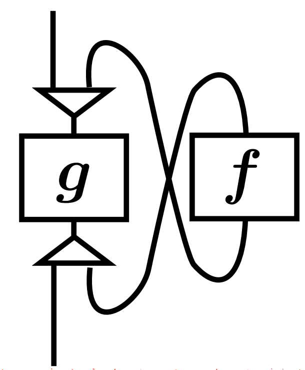


$\in \mathbb{C}(U, U)$

$g \cdot f =$ 

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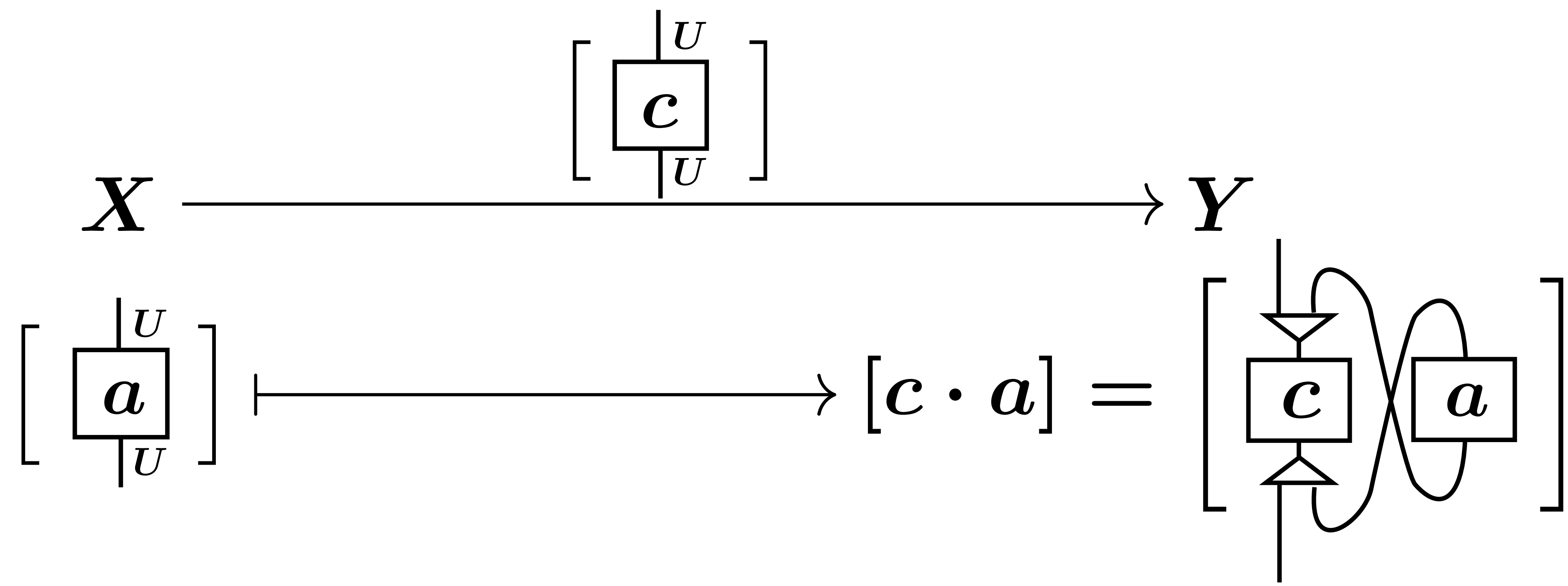

Affine LCA

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Not ⊗,
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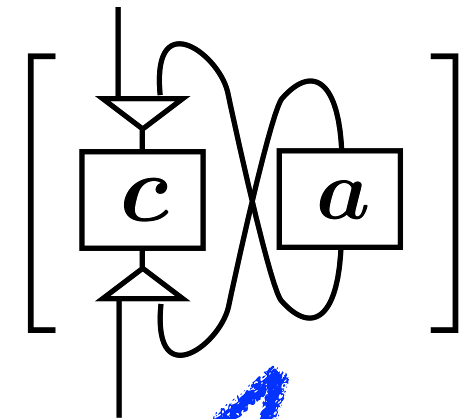
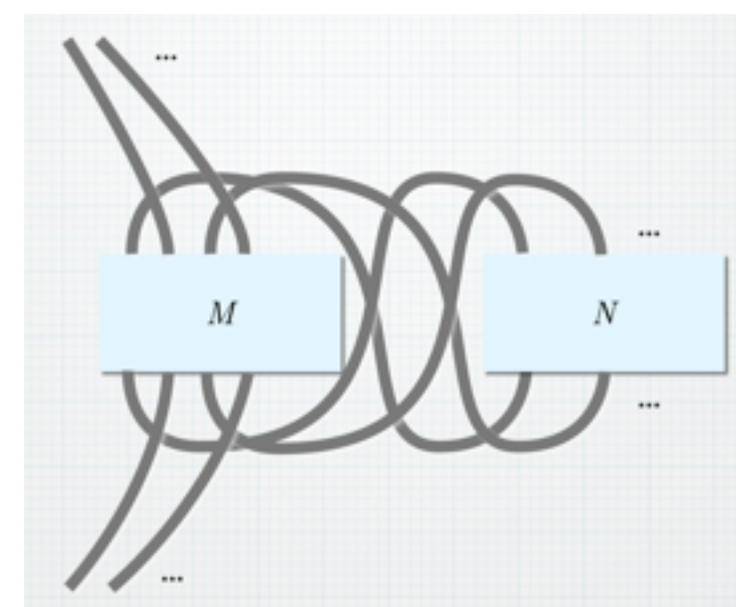
suo (Tokyo)

$$X \xrightarrow{\left[\,\boxed{c}_{U}^{U}\,\right]} Y$$

$$\left[\,\boxed{a}_{U}^{U}\,\right] \longmapsto [c \cdot a] =$$





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Time to Wake Up!!

It's time to save them.

WWF

Part 3

Quantum Computation in 5 min.

What You Need to Know

* Not much, really!
* Our principal reference:
    * M.A. Nielsen and I.L. Chuang. **Quantum Computation and Quantum Information**. CUP, 2000
    * Its Chap. 3 & Chap. 8
        * Hilbert space formulation
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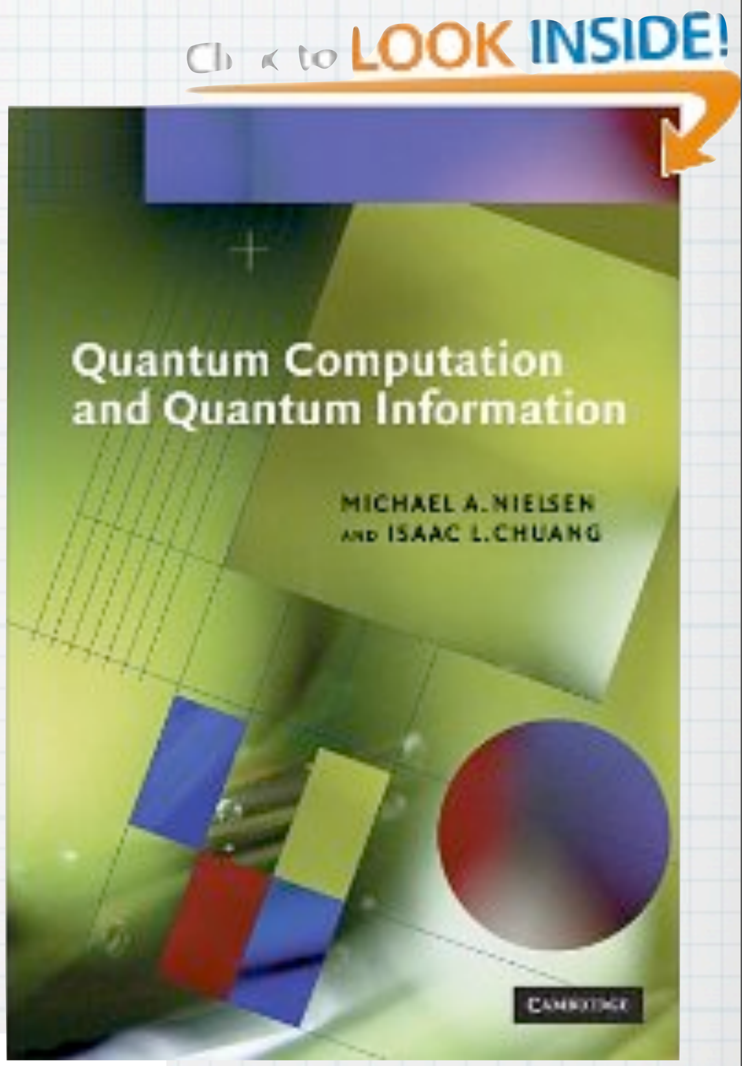
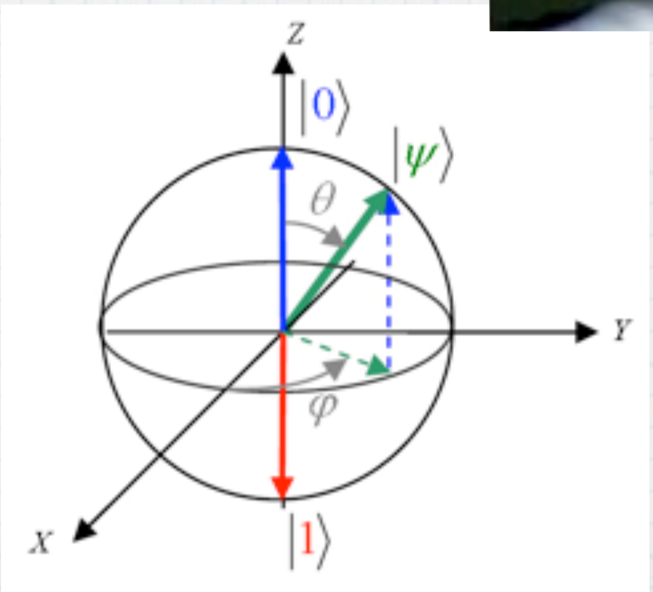
    * No need for the Bloch sphere





Hasuo (Tokyo)

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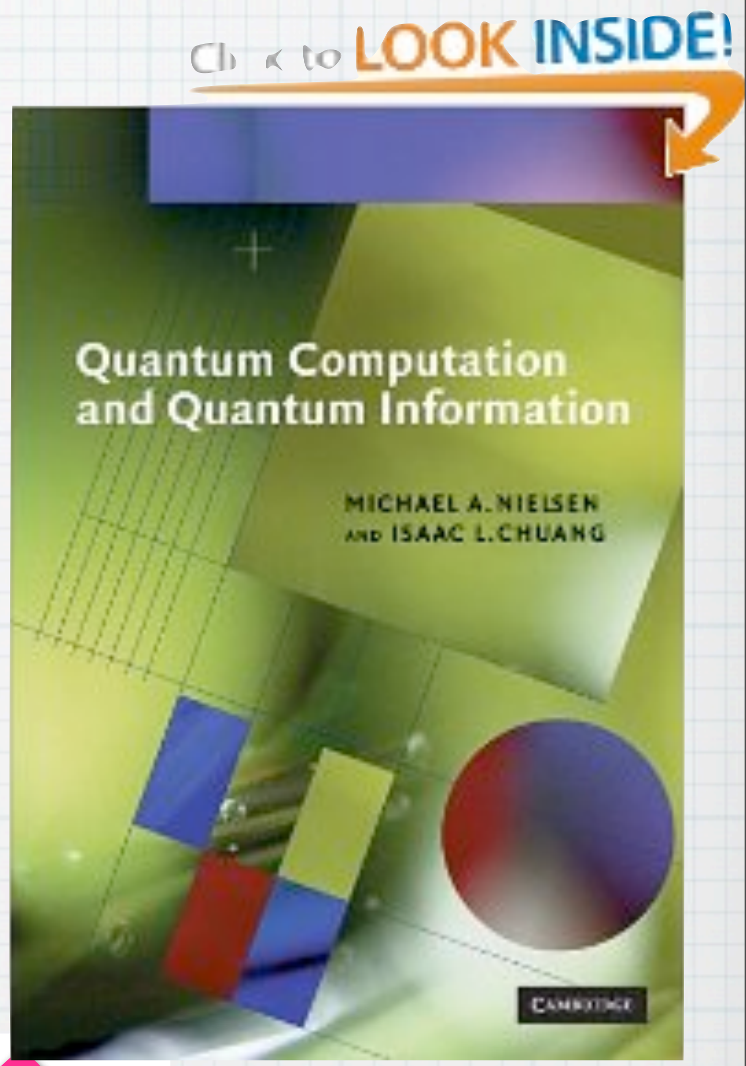

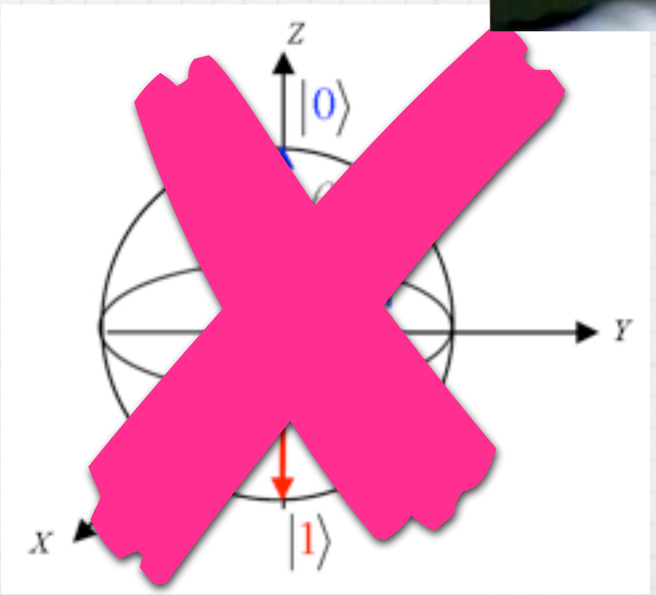

Hasuo (Tokyo)

Some Principles

$\frac{1}{\sqrt{2}}$ 

$+\frac{1}{\sqrt{2}}$ 

* A state of a 1-qubit system = a normalized vector

$$|\varphi\rangle = \alpha|0\rangle + \beta|1\rangle \in \mathbb{C}^2$$

* with $\big\| |\varphi\rangle \big\|^2 = |\alpha|^2 + |\beta|^2 = 1$

* Various notations for base:

$$\{\, |0\rangle, |1\rangle \,\},\ \{\, |+\rangle, |-\rangle \,\},\ \{\, |\uparrow\rangle, |\downarrow\rangle \,\}, \ldots$$

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Some Principles

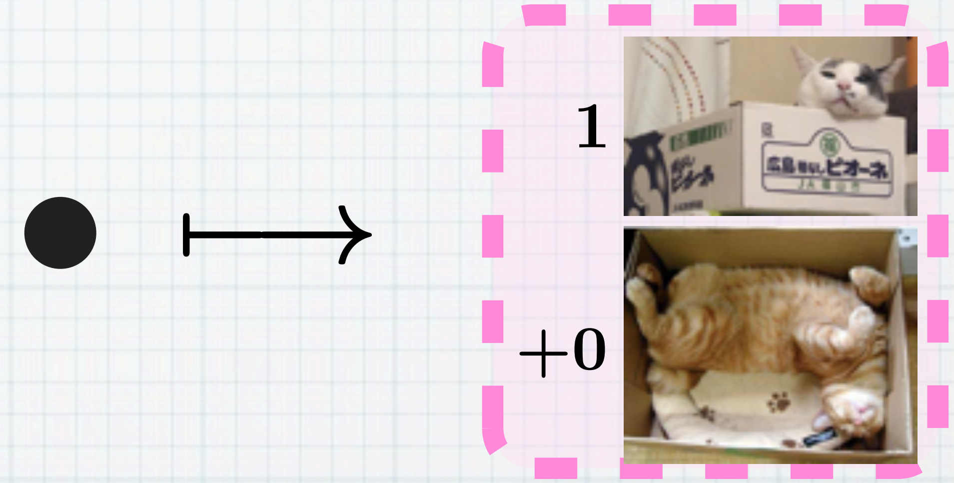
- Composed system: $\otimes$, not $\times$.
  - not $\mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2 \cong \mathbb{C}^6$, with base $\left\{ \begin{array}{ccc} |0_1\rangle & |0_2\rangle & |0_3\rangle \\ |1_1\rangle & |1_2\rangle & |1_3\rangle \end{array} \right\}$
  - but $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C}^8$, with base $\left\{ \begin{array}{cccc} |000\rangle & |001\rangle & |010\rangle & |011\rangle \\ |100\rangle & |101\rangle & |110\rangle & |111\rangle \end{array} \right\}$

Some Principles

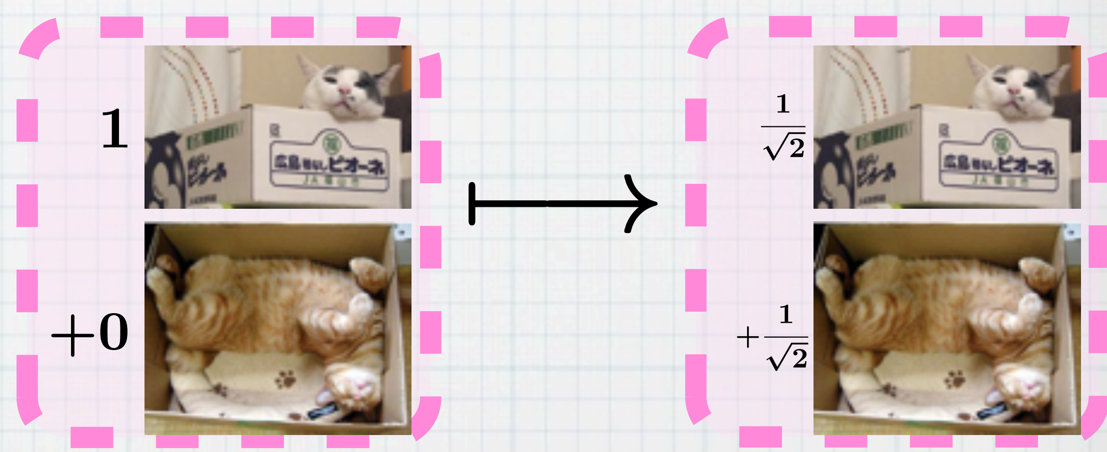
* Composed system: $\otimes$, not $\times$.
  * Source of power of quantum comp./comm.
    * N-qubit ➜ 2^N-dim (not 2N-dim)
    * Entanglement; superposition

Three Quantum Primitives

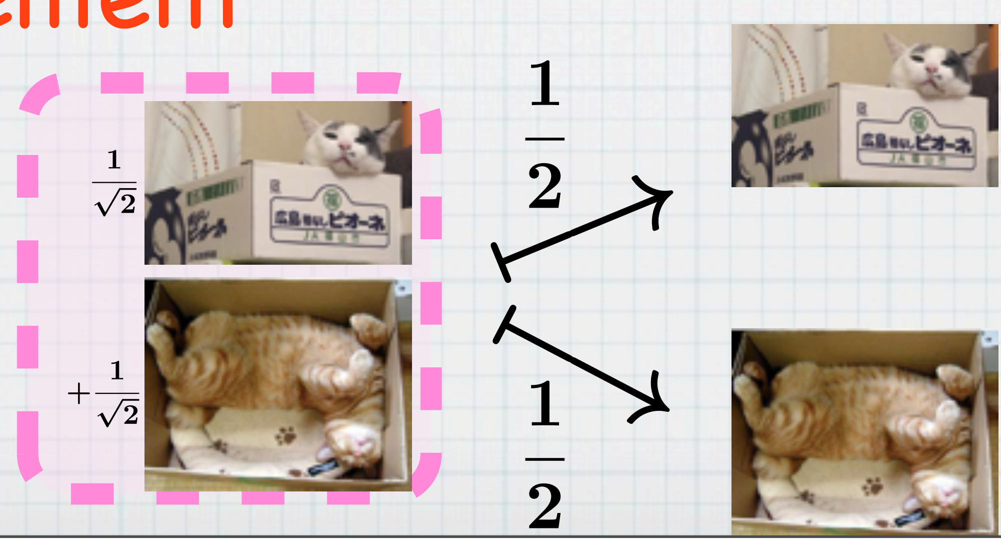
* Preparation



* Unitary transformation



* Measurement



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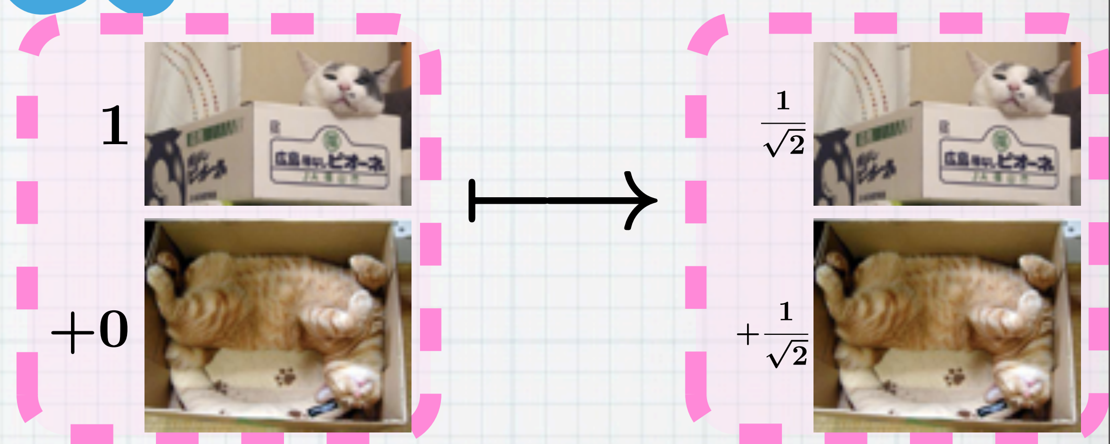
Three Quantum Primitives

$\bullet \longmapsto$

* Preparation
    * Creates/"prepares" a quantum state (typically $|0\rangle$)

Three Quantum Primitives

* Unitary transformation



$$\alpha|0\rangle + \beta|1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad \overset{U}{\longmapsto} \quad U \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

* Unitary matrix: $UU^\dagger = U^\dagger U = \mathcal{I}$
  * Invertible. "Rotation"
* Also for N-dim systems (of course)

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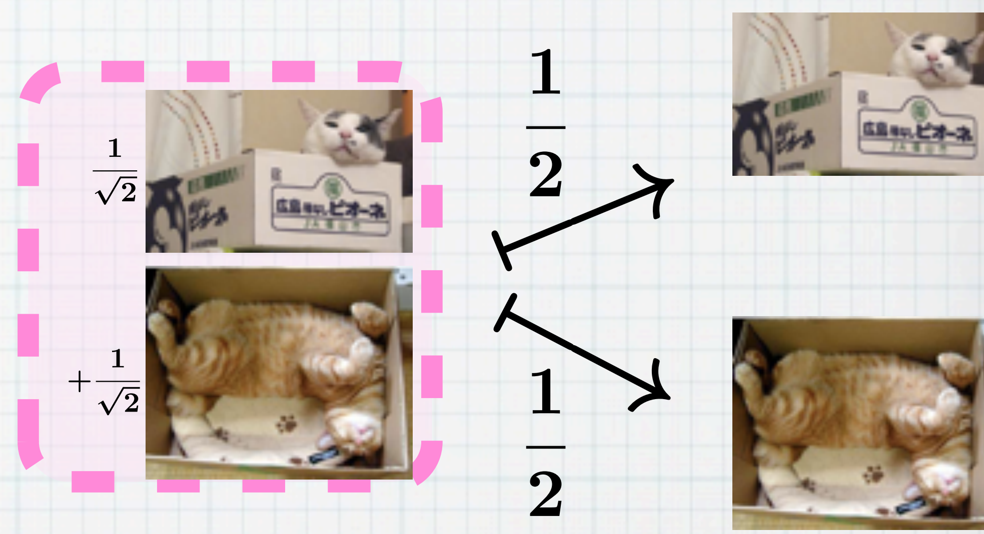
Three Quantum Primitives

* Measurement

  When one measures

$$\alpha|0\rangle + \beta|1\rangle$$



* $|0\rangle$ is observed, and
* the state becomes $|0\rangle$

with prob. $|\alpha|^2$

* $|1\rangle$ is observed, and
* the state becomes $|1\rangle$

with prob. $|\beta|^2$

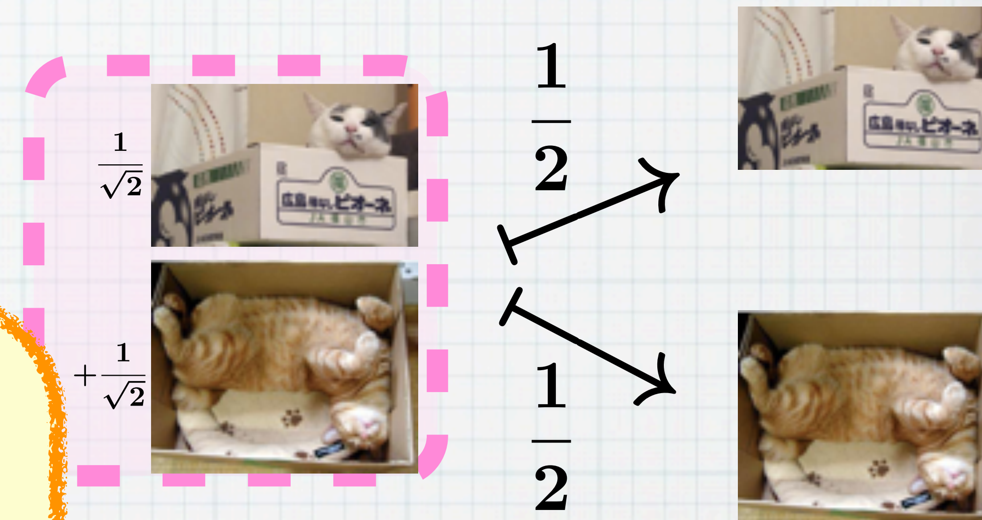
Three Quantum Primitives

* **Measurement**

  When one measures

  $$\alpha|0\rangle + \beta|1\rangle$$

state reduction



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Three Quantum Primitives

* **Measurement**

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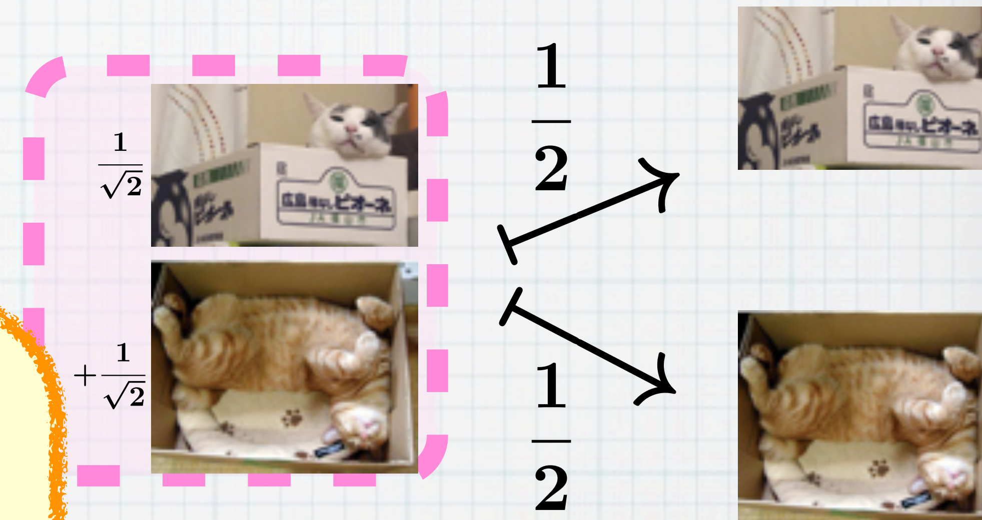

state reduction

  * $|0\rangle$ is observed, and
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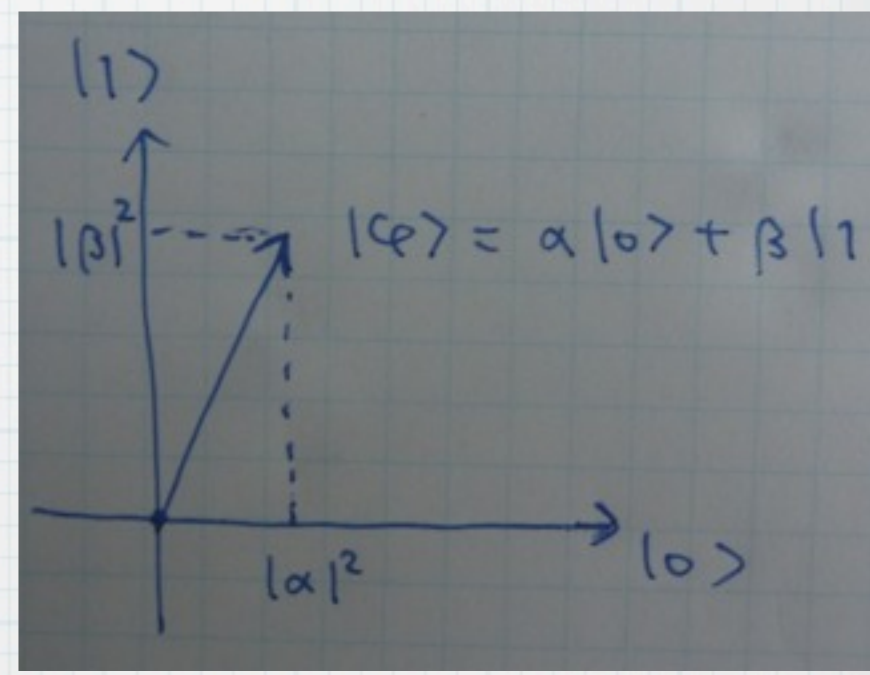
  with prob. $|\beta|^2$



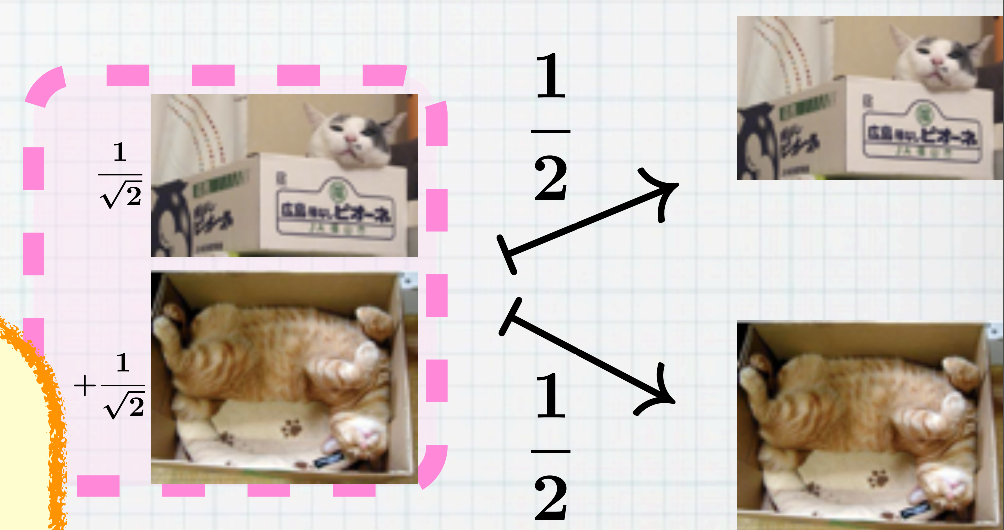
Hasuo (Tokyo)

Three Quantum Primitives

* **Measurement**

  When one measures

  $$\alpha|0\rangle + \beta|1\rangle$$



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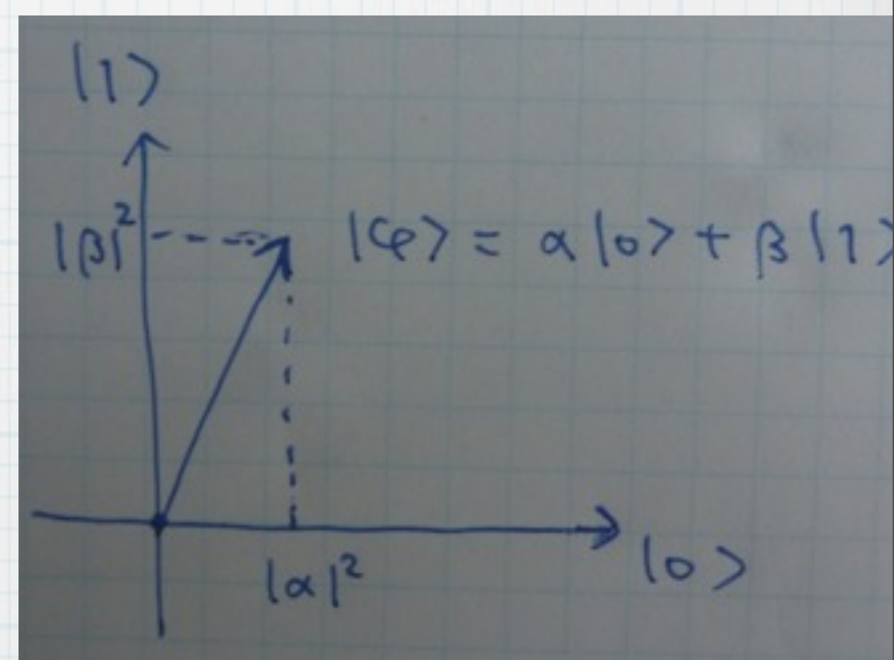
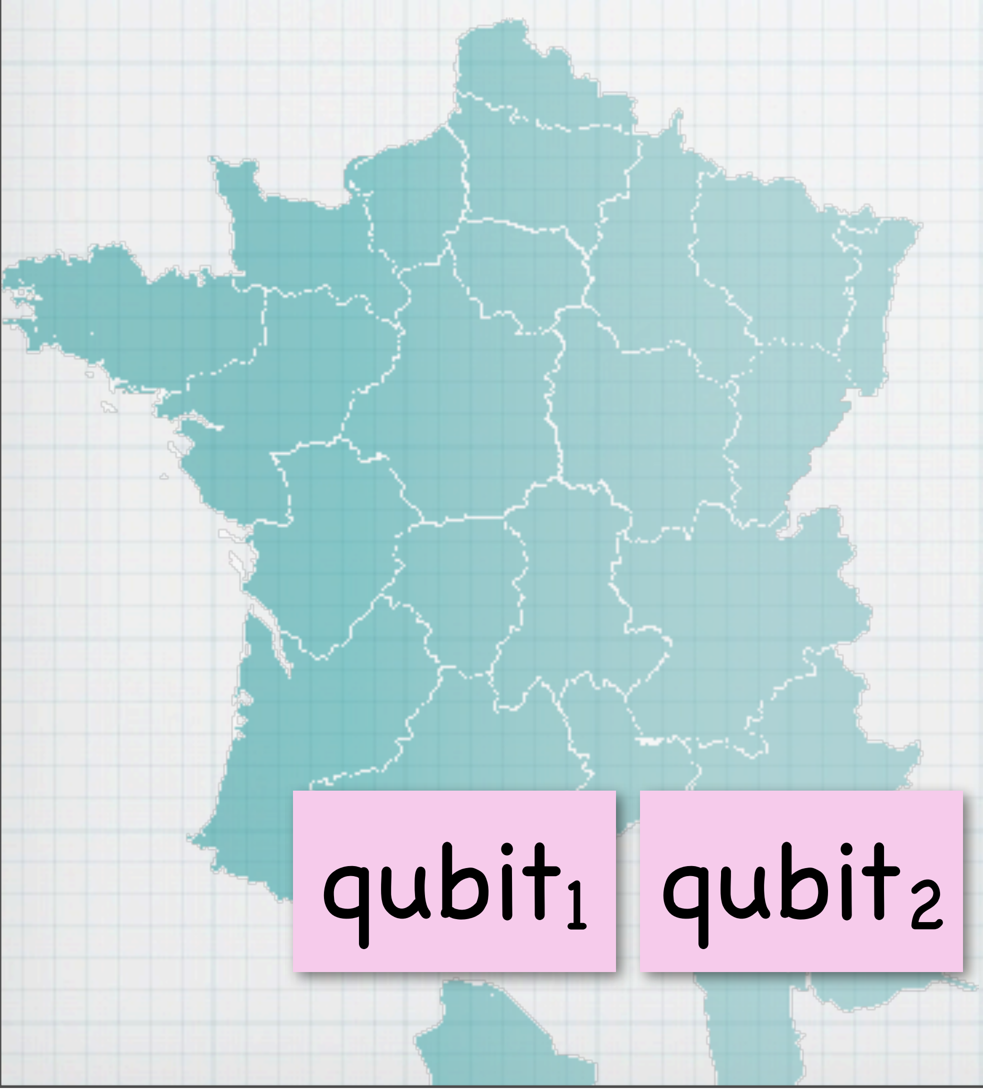


Also: for other dimensions, bases

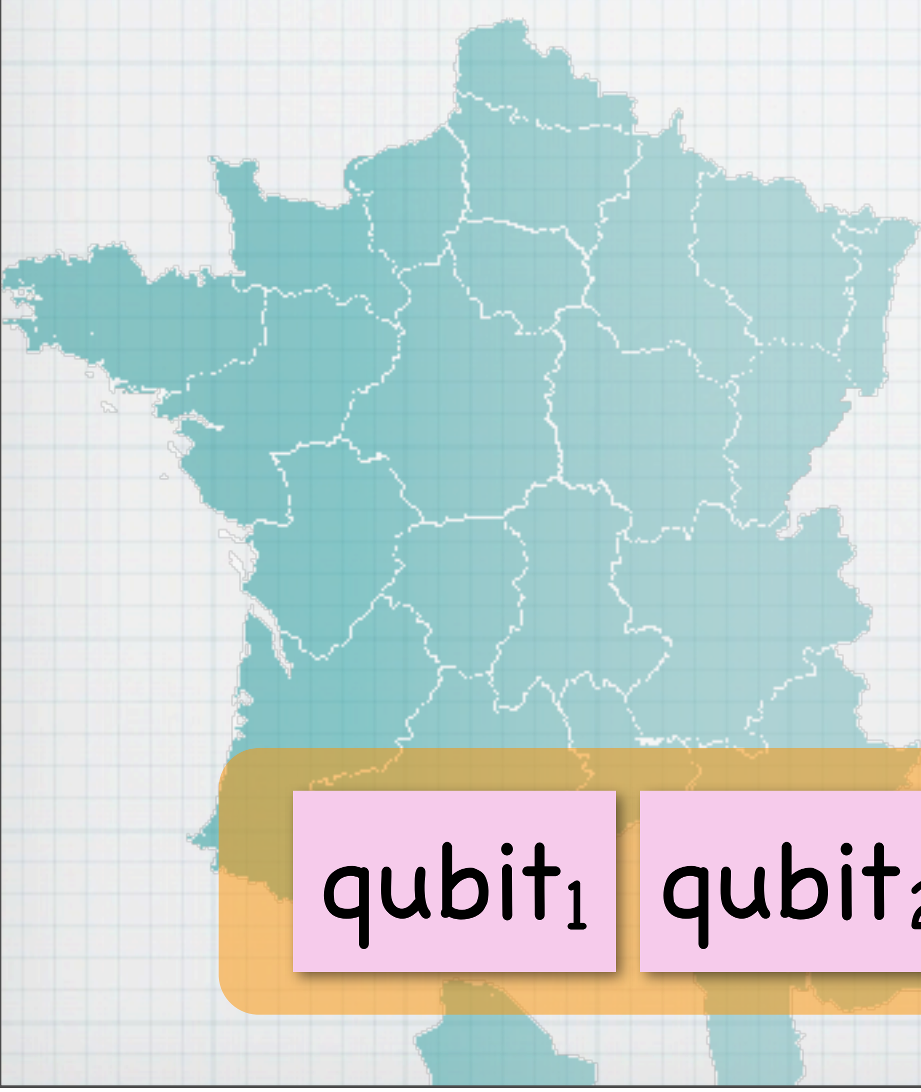
Entanglement

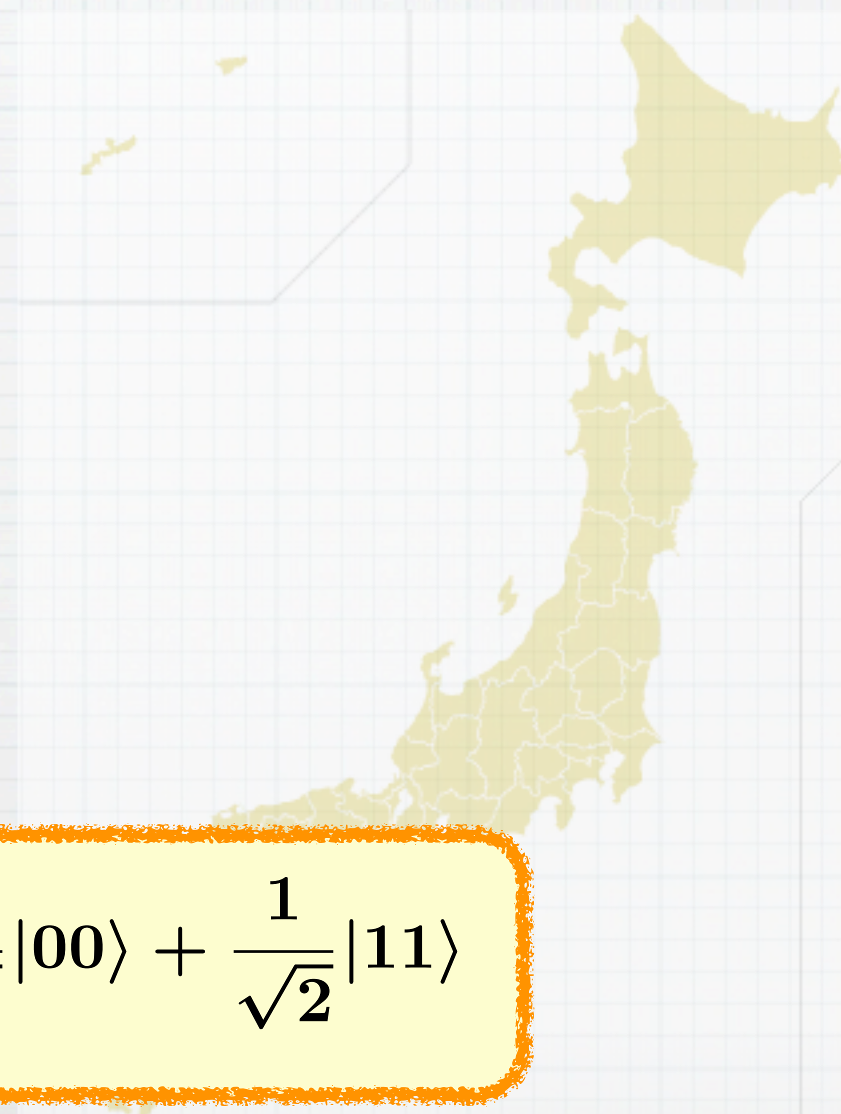


qubit$_1$ qubit$_2$

Hasuo (Tokyo)

Entanglement



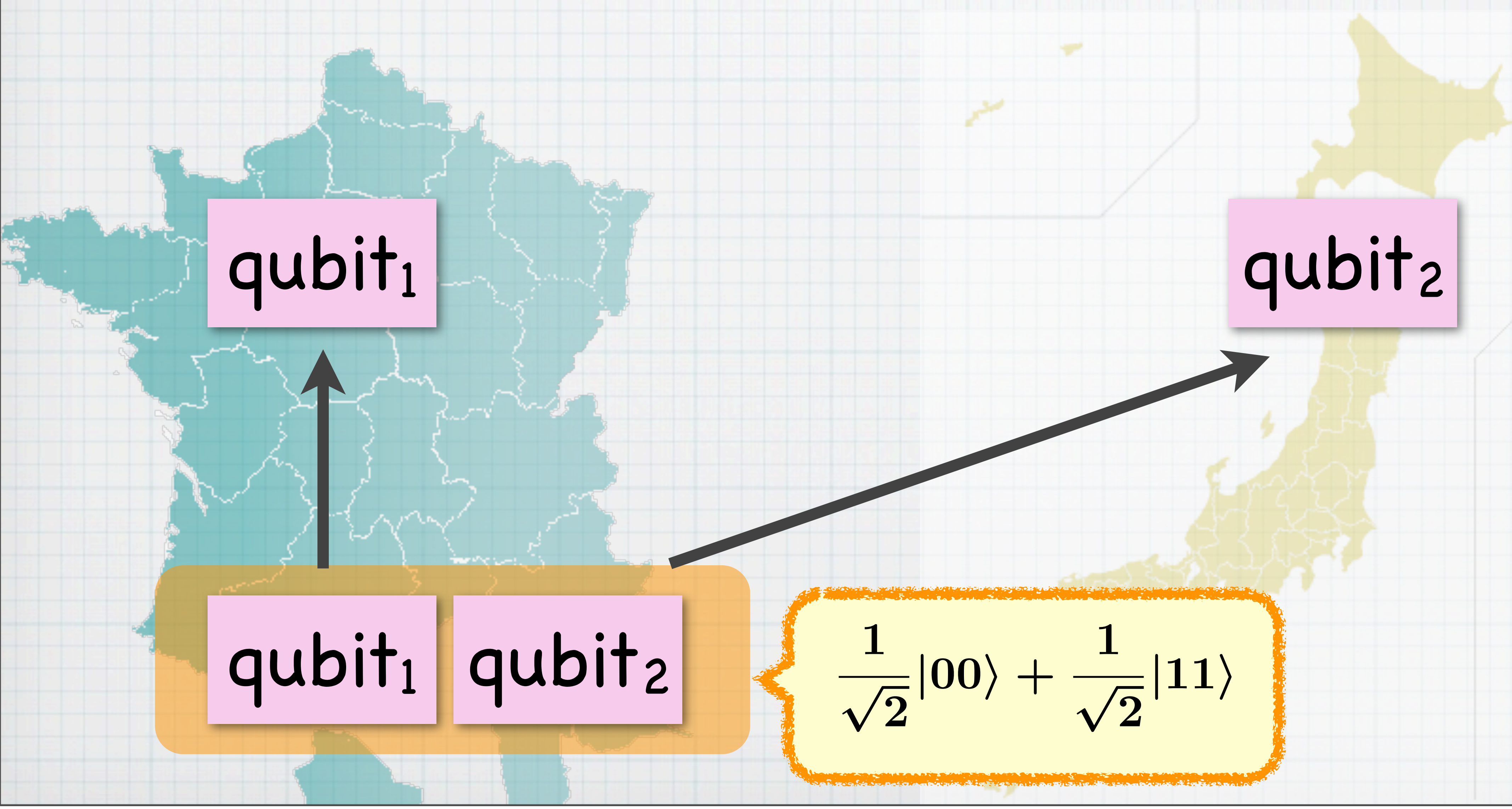


qubit$_1$ qubit$_2$

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

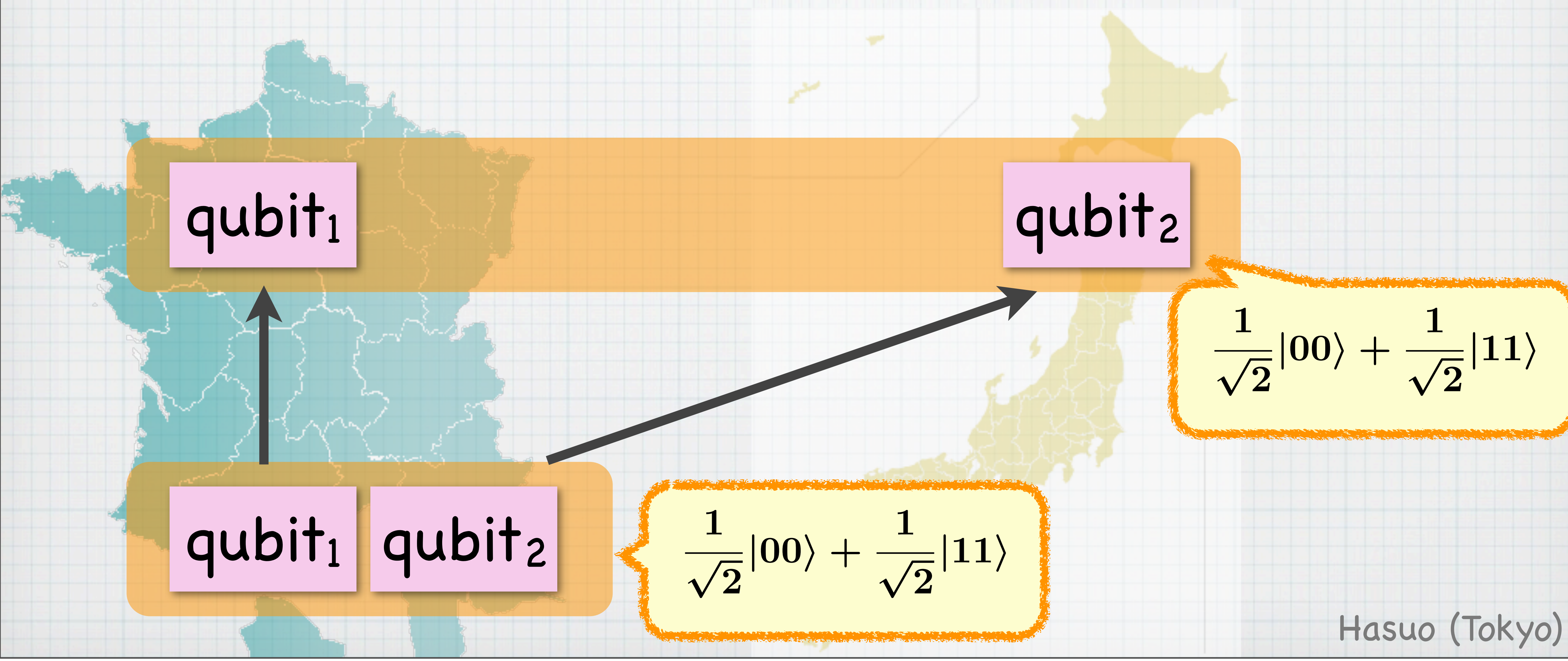
Hasuo (Tokyo)

Entanglement



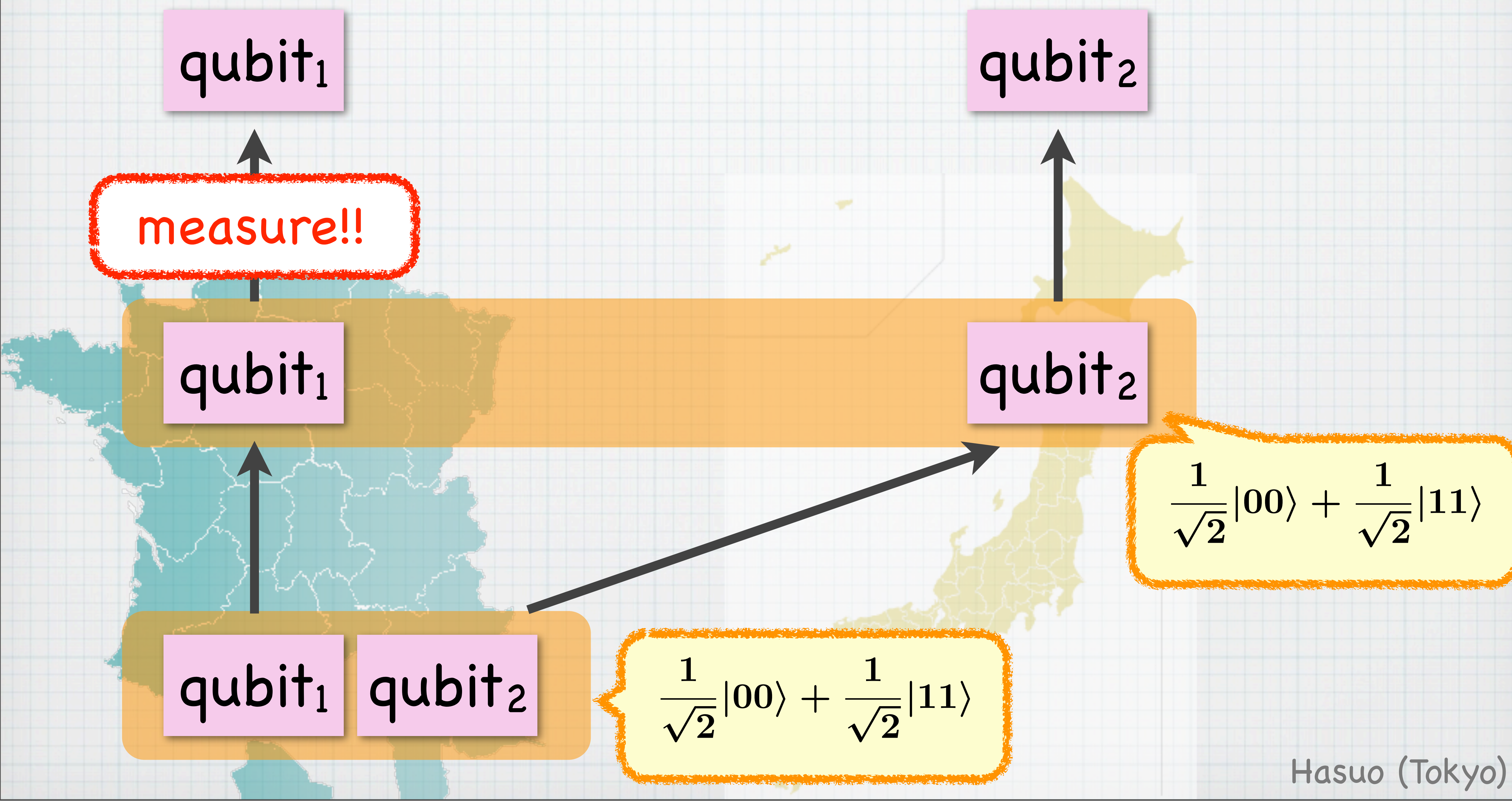
$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

Entanglement



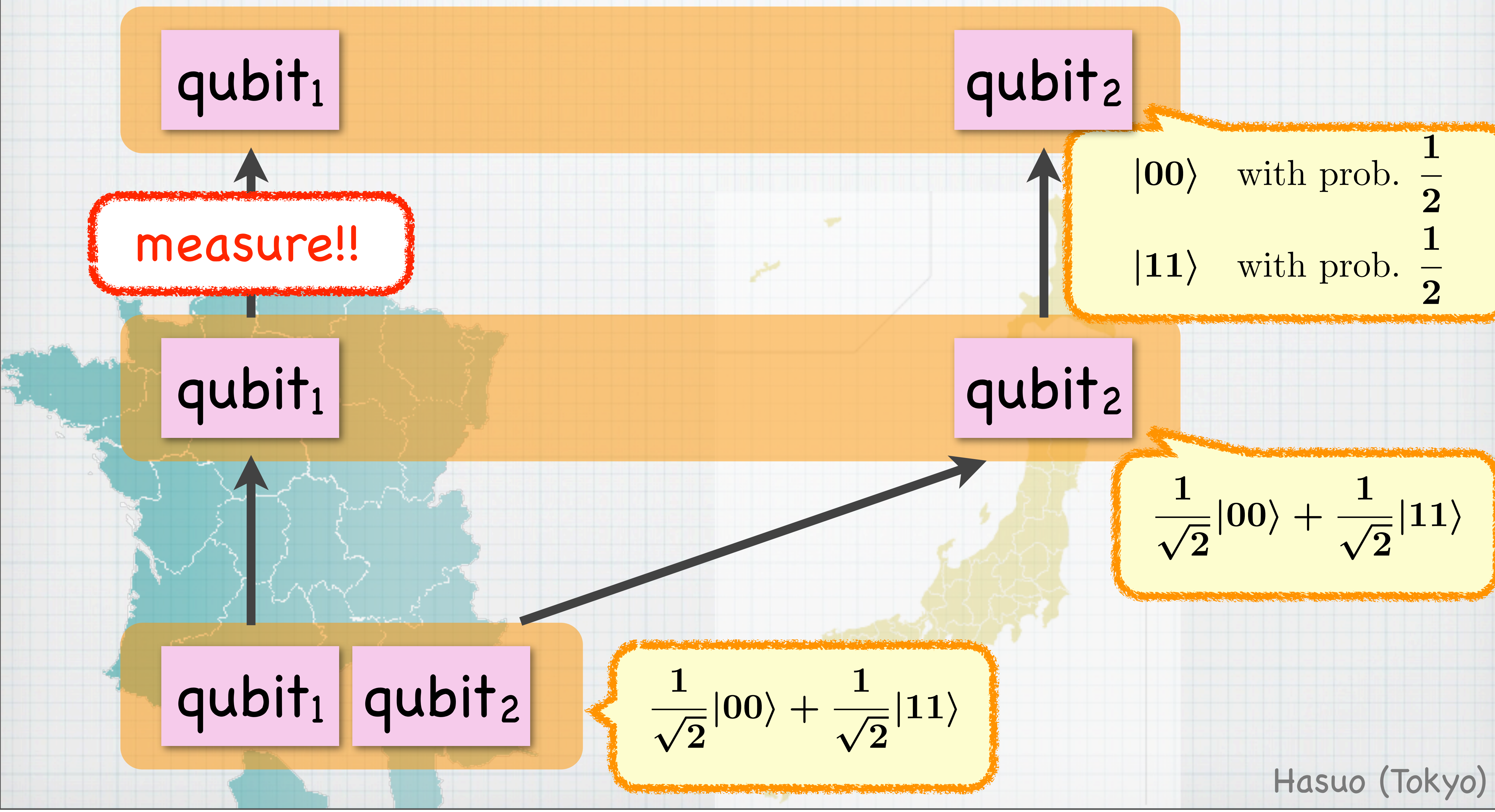
Hasuo (Tokyo)

Entanglement



Hasuo (Tokyo)

Entanglement



Hasuo (Tokyo)

Density Matrix, Quantum Operation

* Advanced, mathematically convenient formalisms
* State vector ➜ density matrix
  * Use $|\varphi\rangle\langle\varphi|$ in place of $|\varphi\rangle$
  * Can also represent **mixed states**, e.g.

$$\begin{array}{ll} |\mathbf{00}\rangle & \text{with prob. } \frac{\mathbf{1}}{\mathbf{2}} \\ |\mathbf{11}\rangle & \text{with prob. } \frac{\mathbf{1}}{\mathbf{2}} \end{array}$$

* Quantum operation (QO) [Kraus]
  * {QOs} = {any combinations of preparation, Unitary transf., measurement}
  * But **no** classical control (like case-distinction)
  * Used in [Selinger,MSCS'04] and other

Hasuo (Tokyo)

Density Matrix, Quantum Operation

Defn.

- An $\boldsymbol{m}$-*dimensional density matrix* is an $\boldsymbol{m} \times \boldsymbol{m}$ matrix $\boldsymbol{\rho} \in \mathbb{C}^{\boldsymbol{m} \times \boldsymbol{m}}$ which is positive and satisfies $\mathbf{tr}(\boldsymbol{\rho}) \in [\mathbf{0}, \mathbf{1}]$.
  - Notation: $\boldsymbol{D_m} = \{\boldsymbol{m}\text{-dim. density matrices}\}$
- A *quantum operation (QO)* is a mapping $\boldsymbol{\mathcal{E}} : \boldsymbol{D_m} \to \boldsymbol{D_n}$ subject to the following axioms.
  1. (Trace condition) $\mathbf{tr}[\boldsymbol{\mathcal{E}}(\boldsymbol{\rho})] \in [\mathbf{0}, \mathbf{1}]$ for any $\boldsymbol{\rho} \in \boldsymbol{D_m}$.
  2. (Linearity) Let $(\boldsymbol{\rho_i})_{\boldsymbol{i} \in \boldsymbol{I}}$ be a family of $\boldsymbol{m}$-dim. density matrices; and $(\boldsymbol{p_i})_{\boldsymbol{i} \in \boldsymbol{I}}$ be a probability subdistribution (meaning $\sum_{\boldsymbol{i}} \boldsymbol{p_i} \leq \mathbf{1}$). Then: $\boldsymbol{\mathcal{E}}\left(\sum_{\boldsymbol{i} \in \boldsymbol{I}} \boldsymbol{p_i \rho_i}\right) = \sum_{\boldsymbol{i} \in \boldsymbol{I}} \boldsymbol{p_i} \boldsymbol{\mathcal{E}}(\boldsymbol{\rho_i})$.
  3. (Complete positivity) An arbitrary "extension" of $\boldsymbol{\mathcal{E}}$ of the form $\boldsymbol{\mathcal{I}_k} \otimes \boldsymbol{\mathcal{E}} : \boldsymbol{M_k} \otimes \boldsymbol{M_m} \to \boldsymbol{M_k} \otimes \boldsymbol{M_n}$ carries a positive matrix to a positive one.
  - Notation: $\mathbf{QO}_{\boldsymbol{m},\boldsymbol{n}} = \{\text{QOs from } \boldsymbol{m}\text{-dim. to } \boldsymbol{n}\text{-dim. }\}$

* For specialists: we allow trace ≤ 1
* So that **probabilities** are implicitly carried by density matrices

Hasuo (Tokyo)

Quantum Computation: Summary

* A quantum state = a vector $|\varphi\rangle$
* Composition by $\otimes$
  ➜ Dimension grows exponentially
* Three primitives:
  * Preparation
  * Unitary transformation
  * Measurement (➜ st. reduction)

Hasuo (Tokyo)

Quantum Computation: Summary

Generalized to
density matrix

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Quantum Computation: Summary

Generalized to density matrix

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  * Unitary transformation
  * Measurement (➜ st. reduction)

Unified to quantum operation (QO)

Hasuo (Tokyo)

Part 4

Quantum GoI

The Language $q\lambda_\ell$

- Roughly: linear λ + quantum primitives
- "Quantum data, classical control"
  - No superposed threads
- Based on [Selinger&Valiron'09]
  - With slight modifications
  - Notably: quantum $\otimes$ vs. linear logic $\boxtimes$
    - The same in [Selinger&Valiron'09]
      ➜ clean type system, aids programming
    - But... problem with GoI-style semantics

The Language qλ_ℓ

The *types* of $\mathbf{q}\boldsymbol{\lambda}_\ell$ are:

$$A, B ::= n\text{-}\mathtt{qbit} \mid\ !\,A \mid A \multimap B \mid \top \mid A \boxtimes B \mid A + B\ ,$$

with conventions $\mathtt{qbit} := \mathbf{1}\text{-}\mathtt{qbit}$ and $\mathtt{bit} := \top + \top$.

The *terms* of $\mathbf{q}\boldsymbol{\lambda}_\ell$ are:

$$\begin{aligned}
&M, N, P ::= \\
&\quad x \mid \lambda x^A.M \mid M N \mid \langle M, N\rangle \mid * \mid \\
&\quad \mathtt{let}\ \langle x^A, y^B\rangle = M\ \mathtt{in}\ N \mid \mathtt{let}\ * = M\ \mathtt{in}\ N \mid \\
&\quad \mathtt{inj}_\ell^B\ M \mid \mathtt{inj}_r^A\ M \mid \\
&\quad \mathtt{match}\ P\ \mathtt{with}\ (x^A \mapsto M \mid y^B \mapsto N) \mid \\
&\quad \mathtt{letrec}\ f^A x = M\ \mathtt{in}\ N \mid \\
&\quad \mathtt{new}\ |\mathbf{0}\rangle \mid \mathtt{meas}_i^{n+1} \mid U \mid \mathtt{cmp}_{m,n}\ ,
\end{aligned}$$

with conventions $\mathtt{tt} := \mathtt{inj}_\ell^\top(*)$ and $\mathtt{ff} := \mathtt{inj}_r^\top(*)$.

Hasuo (Tokyo)

The Langua

Different from quantum ⊗
(Unlike [Selinger-Valiron'09]);
same as the one in **PER**

The *types* of $\mathbf{q}\boldsymbol{\lambda}_\ell$ are:

$$\boldsymbol{A}, \boldsymbol{B} ::= \boldsymbol{n}\text{-}\mathtt{qbit} \mid \boldsymbol{!}\,\boldsymbol{A} \mid \boldsymbol{A} \multimap \boldsymbol{B} \mid \top \mid \boldsymbol{A} \boxtimes \boldsymbol{B} \mid \boldsymbol{A} + \boldsymbol{B} \ ,$$

with conventions $\mathtt{qbit} := \mathbf{1}\text{-}\mathtt{qbit}$ and $\mathtt{bit} := \top + \top$.

The *terms* of $\mathbf{q}\boldsymbol{\lambda}_\ell$ are:

$$\begin{aligned}
&\boldsymbol{M}, \boldsymbol{N}, \boldsymbol{P} ::= \\
&\quad \boldsymbol{x} \mid \boldsymbol{\lambda x^A}.\boldsymbol{M} \mid \boldsymbol{M}\,\boldsymbol{N} \mid \langle \boldsymbol{M}, \boldsymbol{N} \rangle \mid * \mid \\
&\quad \mathtt{let}\,\langle \boldsymbol{x^A}, \boldsymbol{y^B} \rangle = \boldsymbol{M}\ \mathtt{in}\ \boldsymbol{N} \mid \mathtt{let}\, * = \boldsymbol{M}\ \mathtt{in}\ \boldsymbol{N} \mid \\
&\quad \mathtt{inj}_\ell^{\boldsymbol{B}}\ \boldsymbol{M} \mid \mathtt{inj}_r^{\boldsymbol{A}}\ \boldsymbol{M} \mid \\
&\quad \mathtt{match}\ \boldsymbol{P}\ \mathtt{with}\ (\boldsymbol{x^A} \mapsto \boldsymbol{M} \mid \boldsymbol{y^B} \mapsto \boldsymbol{N}) \mid \\
&\quad \mathtt{letrec}\ \boldsymbol{f^A}\,\boldsymbol{x} = \boldsymbol{M}\ \mathtt{in}\ \boldsymbol{N} \mid \\
&\quad \mathtt{new}\,|\mathbf{0}\rangle \mid \mathtt{meas}_i^{\boldsymbol{n+1}} \mid \boldsymbol{U} \mid \mathtt{cmp}_{\boldsymbol{m},\boldsymbol{n}} \ ,
\end{aligned}$$

with conventions $\mathtt{tt} := \mathtt{inj}_\ell^{\top}(*)$ and $\mathtt{ff} := \mathtt{inj}_r^{\top}(*)$.

Hasuo (Tokyo)

The Langua

Different from quantum ⊗
(Unlike [Selinger-Valiron'09]);
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2-qbit $\cong$ qbit $\otimes$ qbit

$$A, B ::= n\text{-}\mathtt{qbit} \mid !A \mid A \multimap B \mid \top \mid A \boxtimes B \mid A + B \ ,$$

with conventions $\mathtt{qbit} := \mathbf{1}\text{-}\mathtt{qbit}$ and $\mathtt{bit} := \top + \top$.

The *terms* of $\mathbf{q}\boldsymbol{\lambda}_\ell$ are:

$$\begin{array}{l} M, N, P ::= \\ \quad x \mid \lambda x^A.M \mid MN \mid \langle M, N\rangle \mid * \mid \\ \quad \mathtt{let}\,\langle x^A, y^B\rangle = M\,\mathtt{in}\,N \mid \mathtt{let}\,* = M\,\mathtt{in}\,N \mid \\ \quad \mathtt{inj}_\ell^B\, M \mid \mathtt{inj}_r^A\, M \mid \\ \quad \mathtt{match}\,P\,\mathtt{with}\,(x^A \mapsto M \mid y^B \mapsto N) \mid \\ \quad \mathtt{letrec}\,f^A x = M\,\mathtt{in}\,N \mid \\ \quad \mathtt{new}\,|\mathbf{0}\rangle \mid \mathtt{meas}_i^{n+1} \mid U \mid \mathtt{cmp}_{m,n} \ , \end{array}$$

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$$\text{with conventions } \mathtt{tt} := \mathtt{inj}_\ell^\top(*) \text{ and } \mathtt{ff} := \mathtt{inj}_r^\top(*)\ .$$

Recursion

Hasuo (Tokyo)

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(Unlike [Selinger-Valiron'09]);
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The *terms* of $\mathbf{q\lambda}_\ell$ are:

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with conventions $\mathtt{tt} := \mathtt{inj}_\ell^\top(*)$ and $\mathtt{ff} := \mathtt{inj}_r^\top(*)$.

Recursion

Quantum primitives

Implicit linearity tracking via subtyping <:

e.g. !A <: A, !A <: !!A

(following [Selinger-Valiron'09])

$$\frac{n=0 \Rightarrow m=0\ (*)}{!^n\ k\text{-qbit} <: !^m\ k\text{-qbit}}\ (k\text{-qbit}) \qquad \frac{n=0 \Rightarrow m=0}{!^n \top <: !^m \top}\ (\top)$$

$$\frac{A_1 <: B_1 \quad A_2 <: B_2 \quad (*)}{!^n(A_1 \boxdot A_2) <: !^m(B_1 \boxdot B_2)}\ (\boxdot) \text{ with } \boxdot \in \{\boxtimes, +\}$$

$$\frac{B_1 <: A_1 \quad A_2 <: B_2 \quad (*)}{!^n(A_1 \multimap A_2) <: !^m(B_1 \multimap B_2)}\ (\multimap)$$

Measurements

$A_{\mathtt{new}\lvert 0\rangle}$	$:=$	$\mathtt{qbit}$
$A_{\mathtt{meas}_i^{n+1}}$	$:=$	$(n+1)\text{-}\mathtt{qbit} \multimap (\mathtt{bit} \boxtimes n\text{-}\mathtt{qbit})$ for $n \geq 1$
$A_{\mathtt{meas}_1^1}$	$:=$	$\mathtt{qbit} \multimap \mathtt{bit}$
A_U	$:=$	$n\text{-}\mathtt{qbit} \multimap n\text{-}\mathtt{qbit}$ for a $2^n \times 2^n$ matrix U
$A_{\mathtt{cmp}_{m,n}}$	$:=$	$(m\text{-}\mathtt{qbit} \boxtimes n\text{-}\mathtt{qbit}) \multimap (m+n)\text{-}\mathtt{qbit}$

Bookkeeping (due to $\otimes$ vs. $\boxtimes$)

$$\frac{A <: A'}{!\Delta, x : A \vdash x : A'}\ (\mathrm{Ax}.1) \qquad \frac{!\,A_c <: A}{!\Delta \vdash c : A}\ (\mathrm{Ax}.2)$$

$$\frac{\Delta \vdash M : !^n A}{\Delta \vdash \mathtt{inj}_\ell^B M : !^n(A+B)}\ (+.\mathrm{I}_1)$$

$$\frac{\Delta \vdash N : !^n B}{\Delta \vdash \mathtt{inj}_r^A N : !^n(A+B)}\ (+.\mathrm{I}_2)$$

$$\frac{!\Delta, \Gamma_1 \vdash P : !^n(A+B) \quad \begin{array}{c} !\Delta, \Gamma_2, x : !^n A \vdash M : C \\ !\Delta, \Gamma_2, y : !^n B \vdash N : C \end{array}}{\begin{array}{l} !\Delta, \Gamma_1, \Gamma_2 \\ \quad \vdash \mathtt{match}\ P\ \mathtt{with}\ (x^{!^n A} \mapsto M \mid y^{!^n B} \mapsto N) : C \end{array}}\ (+.\mathrm{E}), (\dagger)$$

$$\frac{x : A, \Delta \vdash M : B}{\Delta \vdash \lambda x^A.M : A \multimap B}\ (\multimap.\mathrm{I}_1)$$

$$\frac{x : A, !\Delta \vdash M : B}{!\Delta \vdash \lambda x^A.M : !^n(A \multimap B)}\ (\multimap.\mathrm{I}_2)$$

$$\frac{!\Delta, \Gamma_1 \vdash M : A \multimap B \quad !\Delta, \Gamma_2 \vdash N : A}{!\Delta, \Gamma_1, \Gamma_2 \vdash MN : B}\ (\multimap.\mathrm{E}), (\dagger)$$

$$\frac{!\Delta, \Gamma_1 \vdash M_1 : !^n A_1 \quad !\Delta, \Gamma_2 \vdash M_2 : !^n A_2}{!\Delta, \Gamma_1, \Gamma_2 \vdash \langle M_1, M_2\rangle : !^n(A_1 \boxtimes A_2)}\ (\boxtimes.\mathrm{I}), (\dagger)$$

$$\frac{}{!\Delta \vdash * : !^n \top}\ (\top.\mathrm{I})$$

$$\frac{\begin{array}{c} !\Delta, \Gamma_2, x_1 : !^n A_1, x_2 : !^n A_2 \vdash N : A \\ !\Delta, \Gamma_1 \vdash M : !^n(A_1 \boxtimes A_2) \end{array}}{!\Delta, \Gamma_1, \Gamma_2 \vdash \mathtt{let}\ \langle x_1^{!^n A_1}, x_2^{!^n A_2}\rangle = M\ \mathtt{in}\ N : A}\ (\boxtimes.\mathrm{E}), (\dagger)$$

$$\frac{!\Delta, \Gamma_1 \vdash M : \top \quad !\Delta, \Gamma_2 \vdash N : A}{!\Delta, \Gamma_1, \Gamma_2 \vdash \mathtt{let}\ * = M\ \mathtt{in}\ N : A}\ (\top.\mathrm{E}), (\dagger)$$

$$\frac{\begin{array}{c} !\Delta, \Gamma, f : !(A \multimap B) \vdash N : C \\ !\Delta, f : !(A \multimap B), x : A \vdash M : B \end{array}}{!\Delta, \Gamma \vdash \mathtt{letrec}\ f^{A \multimap B} x = M\ \mathtt{in}\ N : C}\ (\mathrm{rec}), (\dagger)$$

Hasuo (Tokyo)

Operational Semantics

$$
\begin{aligned}
&E[\,(\lambda x^A.M)V\,] \to_1 E[\,M[V/x]\,]\\
&E[\,\mathtt{let}\,\langle x^A, y^B\rangle = \langle V, W\rangle\,\mathtt{in}\,M\,] \to_1 E[\,M[V/x, W/y]\,]\\
&E[\,\mathtt{let}\,* = *\,\mathtt{in}\,M\,] \to_1 E[\,M\,]\\
&E[\,\mathtt{match}\,(\mathtt{inj}_\ell^B V)\,\mathtt{with}\,(x^{!^n A} \mapsto M \mid y^{!^n B} \mapsto N)\,]\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \to_1 E[\,M[V/x]\,]\\
&E[\,\mathtt{match}\,(\mathtt{inj}_r^A V)\,\mathtt{with}\,(x^{!^n A} \mapsto M \mid y^{!^n B} \mapsto N)\,]\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \to_1 E[\,N[V/y]\,]\\
&E[\,\mathtt{letrec}\,f^{A\multimap B}x = M\,\mathtt{in}\,N\,]\\
&\qquad\qquad \to_1 E[\,N[\lambda x^A.\mathtt{letrec}\,f^{A\multimap B}x = M\,\mathtt{in}\,M/f]\,]\\
&E[\,\mathtt{meas}_i^{n+1}(\mathtt{new}\,\rho)\,] \to_1 E[\,\langle\, \mathtt{tt}, \mathtt{new}\,\langle 0_i|\rho|0_i\rangle\,\rangle\,]\\
&E[\,\mathtt{meas}_i^{n+1}(\mathtt{new}\,\rho)\,] \to_1 E[\,\langle\, \mathtt{ff}, \mathtt{new}\,\langle 1_i|\rho|1_i\rangle\,\rangle\,]\\
&E[\,\mathtt{meas}_1^{1}(\mathtt{new}\,\rho)\,] \to_{\langle 0|\rho|0\rangle} E[\,\mathtt{tt}\,]\\
&E[\,\mathtt{meas}_1^{1}(\mathtt{new}\,\rho)\,] \to_{\langle 1|\rho|1\rangle} E[\,\mathtt{ff}\,]\\
&E[\,U(\mathtt{new}\,\rho)\,] \to_1 E[\,\mathtt{new}\,(U\rho)\,]\\
&E[\,\mathtt{cmp}_{m,n}\langle \mathtt{new}\,\rho, \mathtt{new}\,\sigma\rangle\,] \to_1 E[\,\mathtt{new}\,(\rho\otimes\sigma)\,]
\end{aligned}
$$

* Standard small-step one, CBV, but with probabilistic branching (measurement)

Hasuo (Tokyo)

The Language qλ_ℓ

* Roughly: linear λ + quantum primitives
* "Quantum data, classical control"
  * No superposed threads
* Based on [Selinger&Valiron'09]
  * With slight modifications
  * Notably: quantum $\otimes$ vs. linear logic $\boxtimes$
    * The same in [Selinger&Valiron'09] ➜ clean type system, aids programming
    * But... problem with GoI-style semantics

Hasuo (Tokyo)

The Categorical GoI Workflow

Branching monad B

Coalgebraic trace semantics

Traced monoidal category $\mathbb{C}$

+ other constructs ➜ "GoI situation" [AHS02]

Categorical GoI [AHS02]

Linear combinatory algebra

Realizability

Linear category

Hasuo (Tokyo)

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Model of quantum language

Hasuo (Tokyo)

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Categorical GoI [AHS02]

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Quantum branching monad

Quantum TSMC

Quantum LCA

Model of quantum language

Different Branching in The GoI Animation

- **Pfn** (partial functions)
  - Pipe can be stuck
- **Rel** (relations)
  - Pipe can branch
- **DSRel**
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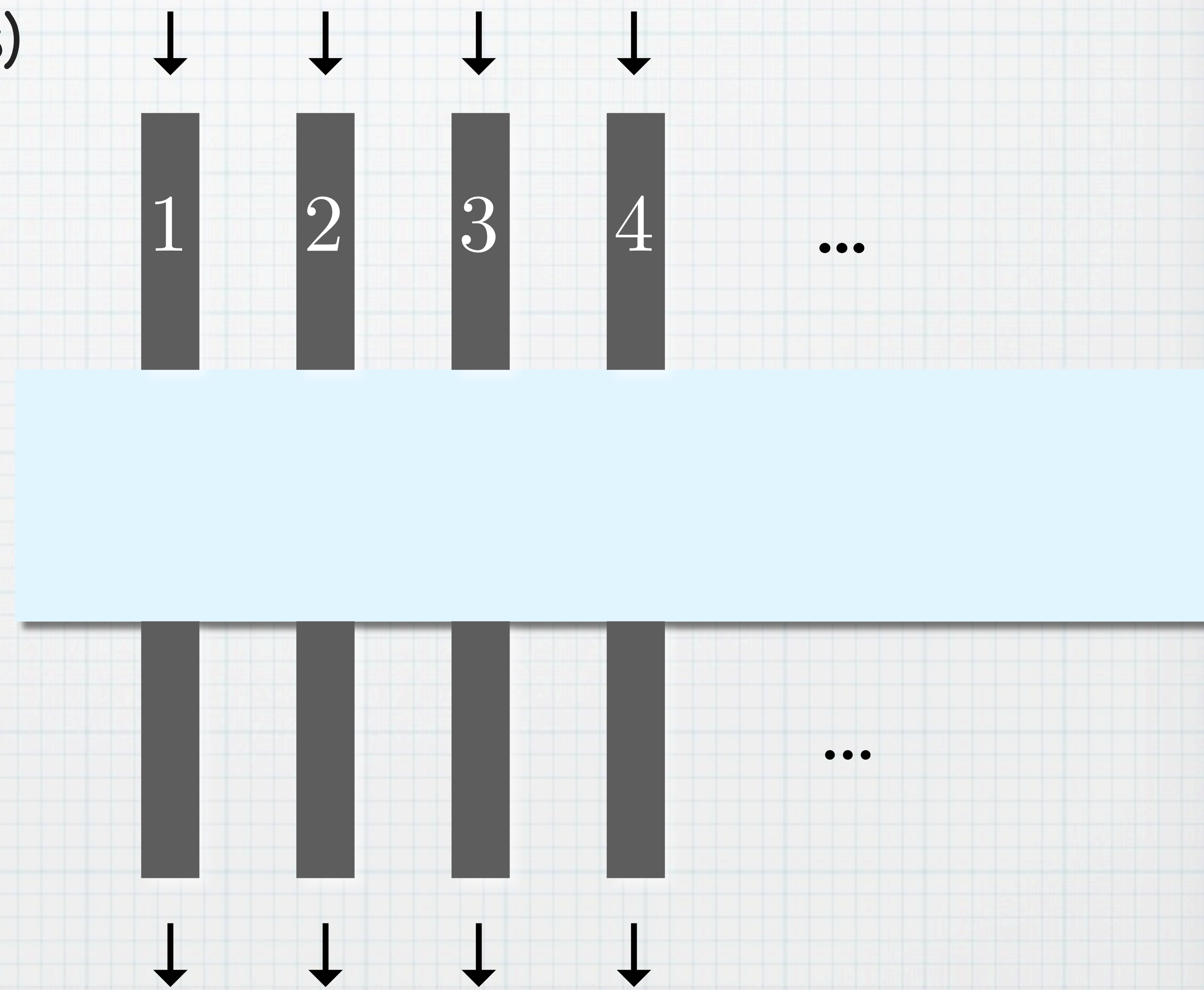

Hasuo (Tokyo)

Different Branching in The GoI Animation

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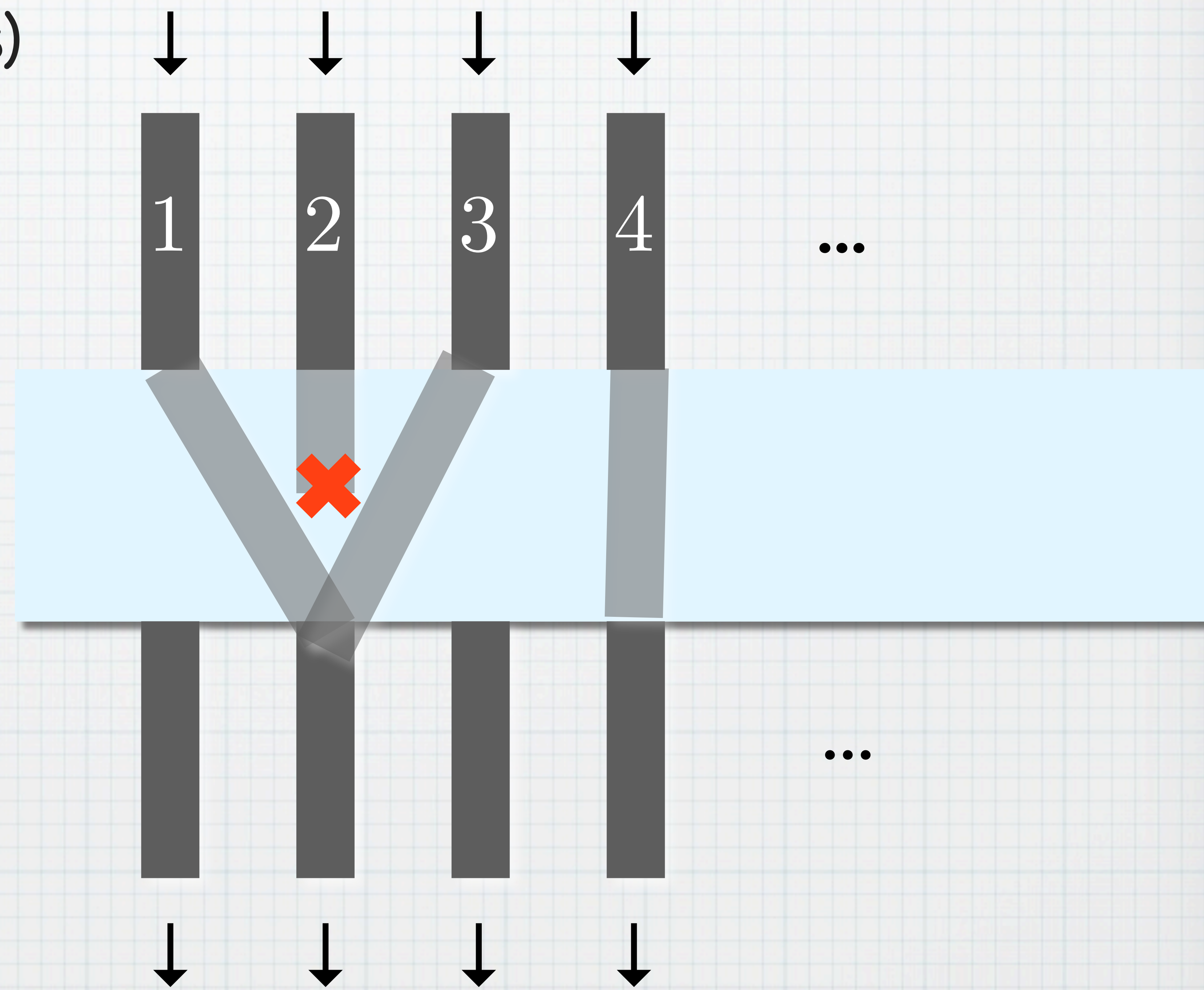

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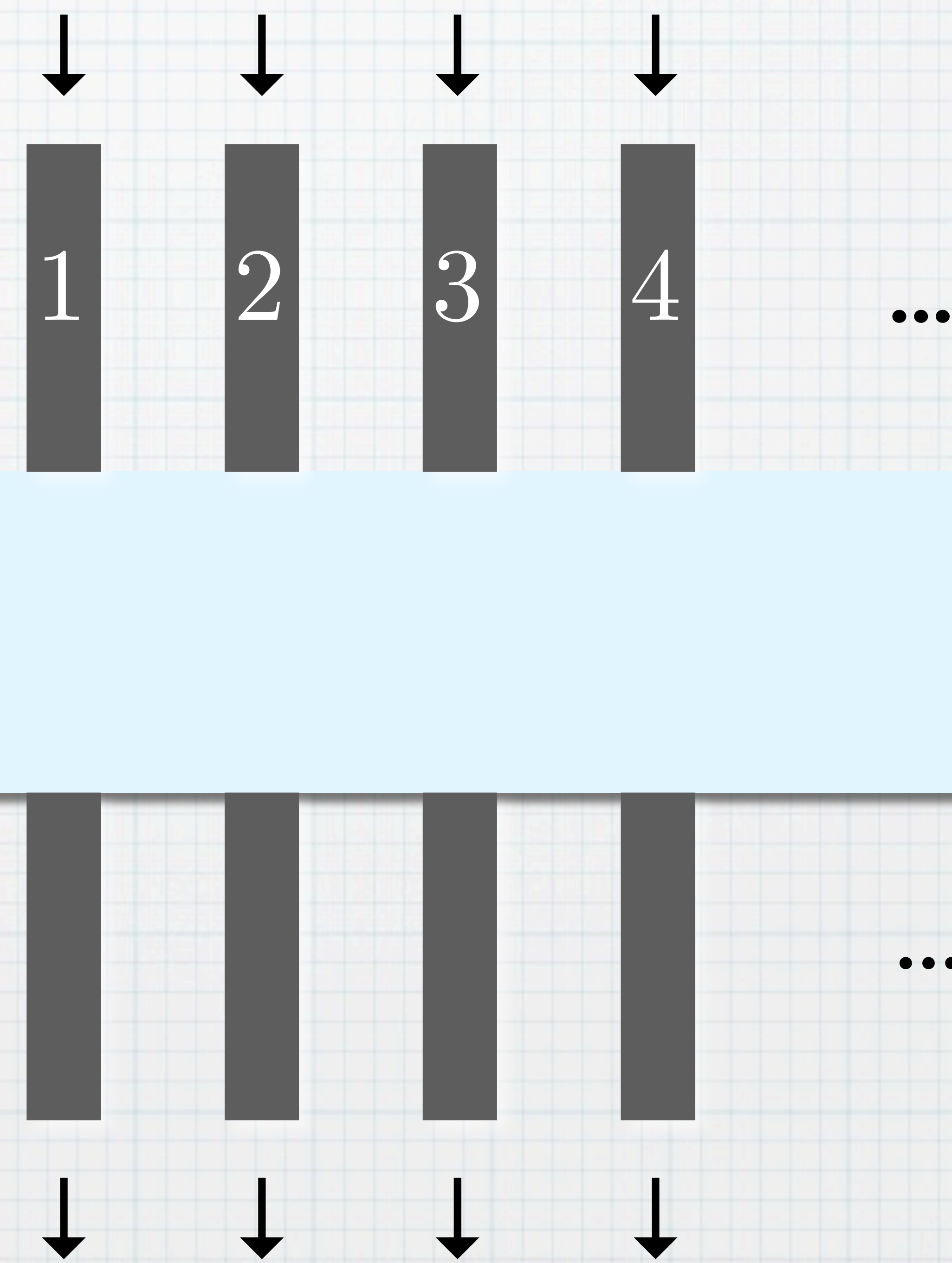

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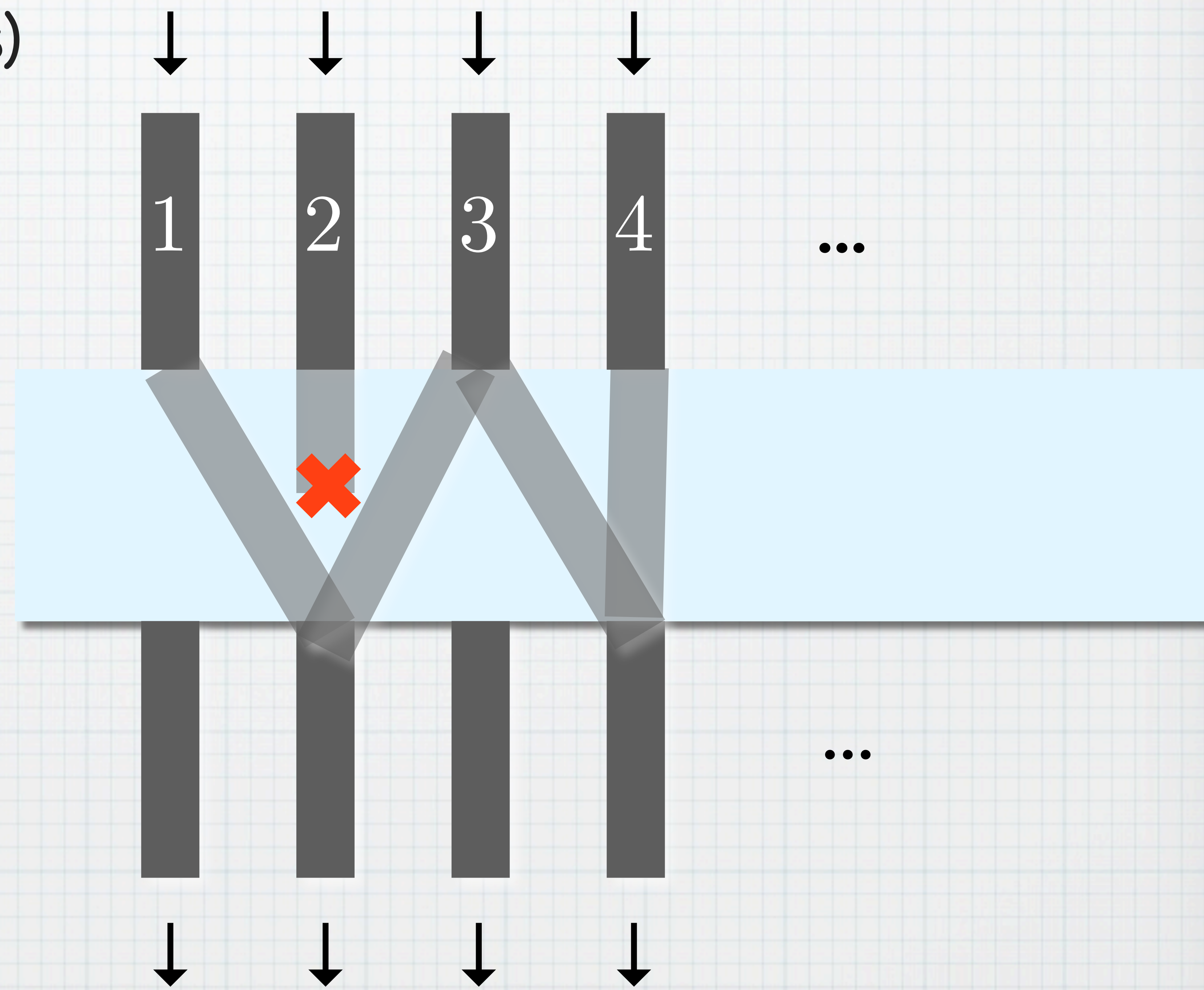

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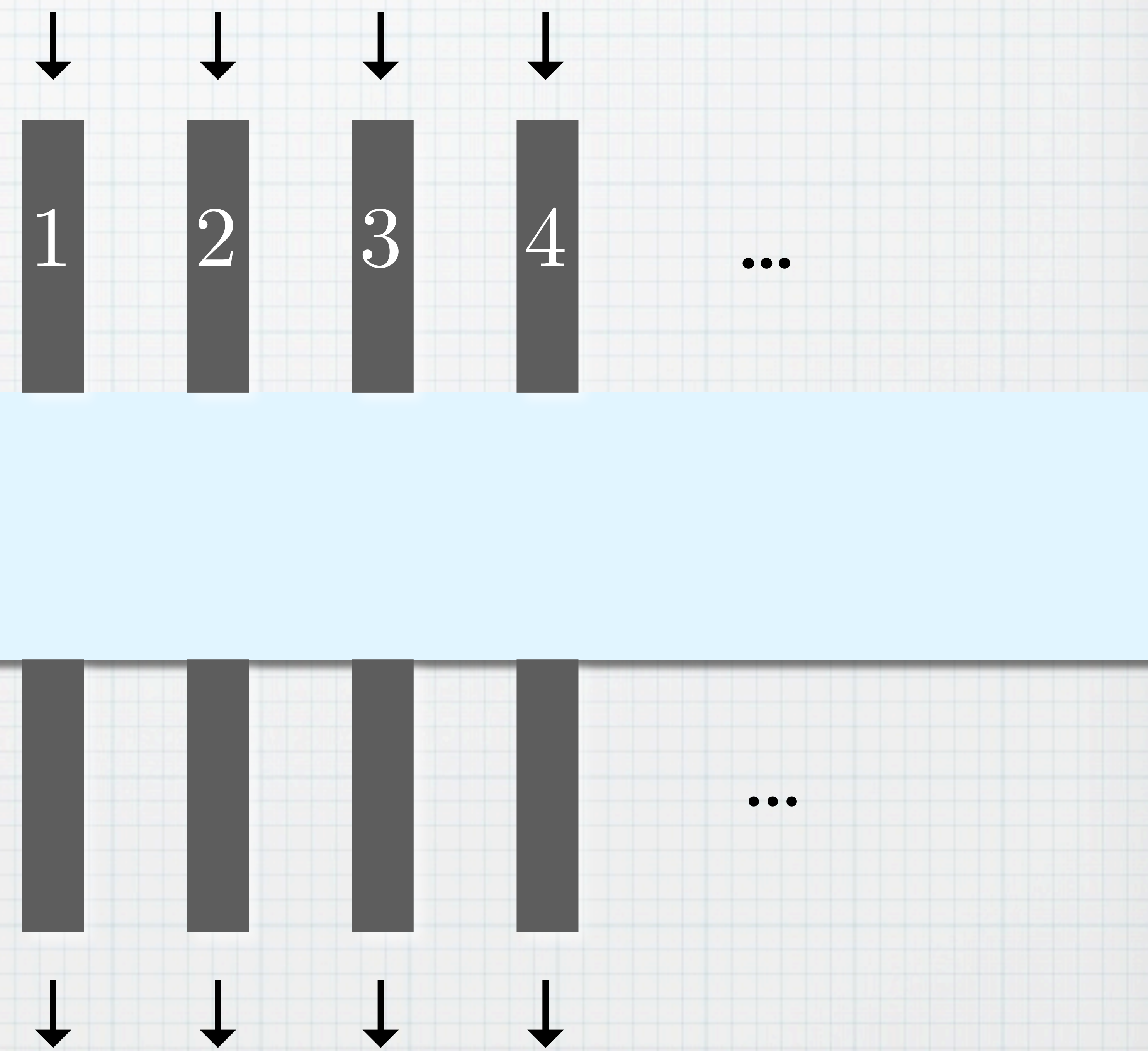

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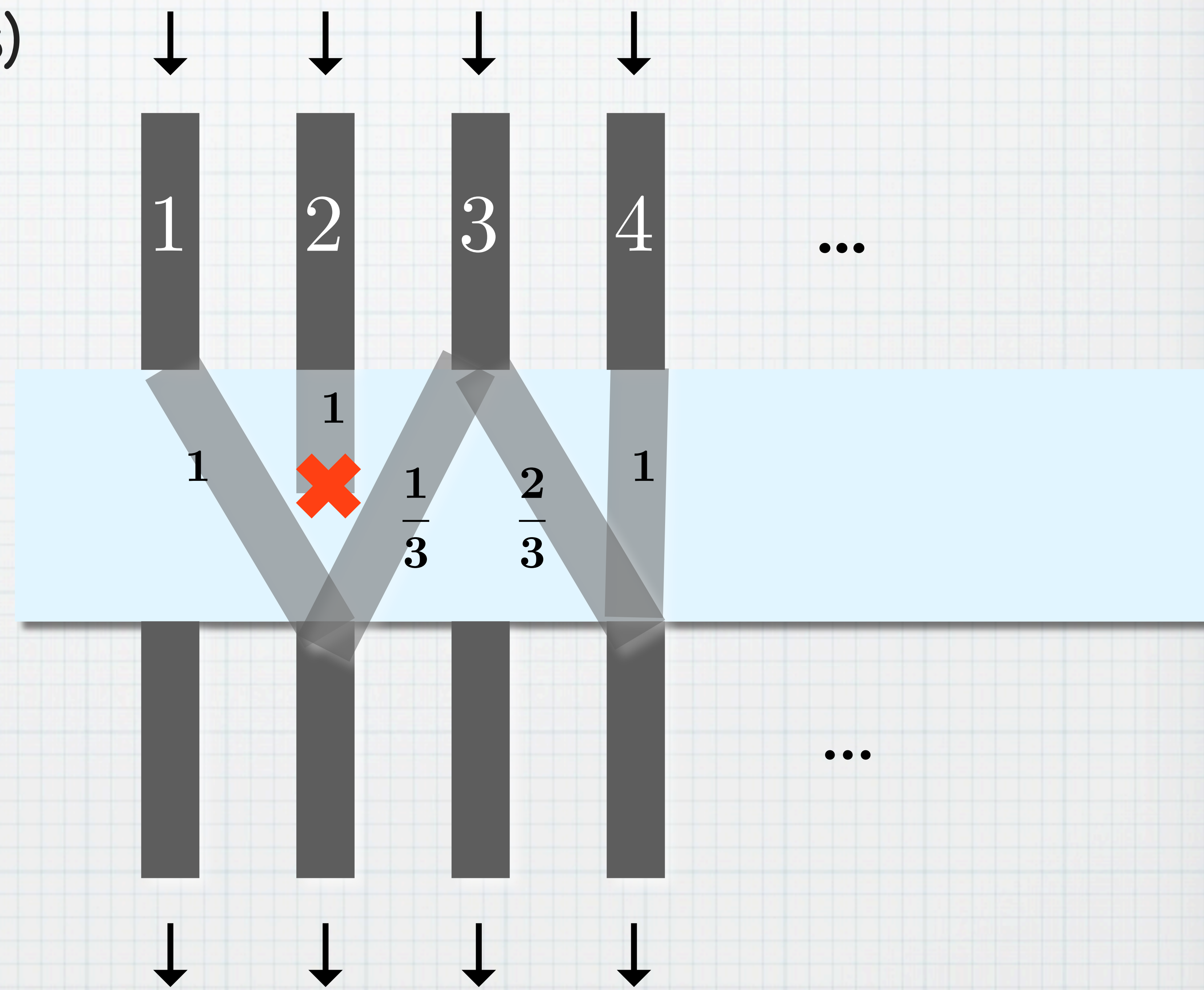

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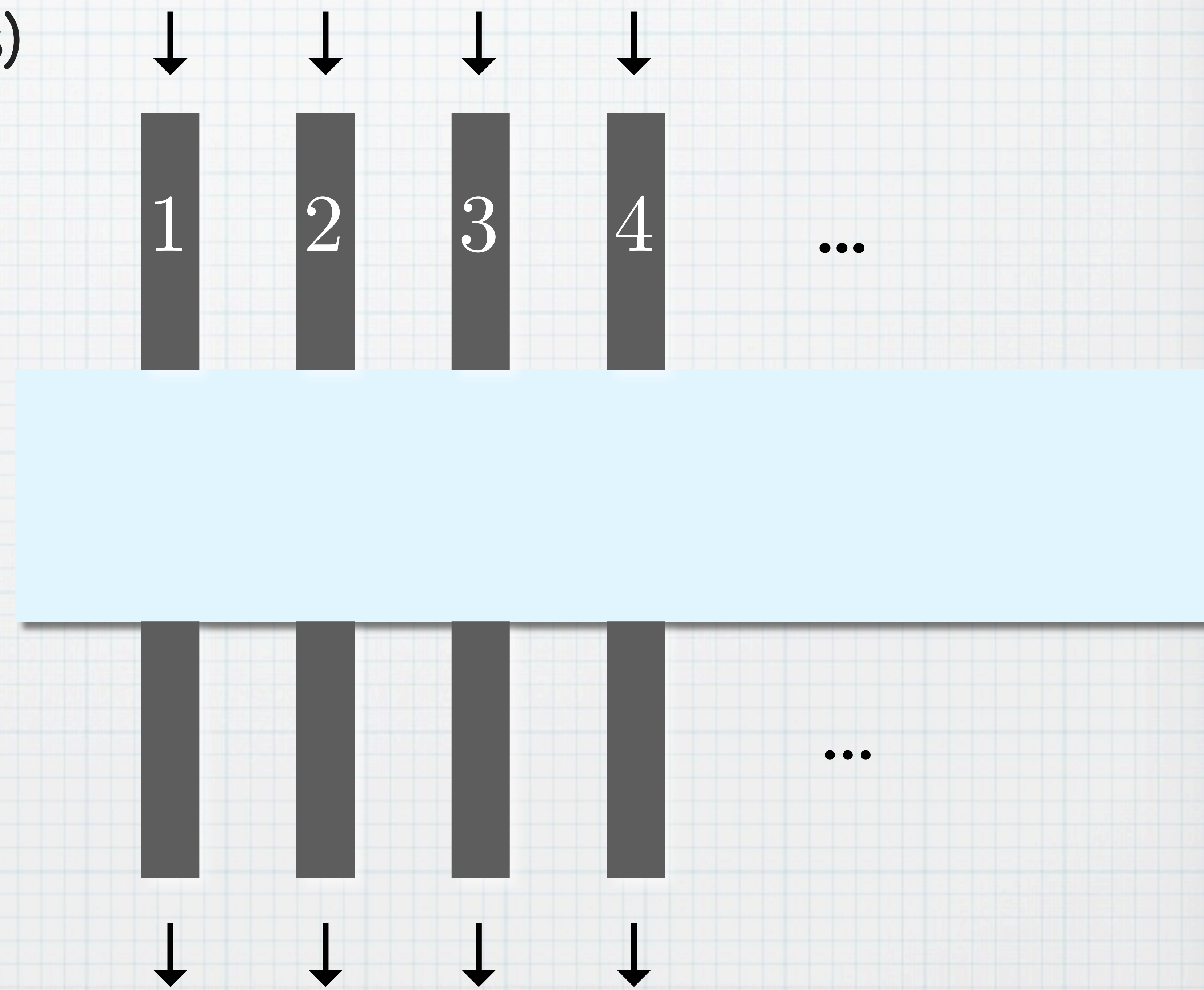

Hasuo (Tokyo)

Different Branching in The

$Kl(\mathcal{L})$, non-termination

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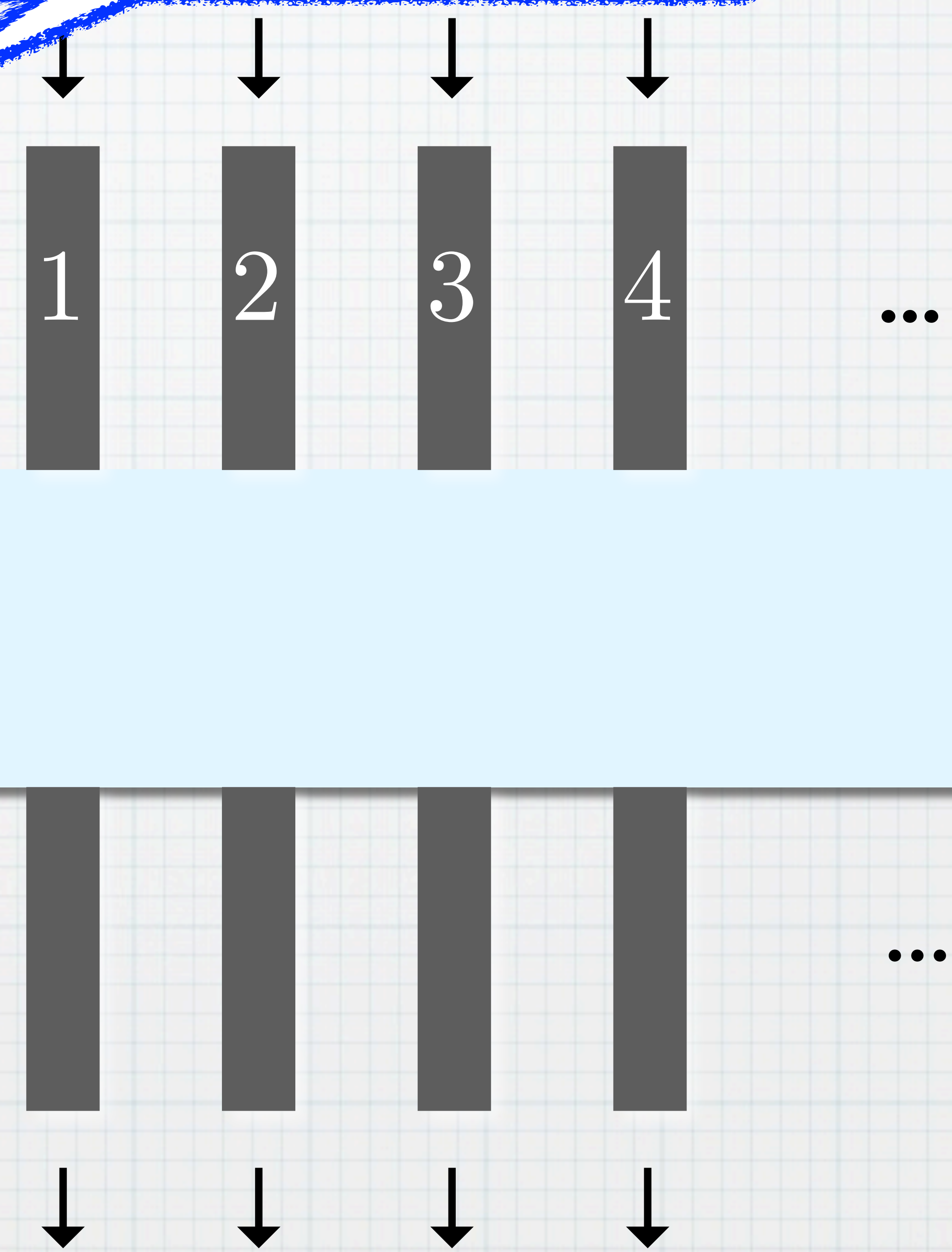

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$Kl(\mathcal{P})$, non-determinism

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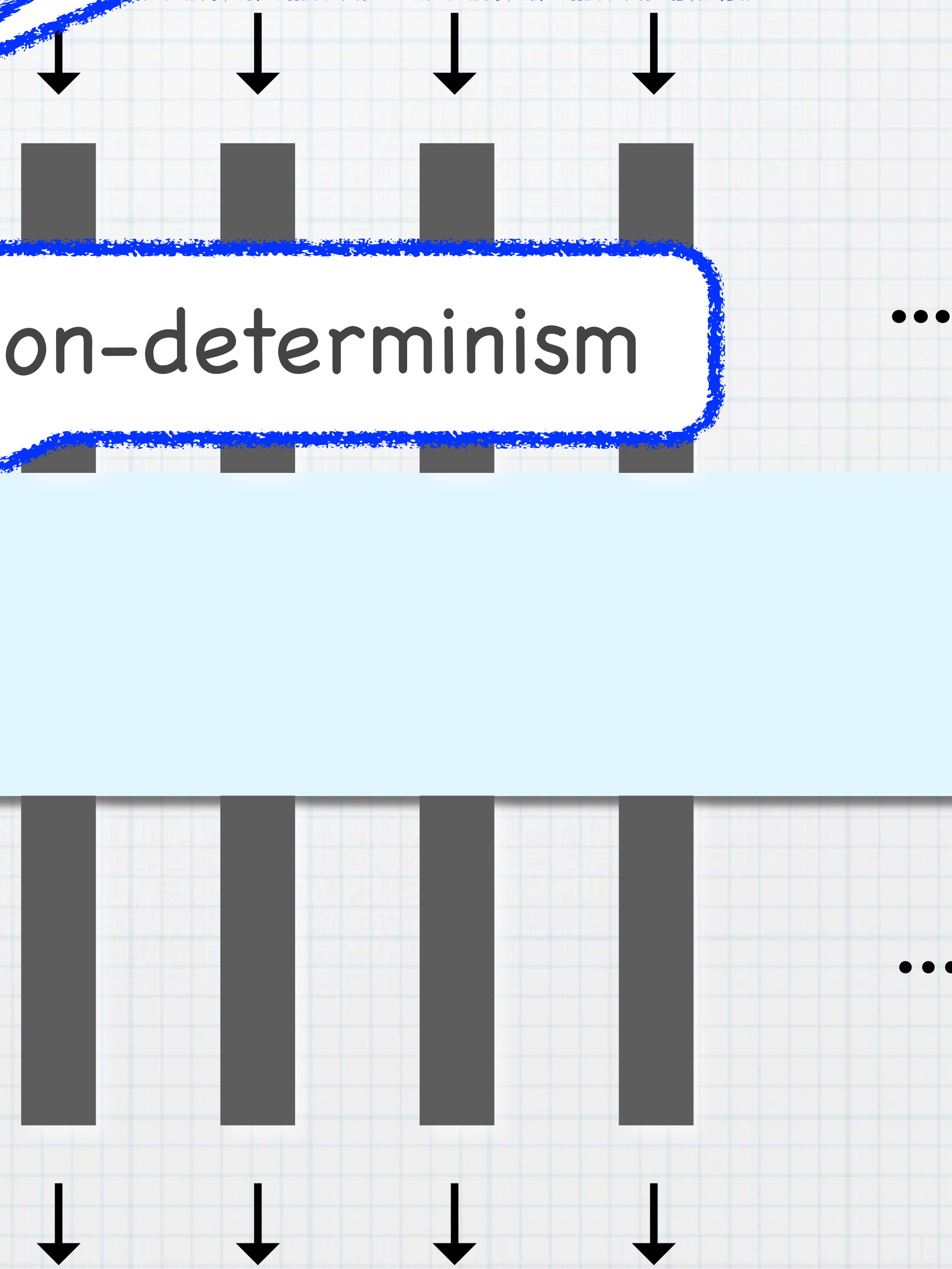

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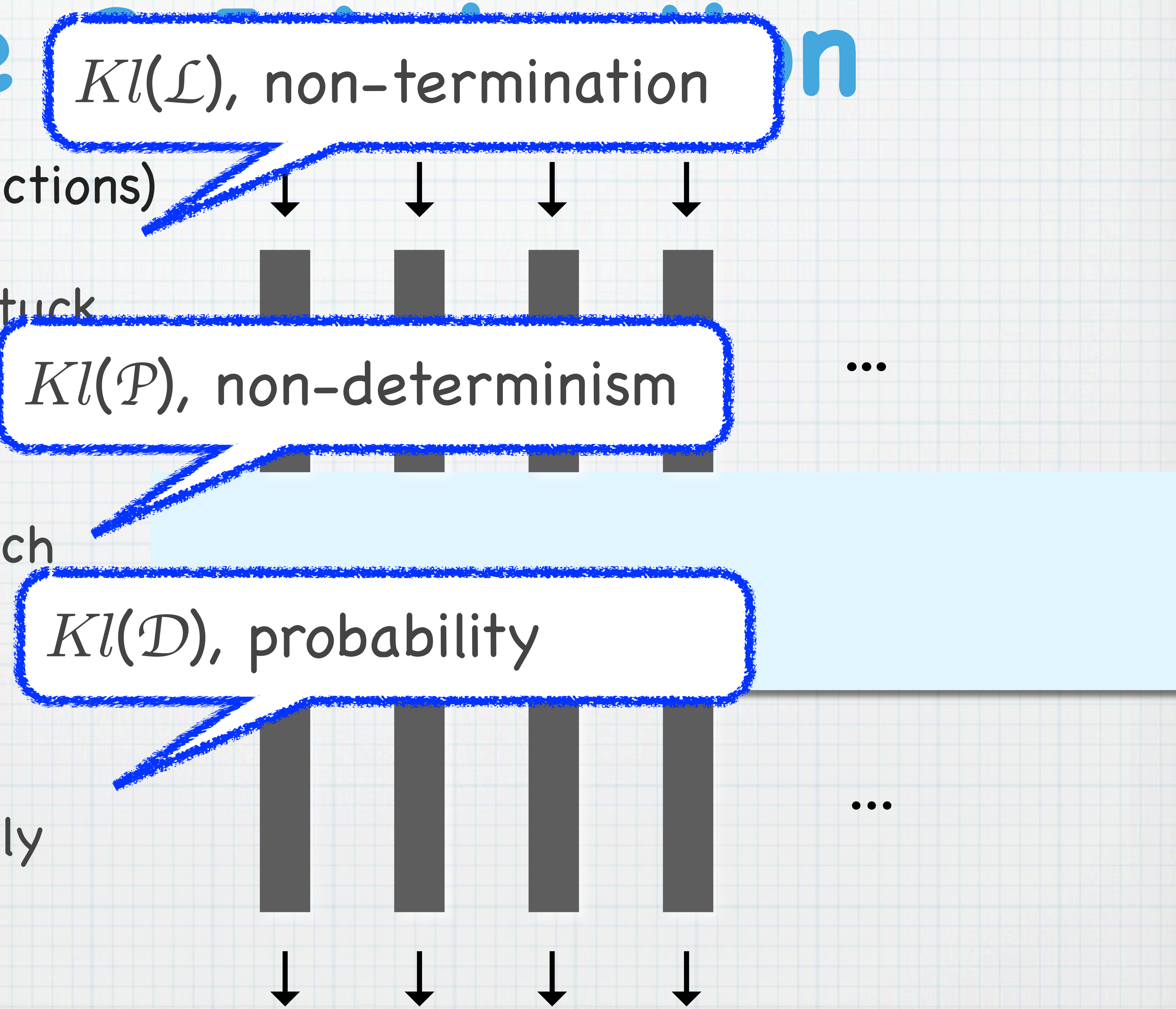
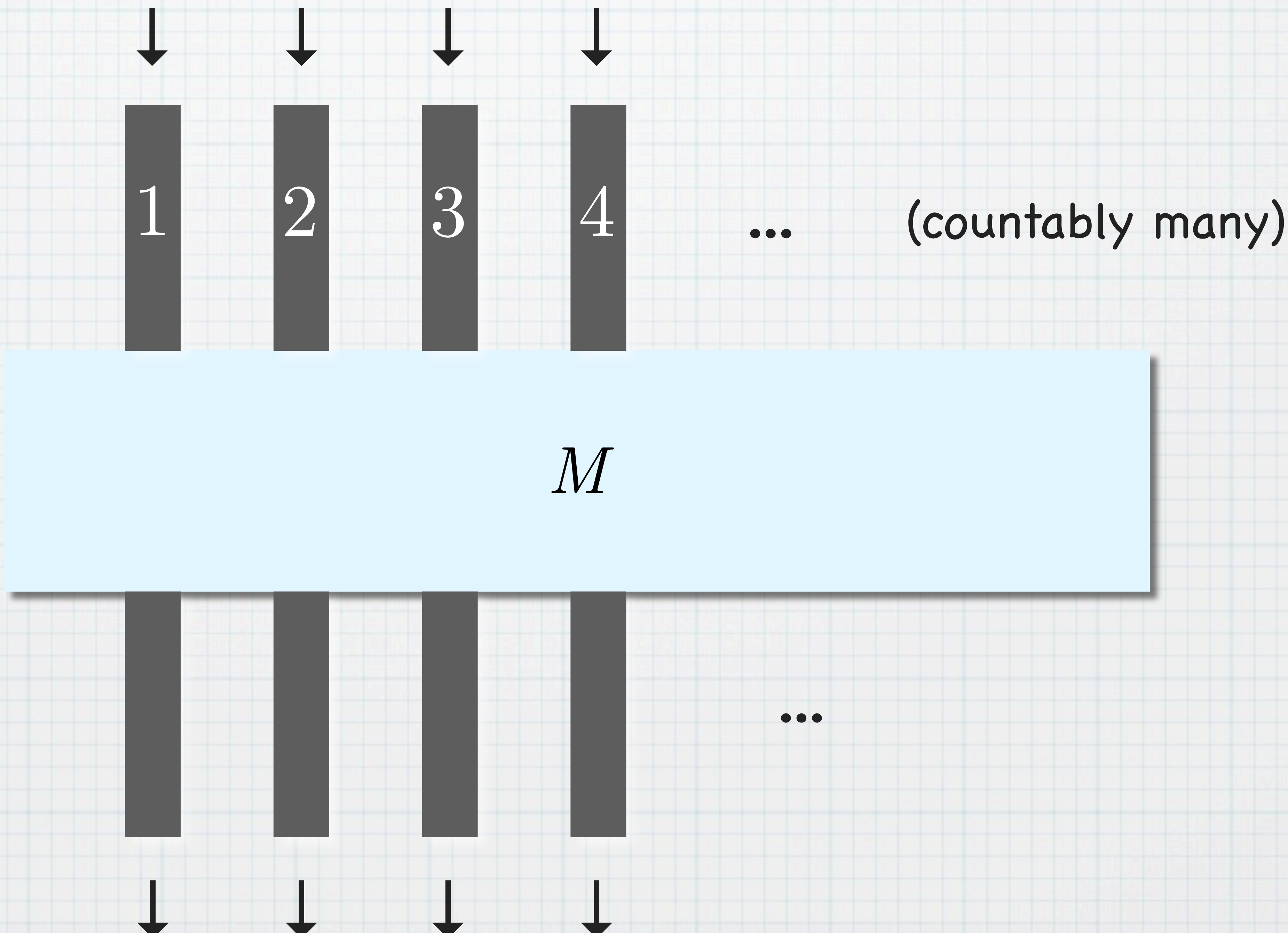


Quantum
Geometry of Interaction

$$[\![M]\!] =$$



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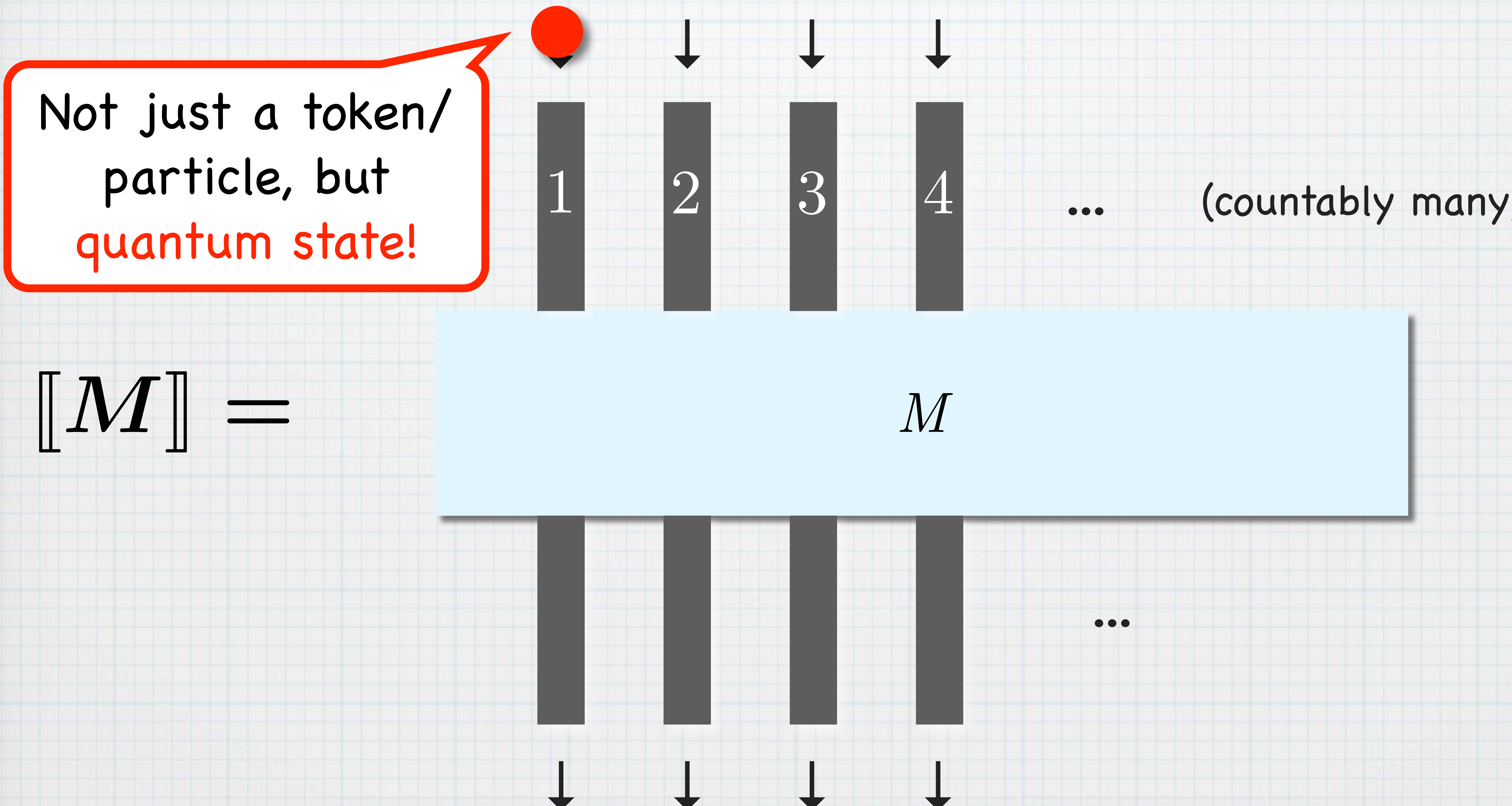

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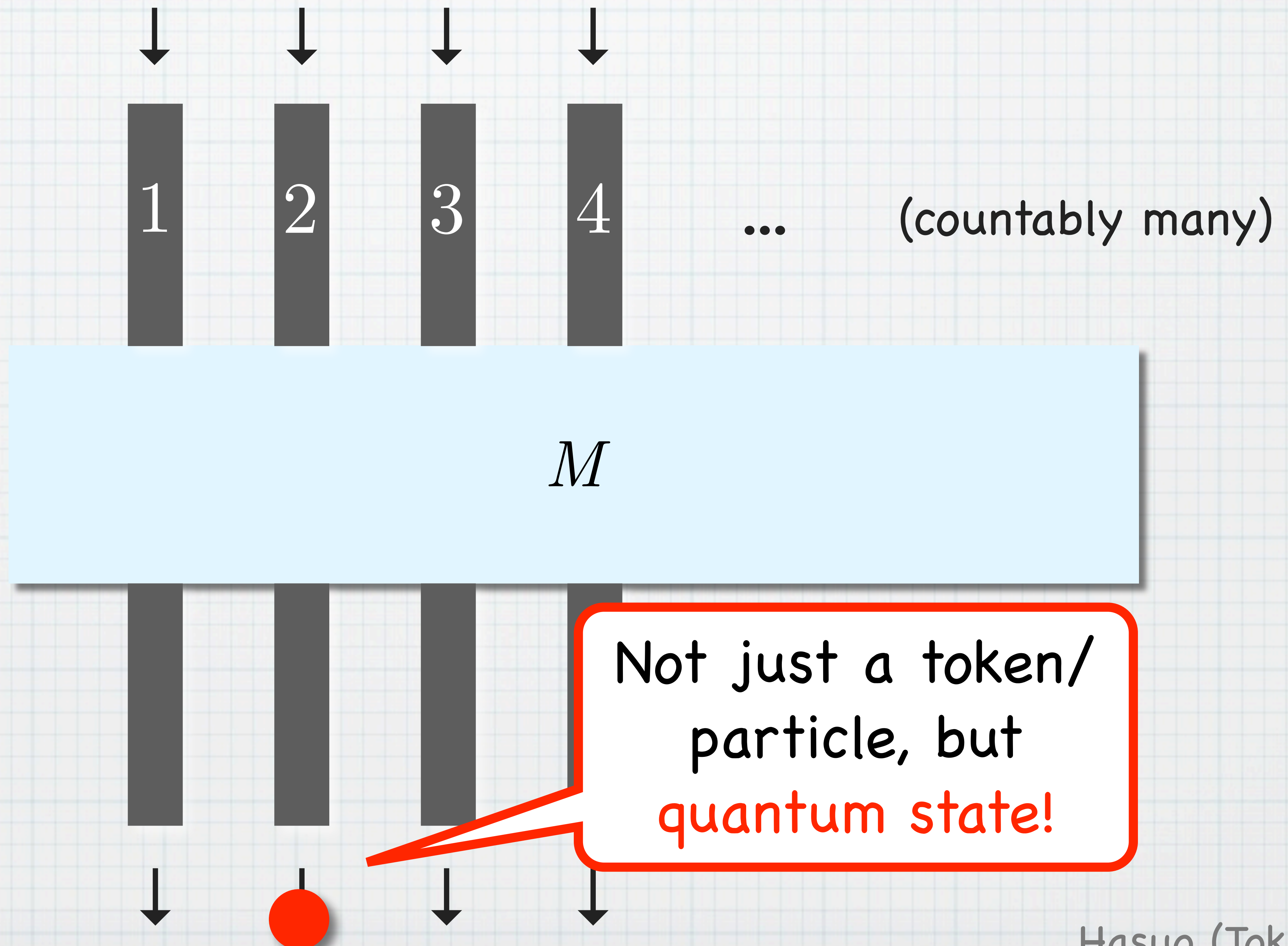

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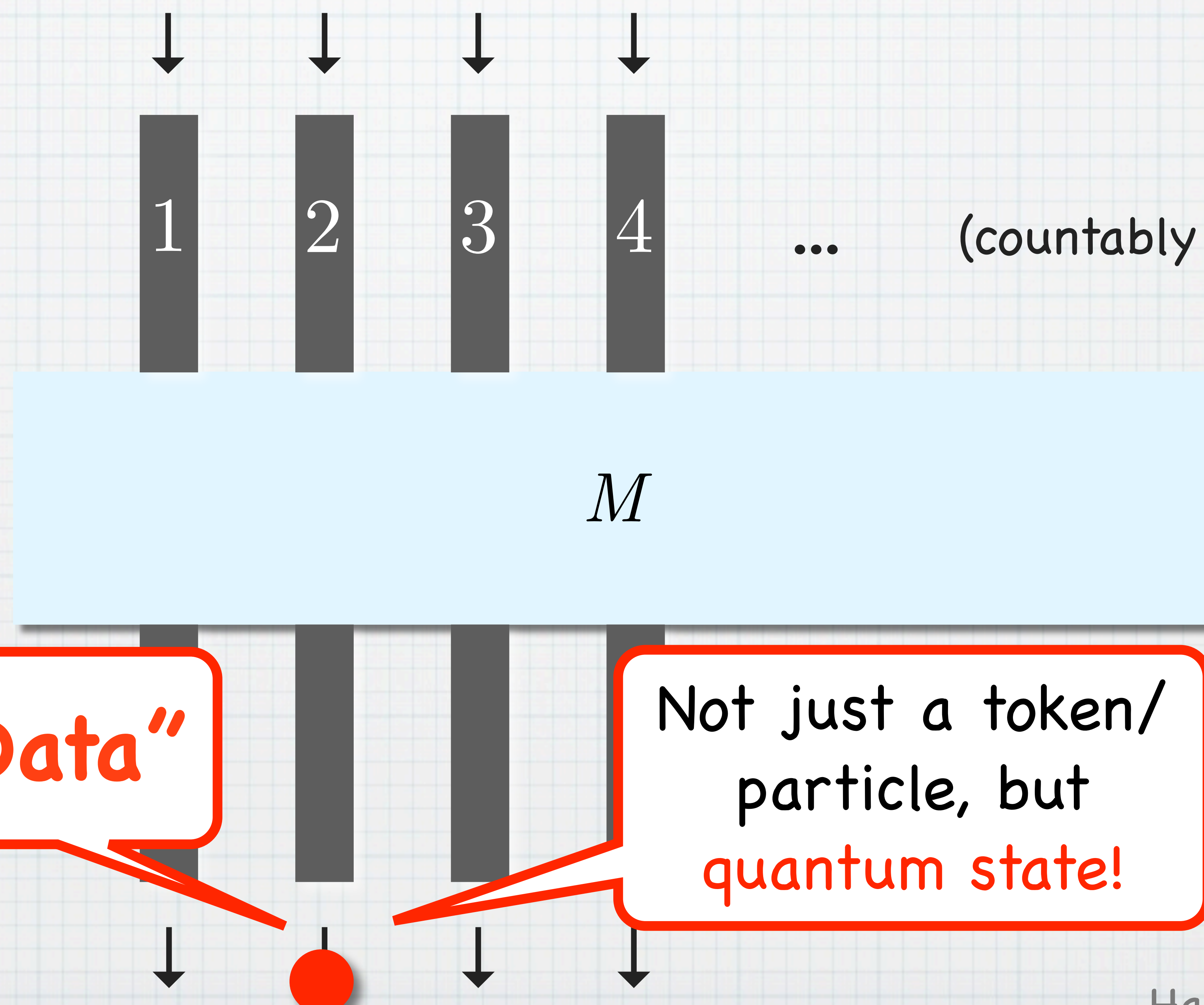

(countably many)

"Quantum Data"

Not just a token/ particle, but quantum state!

Quantum
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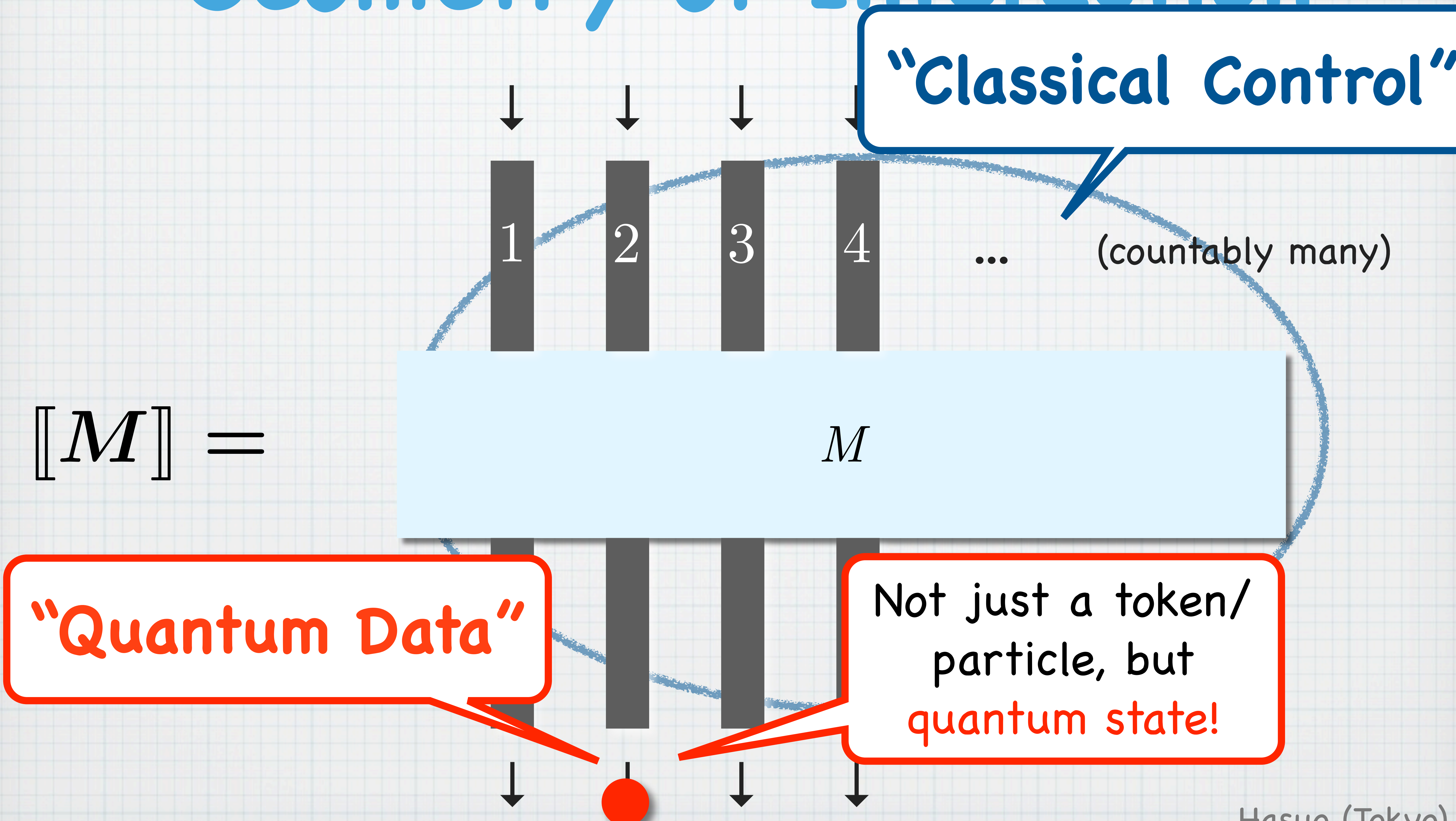

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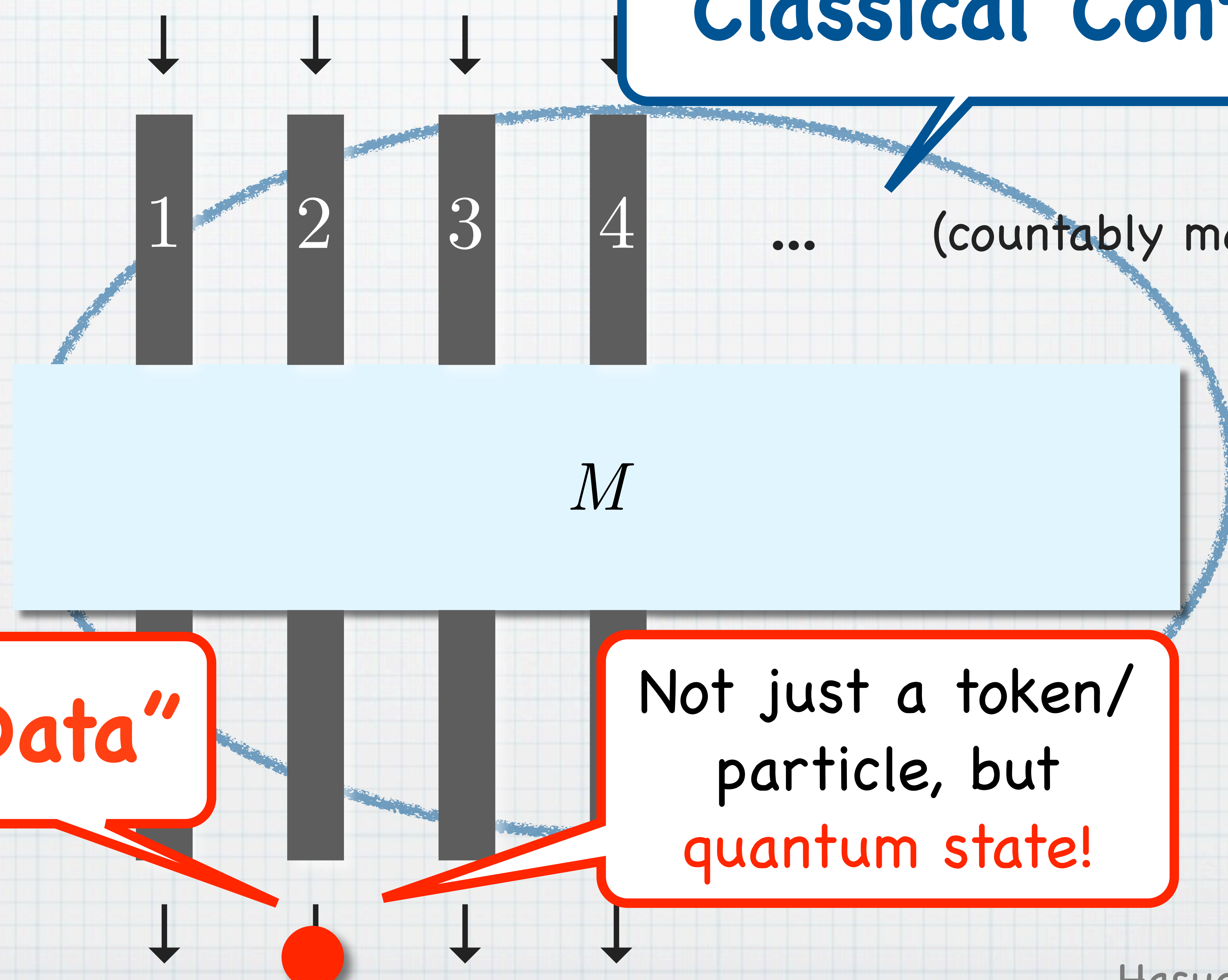

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The Quantum Branching Monad

$$\mathcal{Q}Y = \left\{ c : Y \to \prod_{m,n \in \mathbb{N}} \mathbf{QO}_{m,n} \;\middle|\; \text{the trace condition} \right\}$$

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(trace of matrix ≈ probability)

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* Given $x \in X$, $y \in Y$, $m \in \mathbb{N}$, $n \in \mathbb{N}$ determines a quantum operation

$$\Big(f(x)(y) \Big)_{m,n} \; : \; D_m \to D_n$$

* Subject to the **trace condition**

Any opr. on quantum data:
combination of

- preparation
- unitary transf.
- measurement

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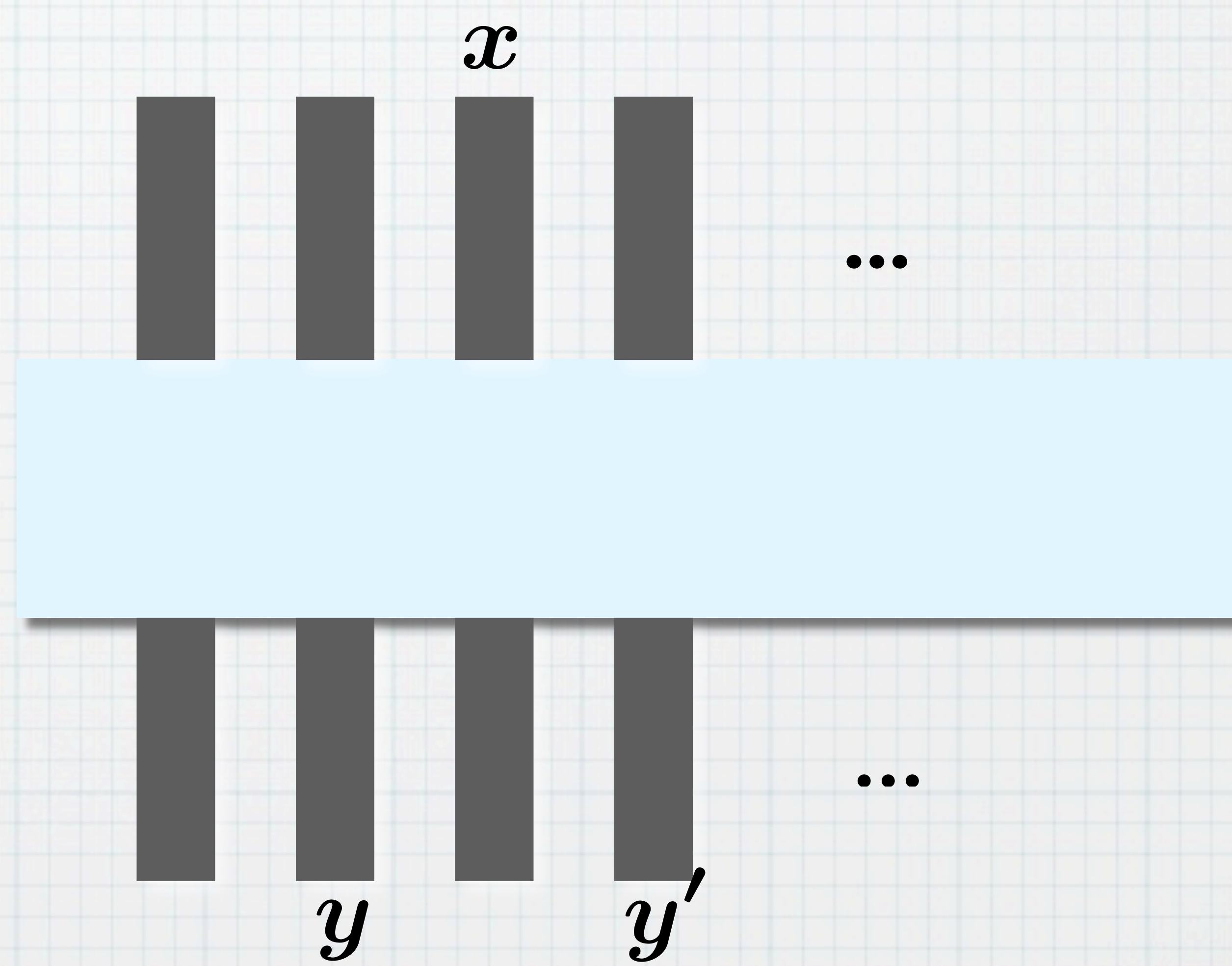

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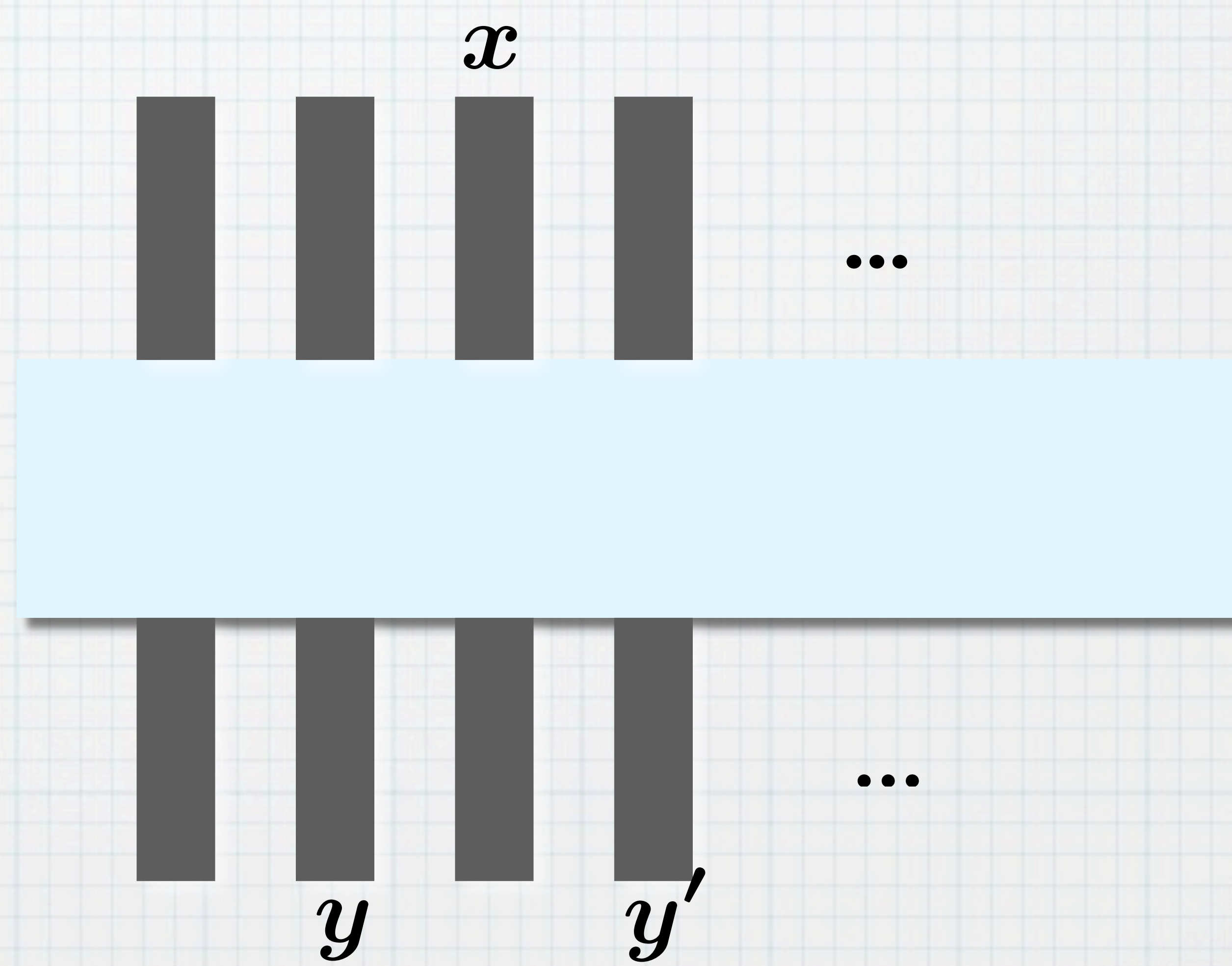

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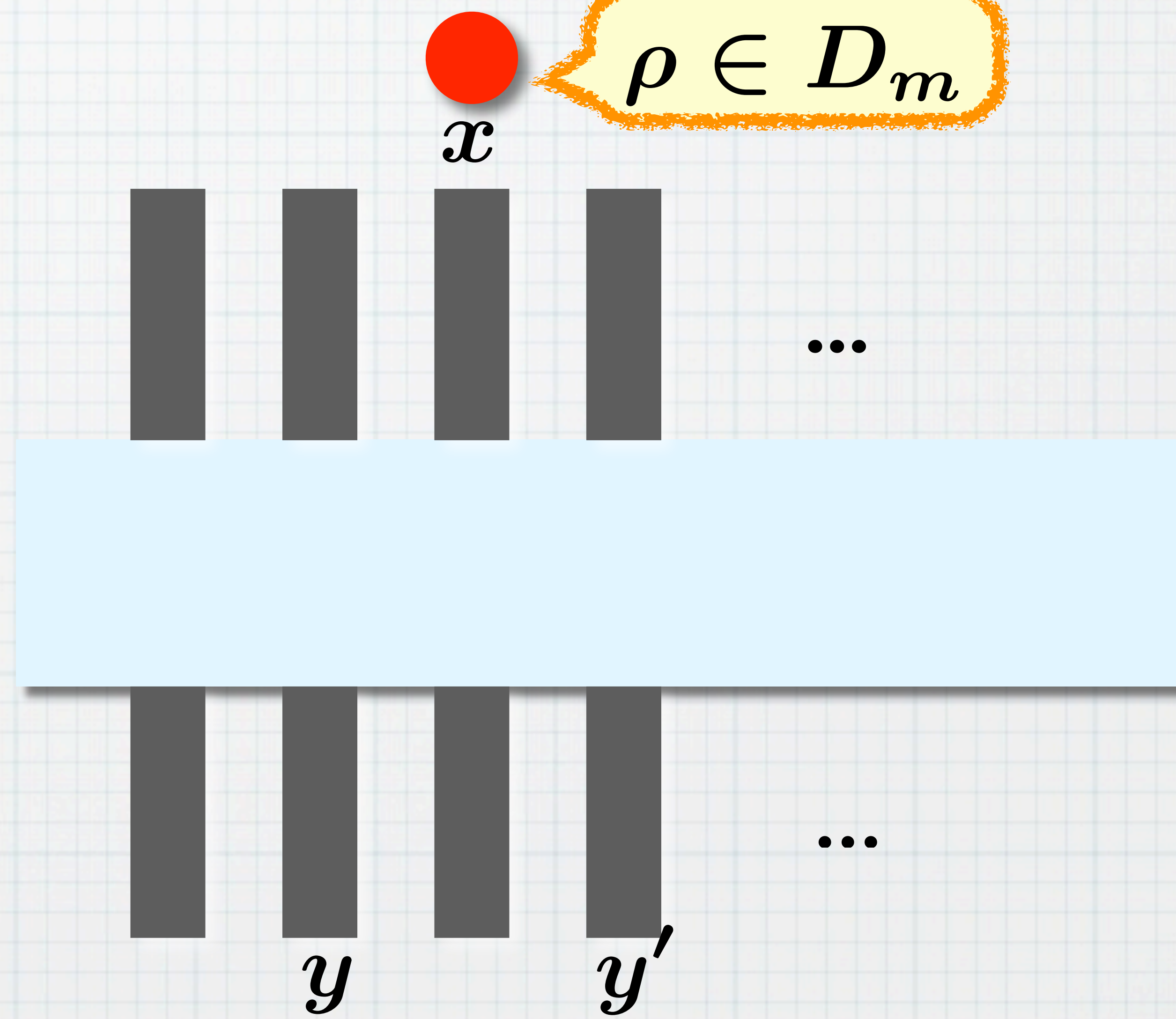

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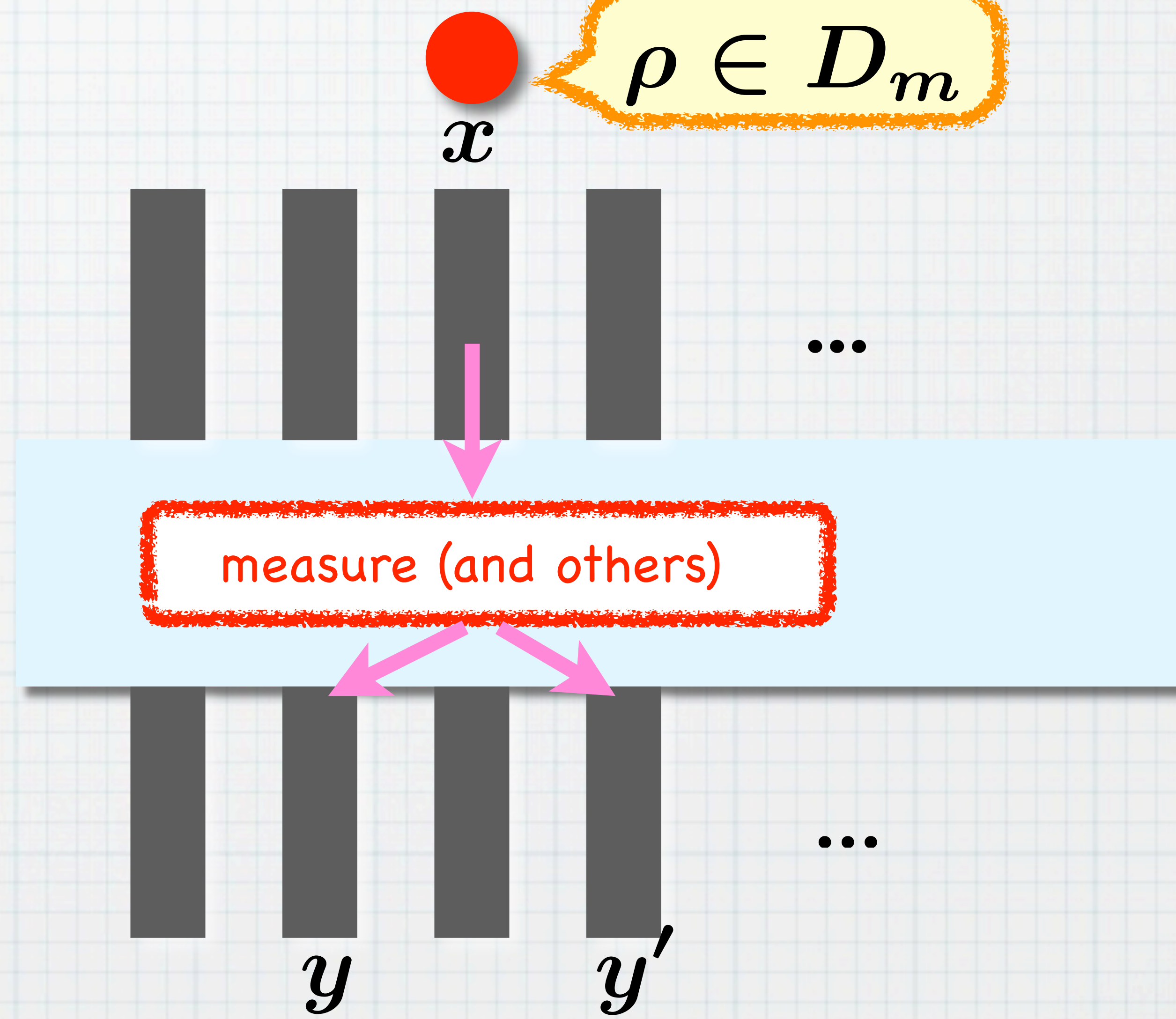
$\rho \in D_m$



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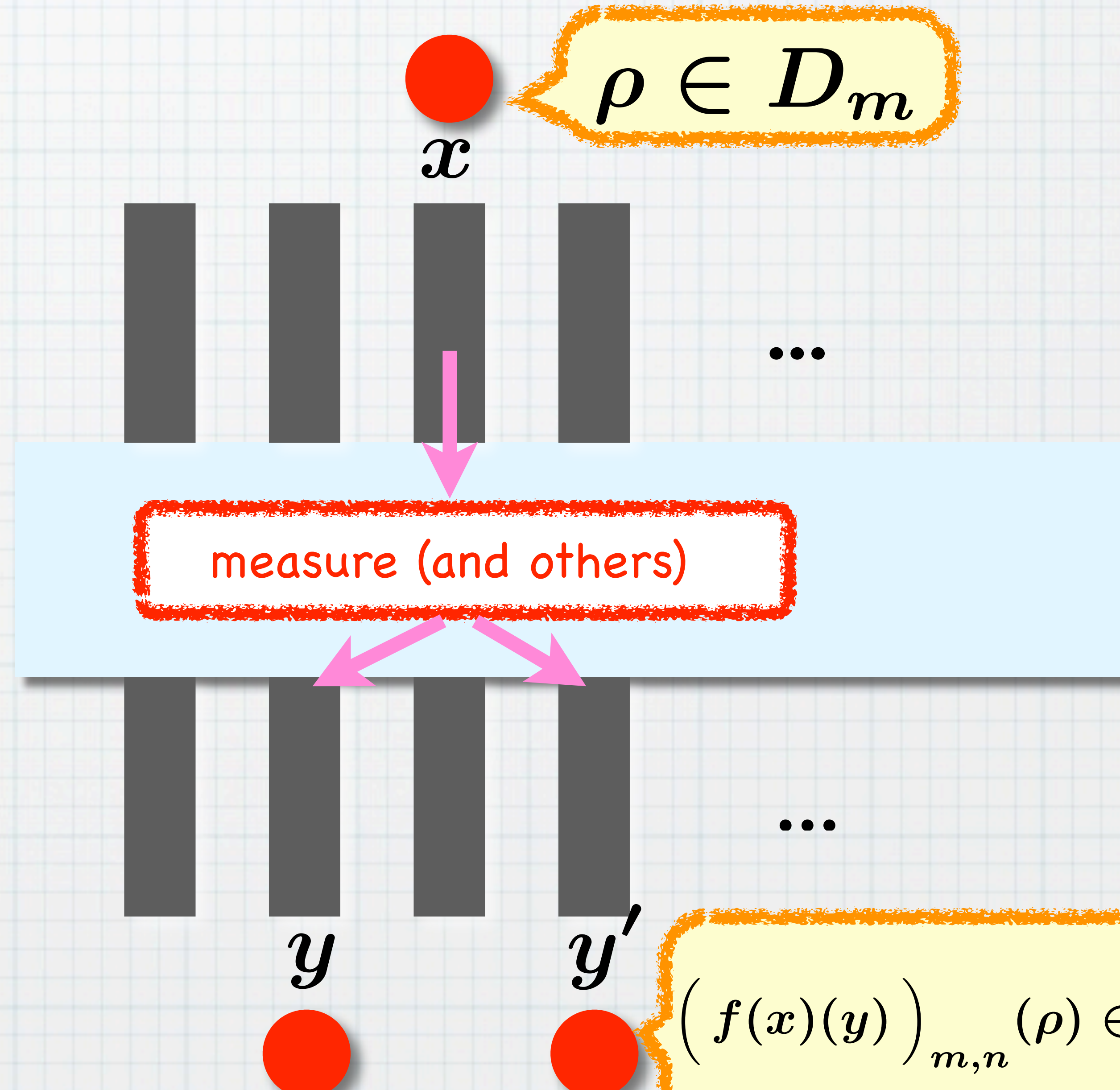

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for each n

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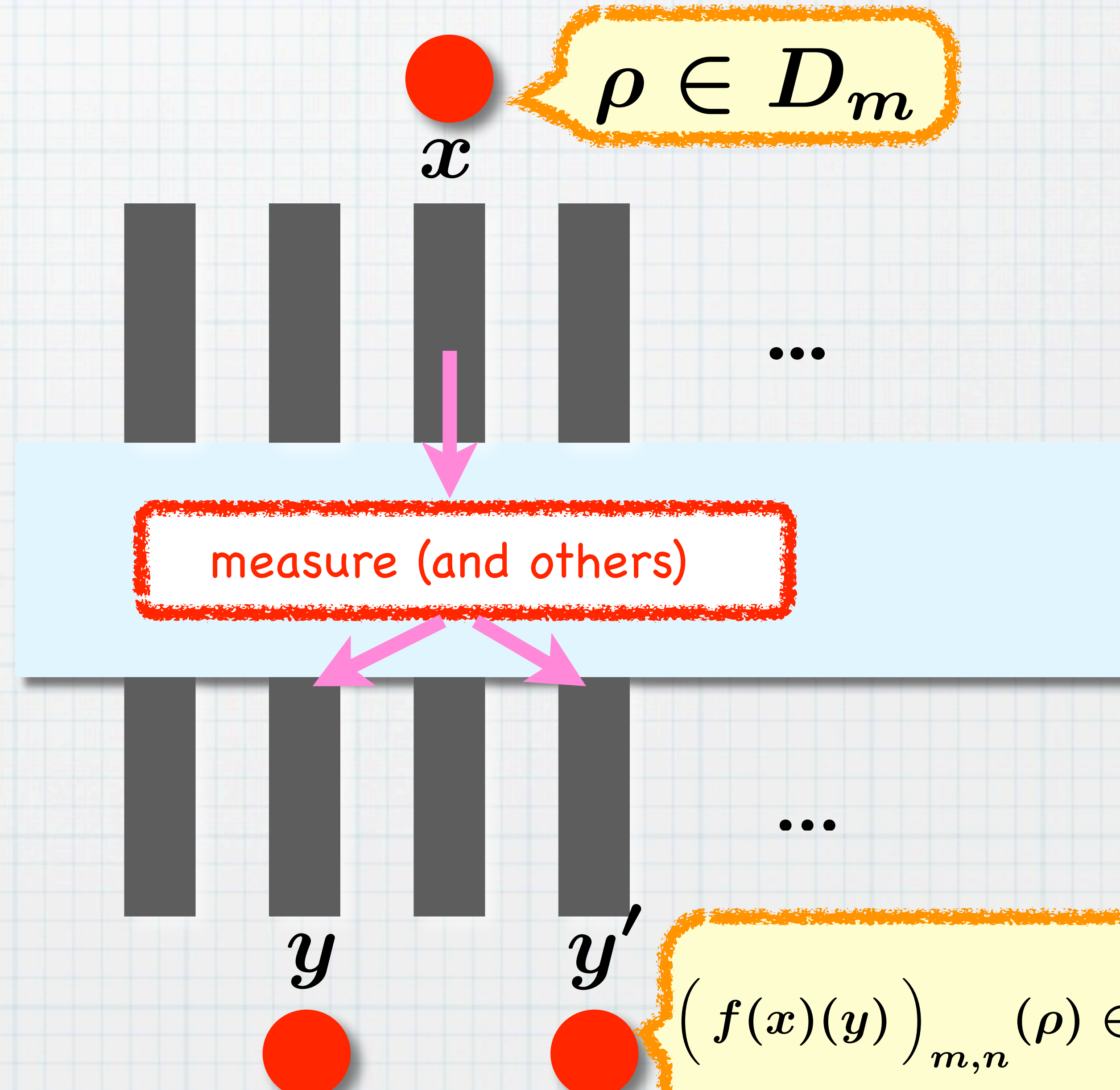
$$\sum_{y,n} \Pr\begin{pmatrix} \text{Token led} \\ \text{to } y \\ \text{with dim. } n \end{pmatrix} \leq 1$$



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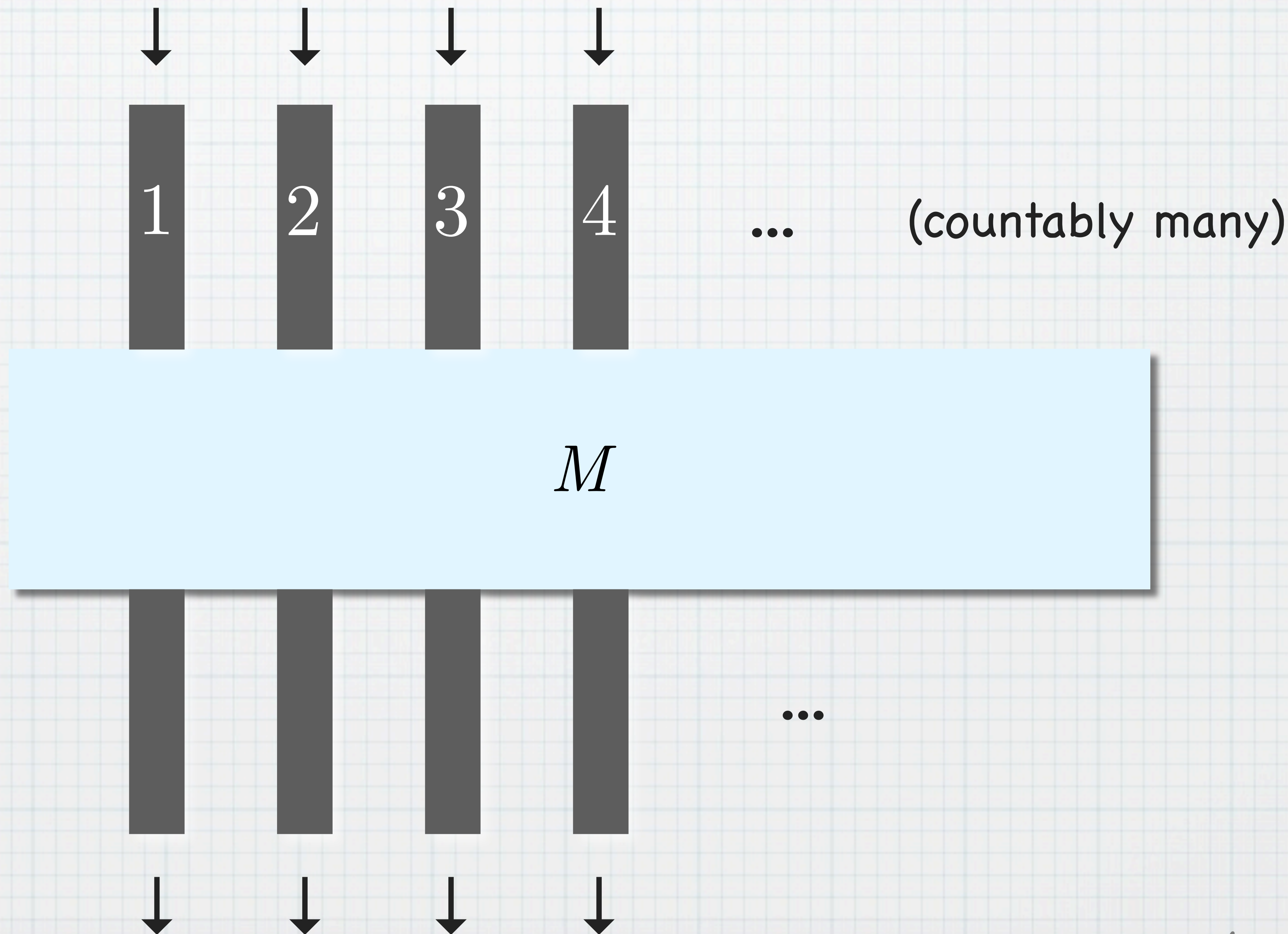
Quantum
Geometry of Interaction

$$[\![M]\!] =$$



Hasuo (Tokyo)

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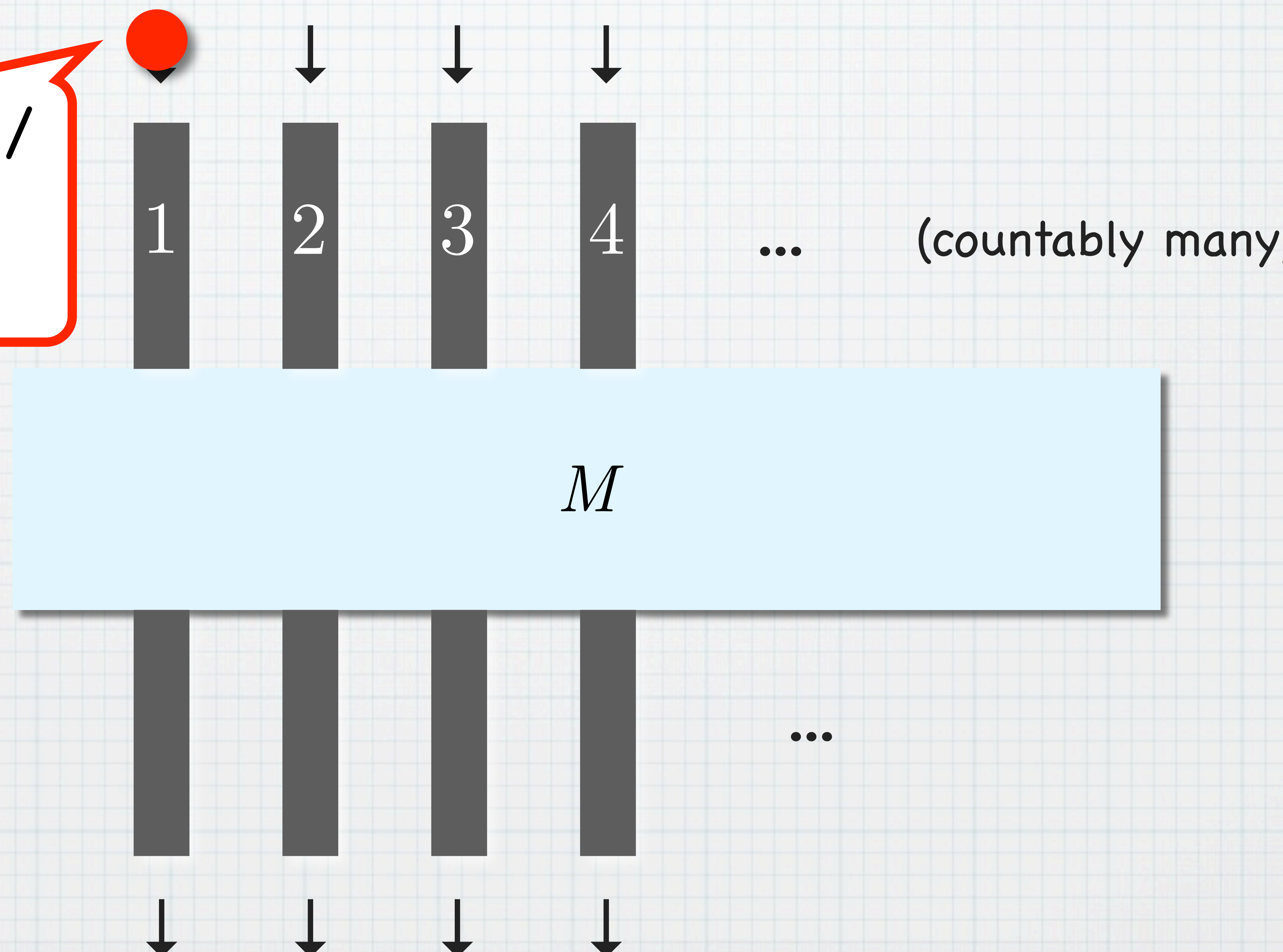

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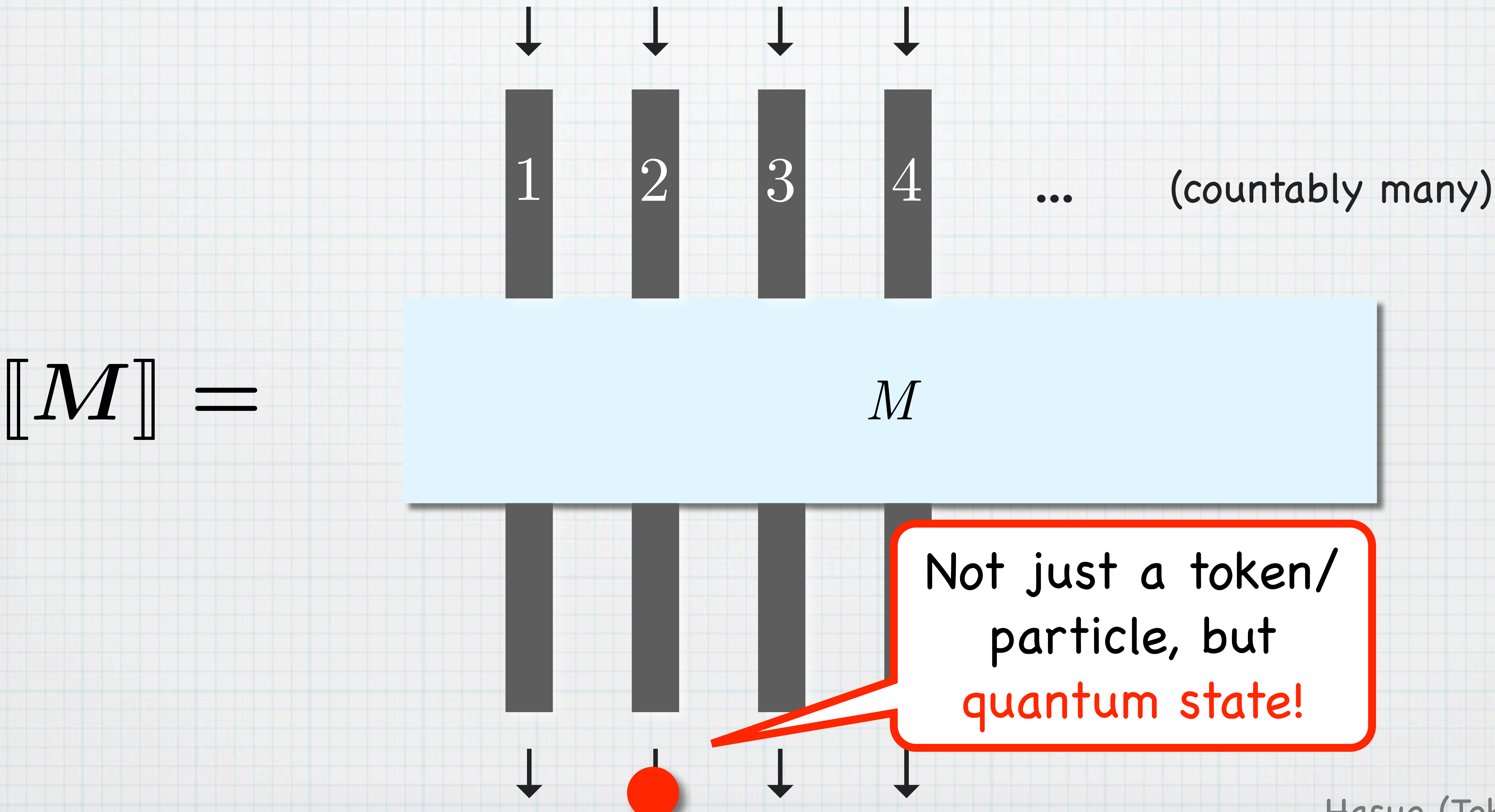

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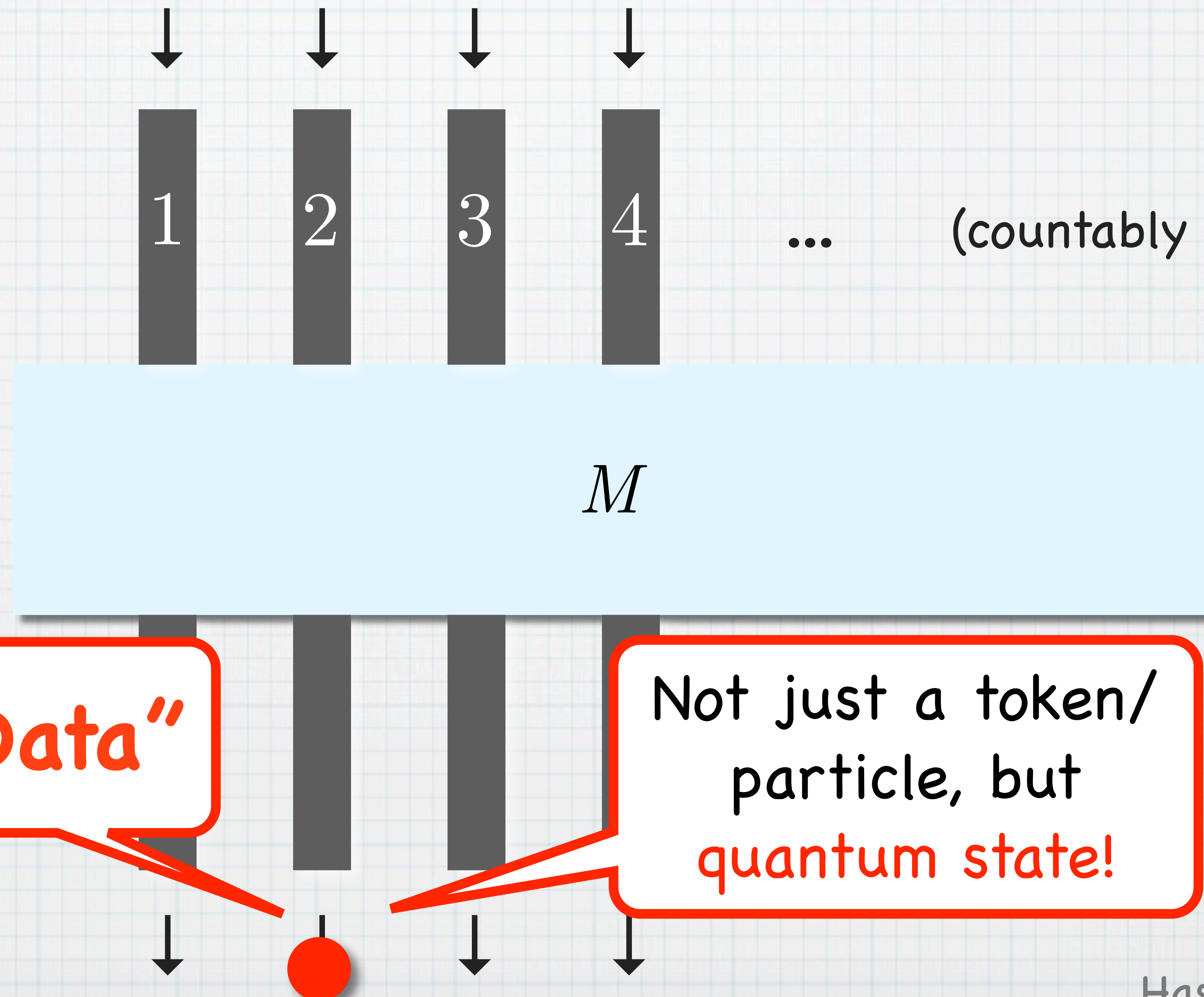

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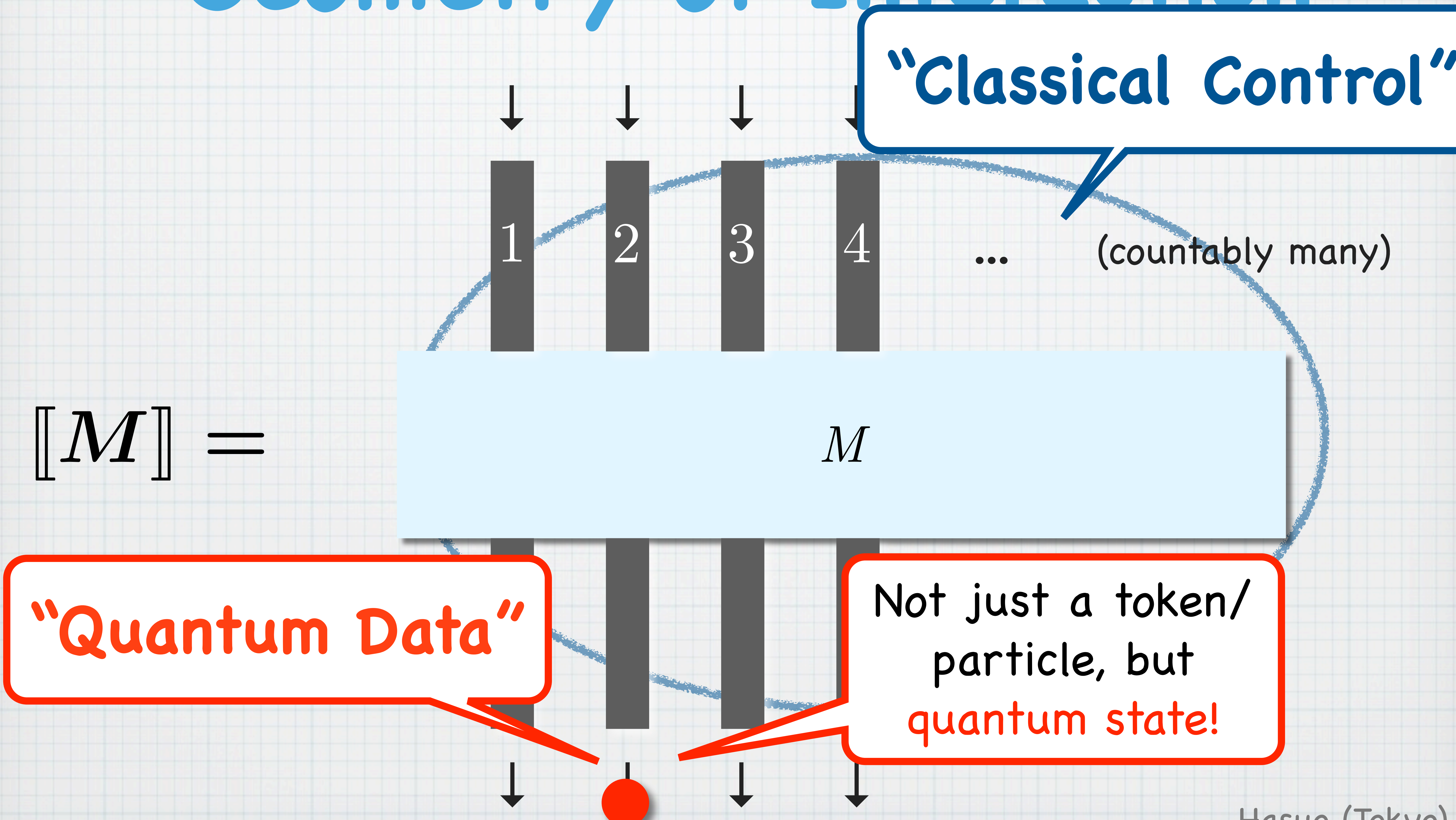

Hasuo (Tokyo)

Quan…
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"Classical Control"

(countably many)

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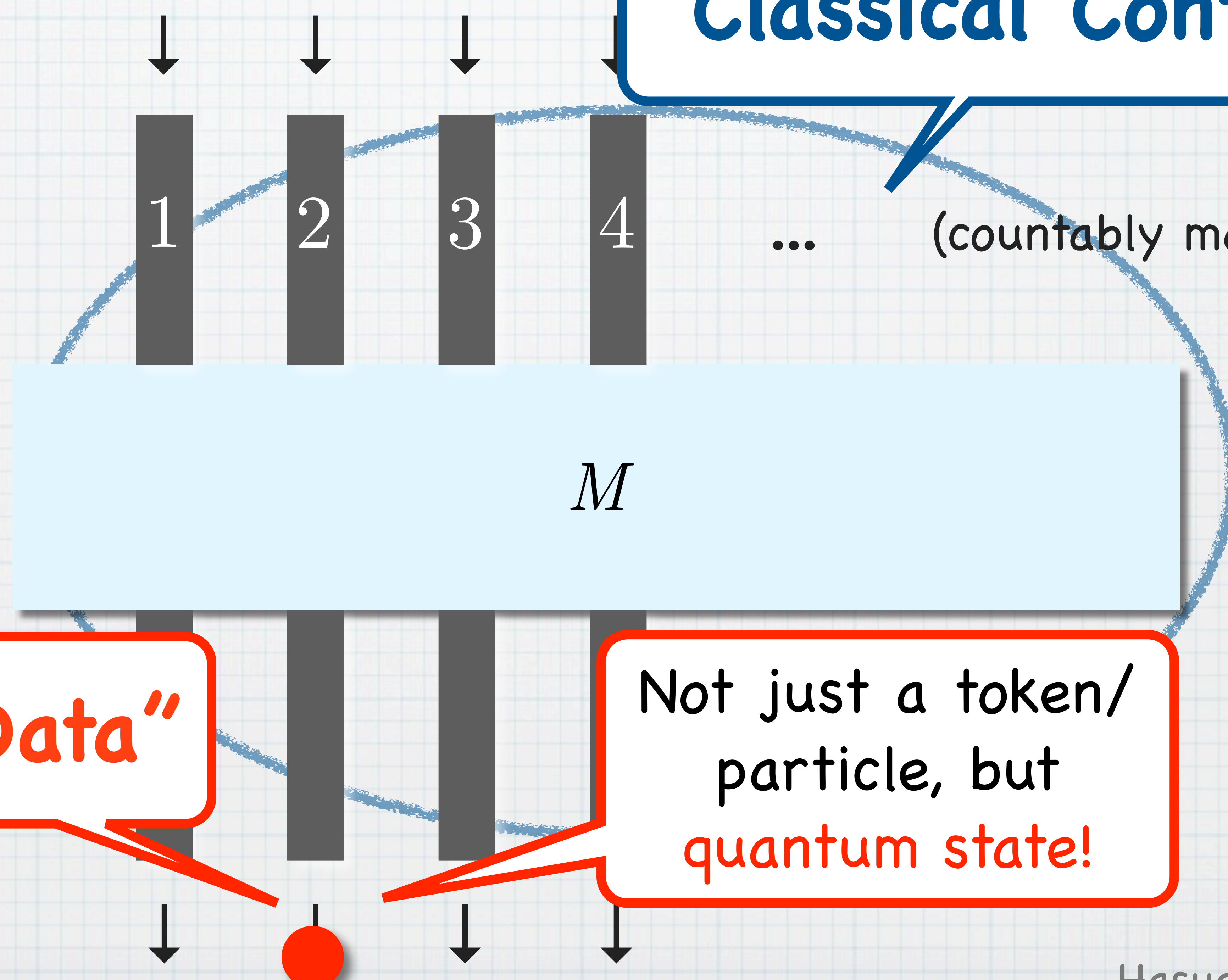

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Hasuo (Tokyo)

Indeed...

- The monad $\mathcal{Q}$ qualifies as a "branching monad"
- The quantum GoI workflow leads to a linear category $\mathbf{PER}_{\mathcal{Q}}$
- From which we construct an adequate denotational model

End of the Story?

* No! All the technicalities are yet to come:
  * CPS-style interpretation (for partial measurement)
  * Result type: a final coalgebra in $\mathbf{PER}_Q$
  * **Admissible PERs** for recursion
  * ...
* On the next occasion :-)

Hasuo (Tokyo)

Conclusion: the Categorical GoI Workflow

Branching monad B

↓ Coalgebraic trace semantics

Traced monoidal category $\mathbb{C}$

+ other constructs ➜ "GoI situation" [AHS02]

↓ Categorical GoI [AHS02]

Linear combinatory algebra

↓ Realizability

Linear category

Quantum branching monad

↓

Quantum TSMC

↓

Quantum LCA

↓

Model of quantum language

Hasuo (Tokyo)

Conclusion: the Cat

Thank you for your attention!
Ichiro Hasuo (Dept. CS, U Tokyo)
http://www-mmm.is.s.u-tokyo.ac.jp/~ichiro/

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