Coinductive Predicates and Final Sequences in a Fibration

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Coinduction

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* In $\mathbb{C}$?

$$\begin{array}{ccc} FX & \dashrightarrow & FZ \\ {\scriptstyle c}\uparrow & & \cong\uparrow{\scriptstyle \text{final}} \\ X & \underset{\mathsf{beh}(c)}{\dashrightarrow} & Z \end{array}$$

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* In a **fibration** $\begin{array}{c}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{array}$!!
  * This work:
    * **final coalgebra** in p;
    * **final sequcence** in p

Fibered Coinduction

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{ F-behaviors } + **coinductive predicate**

Part I:

Coinductive Predicates, Conventionally

Coinductive Predicates

- **Persisting** predicates in dynamical sys.
  - now ✔, next ✔, next^2 ✔, ...
  - ν in the modal μ-calculus
  - **G** in LTL/CTL
  - Expresses **safety**

Coinductive Predicates

$$\begin{array}{c} \mathcal{P}(X) \\ c \uparrow \\ X \end{array}$$

Kripke frame

$$\nu u. \Diamond u$$

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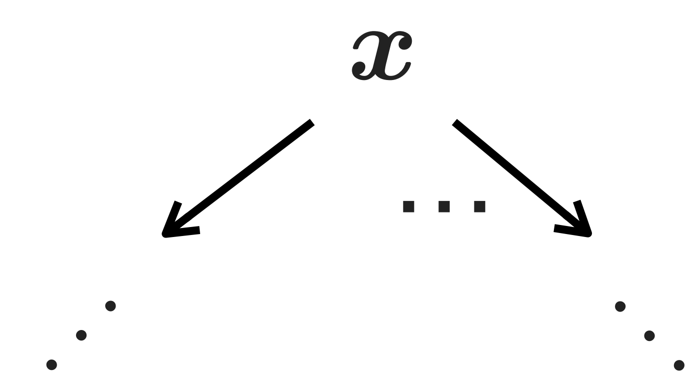

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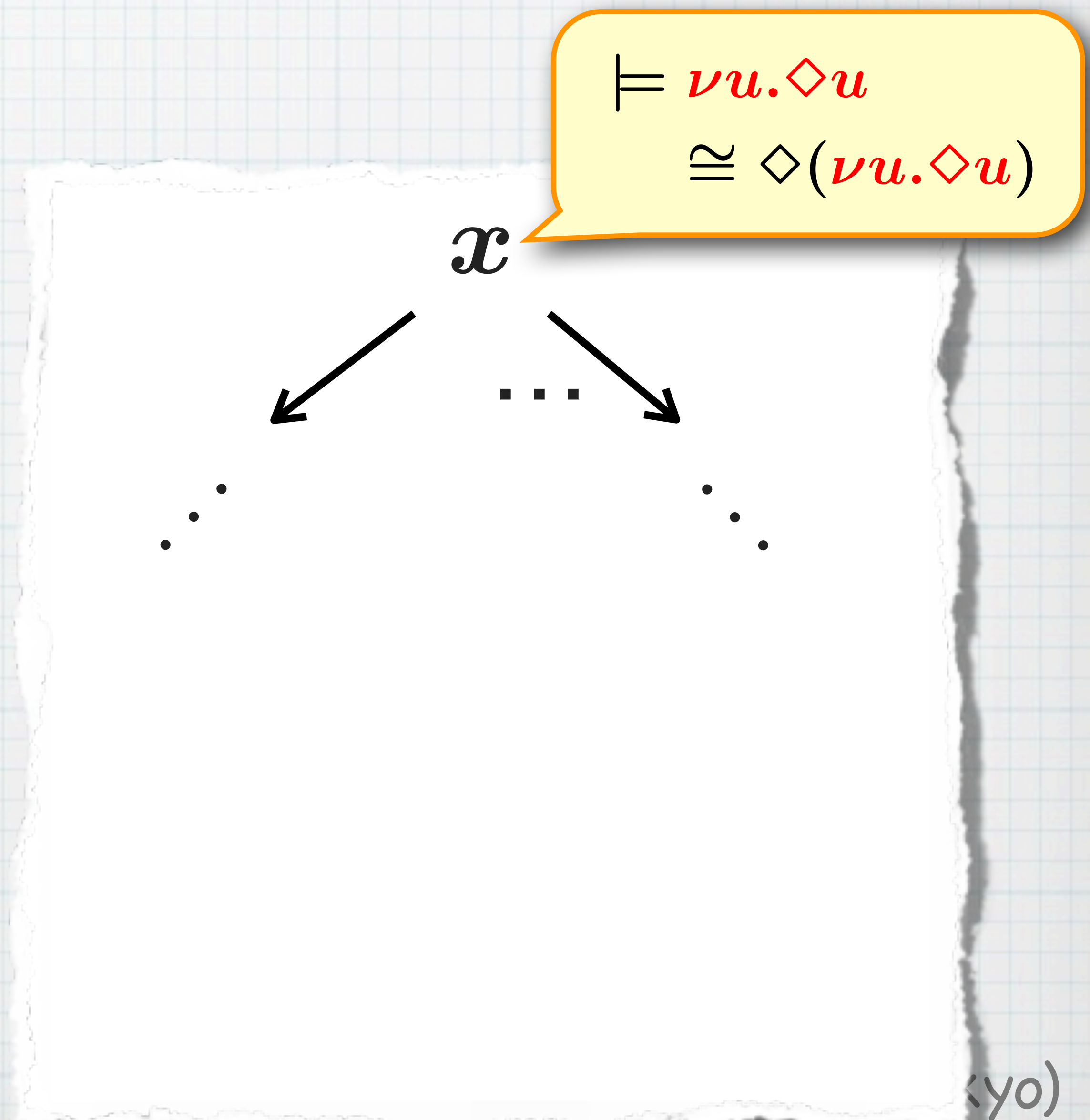

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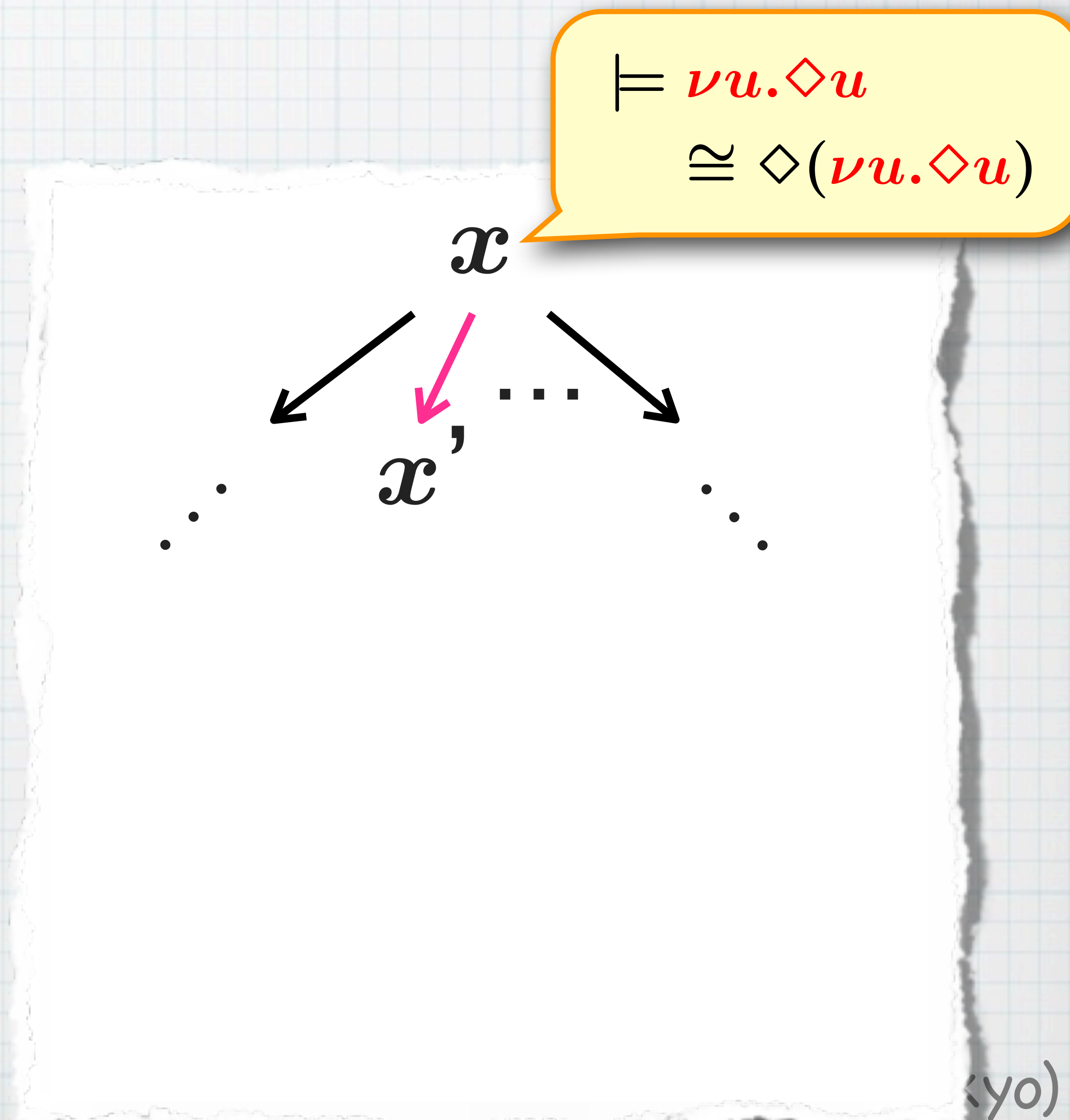

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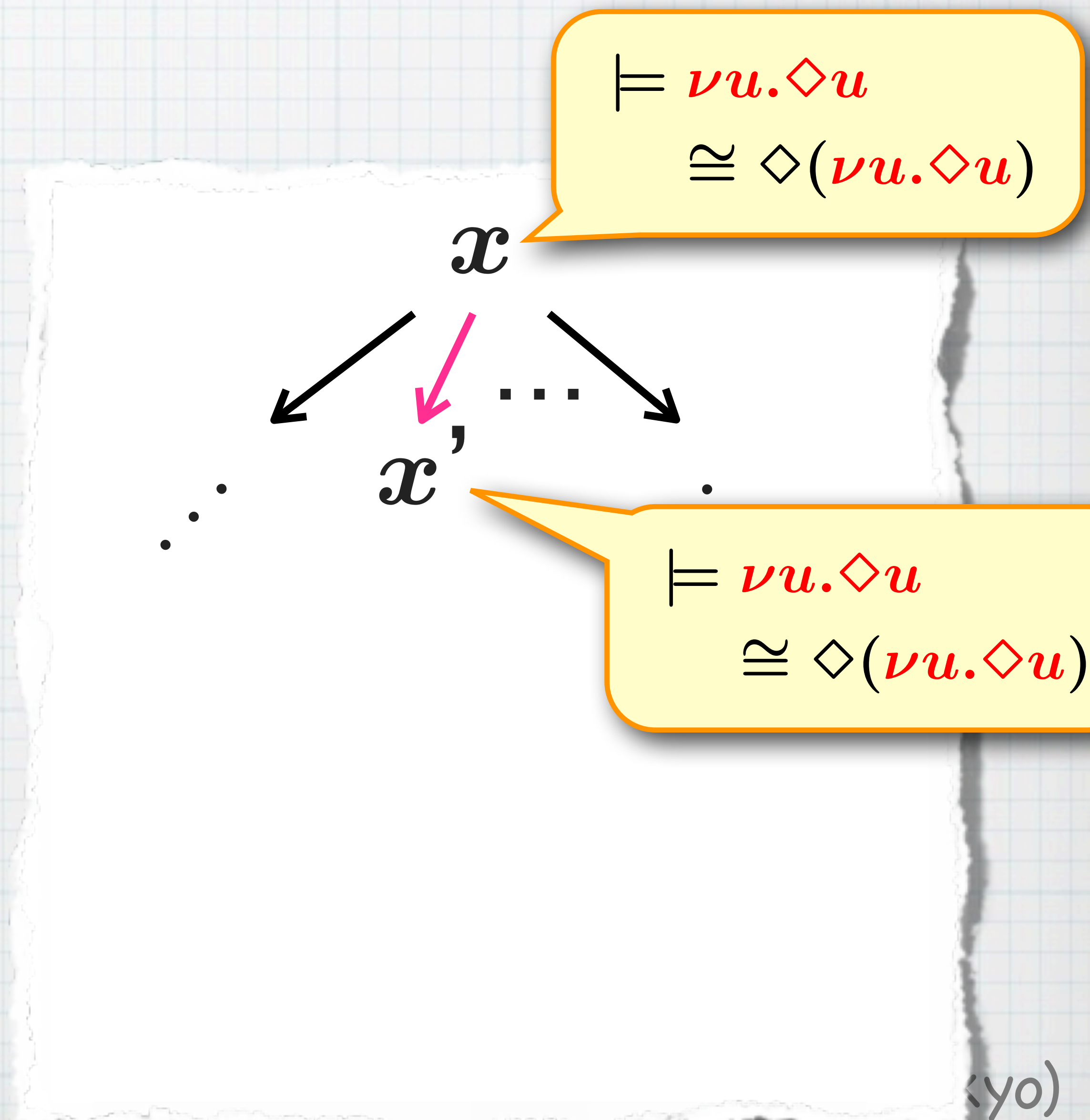

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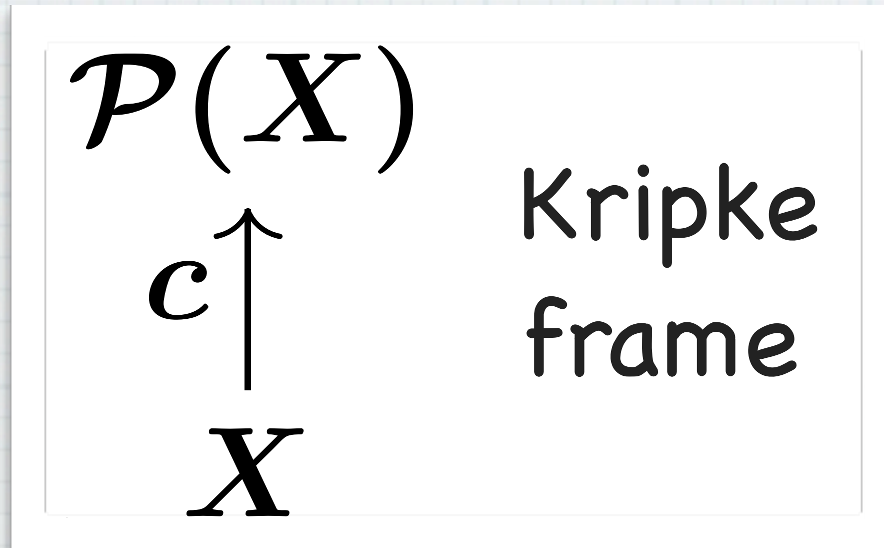

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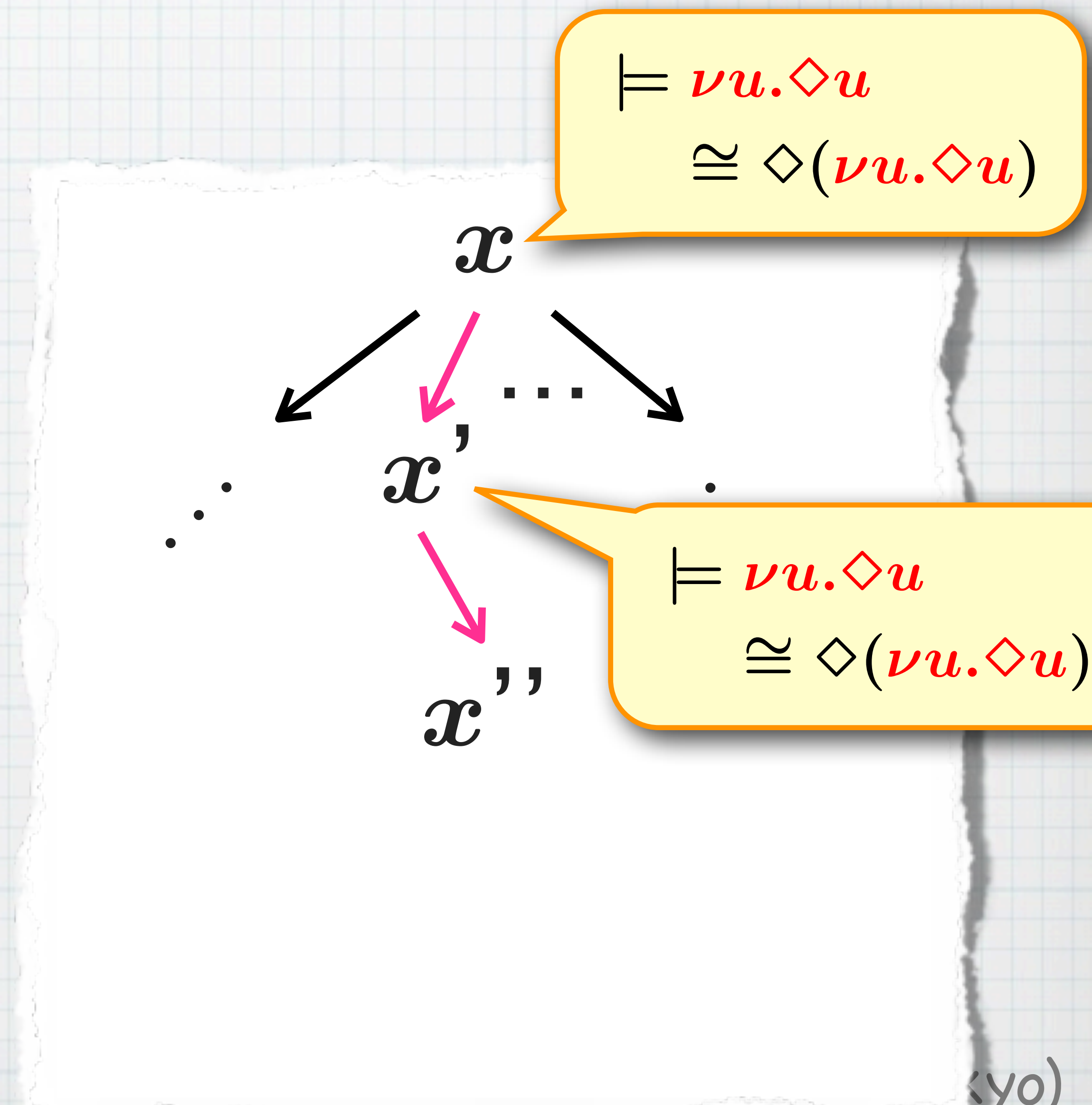

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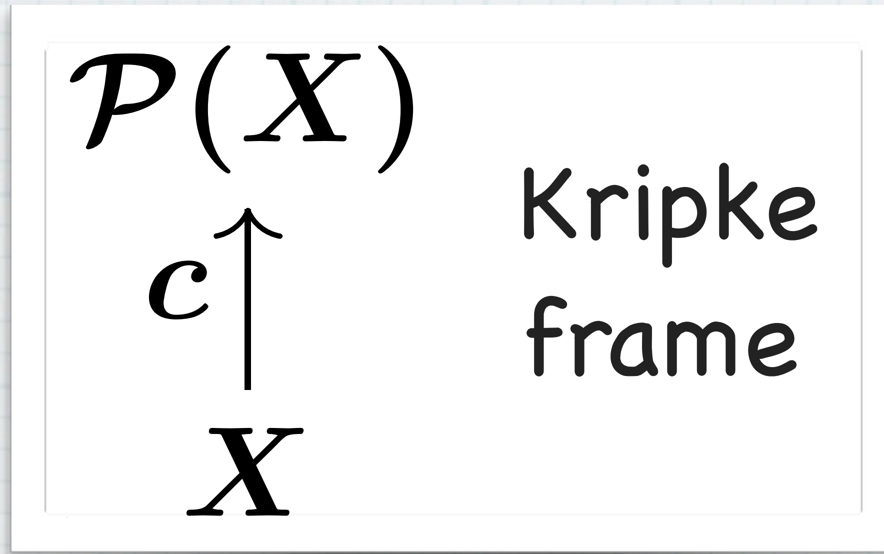

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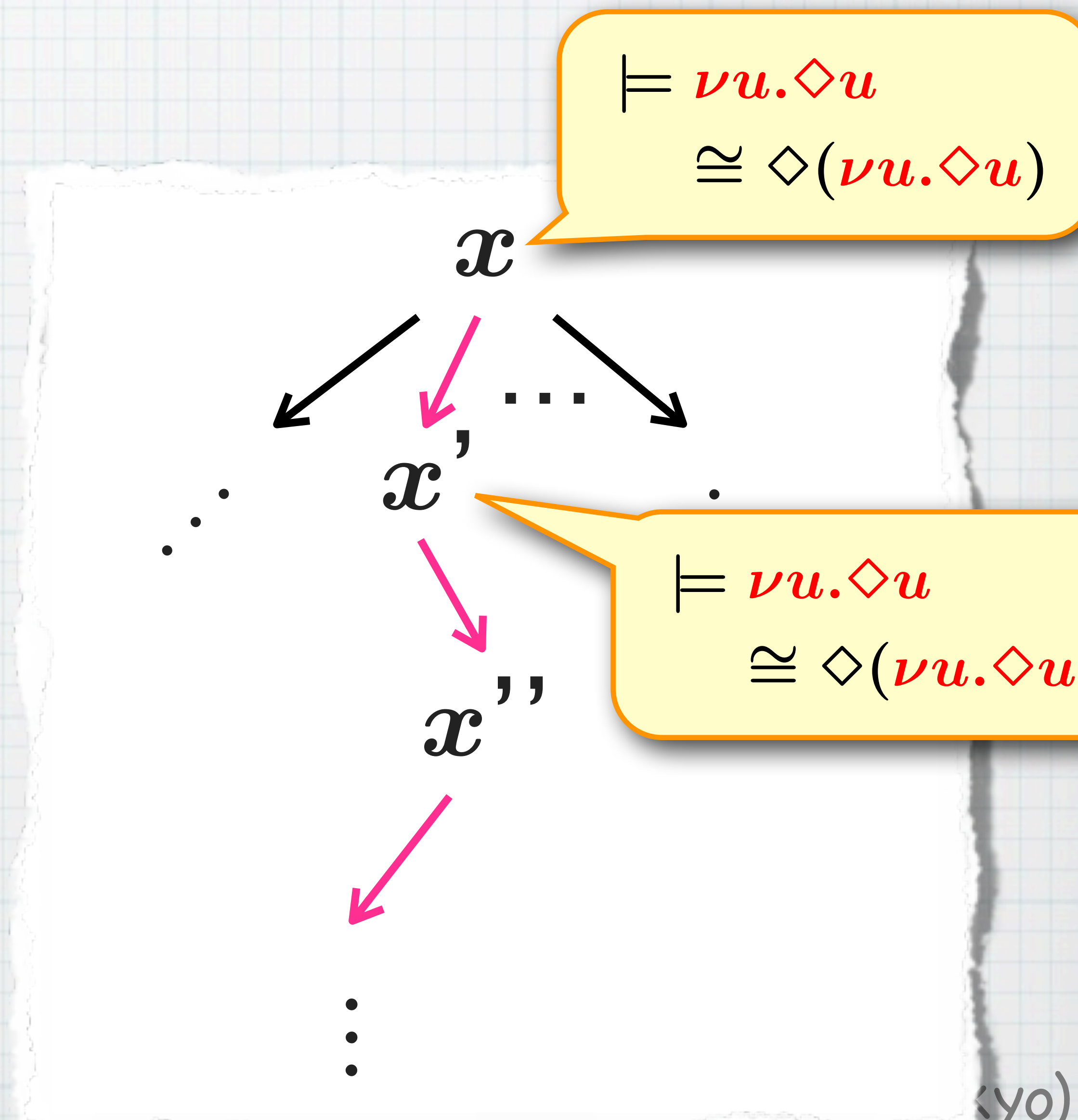

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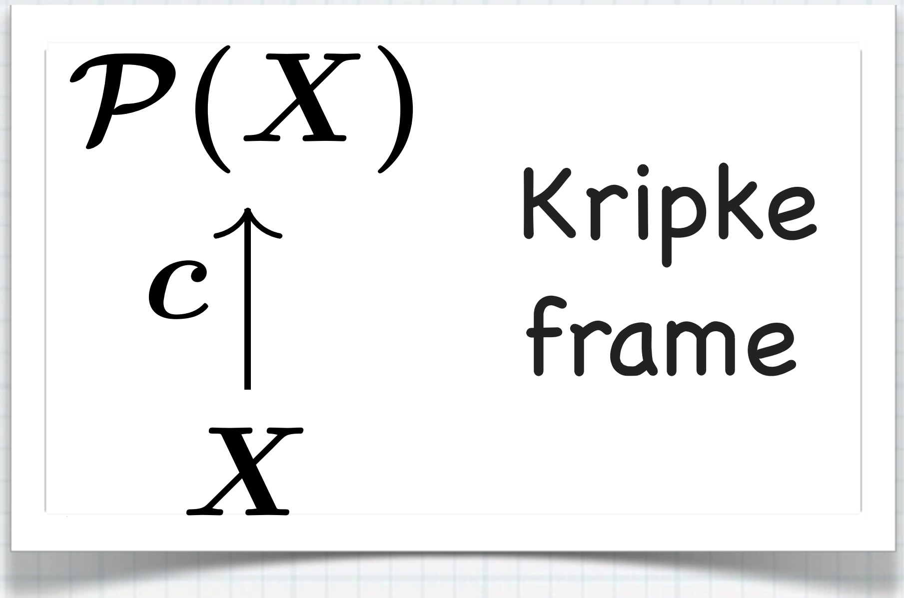
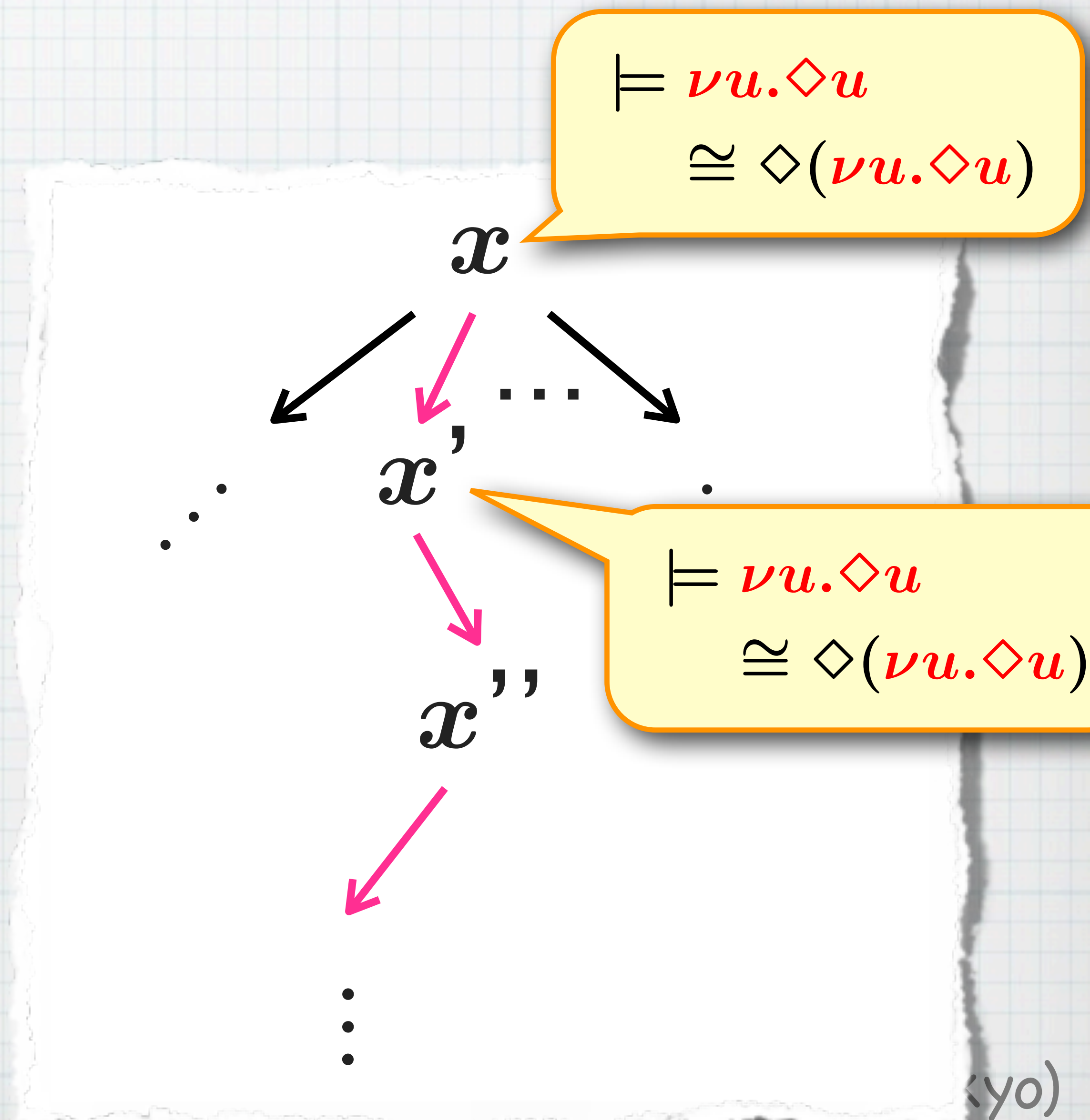


$$\nu u.\,\Diamond u$$

- “There is an infinite path”



Coinductiv[e]

(current st.) $\models P$

witnesses (next st.) $\models P$

$$\begin{array}{c}\mathcal{P}(X)\\ c\uparrow\\ X\end{array}$$

Kripke frame

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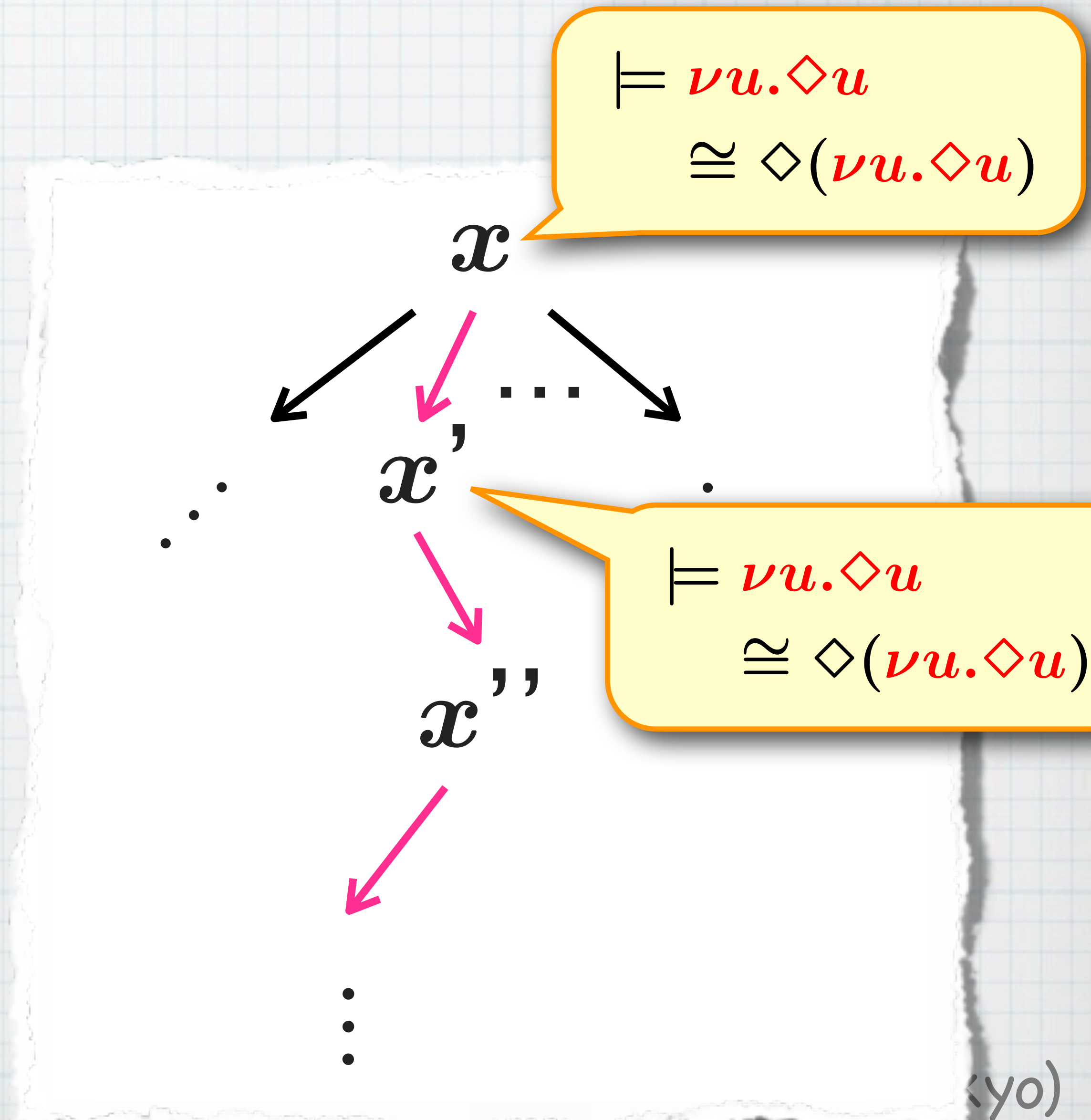

Coinductive Predicates

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Kripke frame

Bisimilarity $\sim$

* $x \sim y, \quad x \to x'$
$\Longrightarrow \quad y \to \exists y' \text{ s.t. } x' \sim y'$

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Coinductive Predicates are HOT!!

* Proof assistants

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  * In Coq; in Agda
    [Giménez, TYPES'95] [Bertot & Komendantskaya, CMCS'08] [Nakano, CPP'12]

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$\mathbf{F}p$

$$\mu u.\, p \vee \mathbf{X}u \;\not\cong\; \neg(\nu u.\, \neg p \wedge \mathbf{X}u)$$

$\neg\mathbf{G}\neg p$

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- Search for useful **proof principles**
  [Hur, Neis, Dreyer & Vafeiadis, POPL'13] [Bonchi & Pous, POPL'13]

Establish/Compute/Construct Coinductive Predicates

$$[\![\nu u.\,\Diamond u]\!]_{\mathcal{P}(X) \atop {c\uparrow \atop X}} \quad = \quad \mathrm{gfp}\left(2^{X} \xrightarrow{\varphi_{\Diamond}} 2^{\mathcal{P}X} \xrightarrow{c^{-1}} 2^{X} \right)$$

$$U \;\longmapsto\; \{X' \subseteq X \mid X' \cap U \neq \emptyset\} \;\longmapsto\; \{x \in X \mid c(x) \cap U \neq \emptyset\}$$

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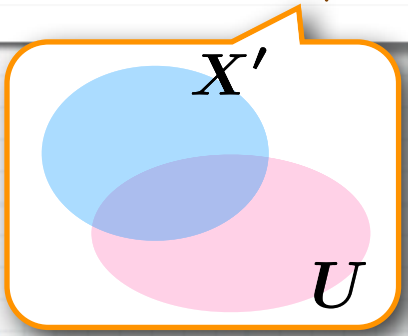

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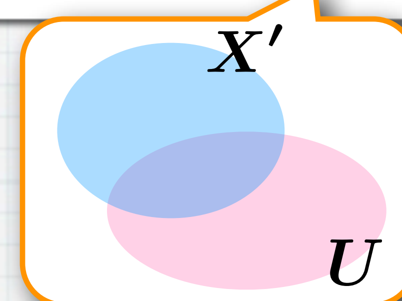

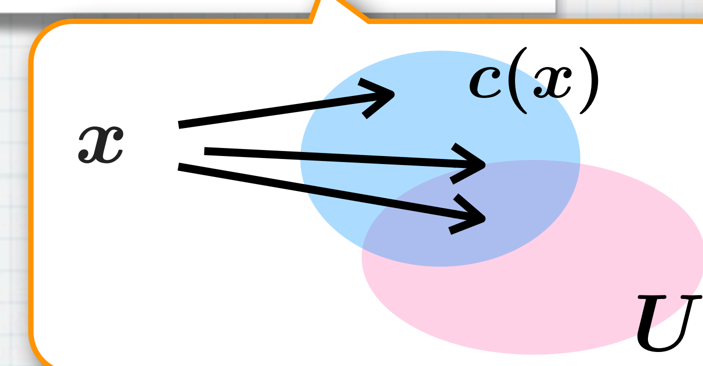

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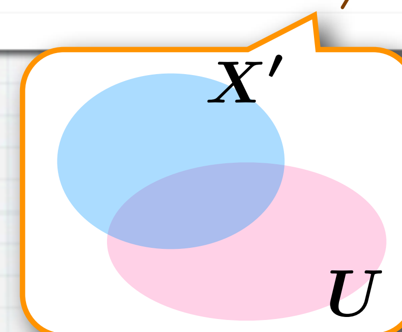

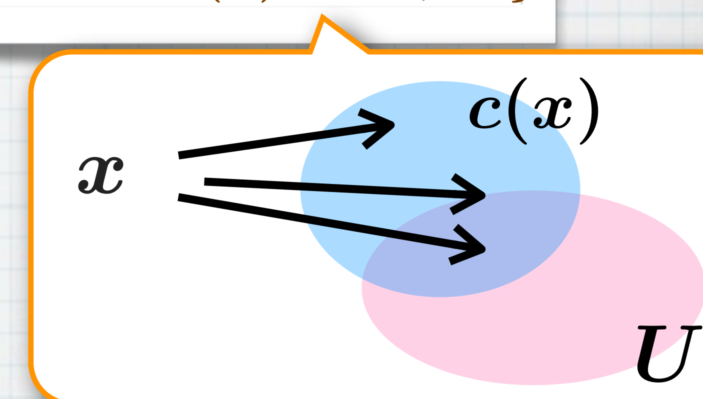

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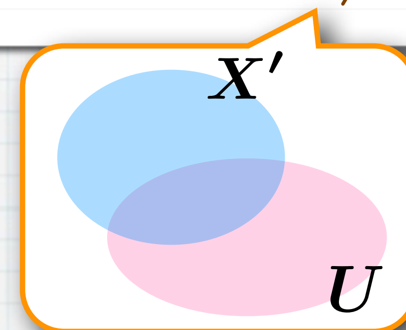

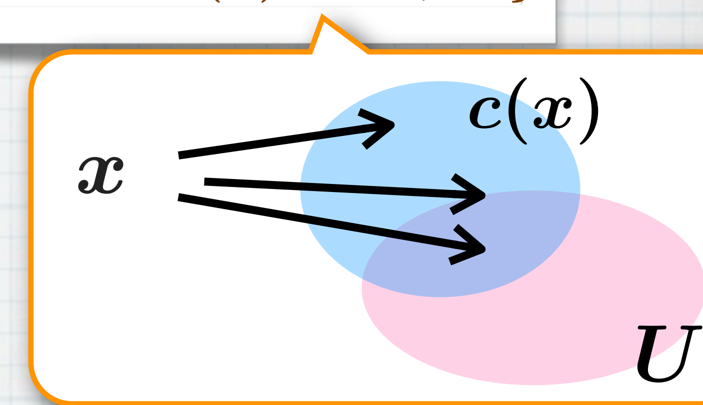

* __1st ans.__ (Knaster-Tarski)
  * $c^{-1} \circ \varphi_\Diamond \;:\; 2^X \longrightarrow 2^X$ is monotone
  * Postfixed points (invariants) $\{U \mid U \subseteq (c^{-1} \circ \varphi_\Diamond)U\}$ form a complete lattice
  * Its maximum (greatest invariant) is the gfp

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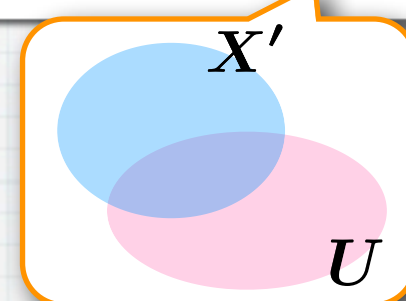

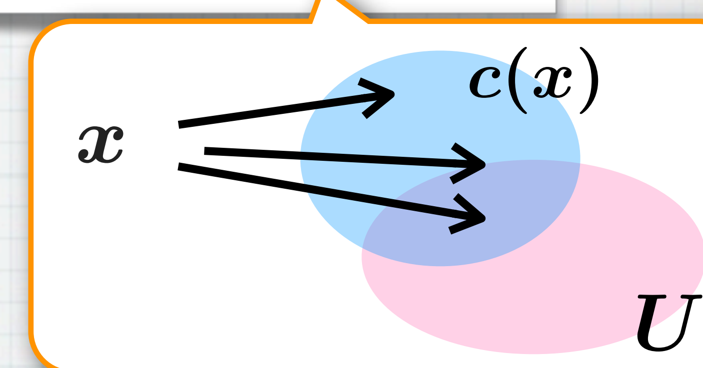

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not really a "construction"…

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* **2nd ans.** (Inductive constr. [Cousot & Cousot '79])

  * Stabilize ➜ gfp
  * But when?
    * ω, if φ is ∩-preserving... not now

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the whole space

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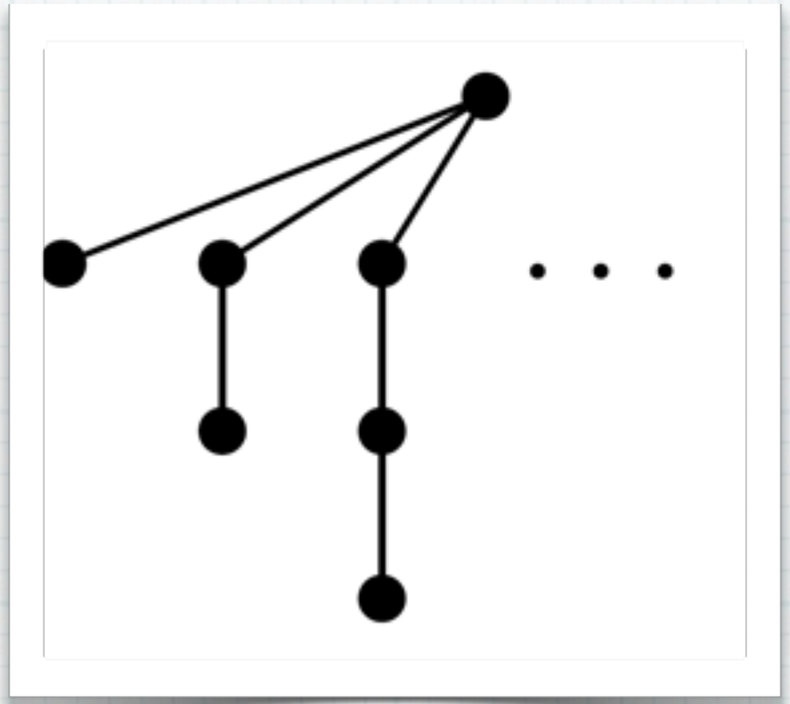

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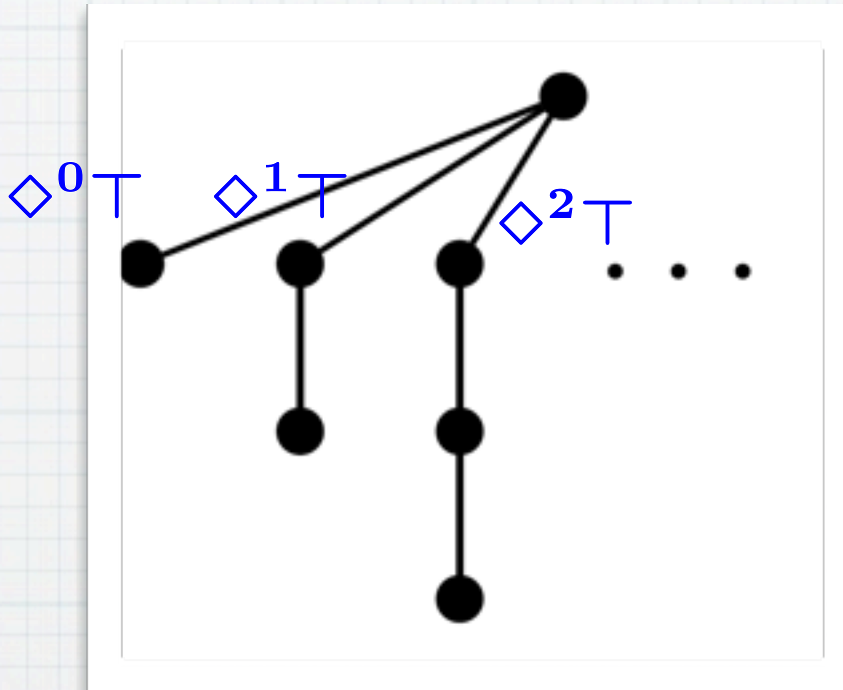

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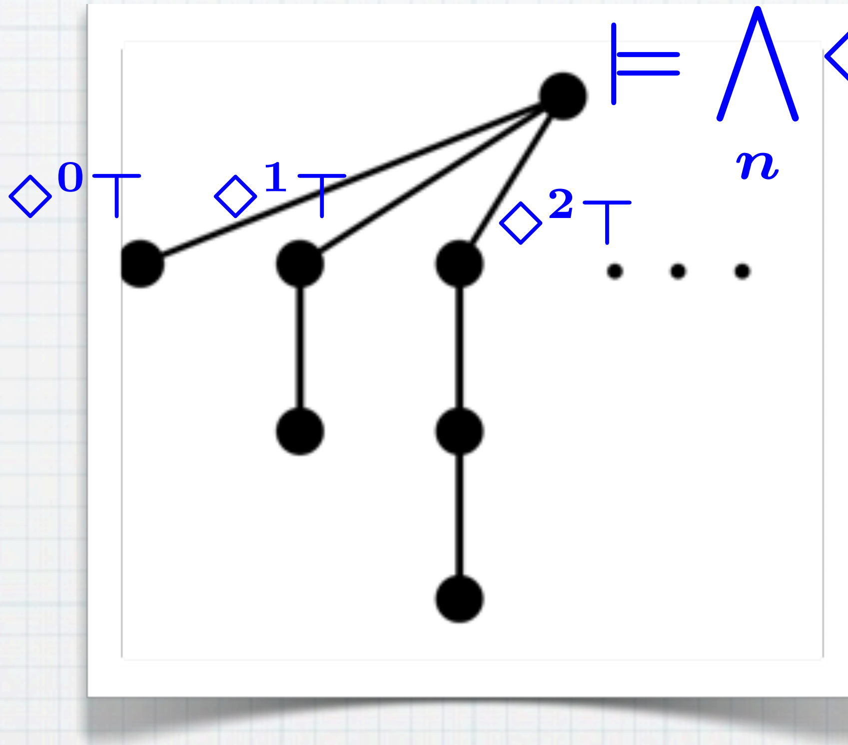

$\models \bigwedge_n \Diamond^n \top$, but $\not\models \nu u. \Diamond u$

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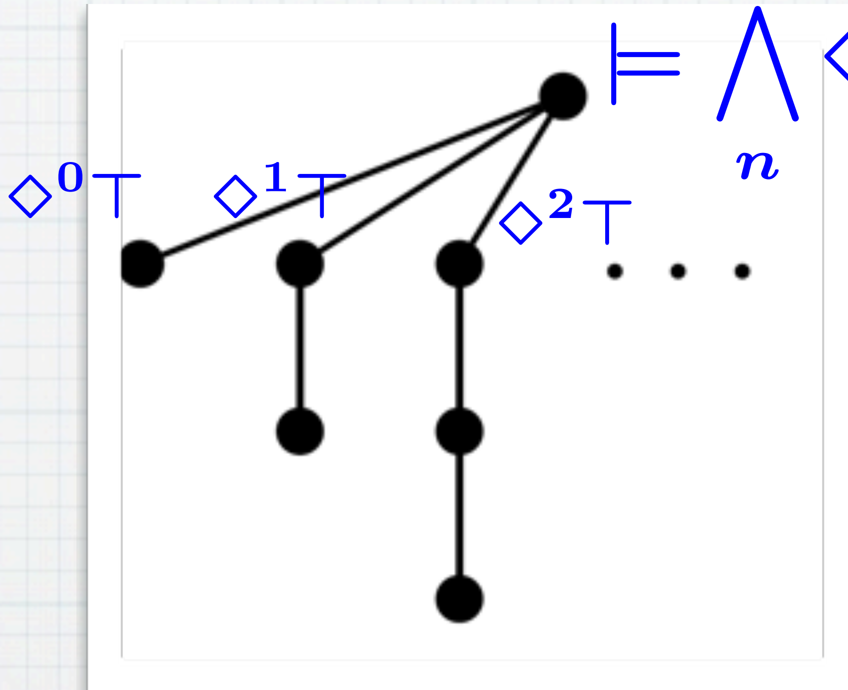

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* <u>**State space bound**</u> [Cousot & Cousot, '79] $|X|$ steps

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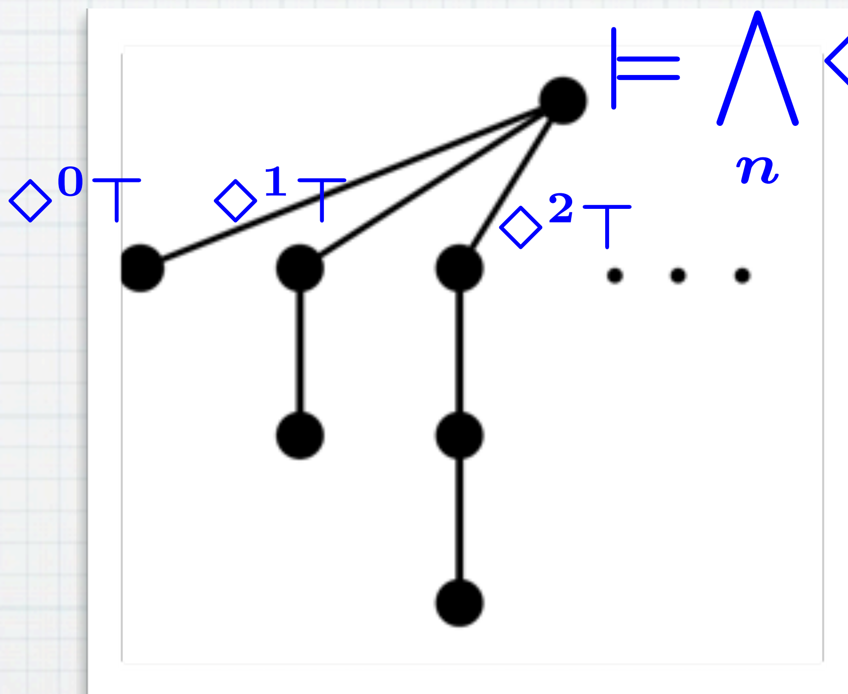

* **State space bound** [Cousot & Cousot, '79] $|X|$ steps
* **"Behavioral bound"** [Hennessy & Milner, '85]
  * ω steps if **finitely branching**!

Behavioral Bound for Computing Coind. Pred.

Theorem. Let a Kripke frame $\begin{matrix}\mathcal{P}(X)\\ {\scriptstyle c}\uparrow\\ X\end{matrix}$ be finitely branching. Then

$$X \supseteq (c^{-1} \circ \varphi_{\diamond})X \supseteq (c^{-1} \circ \varphi_{\diamond})^2 X \supseteq \cdots$$

stabilizes after ω steps.

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- **Proof**: Suffices to show

  $\bigwedge_n \diamond^n \top$ is an invariant.

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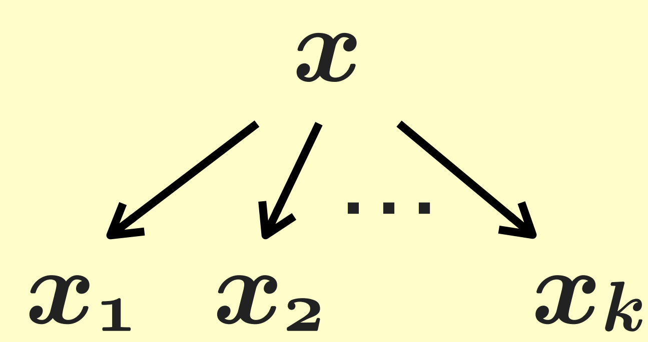

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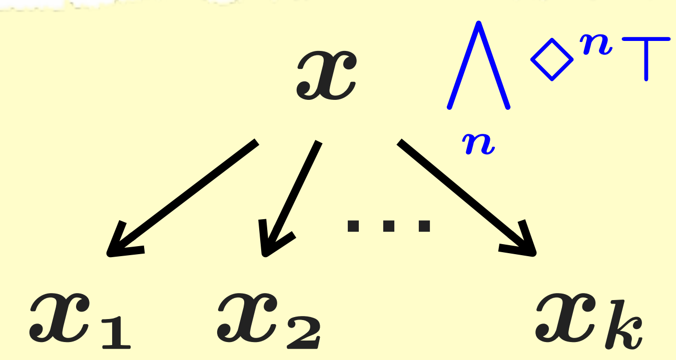

Behavioral Bound for Computing Coind. Pred.

Theorem. Let a Kripke frame $\begin{array}{c}\mathcal{P}(X)\\ c\uparrow\\ X\end{array}$ be finitely branching. Then

$$X \supseteq (c^{-1} \circ \varphi_{\Diamond})X \supseteq (c^{-1} \circ \varphi_{\Diamond})^2 X \supseteq \cdots$$

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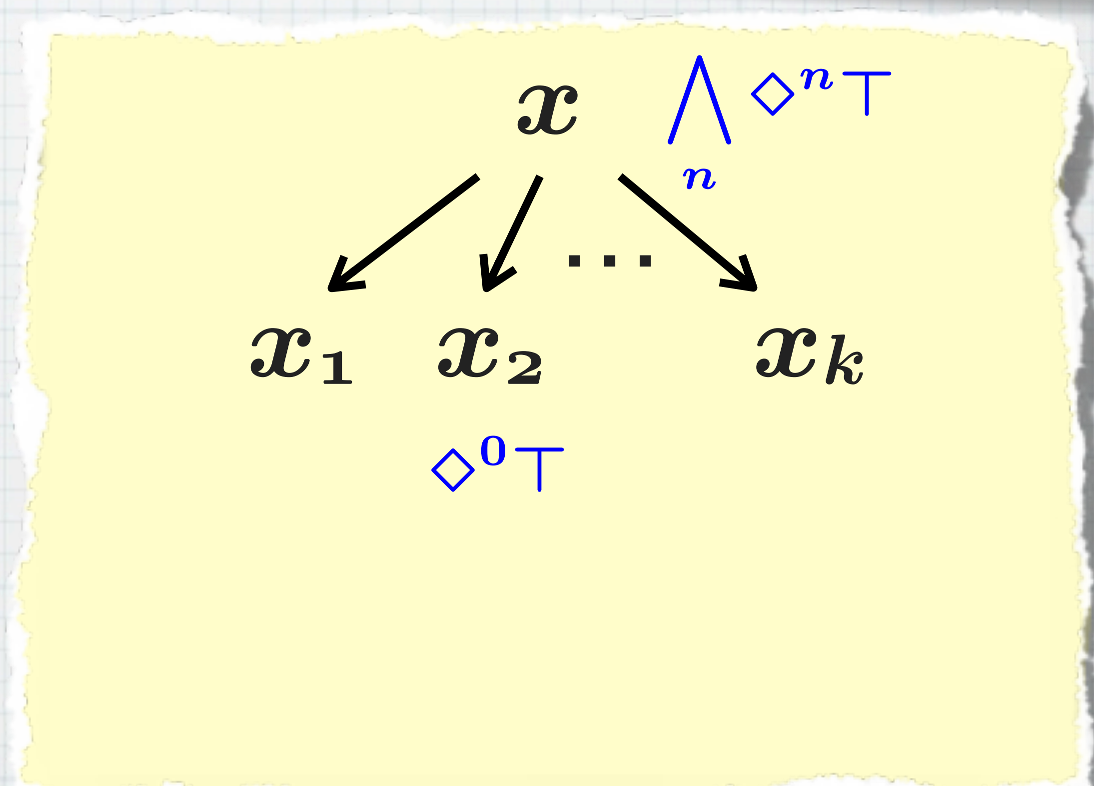

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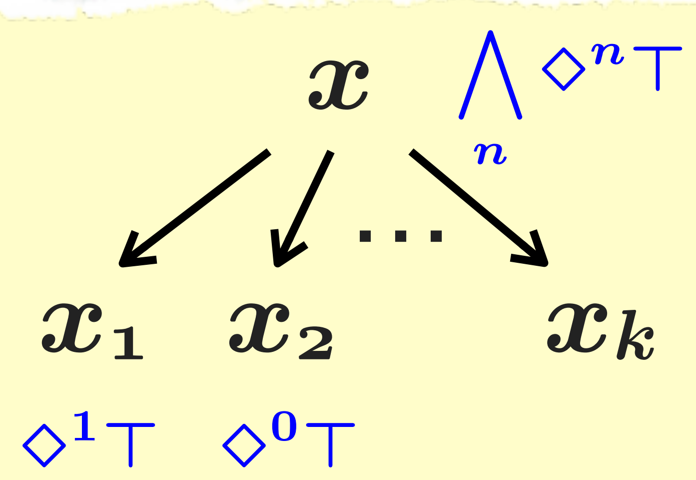

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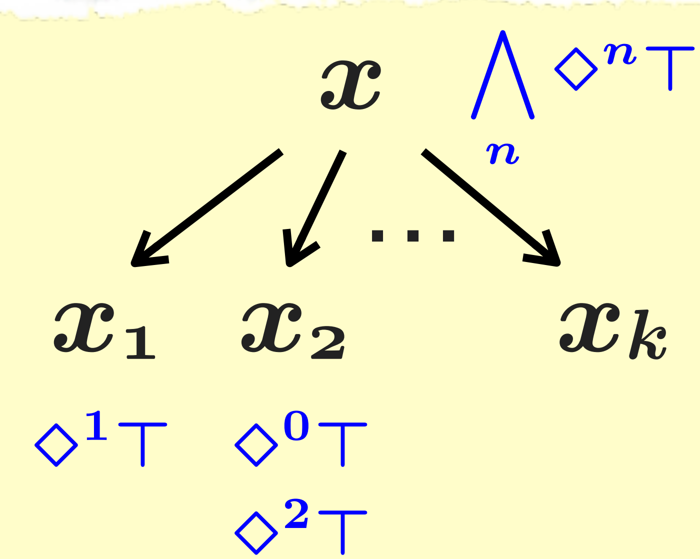

Behavioral Bound for Computing Coind. Pred.

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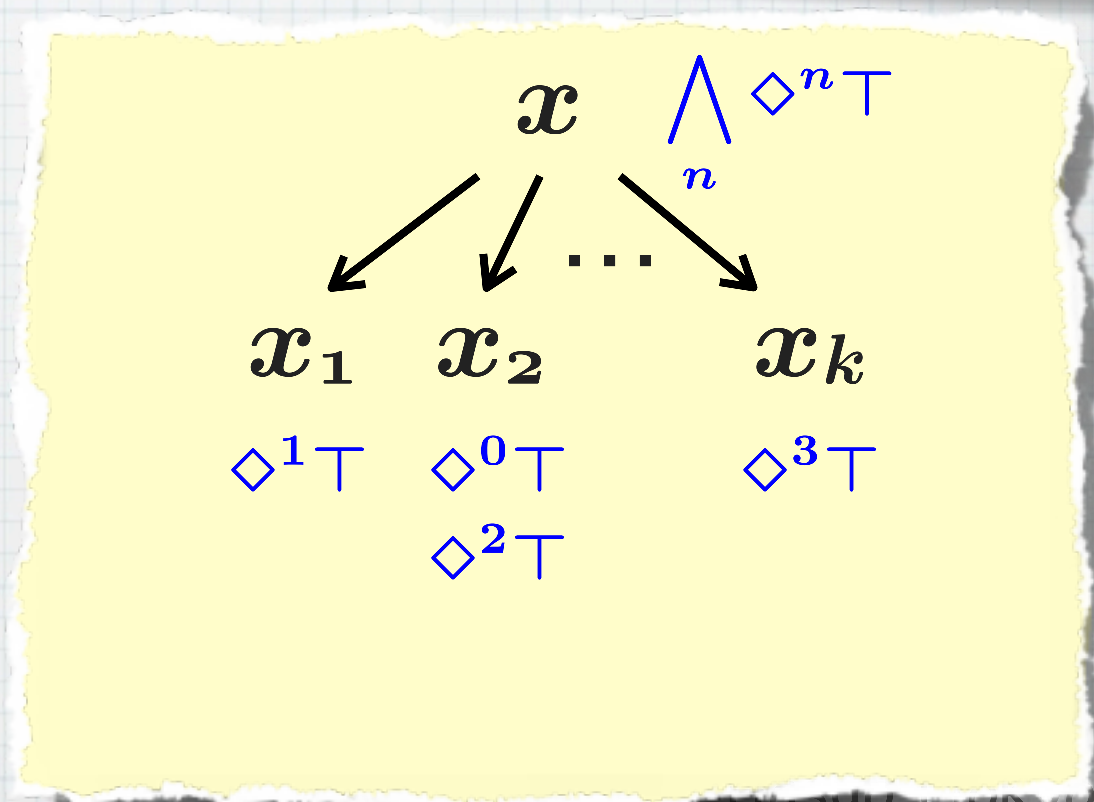

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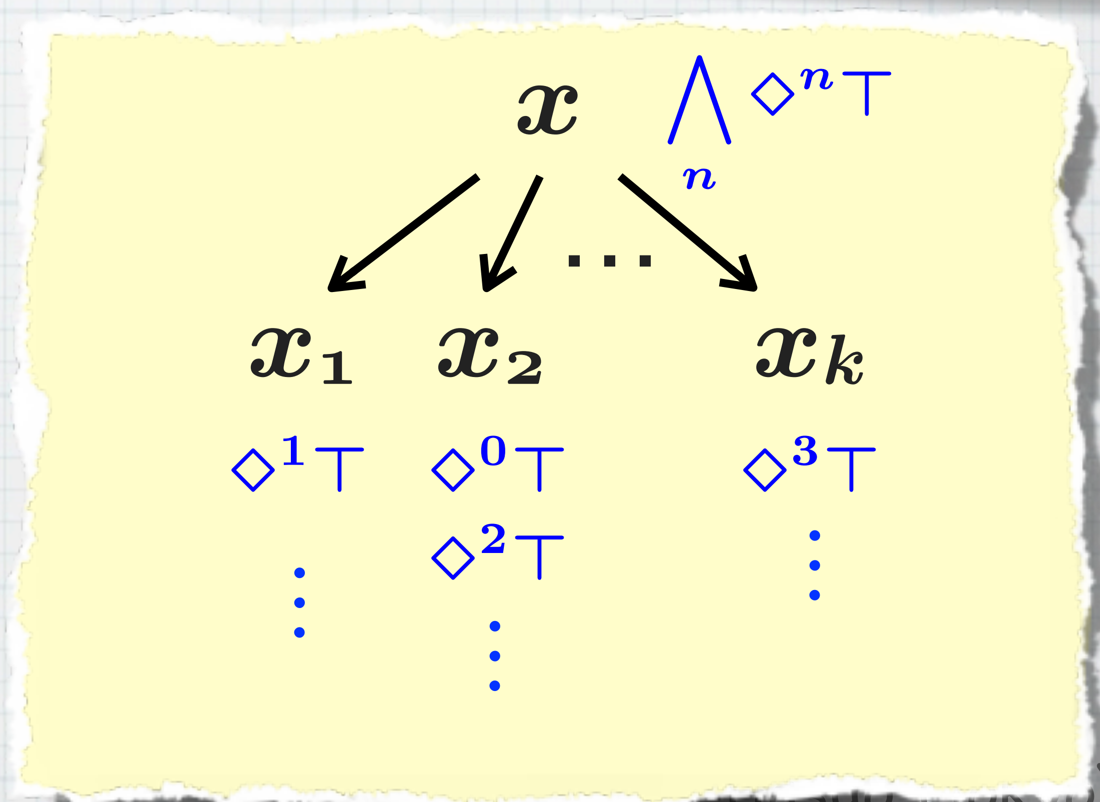

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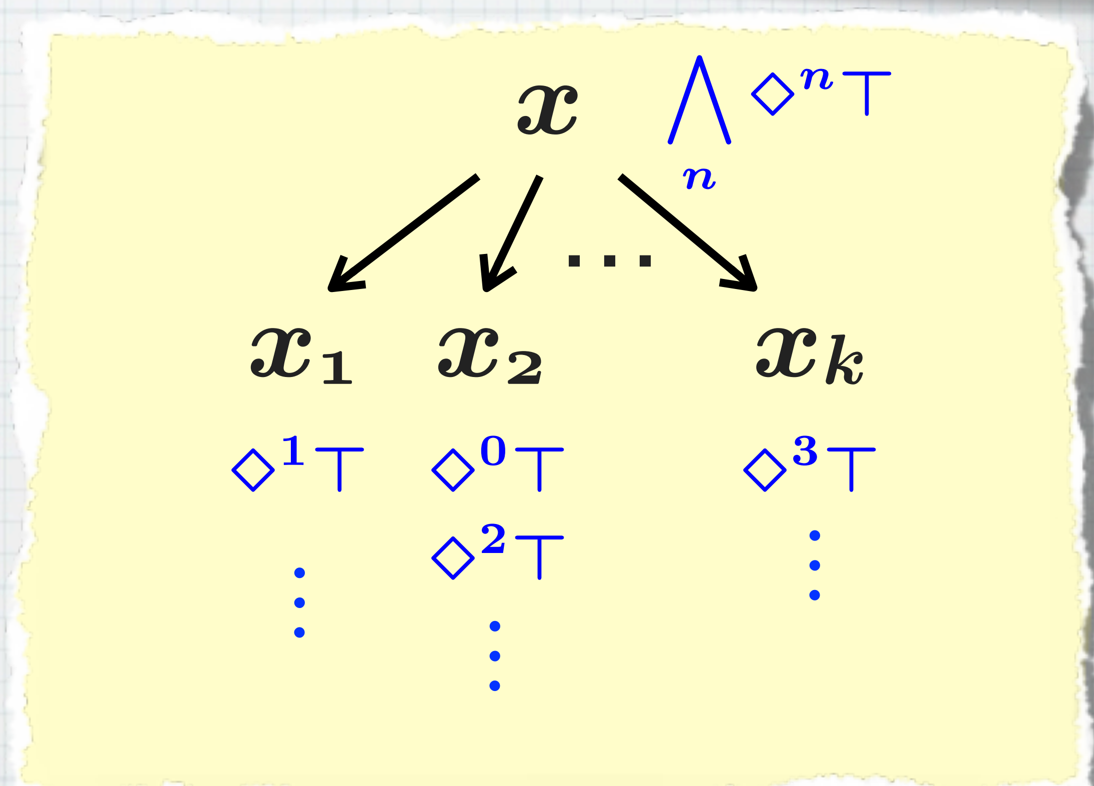
- $\exists i\in[1,k]$ s.t. $x_i \models \Diamond^n\top$ for infinitely many n



Coinductive Predicates Conventionally (Summary)

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* **(current st.)** $\models P$ witnesses **(next st.)** $\models P$

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??

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    - **State space bound** vs. "**behavioral bound**"

Coinductive Predicates Conventionally (Summary)

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current work

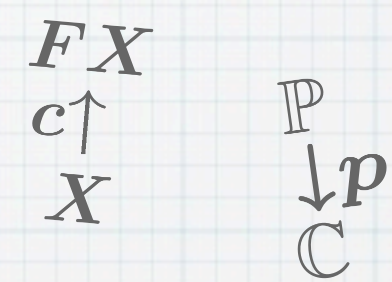
current work

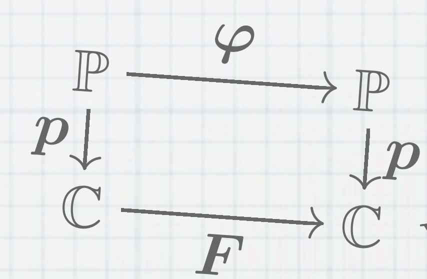
Part II:

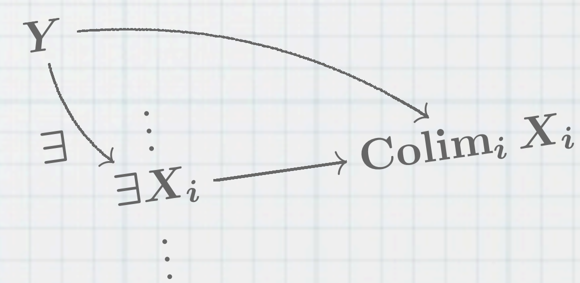
Coinductive Predicates, Categorically

Contributions

- Sufficient condition for **categorical behavioral ω-bound** based on
  - Coalgebra (transition system)
  - Fibration (underlying logic)
  - Predicate lifting (modality)
  - Locally presentable category ("size")







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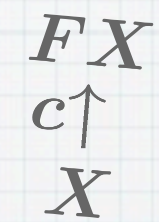
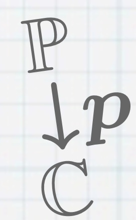
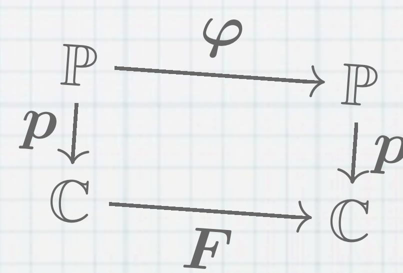
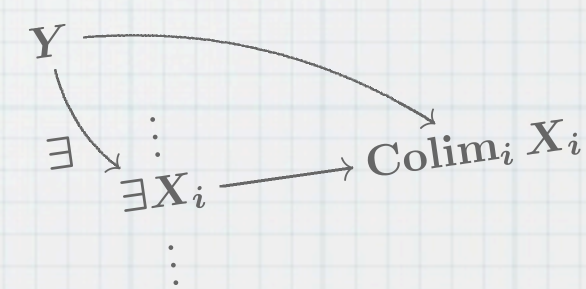

Constr. of final coalg. by
final sequence [Worrell, Adamek]

Contributions

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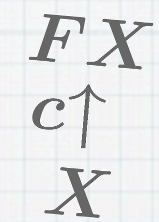
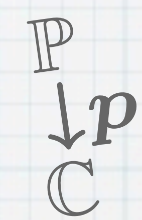
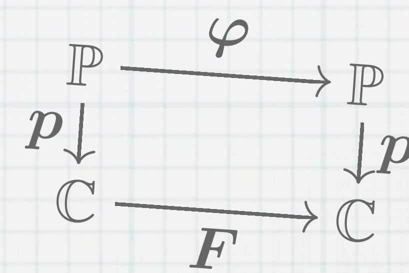




  - Fibration (underlying logic)



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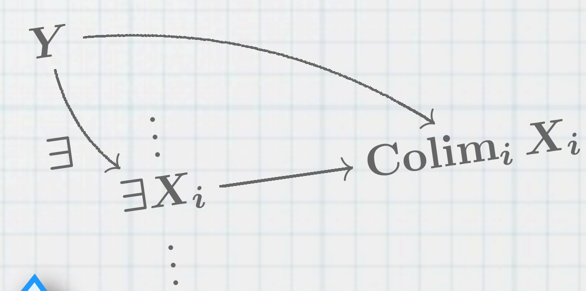
  - Locally presentable category ("size")



Constr. of final coalg. by **final sequence** [Worrell, Adamek]

Coind. predicate as a **final coalgebra** [Hermida, Jacobs]

Contributions

Categorical infrastructure:
fibration and
locally presentable cat.

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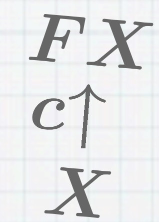

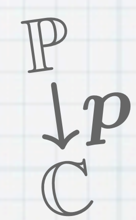

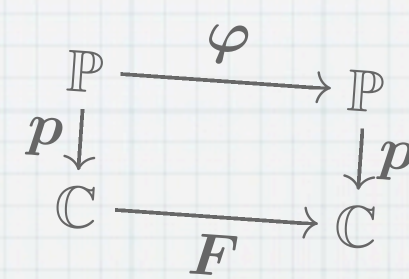

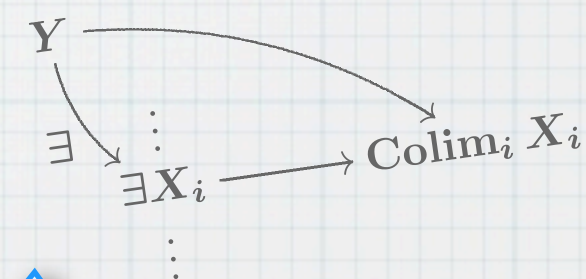

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Contributions

Some math work

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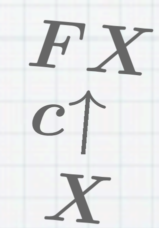

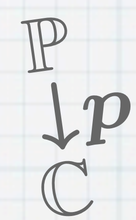

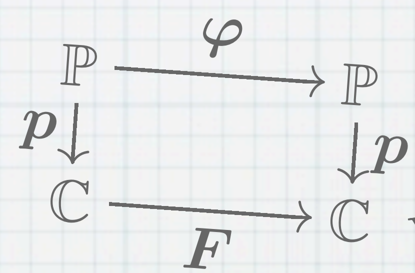

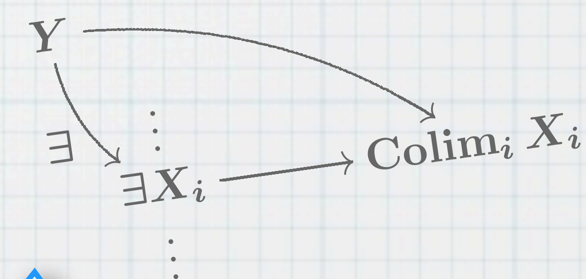

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The Categorical Setup

$$\begin{array}{c}\mathcal{P}_\omega(X)\\ c\uparrow\\ X\end{array}$$

Kripke model

$$\nu u.\,\Diamond u$$

coinductive specification

$$\begin{array}{rccc}\varphi_\Diamond : & 2^X & \longrightarrow & 2^{\mathcal{P}_\omega X}\\ & U & \mapsto & \{X' \subseteq X \mid X' \cap U \neq \emptyset\}\end{array}$$

$$2^X \xrightarrow{\varphi_\Diamond} 2^{\mathcal{P}X} \xrightarrow{c^{-1}} 2^X$$

monotone

$$\begin{array}{c}(c^{-1}\circ\varphi_\Diamond)U\\ \cup\!\!|\\ U\end{array}$$

invariant

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coalgebra

The Categorical Setup

finitely branching

$\mathcal{P}_\omega(X)$
$c\uparrow$
X

Kripke model

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monotone

$(c^{-1} \circ \varphi_\Diamond)U$
$\cup\!|$
U

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FX
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coalgebra

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$$\varphi : \mathbb{P}_X \longrightarrow \mathbb{P}_{FX}$$

predicate lifting

The Categorical Setup

finitely branching

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$$\begin{array}{c} (c^{-1} \circ \varphi_\Diamond)U \\ \cup\!\mid \\ U \end{array}$$ invariant

finitary

$$\begin{array}{c} FX \\ c\uparrow \\ X \end{array}$$ coalgebra

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$$\varphi : \mathbb{P}_X \longrightarrow \mathbb{P}_{FX}$$
predicate lifting

$$\mathbb{P}_X \xrightarrow{\varphi_X} \mathbb{P}_{FX} \xrightarrow{c^*} \mathbb{P}_X$$
endofunctor

The Categorical Setup

finitely branching

finitary

$\begin{array}{c}\mathcal{P}_\omega(X)\\ c\uparrow\\ X\end{array}$ Kripke model	$\begin{array}{c}FX\\ c\uparrow\\ X\end{array}$ coalgebra
$\nu u.\Diamond u$ coinductive specification	$\nu\varphi$ coinductive specification
$\varphi_\Diamond : 2^X \longrightarrow 2^{\mathcal{P}_\omega X}$, $U \mapsto \{X' \subseteq X \mid X' \cap U \neq \emptyset\}$	$\varphi : \mathbb{P}_X \longrightarrow \mathbb{P}_{FX}$ predicate lifting
$2^X \xrightarrow{\varphi_\Diamond} 2^{\mathcal{P}X} \xrightarrow{c^{-1}} 2^X$ monotone	$\mathbb{P}_X \xrightarrow{\varphi_X} \mathbb{P}_{FX} \xrightarrow{c^*} \mathbb{P}_X$ endofunctor
$\begin{array}{c}(c^{-1}\circ\varphi_\Diamond)U\\ \cup\!\mid\\ U\end{array}$ invariant	$\begin{array}{c}(c^*\circ\varphi)P\\ \uparrow\\ P\end{array}$ coalgebra (in a fibr.)

kyo)

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monotone

$$(c^{-1} \circ \varphi_\Diamond)U \;\supseteq\; U$$

invariant

$$(c^{-1} \circ \varphi_\Diamond)[\![\nu u.\, \varphi_\Diamond u]\!]_c = [\![\nu u.\, \varphi_\Diamond u]\!]_c$$

coind. pred.

$X \;\supseteq\; (c^{-1} \circ \varphi_\Diamond)X \;\supseteq\; \cdots$

inductive constr.

$\nu\varphi$ coinductive specification

$\varphi : \mathbb{P}_X \longrightarrow \mathbb{P}_{FX}$

predicate lifting

$\mathbb{P}_X \xrightarrow{\varphi_X} \mathbb{P}_{FX} \xrightarrow{c^*} \mathbb{P}_X$

endofunctor

$$(c^* \circ \varphi)P \;\uparrow\; P$$

coalgebra (in a fibr.)

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monotone

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$(c^{-1} \circ \varphi_\Diamond)[\![\nu u.\varphi_\Diamond u]\!]_c = [\![\nu u.\varphi_\Diamond u]\!]_c$

coind. pred.

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inductive constr.

$\nu\varphi$ coinductive specification

$\varphi : \mathbb{P}_X \longrightarrow \mathbb{P}_{FX}$

predicate lifting

$\mathbb{P}_X \xrightarrow{\varphi_X} \mathbb{P}_{FX} \xrightarrow{c^*} \mathbb{P}_X$

endofunctor

$(c^* \circ \varphi)P \uparrow P$

coalgebra (in a fibr.)

$(c^* \circ \varphi)[\![\nu\varphi]\!]_c \cong\uparrow [\![\nu\varphi]\!]_c$

final coalg. (in a fibr.)

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$$(c^{-1} \circ \varphi_\Diamond)U \;\cup\!\mid\; U$$

invariant

$$(c^{-1} \circ \varphi_\Diamond)[\![\nu u.\, \varphi_\Diamond u]\!]_c \;\|\; [\![\nu u.\, \varphi_\Diamond u]\!]_c$$

coind. pred.

$X \supseteq (c^{-1} \circ \varphi_\Diamond)X \supseteq \cdots$

inductive constr.

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predicate lifting

$\mathbb{P}_X \xrightarrow{\varphi_X} \mathbb{P}_{FX} \xrightarrow{c^*} \mathbb{P}_X$

endofunctor

$$(c^* \circ \varphi)P \;\uparrow\; P$$

coalgebra (in a fibr.)

$$(c^* \circ \varphi)[\![\nu\varphi]\!]_c \;\cong\!\uparrow\; [\![\nu\varphi]\!]_c$$

final coalg. (in a fibr.)

$\top_X \leftarrow (c^* \circ \varphi_X)\top_X \leftarrow \cdots$

final sequence in a fibr.

kyo)

What Categorical Generalization Buys Us

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* Final coalgebra in
  $\mathbb{C}$: (strongly) LFP ($\mathbf{Posets}$, $\mathbf{Graphs}$, $\mathbf{Vec}$, ...) [Adamek '03]

* Coinductive pred. for different $F : \mathbf{Sets} \to \mathbf{Sets}$

  * **Coalg. μ-calculus; coalg. automata**
    [Cirstea, Kupke & Pattinson, CSL'09] [Cirstea & Sadrzadeh, CMCS'08] [Venema, I&C'06]

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* Various "underlying logics" as $\begin{array}{c}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{array}$

$\begin{array}{c}\mathbf{Sub}(\mathbb{C})\\ \downarrow\\ \mathbb{C}\end{array}$ ($\mathbb{C}$: a topos)
Constructive logics

$\begin{array}{c}\mathbf{Sub}(\mathbf{Sets}^{\mathbf{F}})\\ \downarrow\\ \mathbf{Sets}^{\mathbf{F}}\end{array}$
For name-passing

$\begin{array}{c}\mathbf{Rel}\\ \downarrow\\ \mathbf{Sets}\end{array}$
Relations ("binary pred.")

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Coind. relations e.g. **bisimilarity**

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Relations ("binary pred.")

Coindution in a Fibration

conventional $\begin{array}{c}\mathbf{Pred}\\ \downarrow\\ \mathbf{Sets}\end{array}$	**relational** $\begin{array}{c}\mathbf{Rel}\\ \downarrow\\ \mathbf{Sets}\end{array}$	**fibrational** $\begin{array}{c}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{array}$
invariant	bisimulation	coalgebra
coind. pred.	bisimilarity	final coalg.
inductive constr.	partition refinement	final sequence

Part III:

Technical Ingredients

Final Sequence, Fibration, Predicate Lifting,

Locally Finitely Presentable Category, ...

Final Sequence

[Worrell, TCS'05] in $\mathbf{Sets}$
[Adamek, TCS'03] in strongly LFP $\mathbb{C}$

$$\mathbf{1} \xleftarrow{\;!\;} \boldsymbol{F}\mathbf{1} \longleftarrow \cdots \longleftarrow \boldsymbol{F}^{i}\mathbf{1} \longleftarrow \cdots$$

[Worrell, TCS'05] in $\mathbf{Sets}$
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Final Sequence

{ i-step behaviors}

$$\mathbf{1} \xleftarrow{\;!\;} \boldsymbol{F}\mathbf{1} \longleftarrow \cdots \longleftarrow \boldsymbol{F}^i\mathbf{1} \longleftarrow \cdots$$

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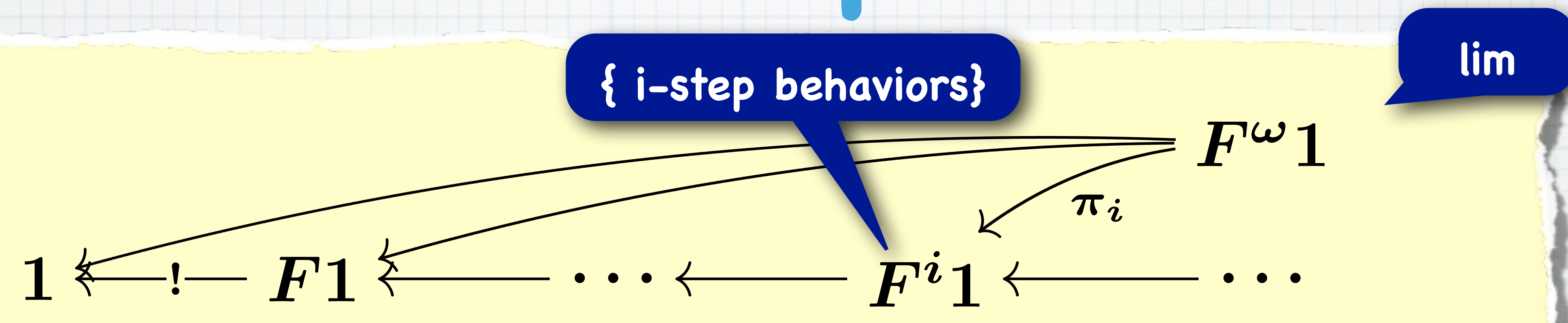
Final Sequence



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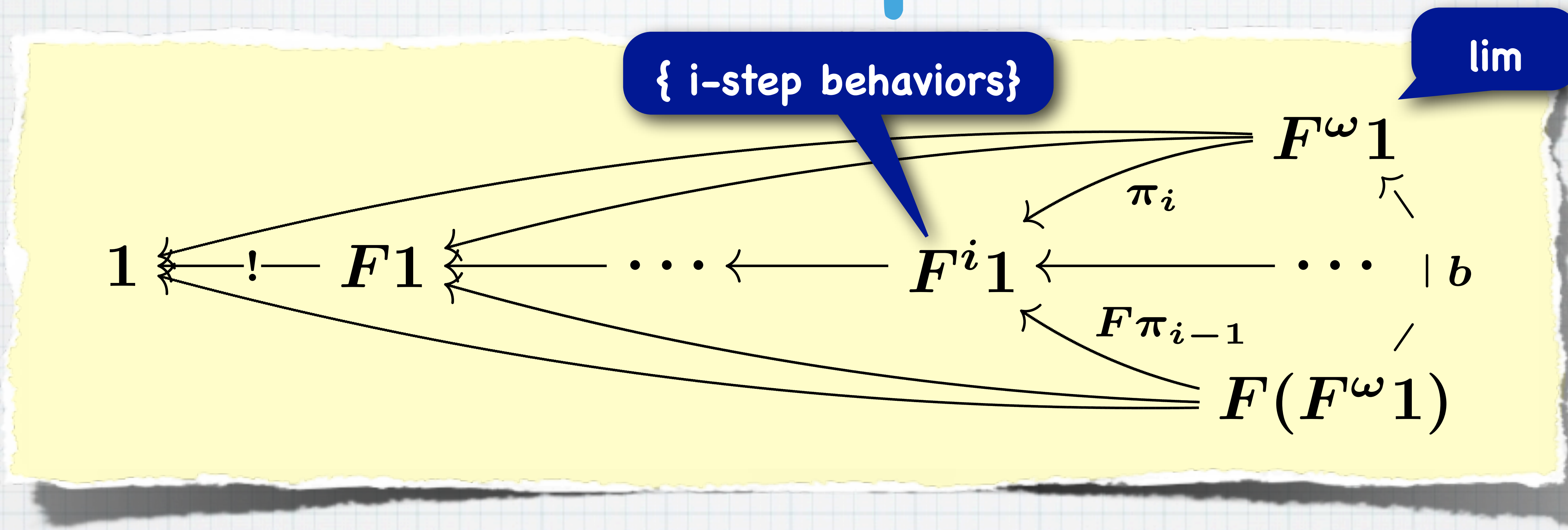

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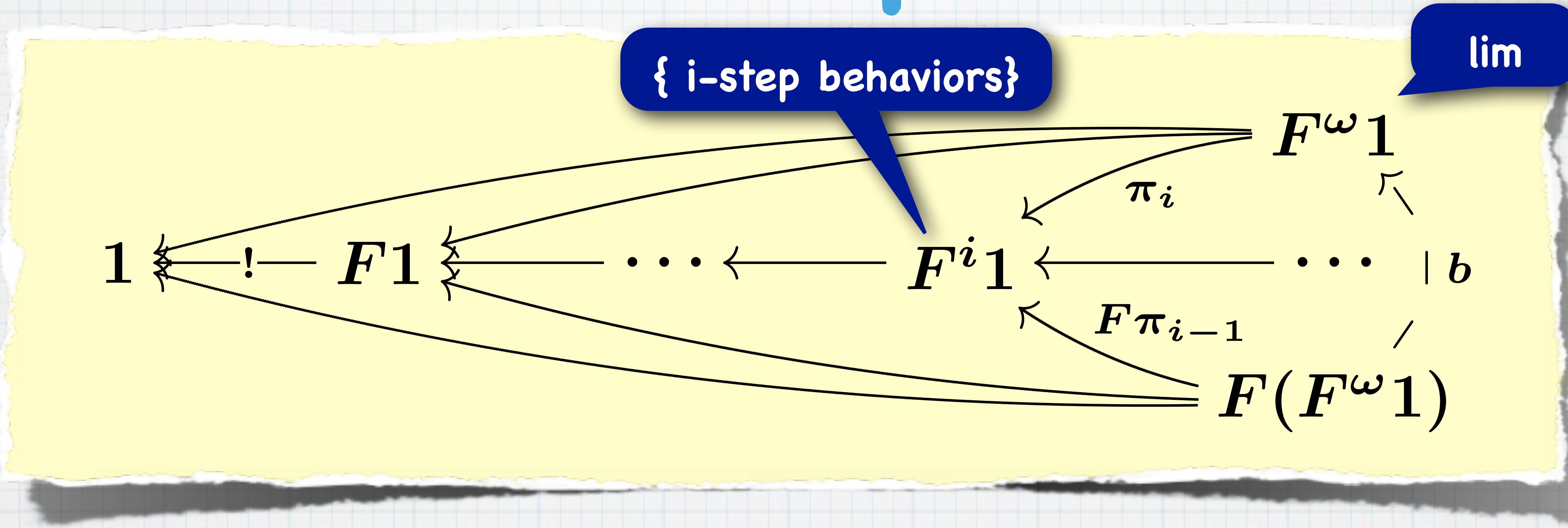

- $F^\omega 1$: a final coalgebra?
  - Yes, when F is limit preserving (b is iso)

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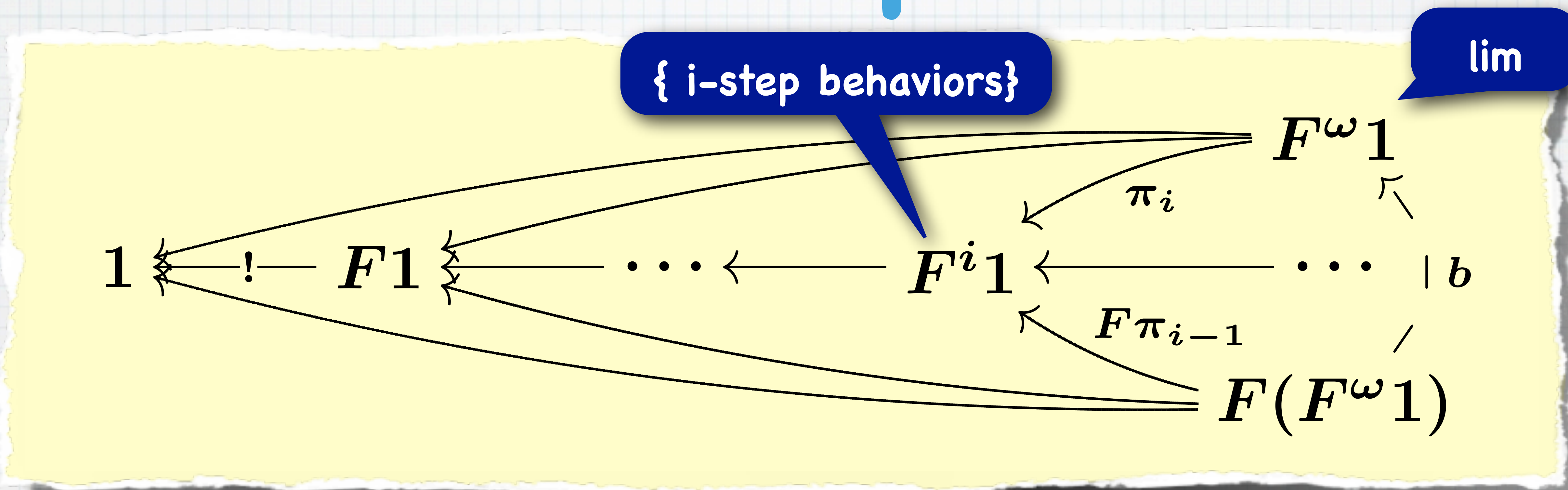


- $F^\omega 1$: a final coalgebra?
  - Yes, when F is limit preserving (b is iso)
  - Almost, when F is **finitary** (b is monic)
    - Quotient modulo beh. eq.
    - Continue till ω+ω [Worrell]

Fibration

$$\begin{array}{c}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{array}$$

$$X \xrightarrow{f} Y \quad \text{in } \mathbb{C}$$

- "Organize indexed entities," categorically
- In particular: categorical model of **predicate logics**

Fibration

$\mathbb{P}$
$\downarrow p$
$\mathbb{C}$

$$X \xrightarrow{f} Y \quad \text{in } \mathbb{C}$$

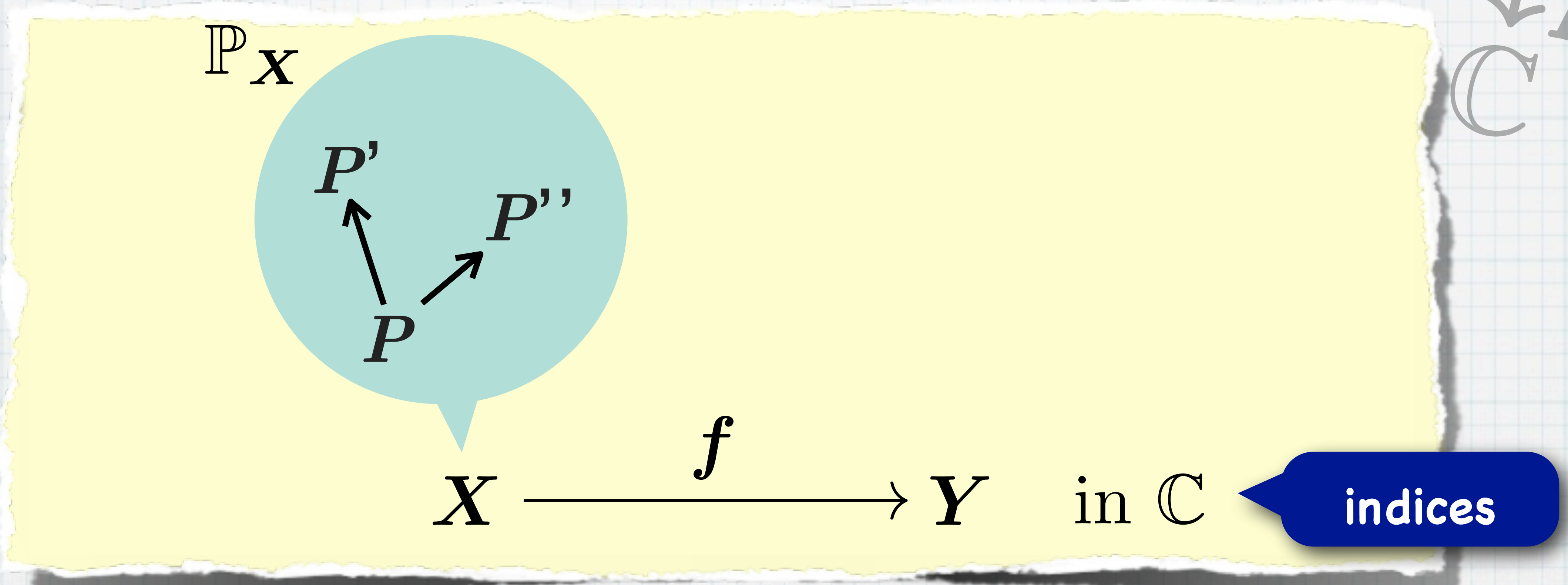
indices

* "Organize indexed entities," categorically

* In particular:
  categorical model of
  predicate logics

Fibration

$\mathbb{P} \downarrow p \ \mathbb{C}$



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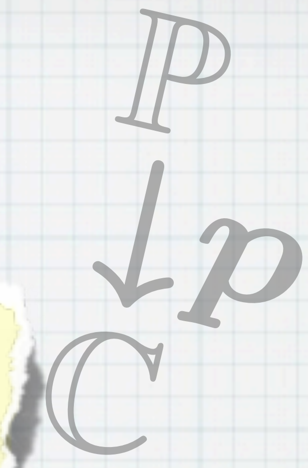

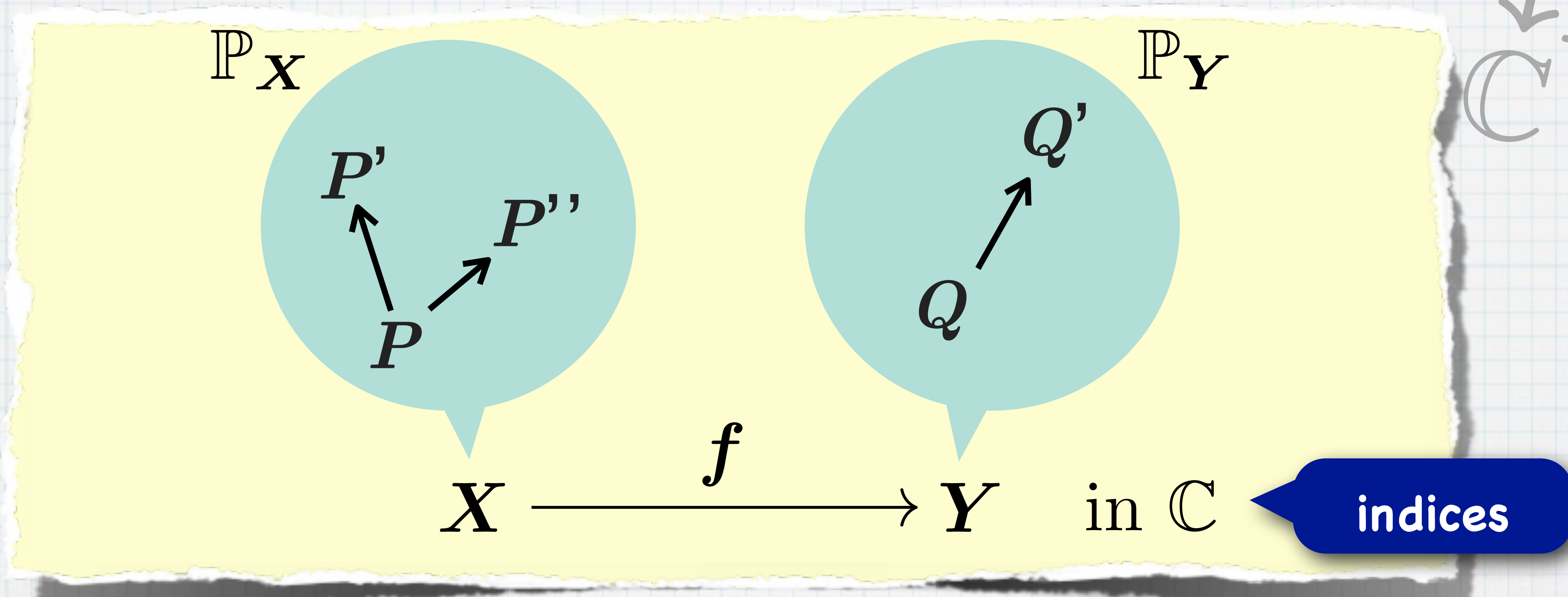

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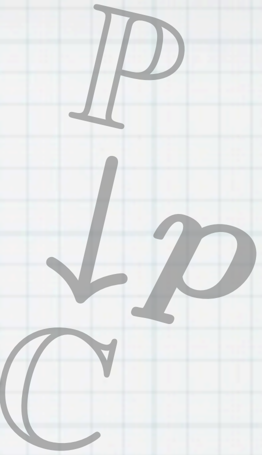

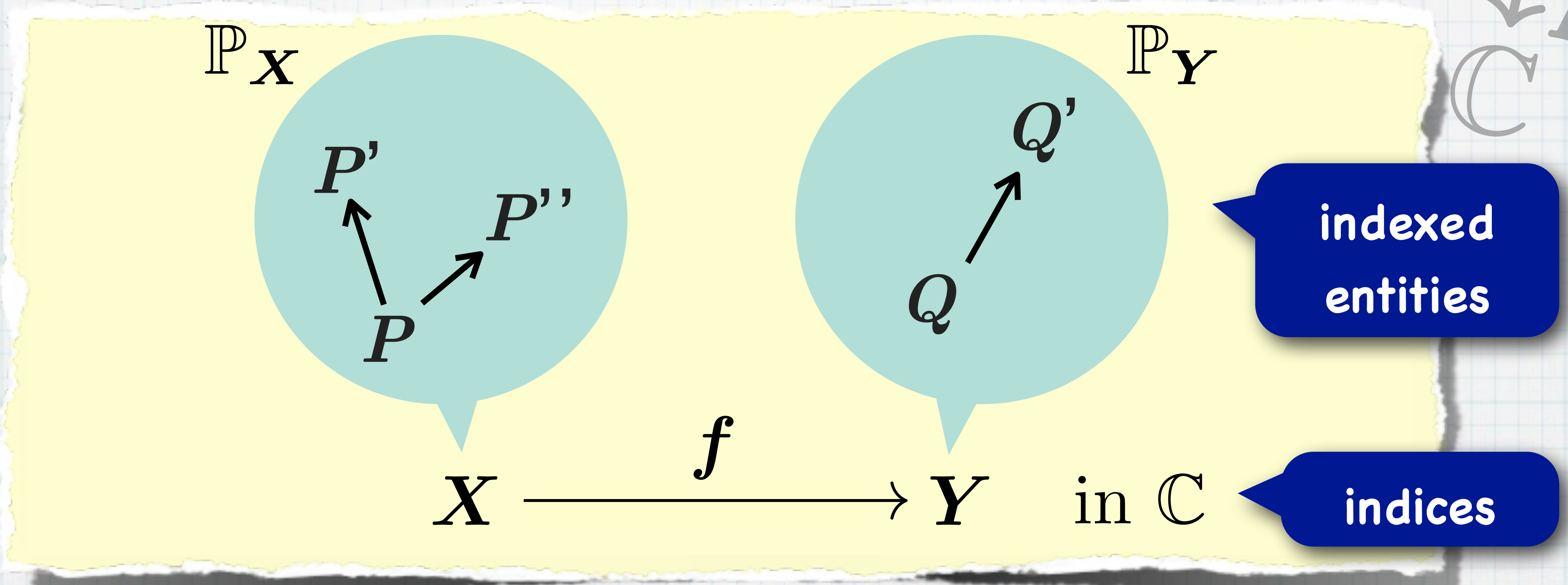

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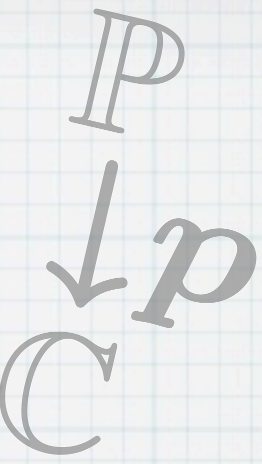

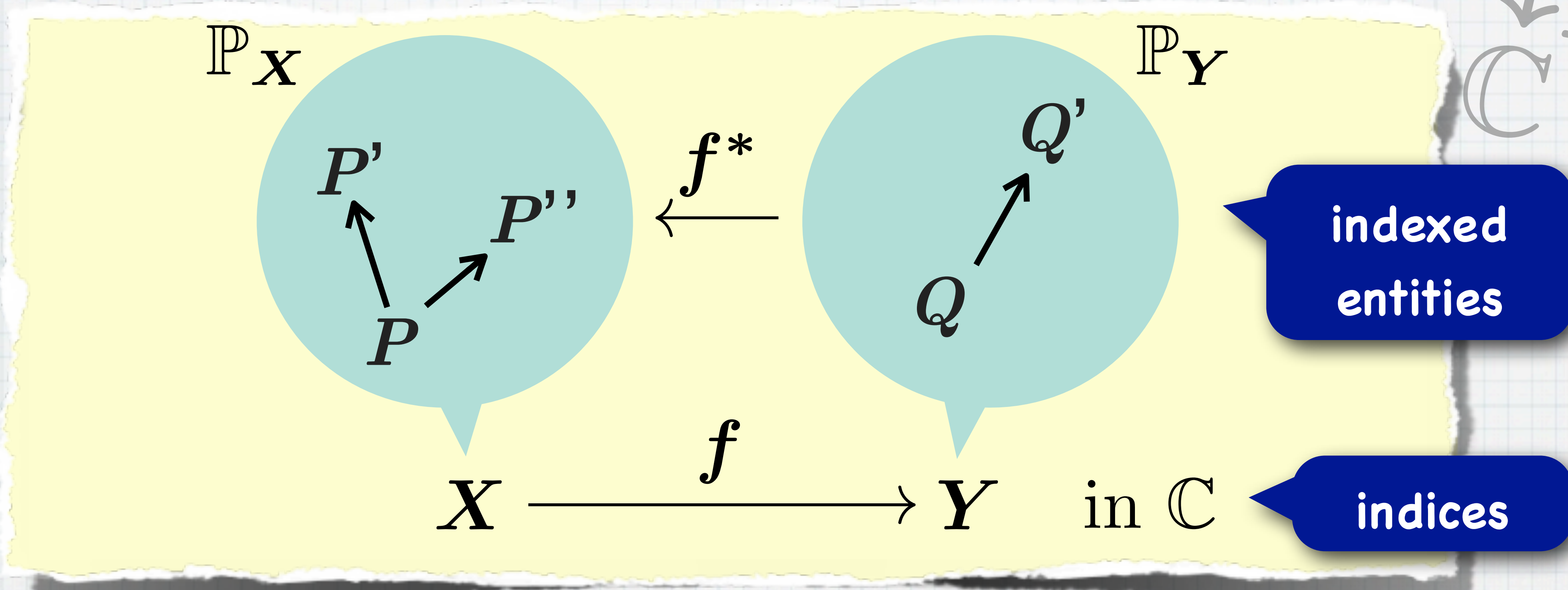

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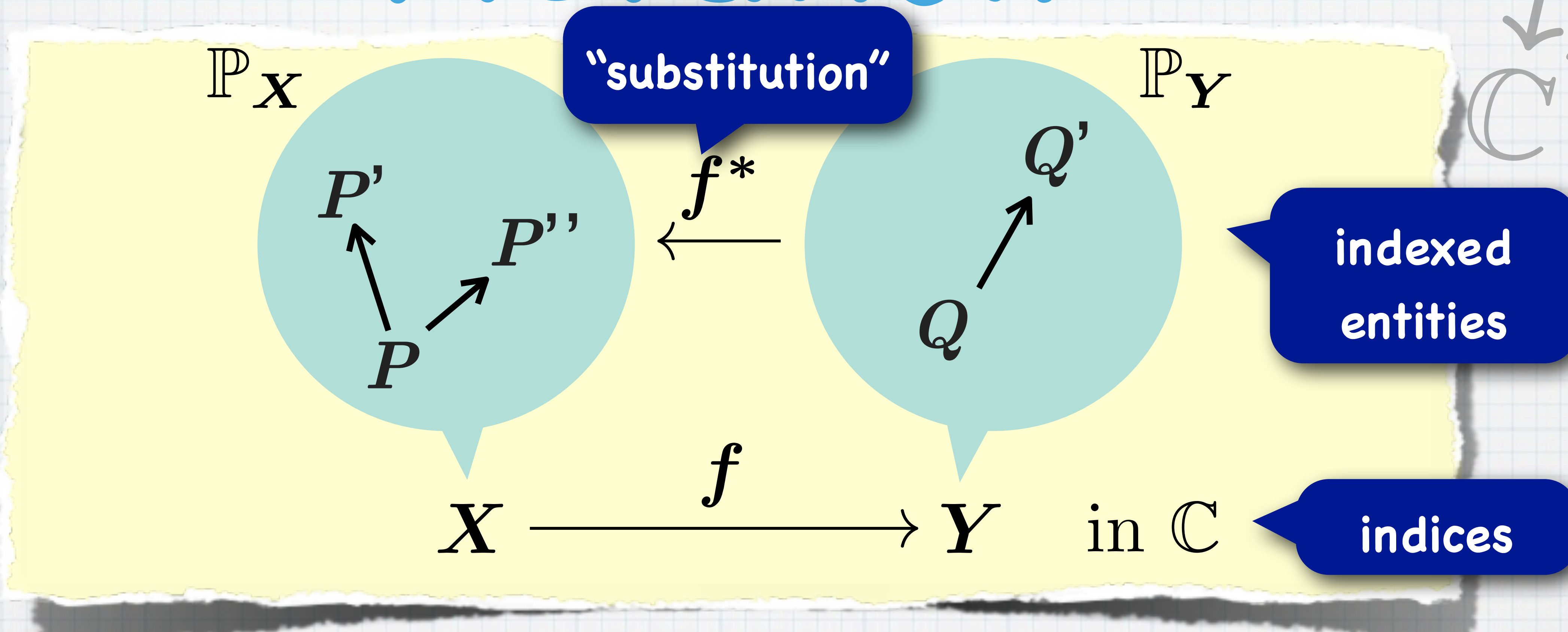

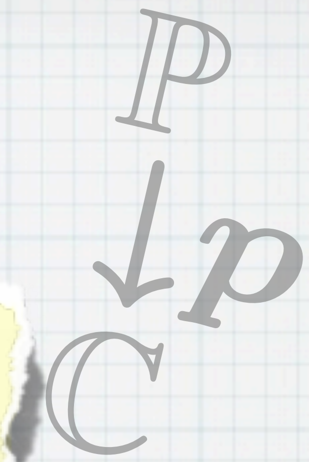

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Fibration

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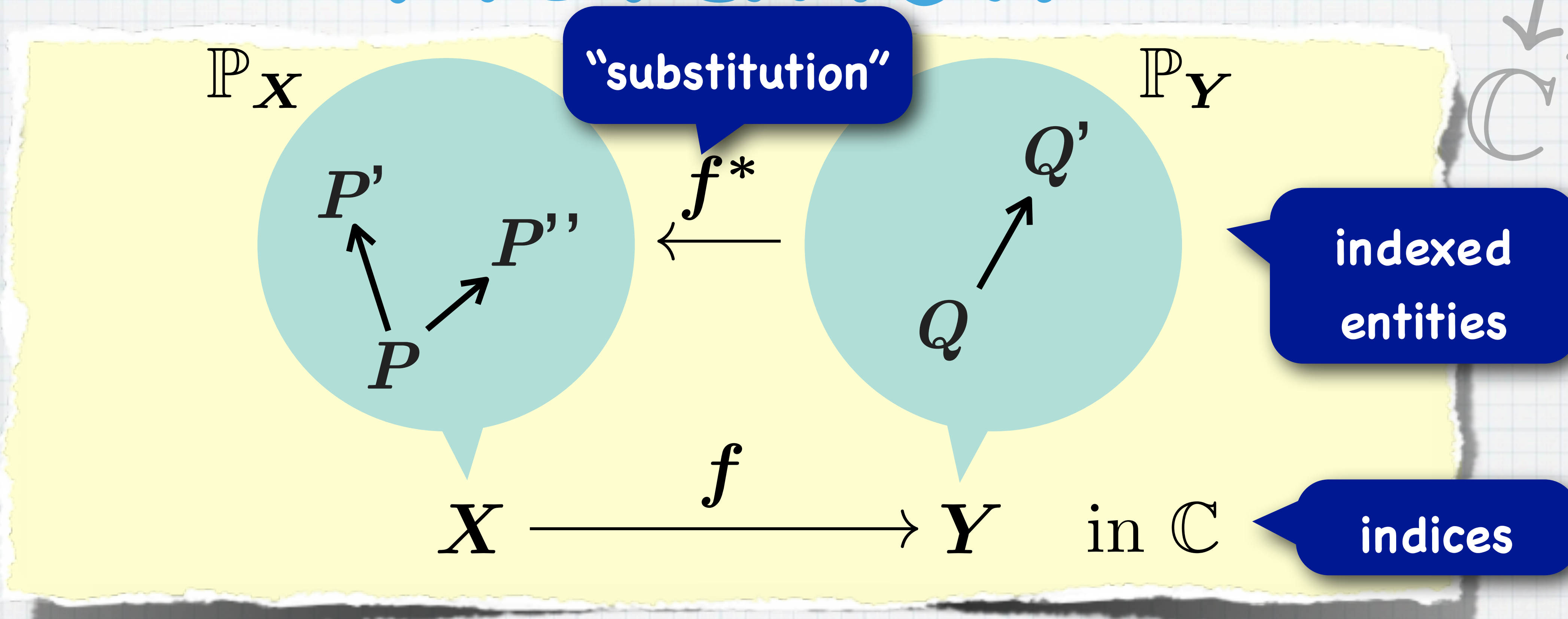

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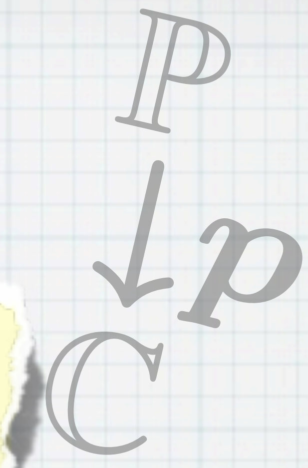
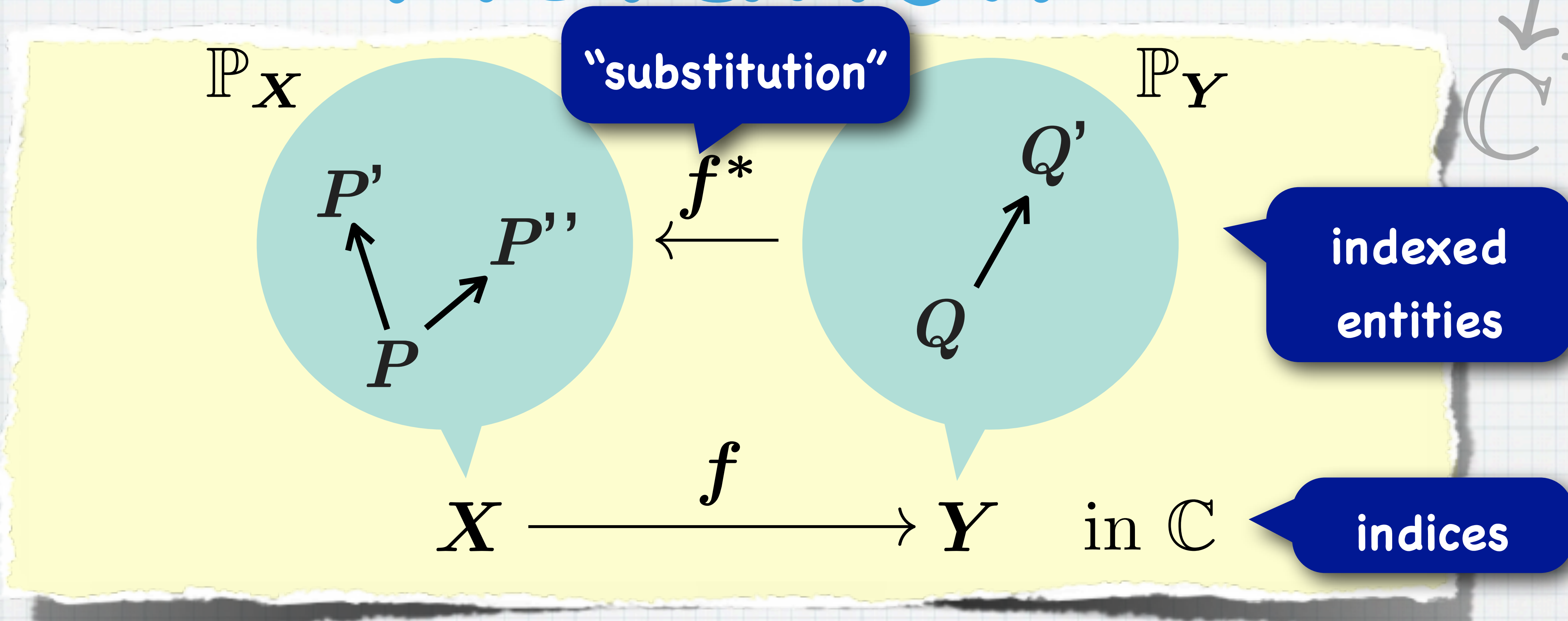
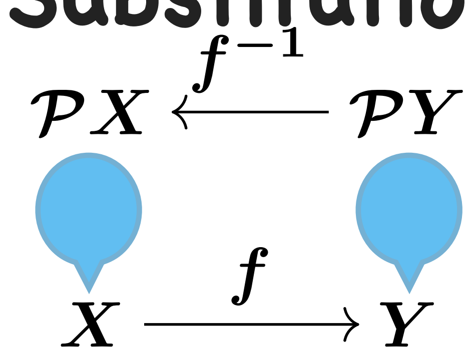




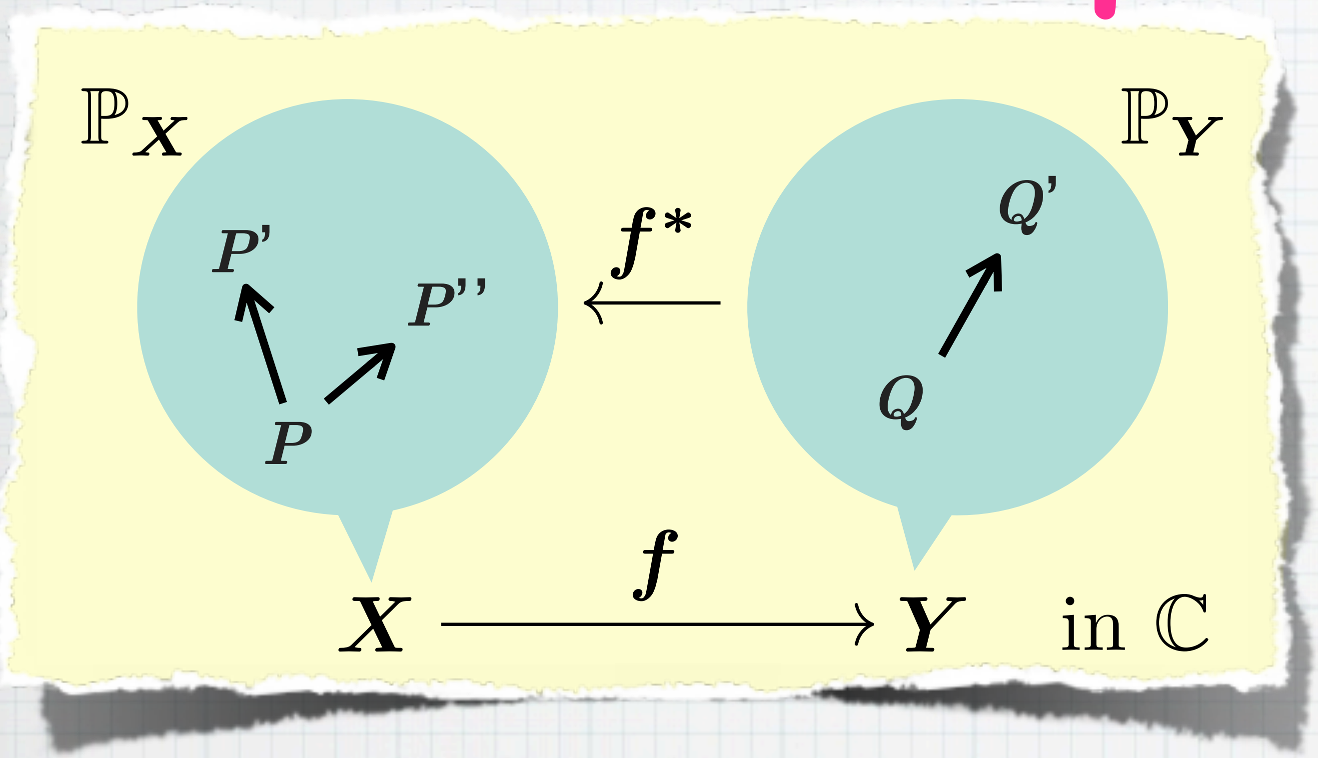
- "Organize indexed entities," categorically
- In particular: categorical model of **predicate logics**

- $(\mathcal{P}X, \subseteq)$: predicates over X
- **Substitution**



$$f^{-1}(V \subseteq Y) = V(f(\_))$$

Fibration: from Pointwise Indexing to Display Indexing



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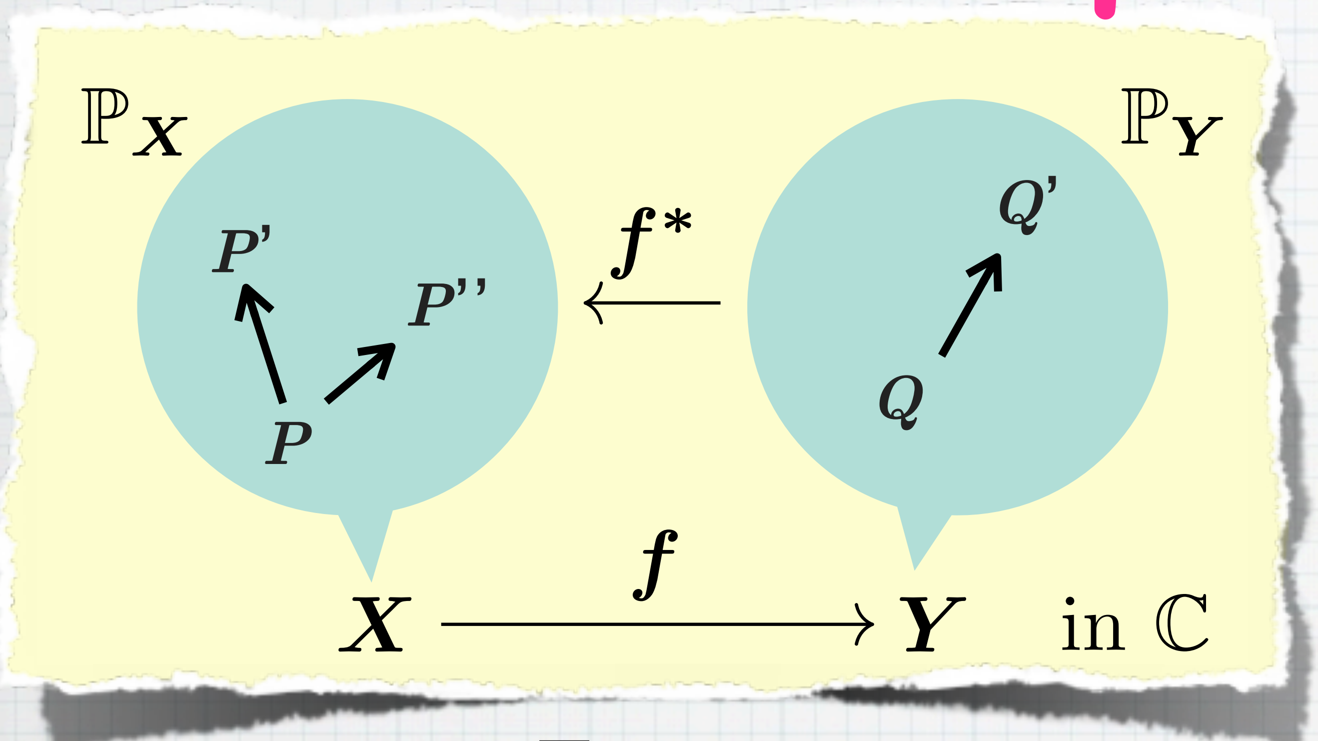
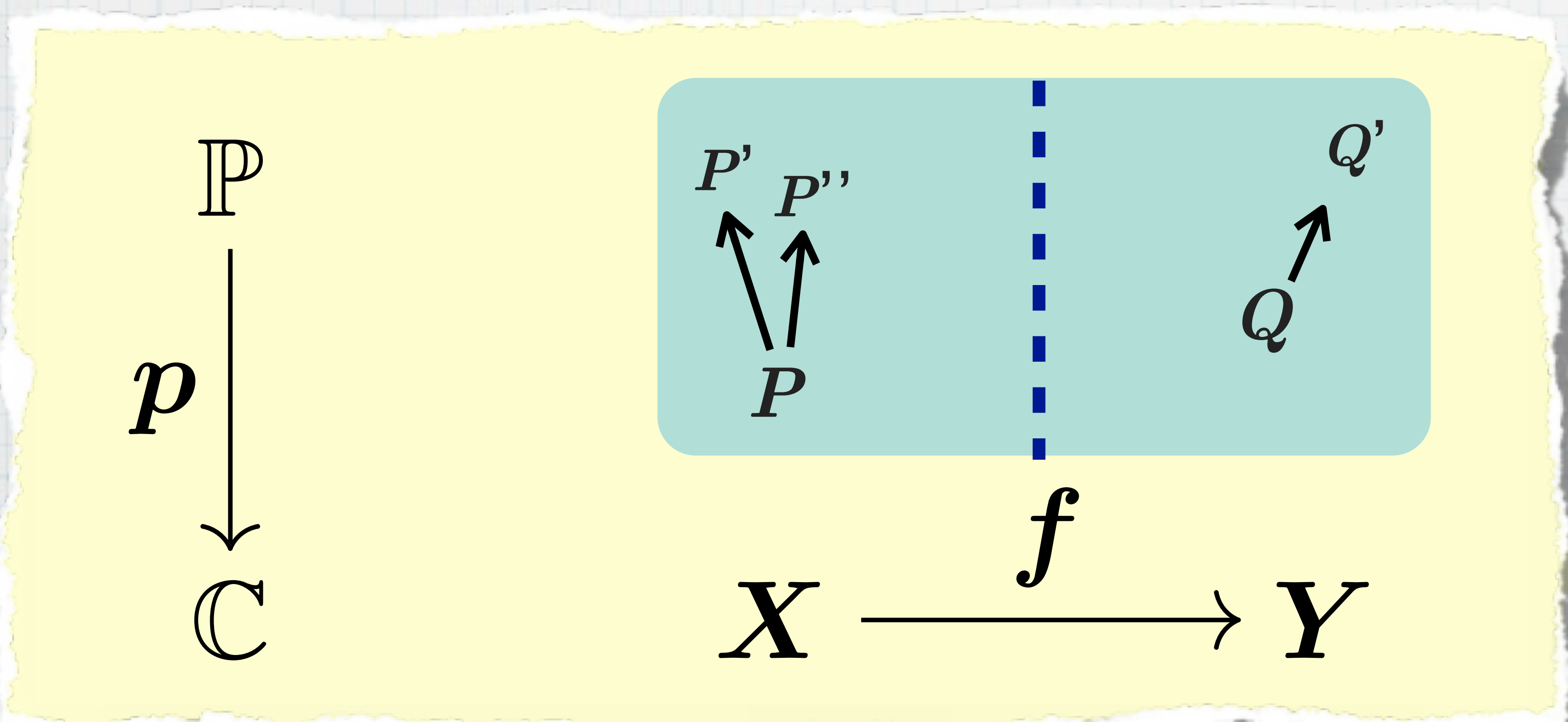


Patch up ↧



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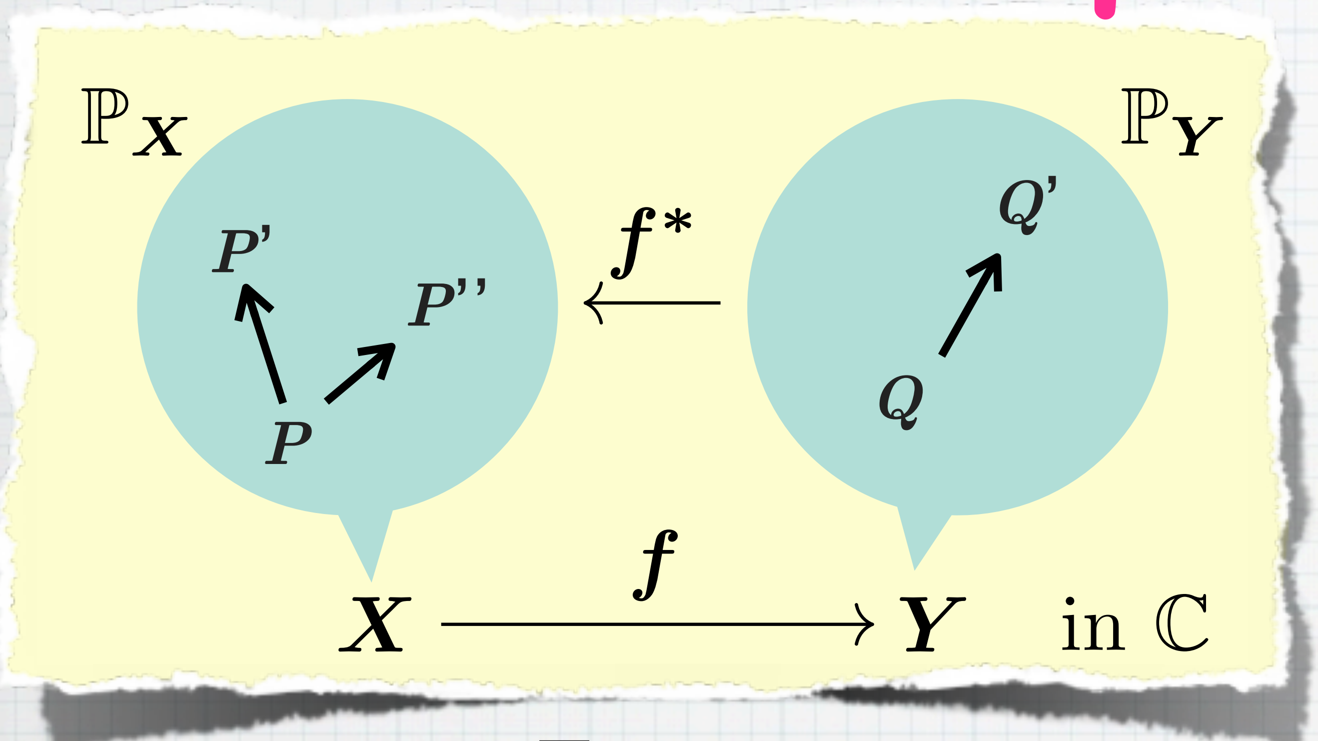

- objects: $|\mathbb{P}| = \coprod_{X \in \mathbb{C}} |\mathbb{P}_X|$
- arrows:

$$\frac{P \longrightarrow Q \quad \text{in } \mathbb{P}}{\left(X \xrightarrow{f} Y \text{ in } \mathbb{C},\ P \to f^*Q \text{ in } \mathbb{P}_X \right)}$$

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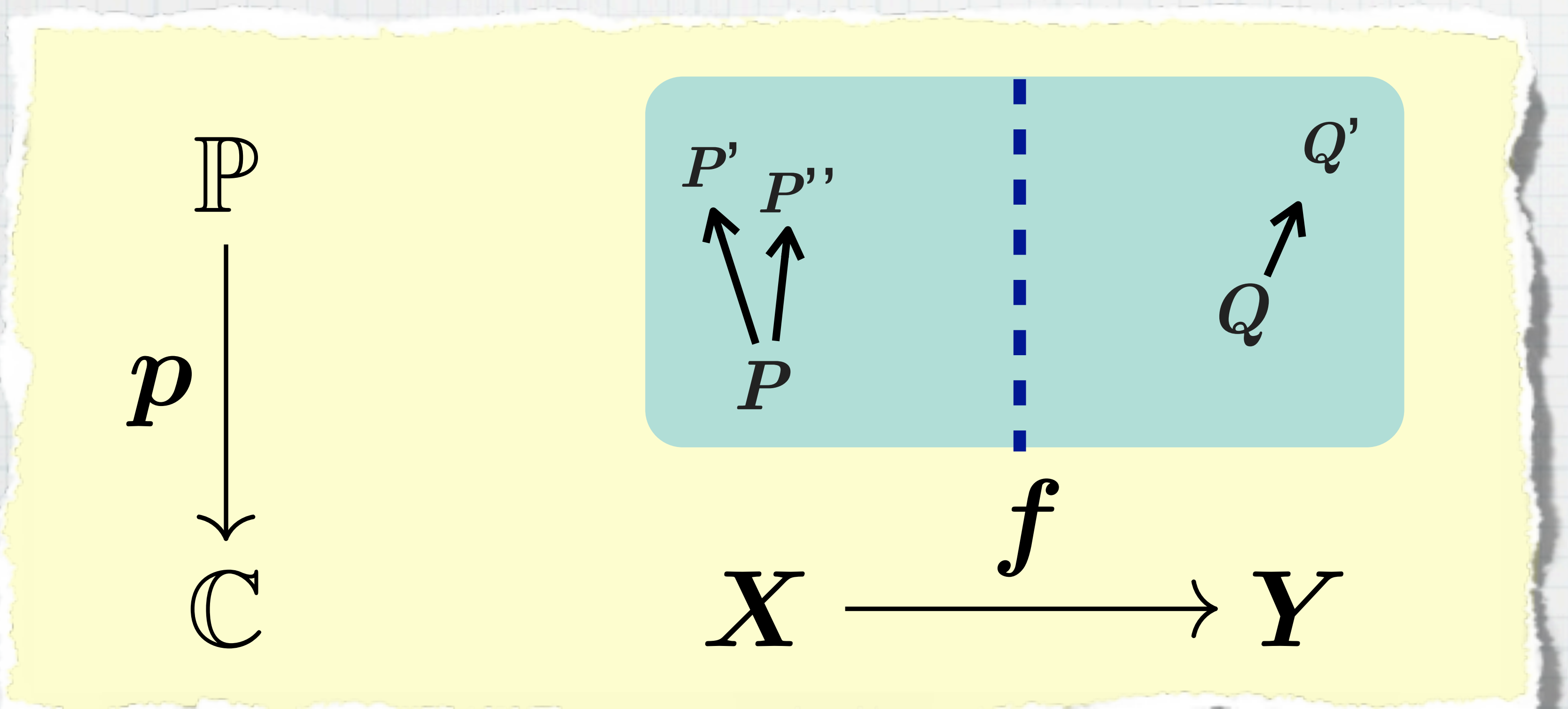

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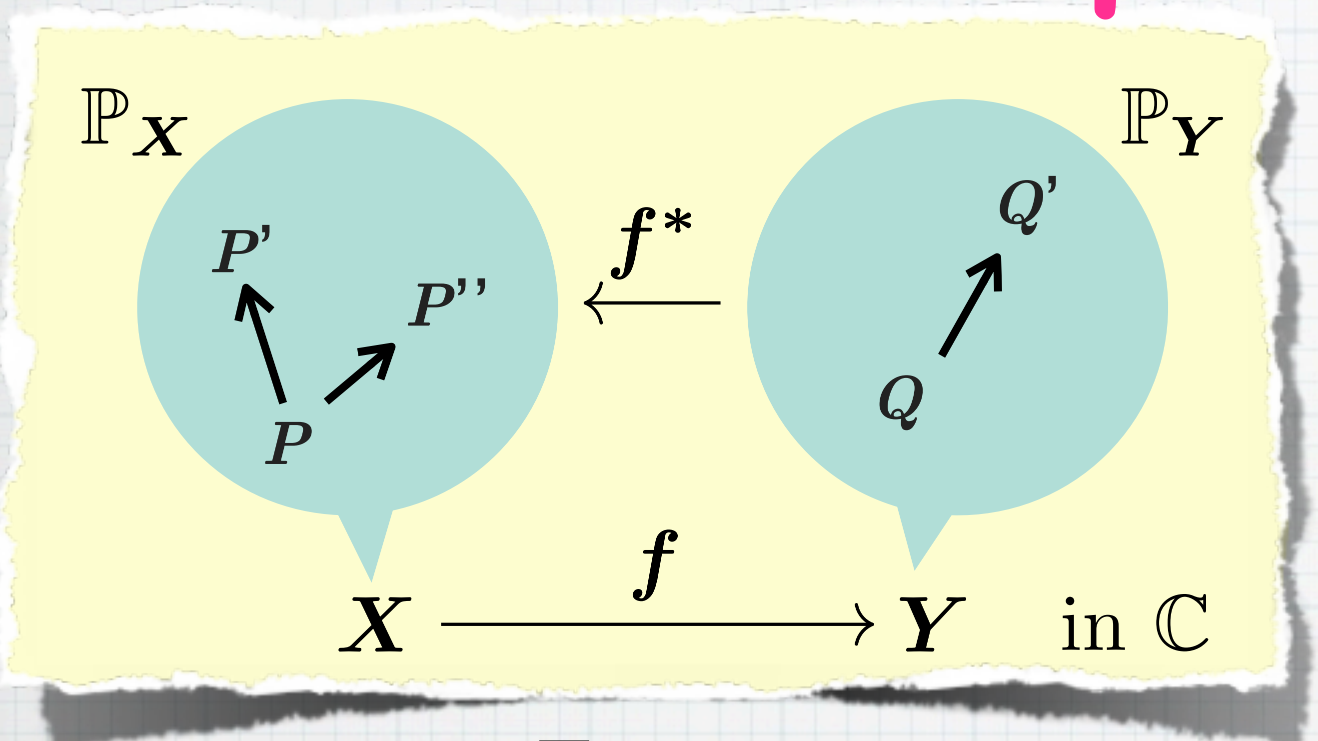

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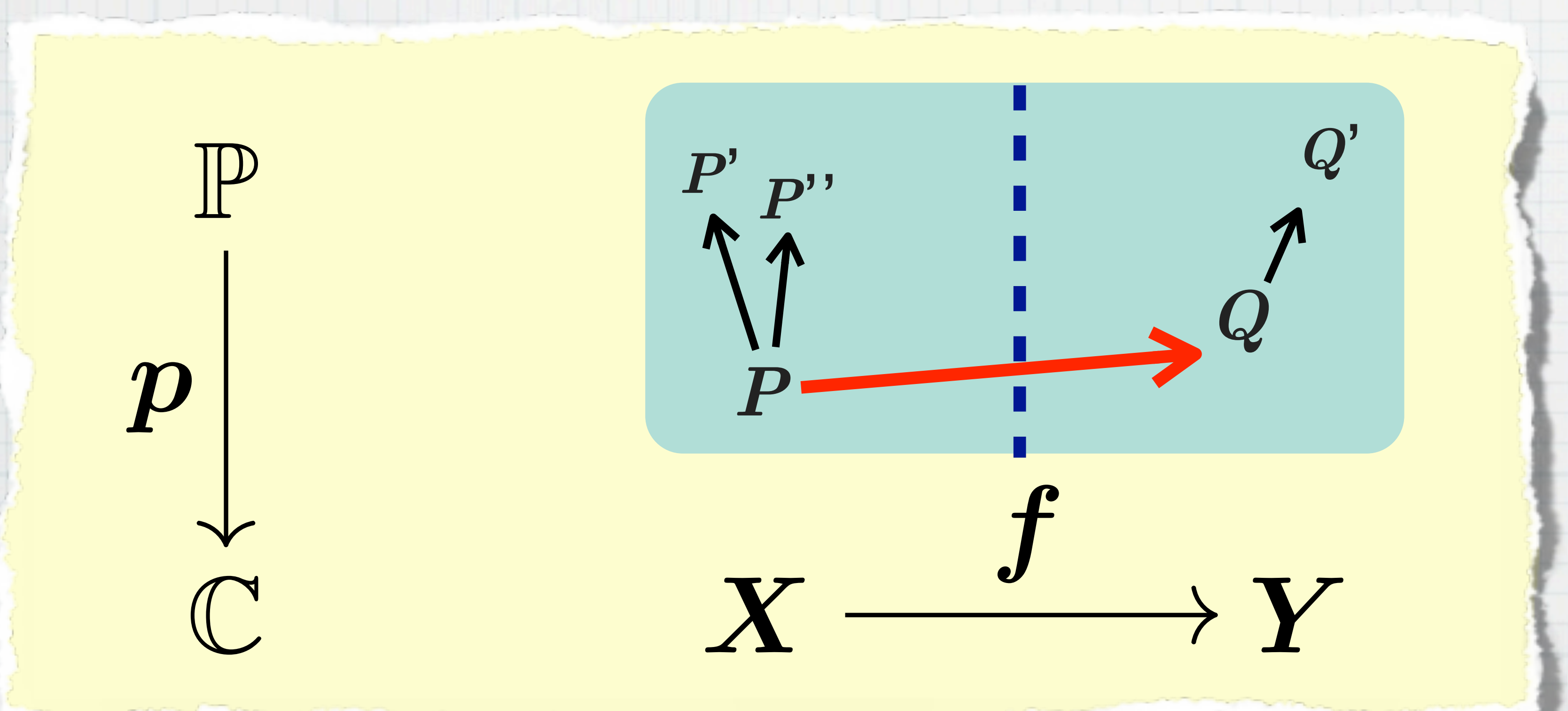

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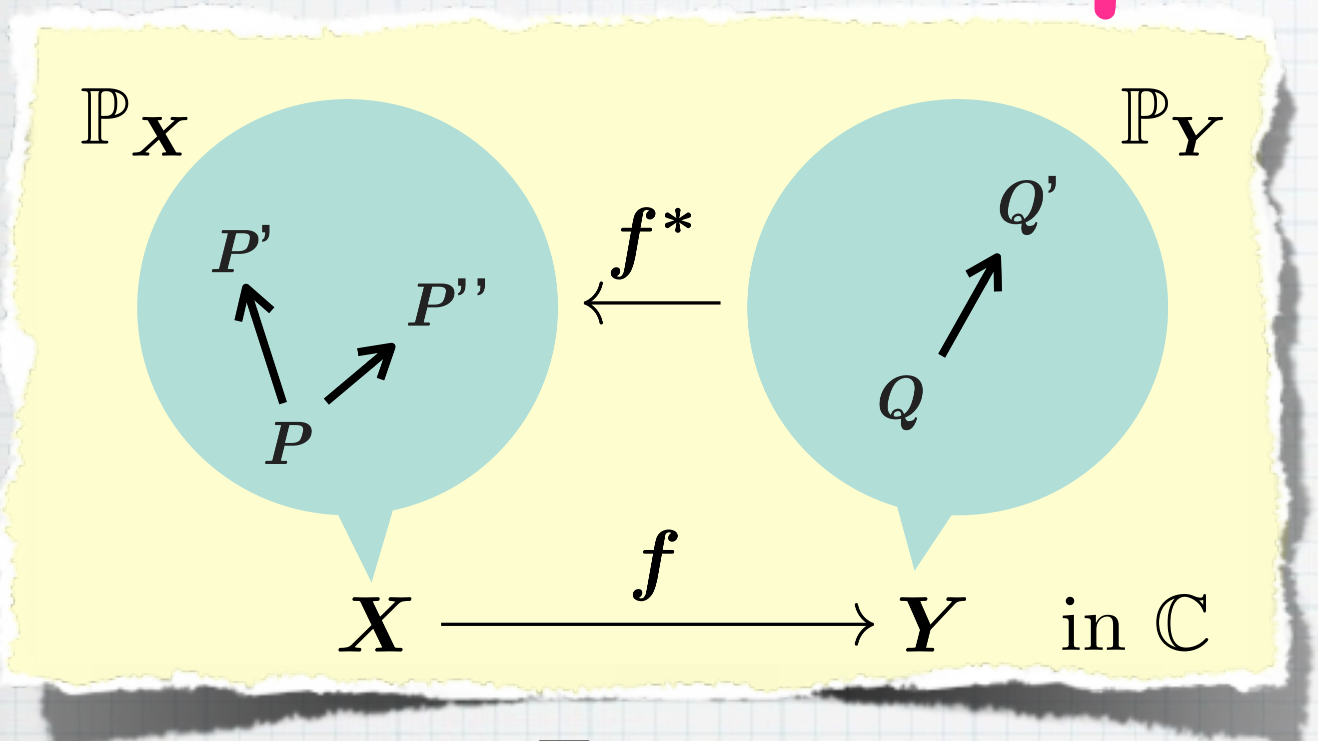

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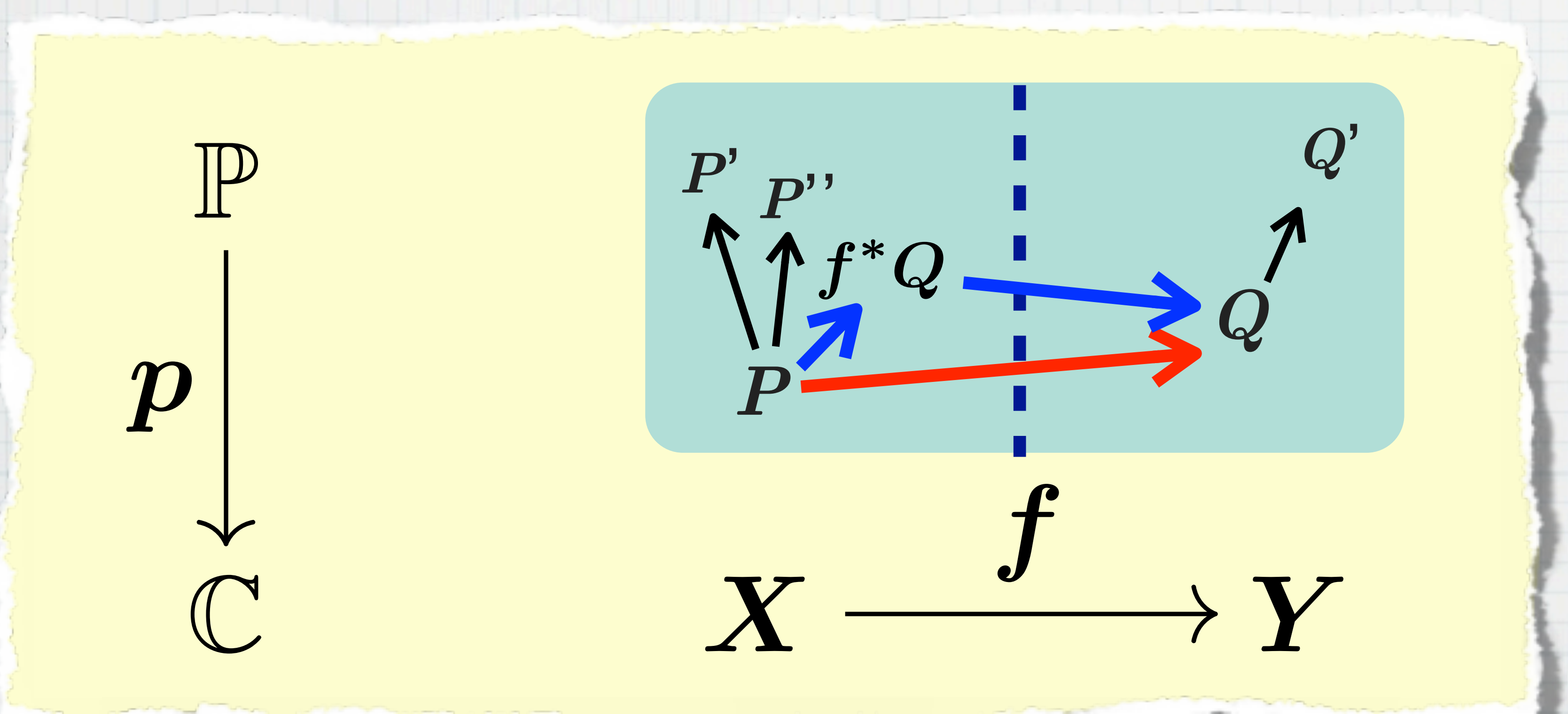
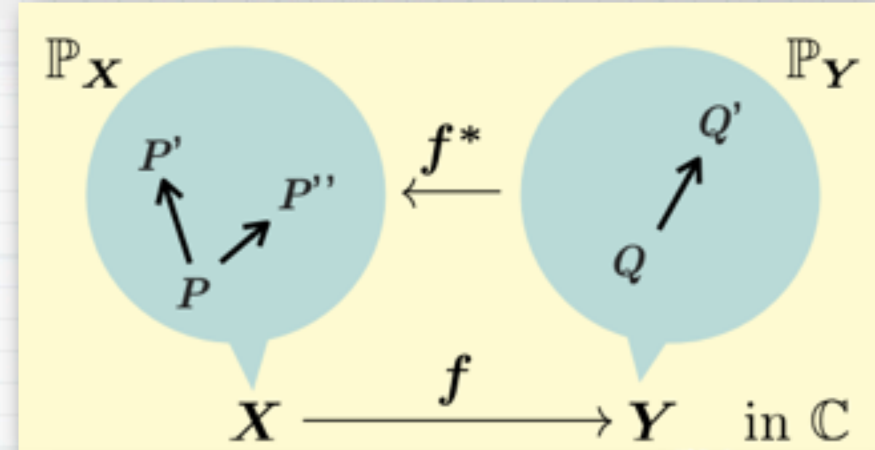
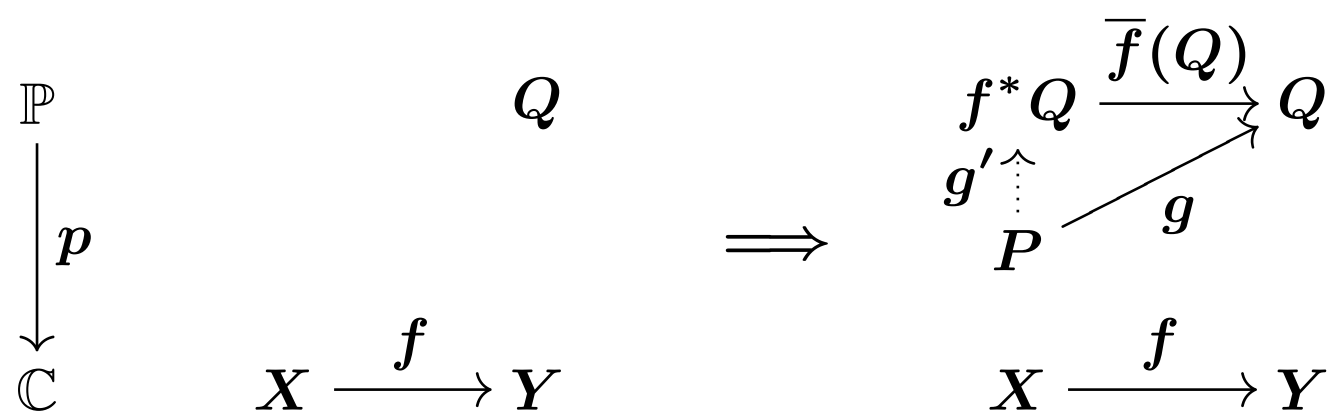


Fibration



Defn. A *(poset) fibration* is a functor $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{smallmatrix}$ such that

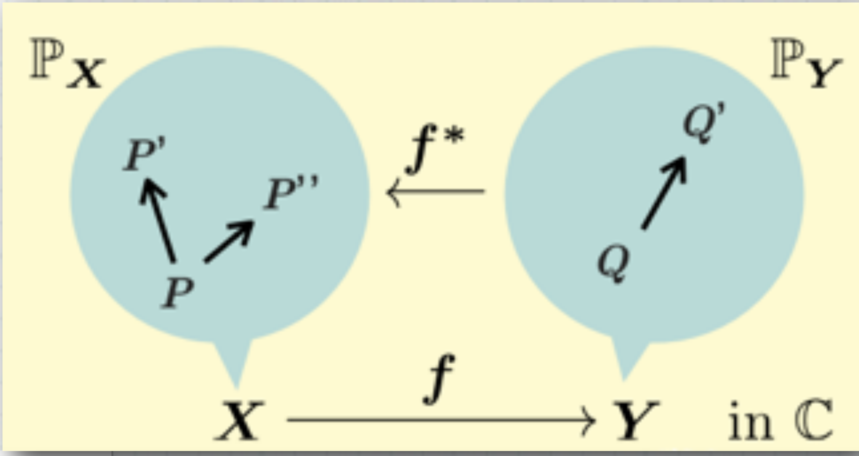
- Each fiber $\mathbb{P}_X$ is a poset.
- For $f : X \to Y$ in $\mathbb{C}$ and $Q \in \mathbb{P}_Y$, a "universal arrow" $\overline{f}Q : f^*Q \to Q$ such that



- The correspondences $(\_)^*$ and $\overline{(\_)}$ are functorial:

$$\mathrm{id}_Y^* Q = Q \ , \qquad (g \circ f)^*(Q) = f^*(g^*Q) \ ,$$
$$\overline{\mathrm{id}_Y}(Q) = \mathrm{id}_Q \ , \qquad \overline{g \circ f}(Q) = \overline{g}Q \circ \overline{f}(g^*Q) \ .$$

Fibration



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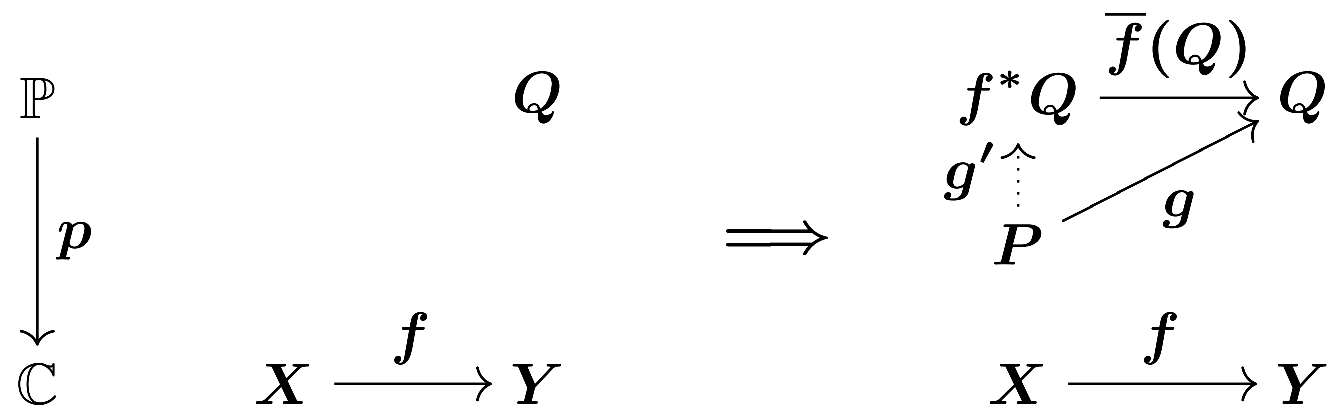

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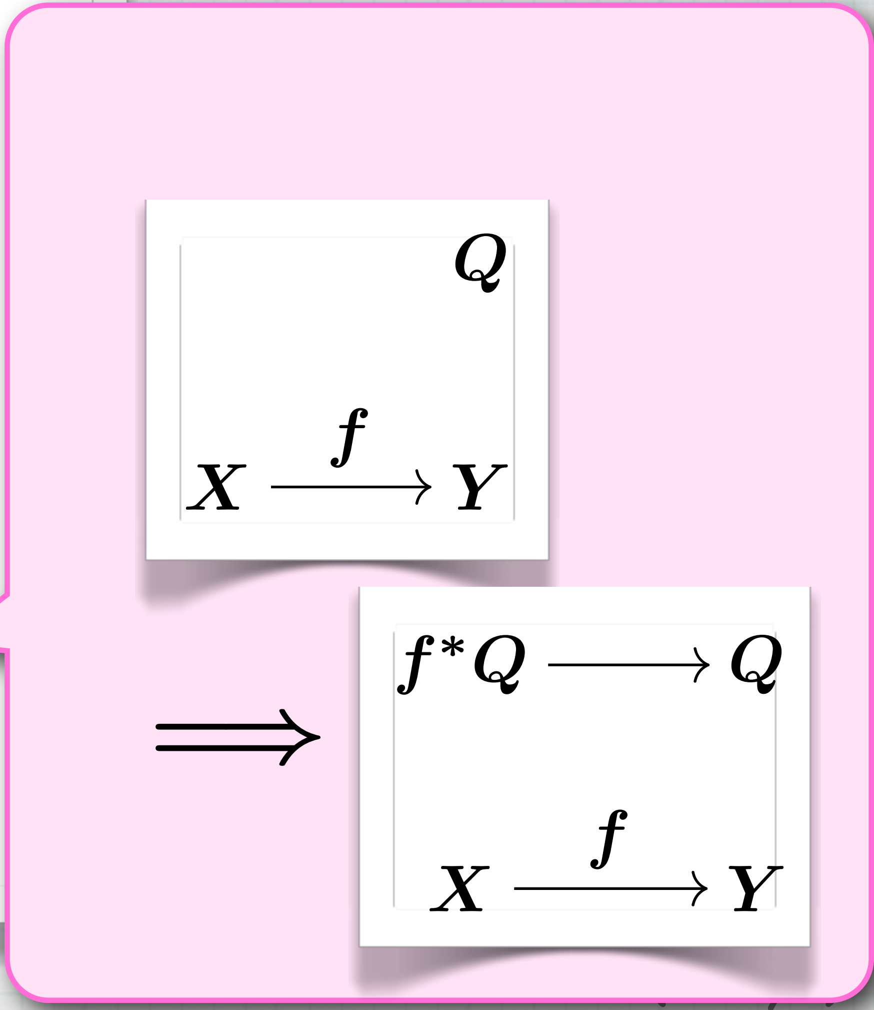
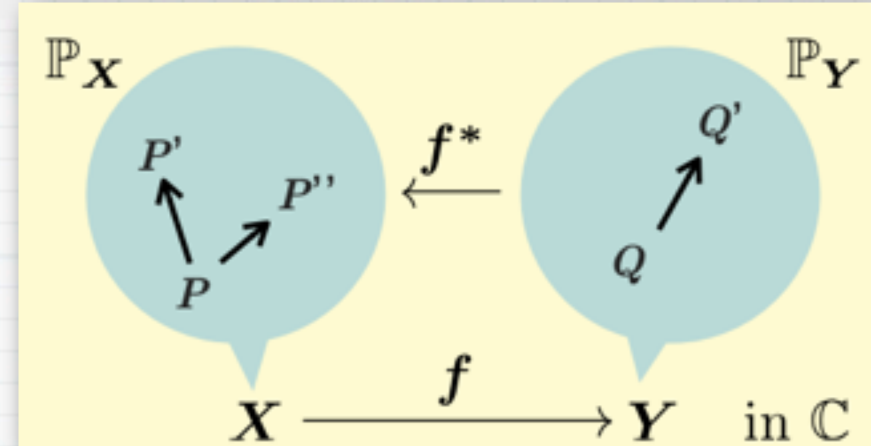


Fibration



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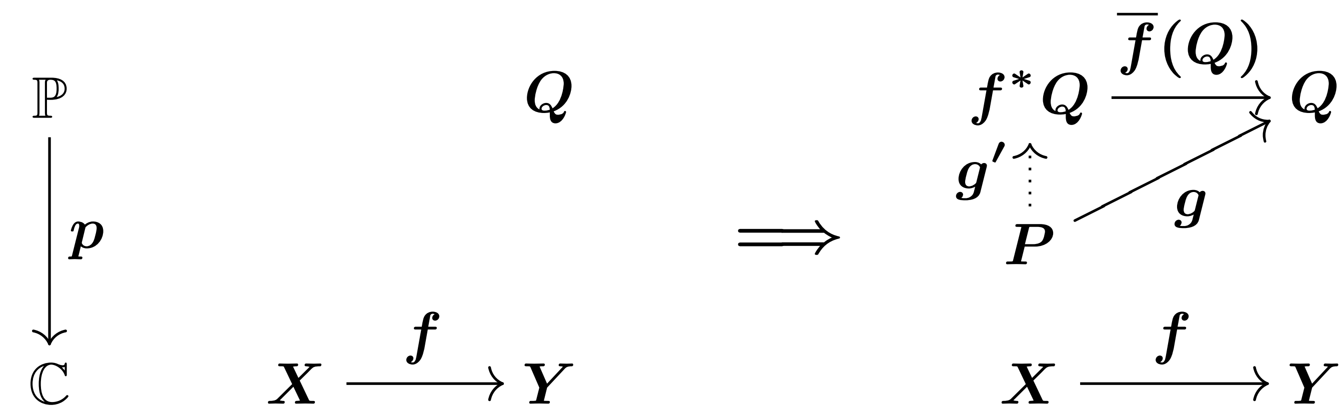

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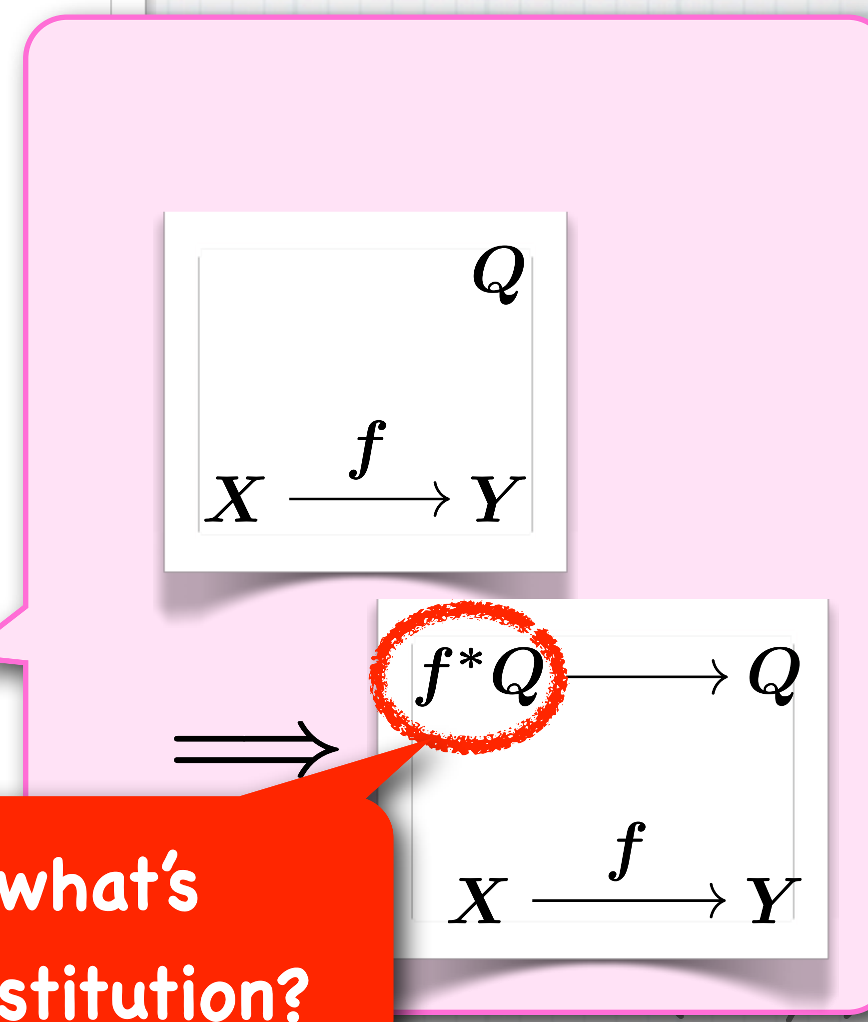

Fibration: Examples

Pred
$\downarrow$
Sets

$$(f^{-1}Q \subseteq X) \longrightarrow (Q \subseteq Y)$$

$$X \xrightarrow{\quad f \quad} Y$$

Fibration: Examples

Pred $\downarrow$ **Sets**

$$(f^{-1}Q \subseteq X) \longrightarrow (Q \subseteq Y)$$

$$X \xrightarrow{\quad f \quad} Y$$

Rel $\downarrow$ **Sets**

$$\begin{pmatrix} (f \times f)^{-1}Q \\ \subseteq X \times X \end{pmatrix} \longrightarrow (Q \subseteq Y \times Y)$$

$$X \xrightarrow{\quad f \quad} Y$$

Fibration: Examples

$$\begin{array}{c}\mathbf{Pred}\\ \downarrow\\ \mathbf{Sets}\end{array}$$

$$(f^{-1}Q \subseteq X) \longrightarrow (Q \subseteq Y)$$

$$X \xrightarrow{\ f\ } Y$$

$$\begin{array}{c}\mathbf{Rel}\\ \downarrow\\ \mathbf{Sets}\end{array}$$

$$\begin{pmatrix}(f \times f)^{-1}Q\\ \subseteq X \times X\end{pmatrix} \longrightarrow (Q \subseteq Y \times Y)$$

$$X \xrightarrow{\ f\ } Y$$

$$\begin{array}{c}\mathbf{Sub}(\mathbb{C})\\ \downarrow\\ \mathbb{C}\end{array} \qquad (\mathbb{C}\text{: a topos})$$

$$\left(\begin{array}{ccc} f^*P & \longrightarrow & P\\ \downarrow & \lrcorner & \downarrow\\ X & \xrightarrow[f]{} & Y\end{array}\right) \longrightarrow \left(\begin{array}{c} P\\ \downarrow\\ Y\end{array}\right)$$

$$X \xrightarrow{\ f\ } Y$$

Fibration: Examples

$$\begin{array}{c}\mathbf{Pred}\\ \downarrow\\ \mathbf{Sets}\end{array}$$

$$(f^{-1}Q \subseteq X) \longrightarrow (Q \subseteq Y)$$

$$X \xrightarrow{\ f\ } Y$$

$$\begin{array}{c}\mathbf{Rel}\\ \downarrow\\ \mathbf{Sets}\end{array}$$

$$\left(\begin{array}{c}(f \times f)^{-1}Q\\ \subseteq X \times X\end{array}\right) \longrightarrow (Q \subseteq Y \times Y)$$

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$$X \xrightarrow{\ f\ } Y$$

$$\begin{array}{c}\mathbf{Sub}(\mathbf{Sets}^{\mathbf{F}})\\ \downarrow\\ \mathbf{Sets}^{\mathbf{F}}\end{array}$$

Predicate Lifting For Modality

Defn.
A *predicate lifting* of $\boldsymbol{F} : \mathbb{C} \to \mathbb{C}$ is $\boldsymbol{\varphi} : \mathbb{P} \to \mathbb{P}$ s.t.

- $$\begin{array}{ccc} \mathbb{P} & \xrightarrow{\boldsymbol{\varphi}} & \mathbb{P} \\ \boldsymbol{p}\downarrow & & \downarrow\boldsymbol{p} \\ \mathbb{C} & \xrightarrow[\boldsymbol{F}]{} & \mathbb{C} \end{array}$$ (hence $\boldsymbol{\varphi_X} : \mathbb{P}_{\boldsymbol{X}} \to \mathbb{P}_{\boldsymbol{FX}}$)
- compatible with substitution.

* For $\begin{array}{c}\mathbf{Pred}\\ \downarrow \\ \mathbf{Sets}\end{array}$, coincides with

$\boldsymbol{\lambda_X} : \mathbf{2}^{\boldsymbol{X}} \Longrightarrow \mathbf{2}^{\boldsymbol{FX}}$, monotone, natural in $\boldsymbol{X}$

Part IV:

Final Sequence in a Fibration

Final Sequence in a Fibration

$$\begin{array}{c}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{array}$$

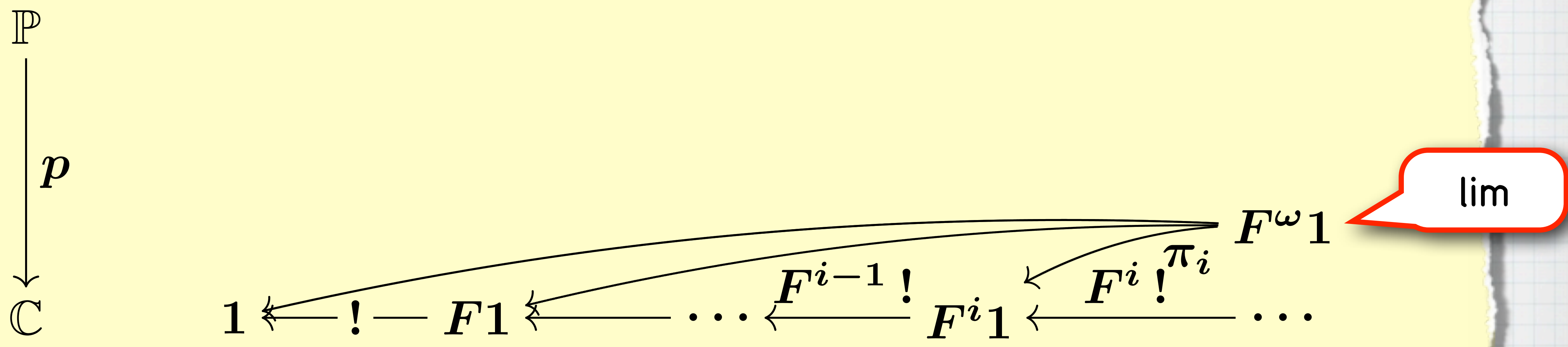
$$\mathbf{1} \xleftarrow{\;!\;} \boldsymbol{F}\mathbf{1} \longleftarrow \cdots \xleftarrow{\boldsymbol{F}^{i-1}!} \boldsymbol{F}^{i}\mathbf{1} \xleftarrow{\boldsymbol{F}^{i}!} \cdots$$

Final Sequence in a Fibration

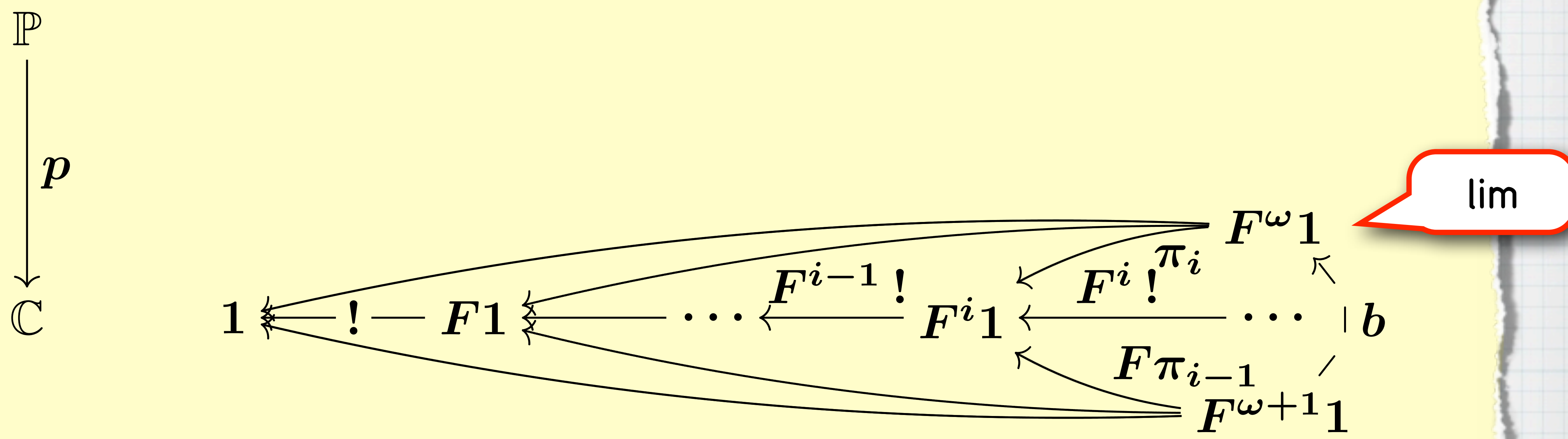
$$\begin{array}{c}\mathbb{P}\\ \big\downarrow p\\ \mathbb{C}\end{array}$$

$$\mathbf{1} \xleftarrow{\;!\;} F\mathbf{1} \longleftarrow \cdots \xleftarrow{F^{i-1}!} F^{i}\mathbf{1} \xleftarrow{F^{i}!} \cdots \qquad F^{\omega}\mathbf{1} \xrightarrow{\pi_i} F^{i}\mathbf{1}$$

Final Sequence in a Fibration



Final Sequence in a Fibration



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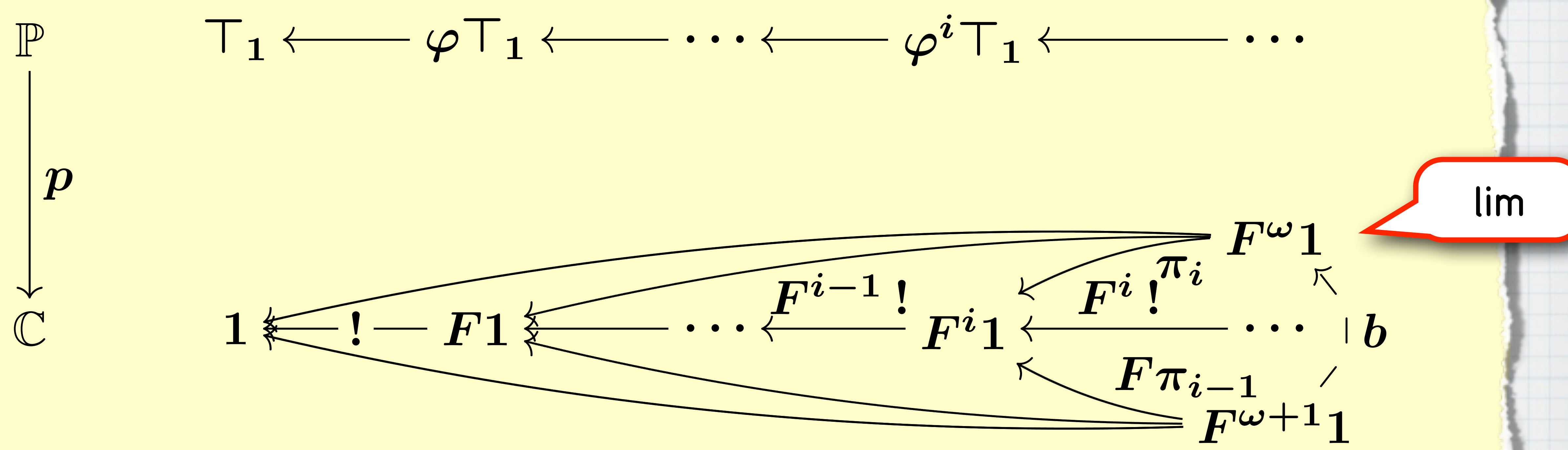
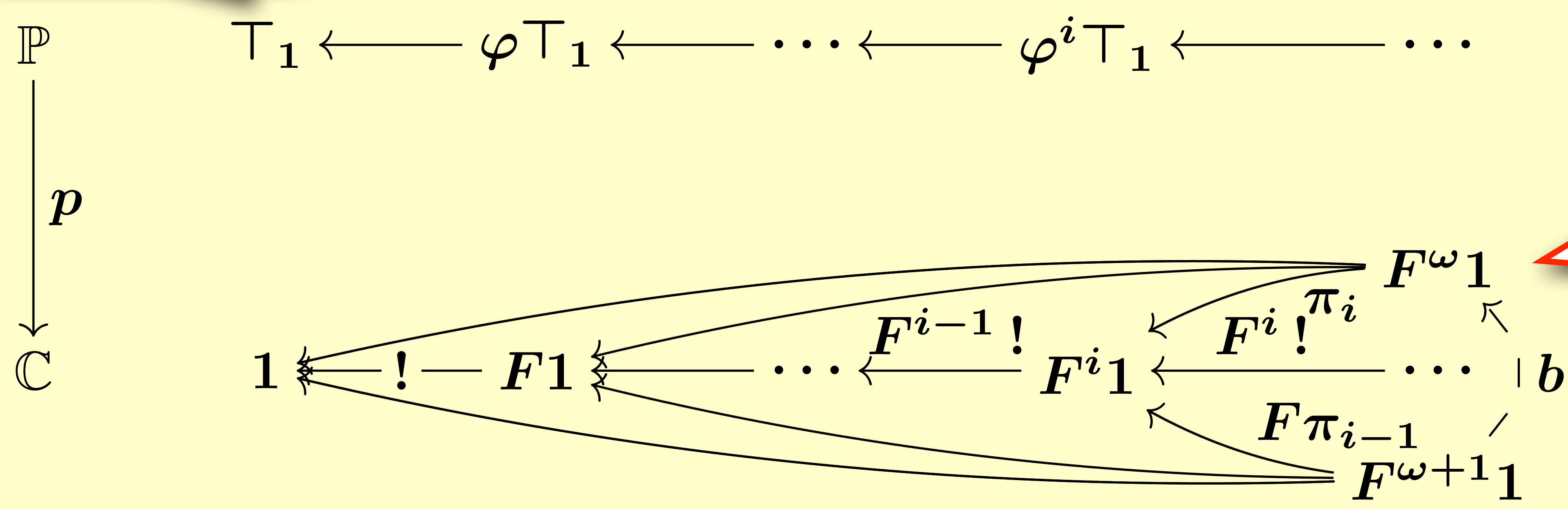


Final Sequence in a Fibration

final in $\mathbb{P}_1$

$\Longrightarrow$ final in $\mathbb{P}$



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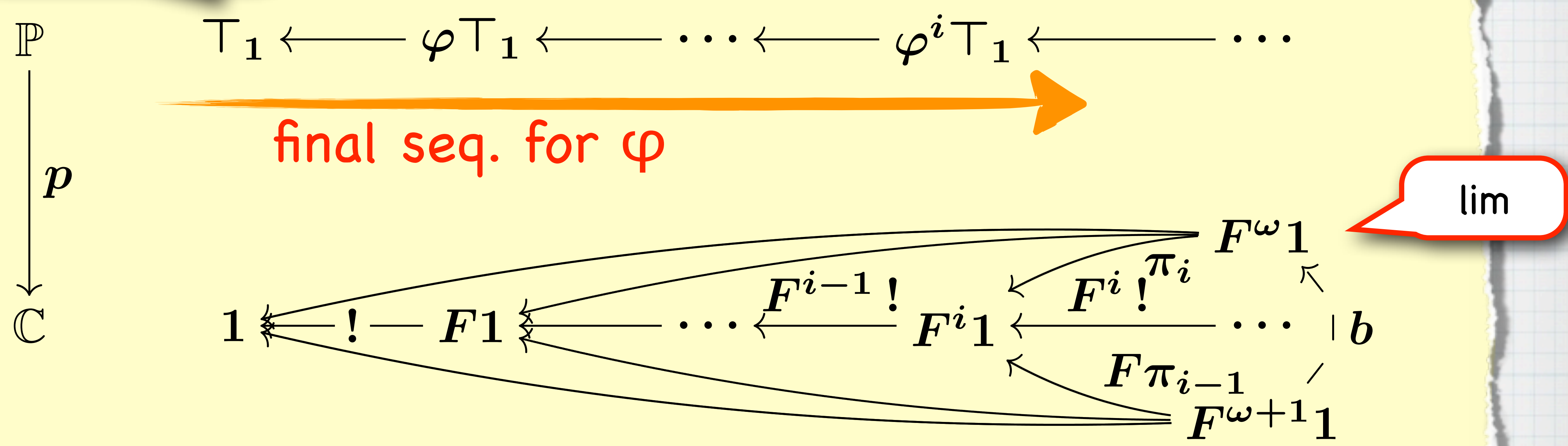
Final Sequence in a Fibration

final in $\mathbb{P}_1$

$\Longrightarrow$ final in $\mathbb{P}$



Final Sequence in a Fibration

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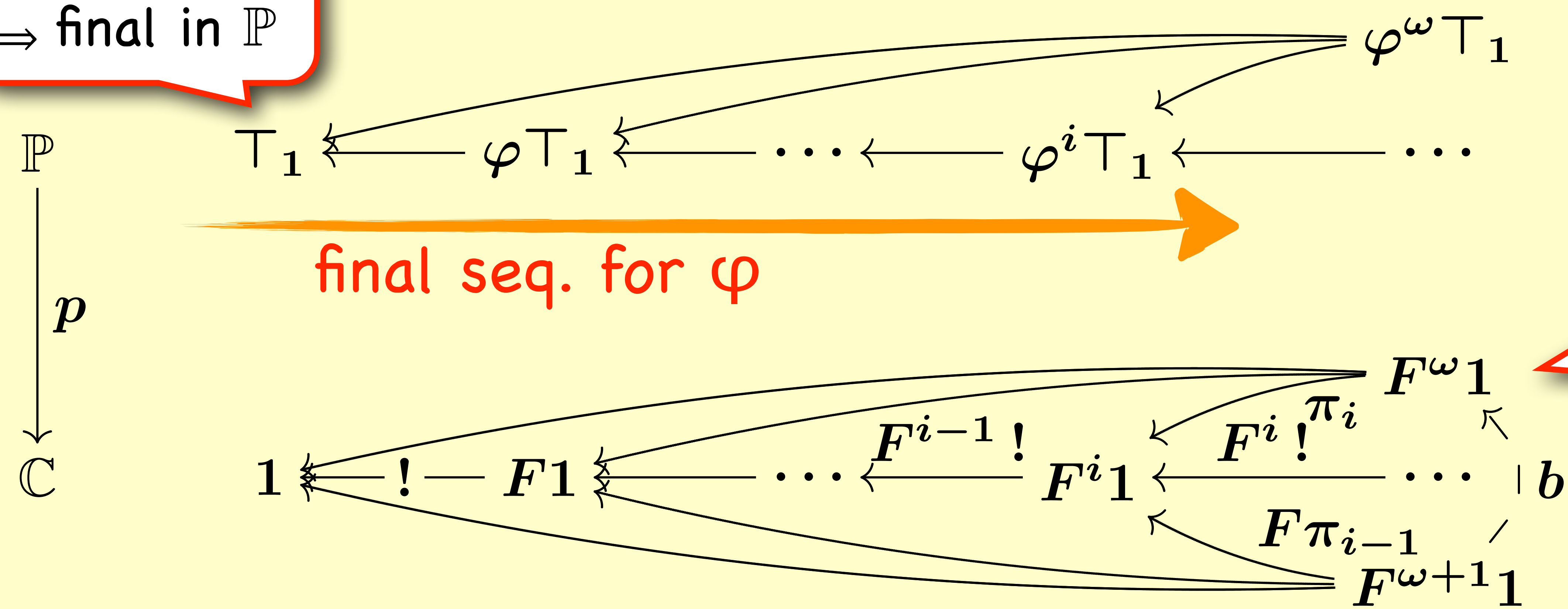

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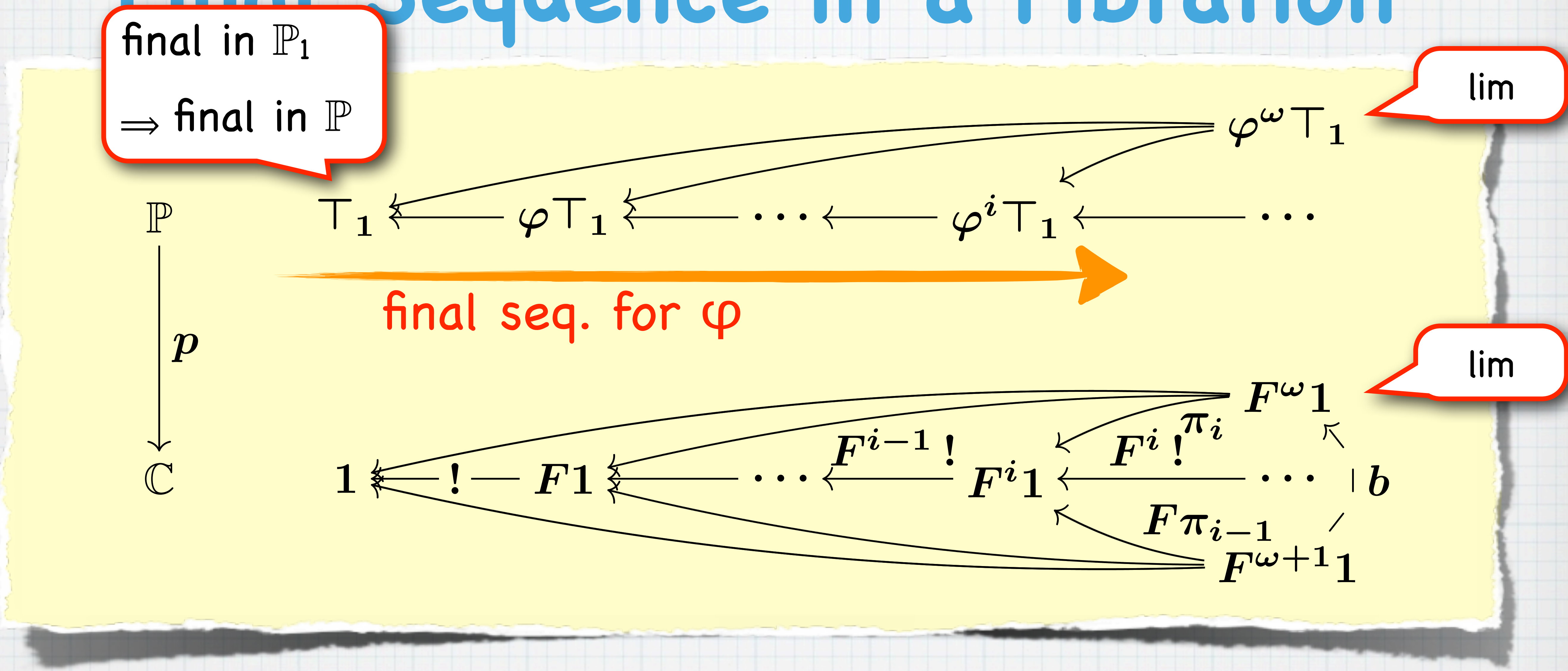

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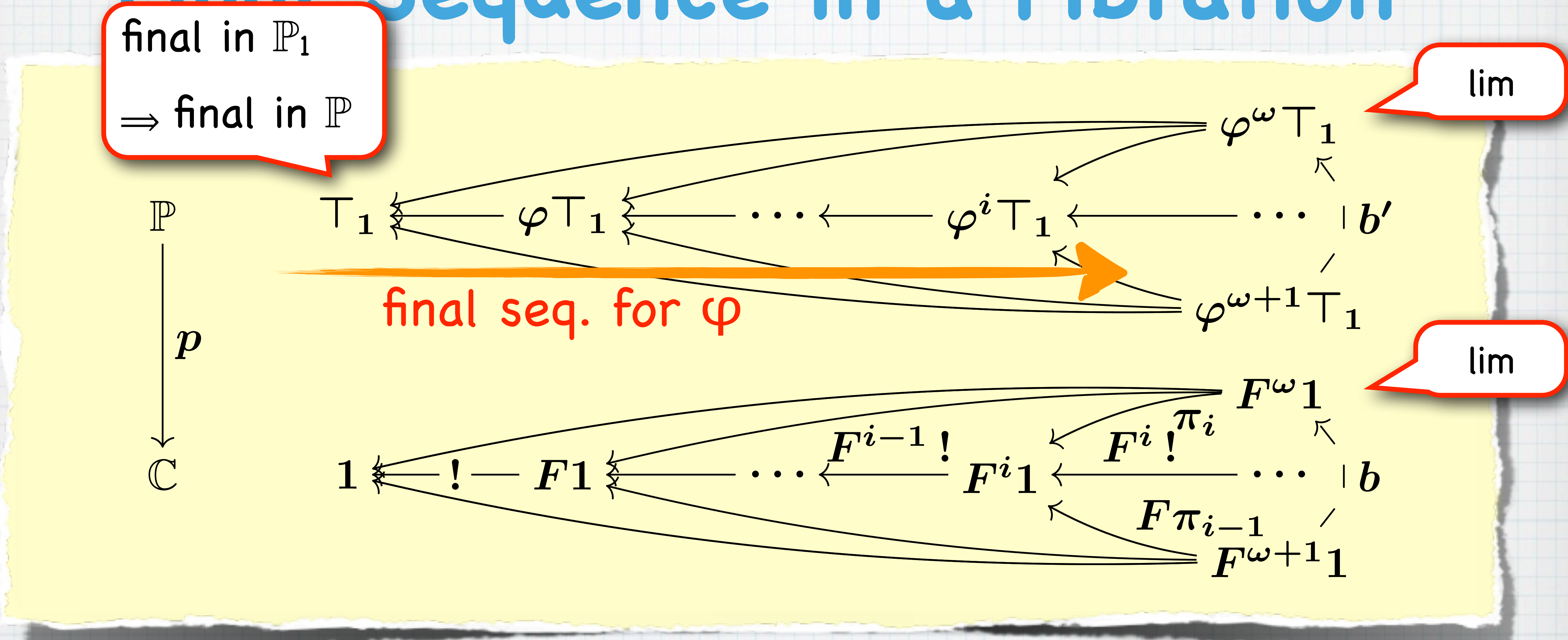

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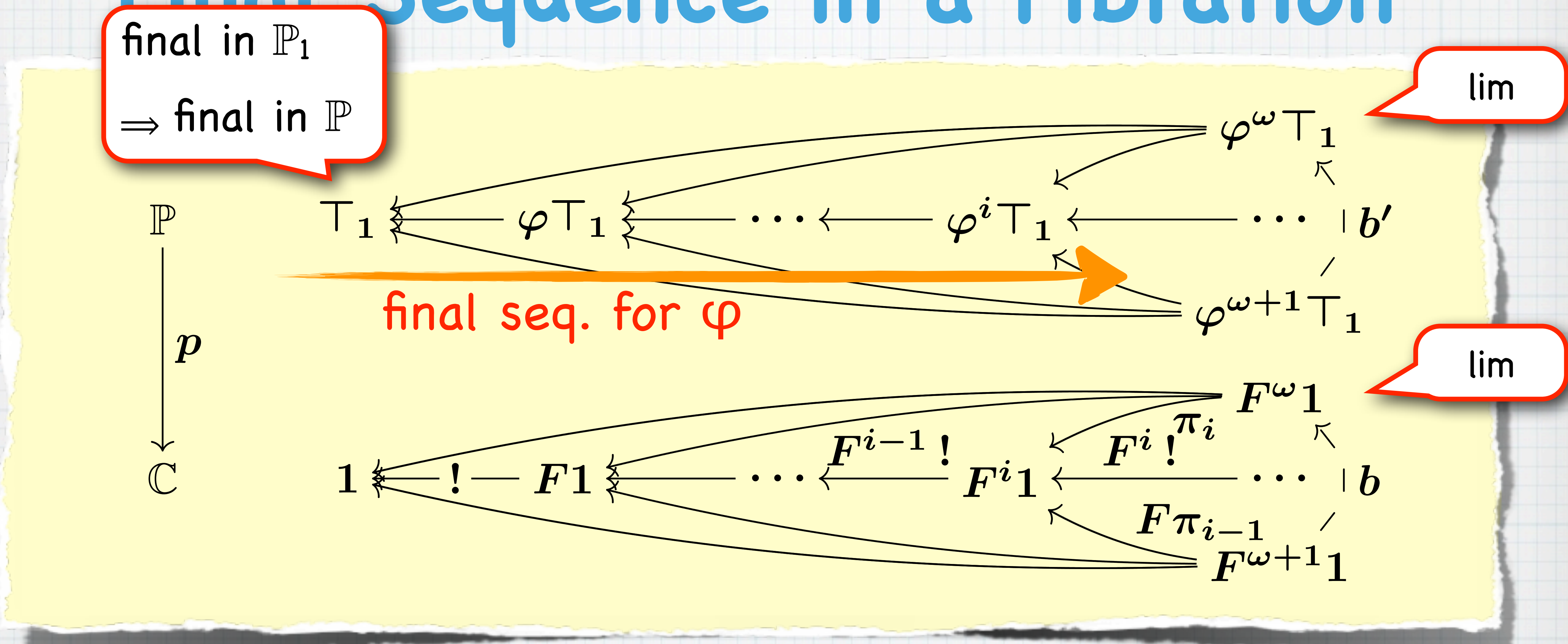

* Assume F: finitary, φ: pred. lifting of F

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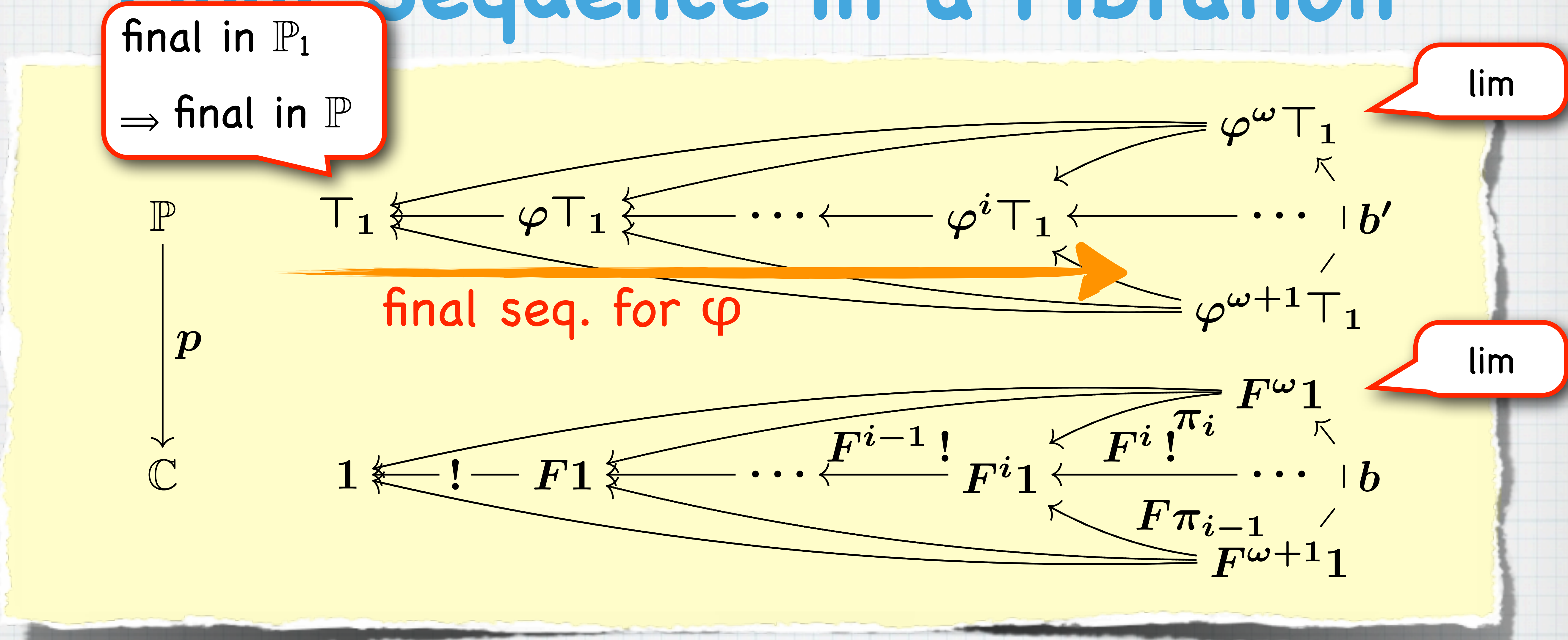

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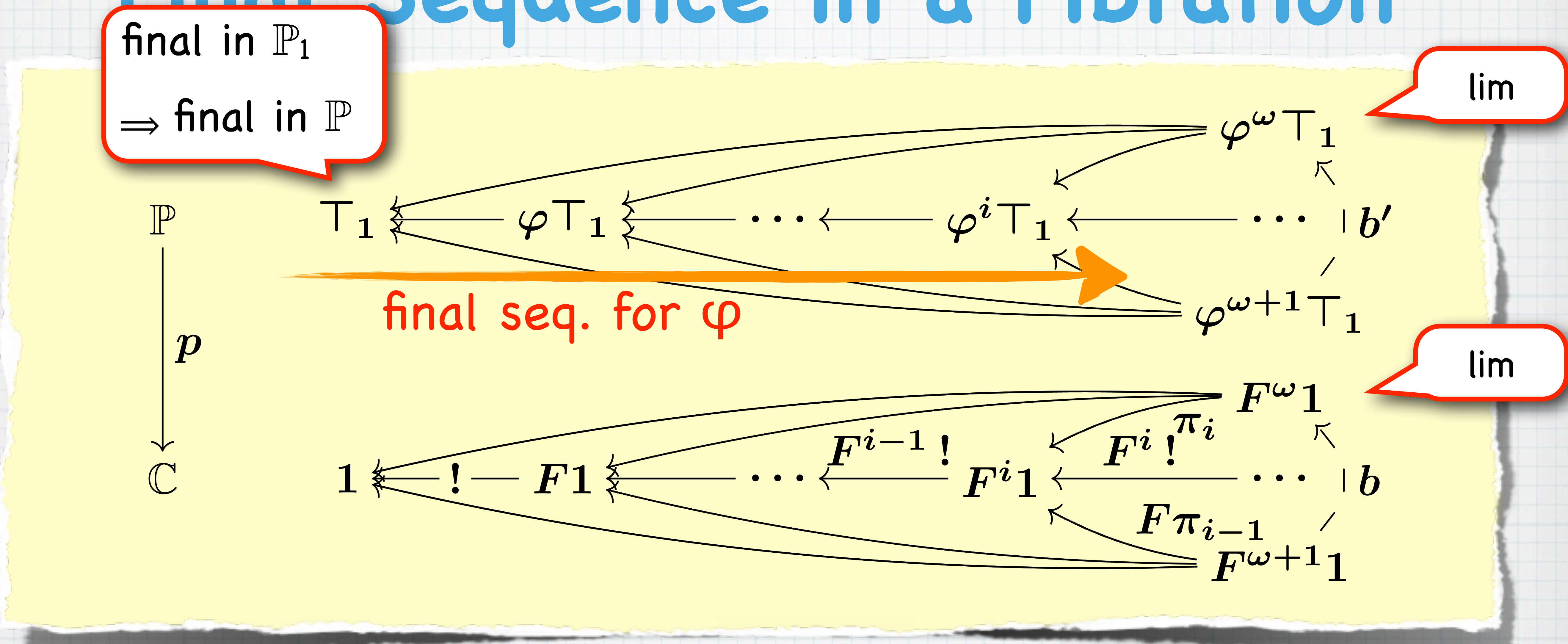
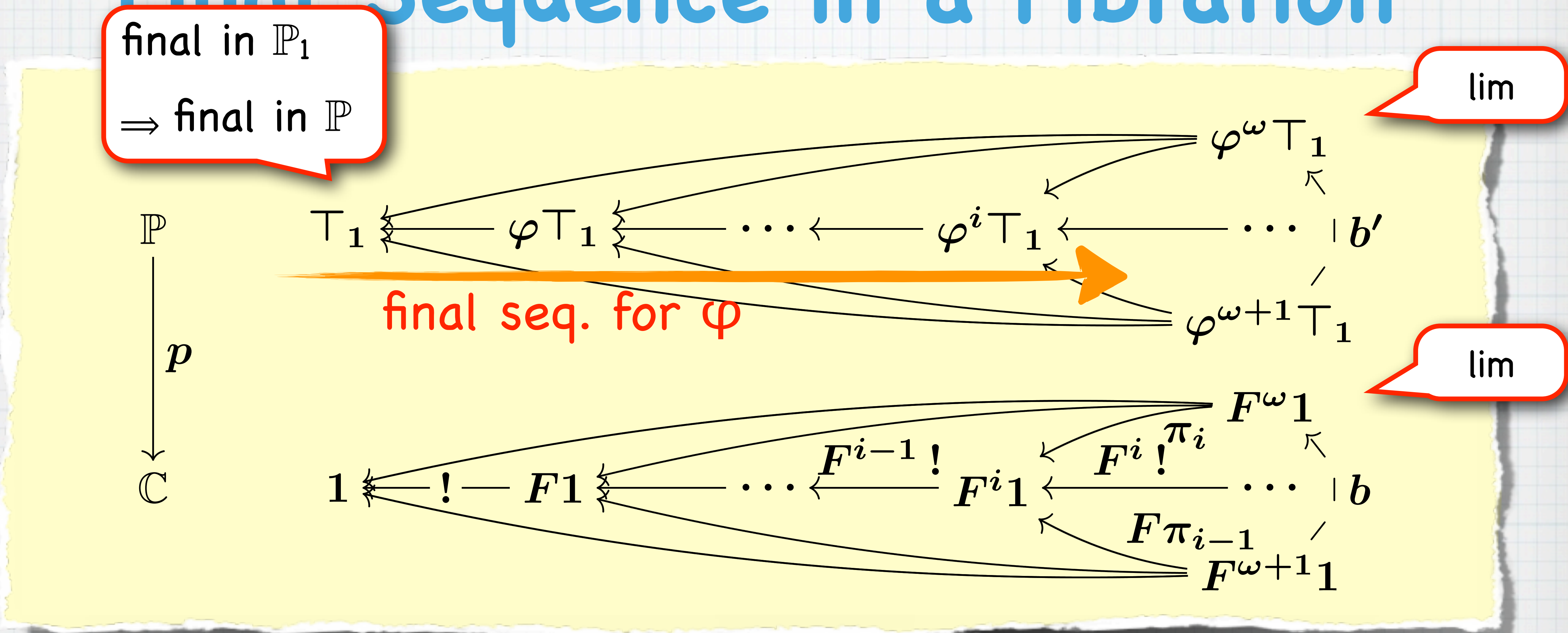


* Assume F: finitary, φ: pred. lifting of F
  * $F^\omega 1$: "almost final coalgebra", prototype of F-behaviors
  * $\varphi^\omega \top_1$: prototype of coind. pred. $[\![\nu\varphi]\!]_{X \xrightarrow{c} FX}$ for each coalgebra $X \xrightarrow{c} FX$

Final Sequence in a Fibration



Key Lemma.

Let $\begin{array}{c}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{array}$ be a well-founded fibration; $F : \mathbb{C} \to \mathbb{C}$ be finitary; and φ be a predicate lifting of F. Then

$$\varphi^{\omega+1}\top_1 = b^*(\varphi^\omega\top_1)$$

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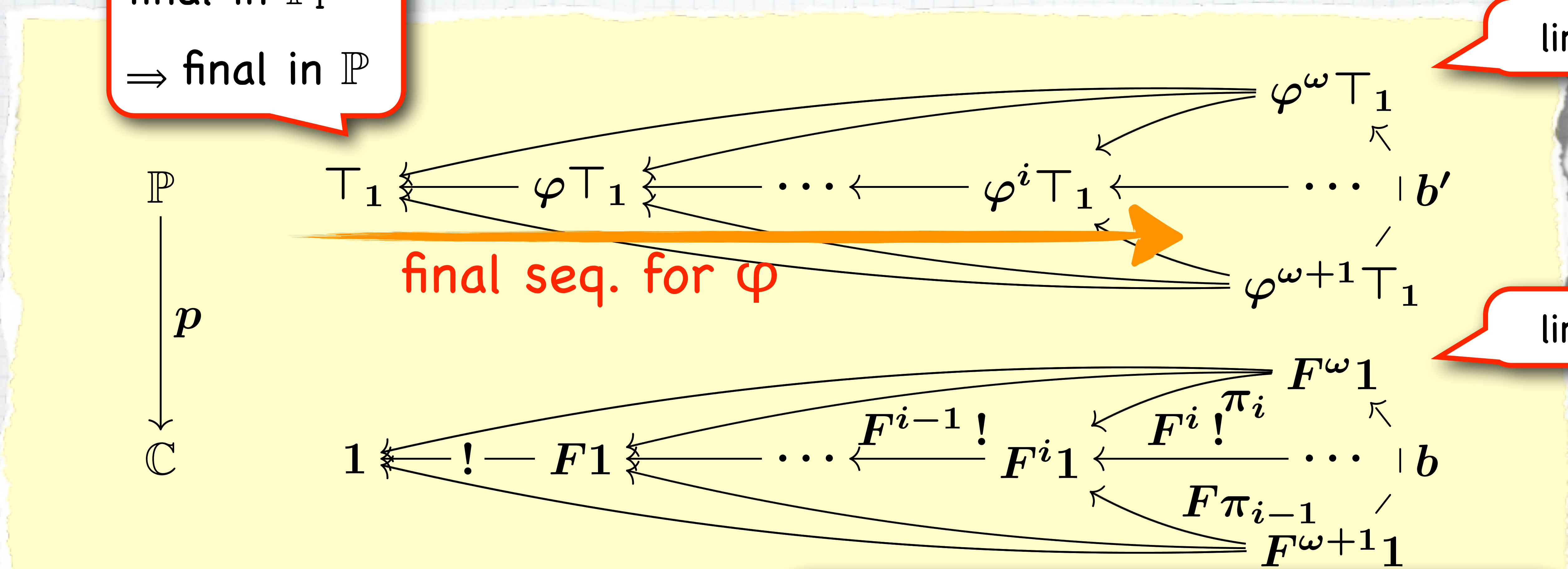

* p is compatible w/ $\mathbb{C}$: LFP
* p itself is "well-fdd"

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Technical Contributions

Definition.

A *finitely determined fibration* $\begin{array}{c}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{array}$ is such that:

1. $\mathbb{C}$ is LFP with $\mathbb{F} = \{\text{FP objects}\}$
2. $\begin{array}{c}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{array}$ has fiberwise (co)limits
3. For each $X \in \mathbb{C}$ and $P, Q \in \mathbb{P}_X$, let $\{X_I \xrightarrow{\kappa_I} X\}_I$ be the canonical diagram from $\mathbb{F}$ to X. Then

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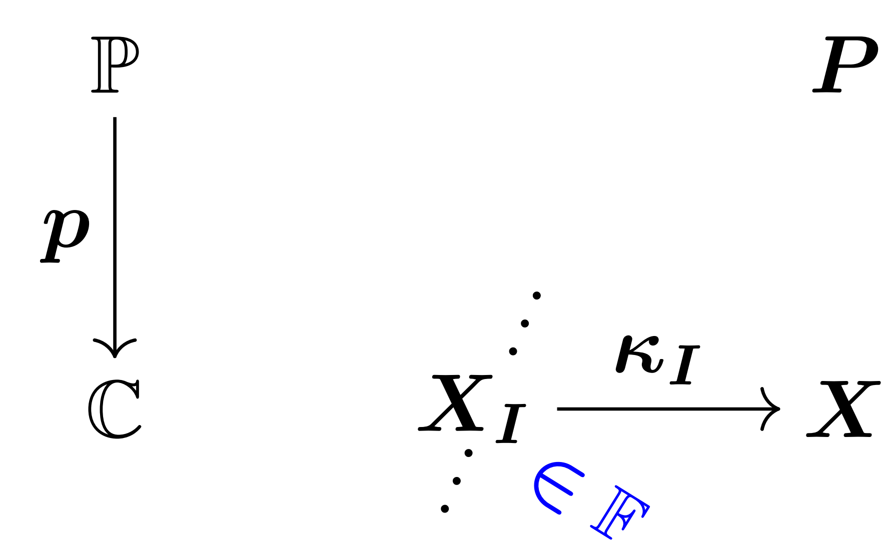

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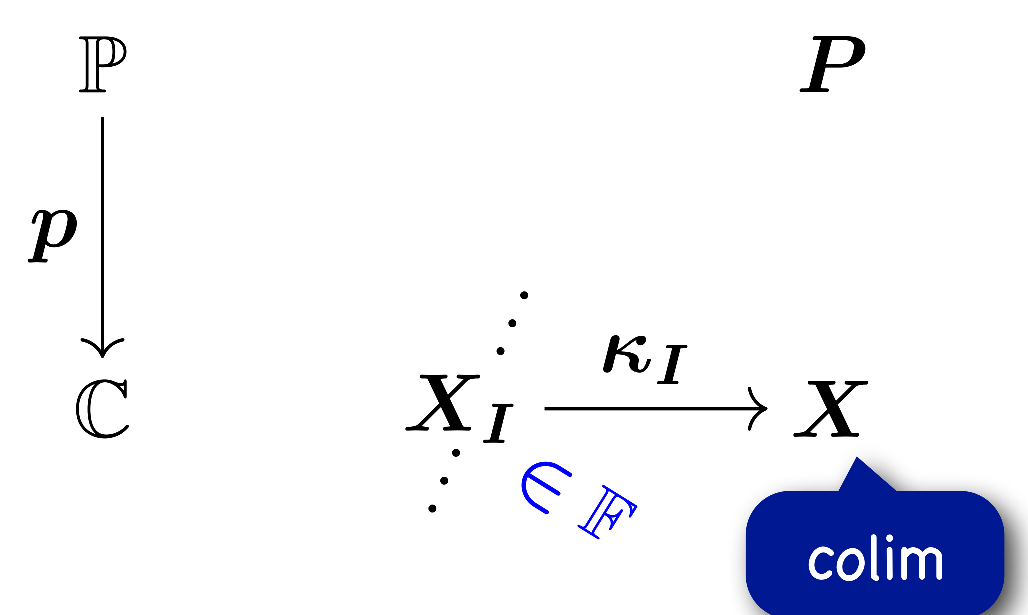

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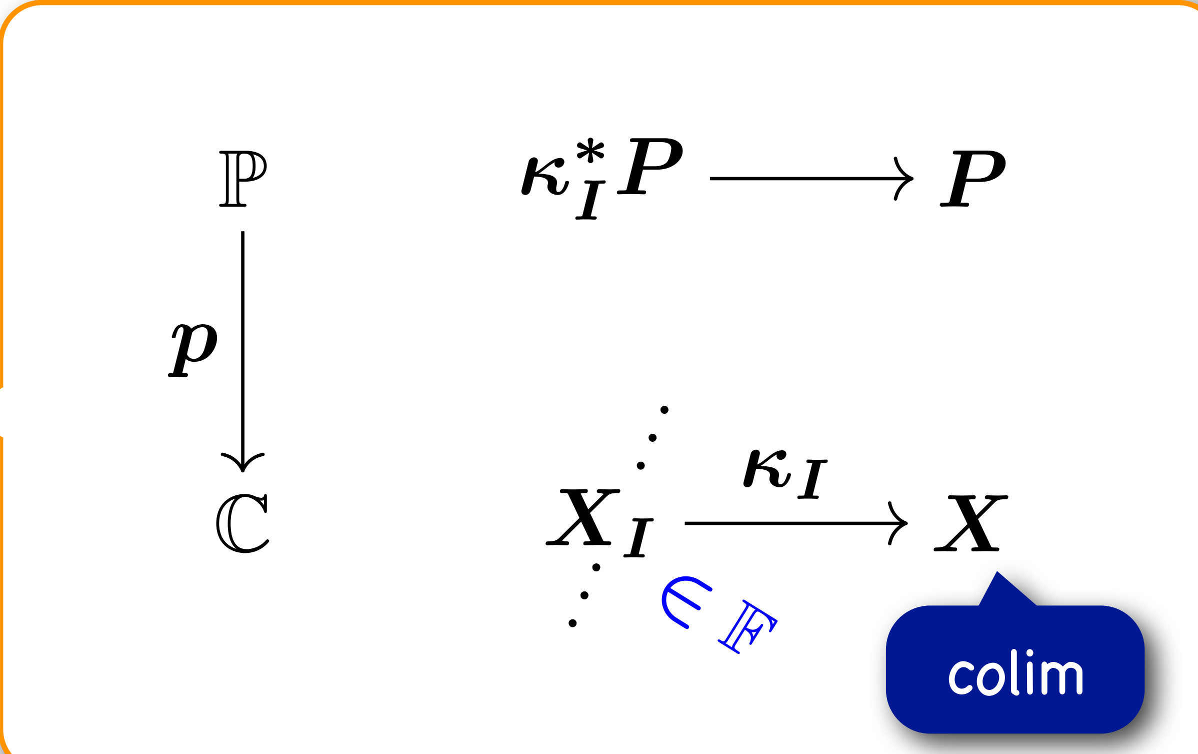

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- $\begin{smallmatrix}FX\\ c\uparrow\\ X\end{smallmatrix}$, a coalgebra
- $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{smallmatrix}$ is a well-founded fibration
- $F : \mathbb{C} \to \mathbb{C}$, finitary
- $\begin{array}{ccc}\mathbb{P} & \xrightarrow{\varphi} & \mathbb{P}\\ p\downarrow & & \downarrow p\\ \mathbb{C} & \xrightarrow[F]{} & \mathbb{C}\end{array}$, predicate lifting

Then the sequence

$$\top_X \leftarrow (c^* \circ \varphi_X)\top_X \longleftarrow (c^{-1} \circ \varphi_\diamond)^2 X \longleftarrow \cdots$$

stablizes after ω steps, yielding $[\![\nu\varphi]\!]_{\begin{smallmatrix}FX\\ c\uparrow\\ X\end{smallmatrix}}$ as its limit.

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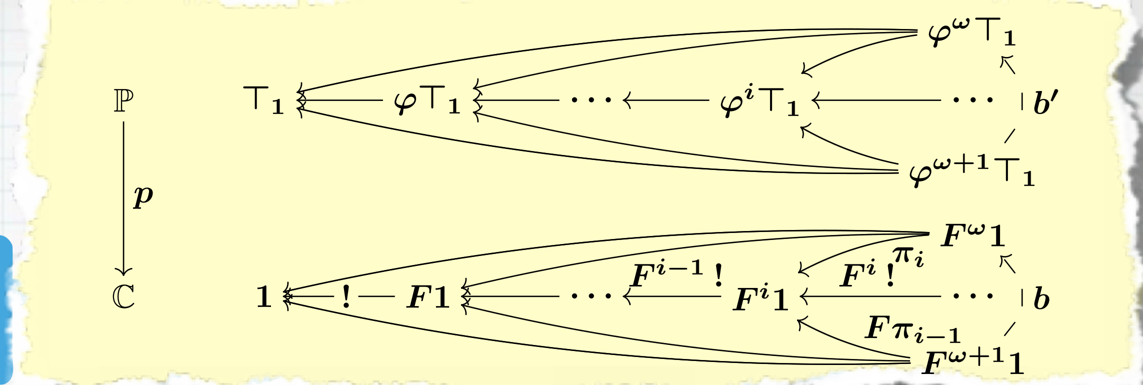

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Examples (or: Fibrations vs LFP)

* **Finitely determined**: very often

Prop. Assume $\mathbb{C}$ is LFP and LCCC. Then

- $\mathbf{Sub}(\mathbb{C})$ is LFP; and
- $\begin{array}{c}\mathbf{Sub}(\mathbb{C})\\ \downarrow\\ \mathbb{C}\end{array}$ is finitely determined.

Prop. Assume $\mathbf{\Omega}$ is an algebraic lattice.

Consider $\begin{array}{c}\mathbf{Fam}(\mathbf{\Omega})\\ \downarrow\\ \mathbf{Sets}\end{array}$; then

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Algebraic lattice:
* every elem. is a sup of compact elem's
* "LFP poset"

Examples

* **Well-founded**: depends

Definition.
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Part IV:

Conclusions & Future Work

Fibered Coinduction

* In $\mathbb{C}$?

$$\begin{array}{ccc} FX & \dashrightarrow & FZ \\ {\scriptstyle c}\uparrow & & \uparrow{\scriptstyle \cong}\ \text{final} \\ X & \underset{\mathsf{beh}(c)}{\dashrightarrow} & Z \end{array}$$

* In a **fibration** $\begin{array}{c}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{array}$!!

  * This work:

    * **final coalgebra** in p;
    * **final sequence** in p

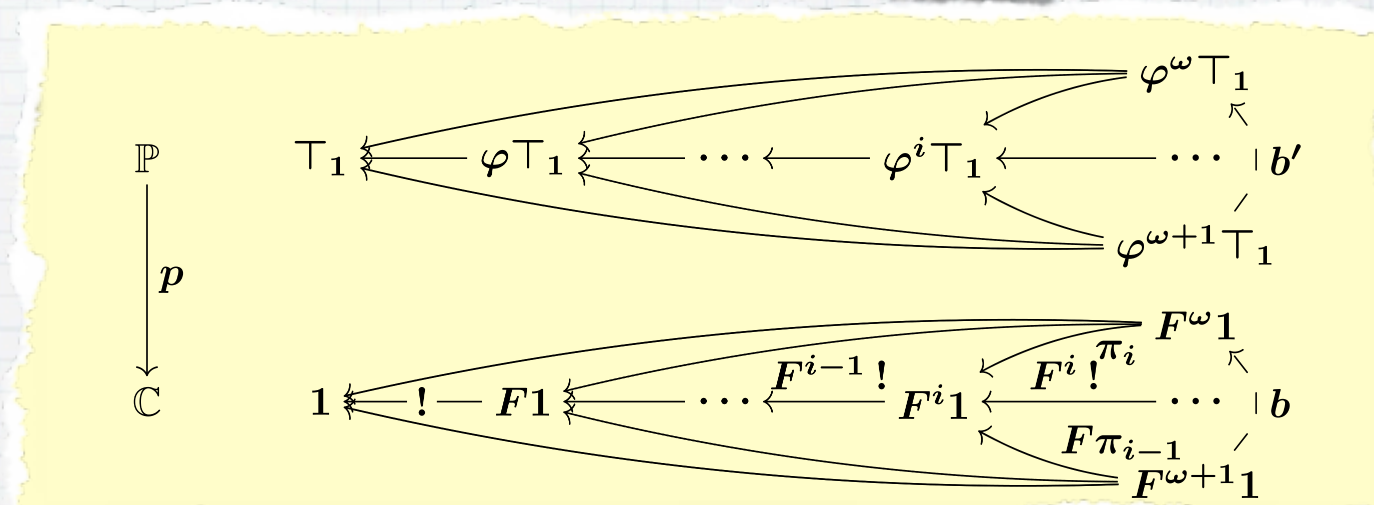
{ F-behaviors } + **coinductive predicate**

Conclusions

* Inductive construction [Cousot & Cousot, '79]

$$X \supseteq (c^{-1} \circ \varphi_\diamond)X \supseteq (c^{-1} \circ \varphi_\diamond)^2 X \supseteq \cdots$$

* Final sequence in a **fibration**



* **behavioral ω-bound**: conditions formulated in **LFP** terms

* Covers various logics

relations, constructive, name-passing, ...

Conclusions

conventional $\begin{array}{c}\mathbf{Pred}\\ \downarrow\\ \mathbf{Sets}\end{array}$	relational $\begin{array}{c}\mathbf{Rel}\\ \downarrow\\ \mathbf{Sets}\end{array}$	fibrational $\begin{array}{c}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{array}$
invariant	bisimulation	coalgebra
coind. pred.	bisimilarity	final coalg.
inductive constr.	partition refinement	final sequence

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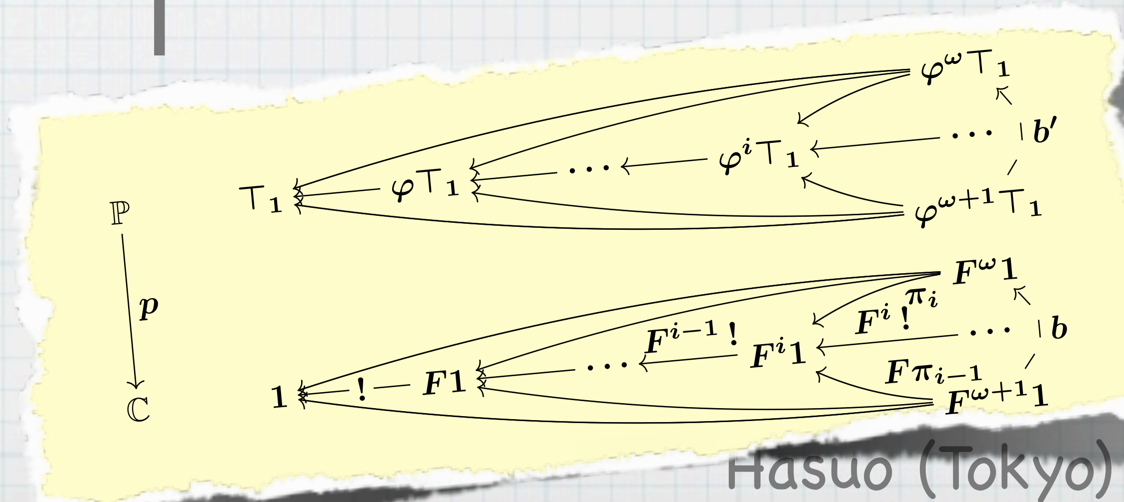

Future Work

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  - Bisimulation up-to [Bonchi & Pous, POPL'13]
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  - Current result: semidecidability
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  - Coalg. μ-calculus, coalg. automata, … **fibrationally**
  - Model checking algorithms
  - Combine with bialgebraic SOS
  - Games ↔ automata ↔ fixedpoint logic

Proof assistants

Much like appl. of final sequences

Thank you for your attention!

Ichiro Hasuo (Dept. CS, U Tokyo)
http://www-mmm.is.s.u-tokyo.ac.jp/~ichiro/

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