

# Freyd is Kleisli, for Arrows

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- Interfaces for ***structured computations*** (as opposed to ***pure functions***):

- monads*** [Moggi'91] for structured output
- comonads*** for structured input
- (monad + comonad + distr. law)*** for structured input/output

- **Arrows** [Hughes'00] generalize them

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All come with notions of

- Kleisli categories
- Eilenberg-Moore algebras

- **Arrows** [Hughes'00] generalize them

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All come with notions of

- Kleisli categories
- Eilenberg-Moore algebras

- **Arrows** [Hughes'00] generalize them

## Question

What are  $\left\{ \begin{array}{l} \text{Kleisli categories} \\ (\text{Eilenberg-Moore}) \text{ algebras} \end{array} \right\}$  for Arrows?

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## ■ Kleisli for Arrows:

**Freyd categories**

[Power, Robinson, Thielecke]

{Arrows on  $\mathbb{C}$ }

$\downarrow \cong$  [Heunen&Jacobs, MFPS'06]

{Freyd categories on  $\mathbb{C}$ }

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## ■ **Kleisli** for Arrows:

**Freyd categories**

[Power, Robinson, Thielecke]

{Arrows on  $\mathbb{C}$ }

**Kleisli!** [This paper]

$\downarrow \cong$  [Heunen&Jacobs, MFPS'06]

{Freyd categories on  $\mathbb{C}$ }

□ 2-categorical characterization (Cf. [Street'72])

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## ■ Kleisli for Arrows:

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**Kleisli!** [This paper]

$\downarrow \cong$  [Heunen&Jacobs, MFPS'06]

{Freyd categories on  $\mathbb{C}$ }

2-categorical characterization (Cf. [Street'72])

## ■ **(Eilenberg-Moore) algebras** for Arrows:

our notion is comparable to monad-algebras

$T$   
 $\downarrow$   
id

Not carried by specific object(s)

# Introducing Arrows

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[Moggi'91]

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Strong monad

Use of comonads,  
(monad + comonad +  
distr. law)

Arrows [Hughes'00]

Arrows generalize  
(co)monads

Comparing strong  
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Arrows, like Monads,  
are monoids [He-  
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# Use of monads [Moggi'91]

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$\mathbb{C}$ : category of types and pure functions.  $\mathbb{C} = \text{Sets}, \text{Cpo}$ , etc.

■ **Monad  $T : \mathbb{C} \rightarrow \mathbb{C}$**  is a functor equipped with:

**unit**  $X \xrightarrow{\eta_X} TX$  for each  $X$

**extension**  $(X \xrightarrow{f} TY) \xrightarrow{(-)^*} (TX \xrightarrow{f^*} TY)$

# Use of monads [Moggi'91]

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**unit**  $X \xrightarrow{\eta_X} TX$  for each  $X$

**extension**  $(X \xrightarrow{f} TY) \xrightarrow{(-)^*} (TX \xrightarrow{f^*} TY)$

■ Useful as an interface for computations **with structured output**, or  
computations **with effect**  $X \xrightarrow{\quad} TY$ . We can...

**embed pure functions**  
due to unit

$$X \xrightarrow{p} Y \quad TY$$
$$\uparrow \eta_Y$$

**compose computations**  
due to extension

$$X \xrightarrow{f} TY \quad \text{and} \quad Y \xrightarrow{g} TZ$$
$$\xrightarrow{\text{compose}} X \xrightarrow{f} TY \xrightarrow{g^*} TZ$$

# Kleisli category for monads

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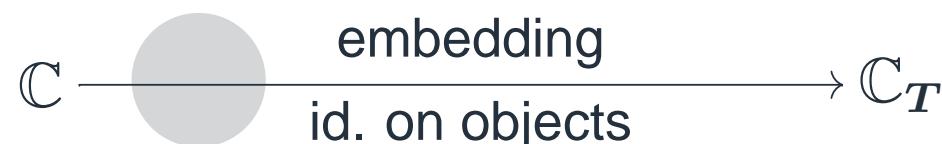
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A monad  $T$  gives rise to a **Kleisli category**.



base category  
cat. of **pure functions**

**Kleisli category**

cat. of **computations with str. output**

$$\frac{X \longrightarrow Y \text{ in } \mathbb{C}_T}{X \longrightarrow TY \text{ in } \mathbb{C}}$$

- Embedding is due to unit.
- $\mathbb{C}_T$  is a category.
  - In particular, composition of arrows is due to extension  $(-)^*$ .

# Strong monad

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Assume  $\mathbb{C}$  has  $\times$ .

**Strong monad** = monad + compatibility with  $\times$

■ **Defn.** A strong monad  $T$  is a monad with **strength**

$$(TX) \times Y \xrightarrow{\text{st}_{X,Y}} T(X \times Y)$$

■ Allows us to add **environments**  $(-) \times Z$ :

$$X \xrightarrow{f} TY$$

Computation from  $X$  to  $Y$

add env.  
 $\Rightarrow$

$$X \times Z \xrightarrow{f \times Z} (TY) \times Z \xrightarrow{\text{st}_{Y,Z}} T(Y \times Z)$$

Computation from  $X \times Z$  to  $Y \times Z$

# Use of comonads, (monad + comonad + distr. law)

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Similarly,

- A **comonad  $M$**  is an interface for computation with **structured input  $MX \rightarrow Y$**

- Brookes & Geva '92, Kieburz '99, Uustalu & Vene '05 embedding
- 

$$\mathbb{C} \xrightarrow{\text{embedding}} \mathbb{C}_M$$

base category  
cat. of **pure functions**

**co-Kleisli** category  
cat. of **computations with str. input**

$$\frac{X \rightarrow Y \text{ in } \mathbb{C}_M}{MX \rightarrow Y \text{ in } \mathbb{C}}$$

# Use of comonads, (monad + comonad + distr. law)

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- Monad  $T$  + comonad  $M$  + distributive law  $MT \xrightarrow{\lambda} TM$   
is an interface for  
computation with **structured input & output**  $MX \longrightarrow TY$

Uustalu & Vene, '05

$$\mathbb{C} \xrightarrow{\text{embedding}} \mathbb{C}_{T,M,\lambda}$$

base category  
cat. of **pure functions**

**bi-Kleisli** category  
cat. of **computations with str. I/O**  
$$\frac{X \longrightarrow Y \text{ in } \mathbb{C}_{T,M,\lambda}}{MX \longrightarrow TY \text{ in } \mathbb{C}}$$

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An **Arrow** on  $\mathbb{C}$  is...

- a bifunctor  $A(-, +) : \mathbb{C}^{\text{op}} \times \mathbb{C} \rightarrow \text{Sets}$ ,
- equipped with

$$\text{arr}_{X,Y} : \text{Hom}_{\mathbb{C}}(X, Y) \rightarrow A(X, Y)$$

$$\ggg_{X,Y,Z} : A(X, Y) \times A(Y, Z) \rightarrow A(X, Z)$$

$$\text{first}_{X,Y,Z} : A(X, Y) \rightarrow A(X \times Z, Y \times Z)$$

- with coherence conditions

$$(a \ggg b) \ggg c = a \ggg (b \ggg c),$$

$$\text{arr}(g \circ f) = \text{arr}(f) \ggg \text{arr}(g),$$

$$\text{arr id} \ggg a = a = a \ggg \text{arr id},$$

$$\text{first}(a) \ggg \text{arr}(\pi_1) = \text{arr}(\pi_1) \ggg a,$$

$$\text{first}(a) \ggg \text{arr}(\text{id} \times f) = \text{arr}(\text{id} \times f) \ggg \text{first}(a),$$

$$\text{first}(a) \ggg \text{arr}(\alpha) = \text{arr}(\alpha) \ggg \text{first}(\text{first}(a)),$$

$$\text{first}(\text{arr}(f)) = \text{arr}(f \times \text{id})$$

$$\text{first}(a \ggg b) = \text{first}(a) \ggg \text{first}(b)$$

# Arrows [Hughes'00]

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An **Arrow** on  $\mathbb{C}$  is...

- a bifunctor  $A(-, +) : \mathbb{C}^{\text{op}} \times \mathbb{C} \rightarrow \text{Sets}$ ,

$A(X, Y)$ : set of structured computations from  $X$  to  $Y$

- equipped with

$$\text{arr}_{X,Y} : \text{Hom}_{\mathbb{C}}(X, Y) \rightarrow A(X, Y)$$

embeds pure functions

$$\ggg_{X,Y,Z} : A(X, Y) \times A(Y, Z) \rightarrow A(X, Z)$$

composes structured computations

$$\text{first}_{X,Y,Z} : A(X, Y) \rightarrow A(X \times Z, Y \times Z)$$

allows for handling environment

- Straightforward to **enrich** the whole setting,  
by replacing **Sets** by symmetric monoidal  $\mathbb{V}$

# Arrows generalize (co)monads

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- Strong monad  $T$  gives rise to an Arrow  $A_T$  by

$$A_T(X, Y) = \text{Hom}_{\mathbb{C}}(X, TY)$$

- Comonad  $M$  gives rise to an Arrow  $A_M$  by

$$A_M(X, Y) = \text{Hom}_{\mathbb{C}}(MX, Y)$$

- Monad  $T$  + comonad  $M$  + distributive law  $MT \xrightarrow{\lambda} TM$   
gives rise to an Arrow  $A_{T,M,\lambda}$  by

$$A_{T,M,\lambda}(X, Y) = \text{Hom}_{\mathbb{C}}(MX, TY)$$

# Comparing strong monad vs. Arrow

	Strong monad $\mathbf{T}$	Arrow $A(-, +)$
structured computation from $X$ to $Y$	$X \rightarrow TY$ in $\mathbb{C}$	$a \in A(X, Y)$
pure function is embedded due to	unit $\begin{array}{c} TY \\ \uparrow \eta_Y \\ Y \end{array}$	$\text{Hom}_{\mathbb{C}}(X, Y) \xrightarrow{\text{arr}} A(X, Y)$
composition of computation due to	extension $\begin{array}{c} (X \xrightarrow{f} TY) \\ \downarrow (-)^* \\ (TX \xrightarrow{f^*} TY) \end{array}$	$\begin{array}{c} A(X, Y) \times A(Y, Z) \\ \downarrow \ggg \\ A(X, Z) \end{array}$
compatible with $\times$ (i.e. environment) due to	strength $\begin{array}{c} (TX) \times Y \\ \downarrow \text{st} \\ T(X \times Y) \end{array}$	$\begin{array}{c} A(X, Y) \\ \downarrow \text{first} \\ A(X \times Z, Y \times Z) \end{array}$

# Arrows, like Monads, are monoids

## [Heunen&Jacobs, MFPS'06]

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This similarity can be put more formally:

- Monad      with  $\eta, (-)^*$   
= **monoid** in functor category  $[\mathbb{C}, \mathbb{C}]$

$[\mathbb{C}, \mathbb{C}]$ : monoidal

tensor:  $F \otimes G = F \circ G$

unit: **id**

- Strong monad      with  $\eta, (-)^*, \text{st}$   
= **monoid** in  $[\mathbb{C}, \mathbb{C}]$   
+ (compatibility with  $\times$ )

- Arrow      with **arr**,  $\ggg$ , **first**  
= **monoid** in  $[\mathbb{C}^{\text{op}} \times \mathbb{C}, \text{Sets}]$   
+ (compatibility with  $\times$ )

$[\mathbb{C}^{\text{op}} \times \mathbb{C}, \text{Sets}]$ : monoidal

tensor: like composition

unit: **Hom** $_{\mathbb{C}}$

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# Kleisli categories for Arrows

# Kleisli categories for Arrows

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Obvious definition:

**Kleisli category  $\mathbb{C}_A$**  for  $A$  is

$$\text{arrow} \quad \frac{X \xrightarrow{a} Y \quad \text{in } \mathbb{C}_A}{a \in A(X, Y)}$$

obj.

$$\frac{X \in \mathbb{C}_A}{X \in \mathbb{C}}$$

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$$\text{arrow} \quad \frac{X \xrightarrow{a} Y \quad \text{in } \mathbb{C}_A}{a \in A(X, Y)}$$

$$\text{obj.} \quad \frac{X \in \mathbb{C}_A}{X \in \mathbb{C}}$$

- Inclusion functor

$$\begin{aligned} \mathbb{C} &\longrightarrow \mathbb{C}_A \\ X \xrightarrow{f} Y &\longmapsto \frac{\text{arr}(f) \in A(X, Y)}{X \xrightarrow{\text{arr}(f)} Y} \end{aligned}$$

- $\mathbb{C}_A$  is a category.

In particular, composition is due to  $\ggg$ .

Cf.  
 $\text{Hom}_{\mathbb{C}}(X, Y)$

$$\downarrow \text{arr} \\ A(X, Y)$$

$$A(X, Y) \times A(Y, Z)$$

$$\downarrow \ggg$$

$$A(X, Z)$$

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Let's look at  $\mathbb{C}_A$ ...

■  $\mathbb{C}_A$  is ***pre-monoidal*** (which is, almost monoidal)

- Instead of a bifunctor  $\otimes : \mathbb{C}_A \times \mathbb{C}_A \rightarrow \mathbb{C}_A$ ,  
we have two functors for each  $Z$

$$(-) \times Z : \mathbb{C}_A \rightarrow \mathbb{C}_A$$

$$Z \times (-) : \mathbb{C}_A \rightarrow \mathbb{C}_A$$

given by

$$\frac{X \xrightarrow{a} Y \quad \text{in } \mathbb{C}_A}{\begin{array}{c} X \times Z \longrightarrow Y \times Z \quad \text{in } \mathbb{C}_A \\ \text{first}(a) \\ \parallel \\ a \times Z \end{array}}$$

Cf.

$$\begin{array}{c} A(X, Y) \\ \downarrow \text{first} \\ A(X \times Z, Y \times Z) \end{array}$$

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Let's look at  $\mathbb{C}_A$ ...

■  $\mathbb{C}_A$  is ***pre-monoidal*** (which is, almost monoidal)

□ But not quite monoidal. If it were,

$$\begin{array}{ccc}
 X \times Z & \xrightarrow{X \times g} & X \times W \\
 f \times Z \downarrow & \searrow f \otimes g & \downarrow f \times W \\
 Y \times Z & \xrightarrow{Y \times g} & Y \times W
 \end{array}$$

should commute,

which is unlikely.

E.g.  $A(X, Y)$  : “state-based computations from  $X$  to  $Y$ ”

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In fact,  $\mathbb{C} \rightarrow \mathbb{C}_A$  is a **Freyd category**.

- A Freyd category on  $\mathbb{C}$  is: [Power, Robinson, Thielecke]

$$\mathbb{C} \xrightarrow{\text{id. on objects}} \mathbb{K},$$

with finite products

pre-monoidal

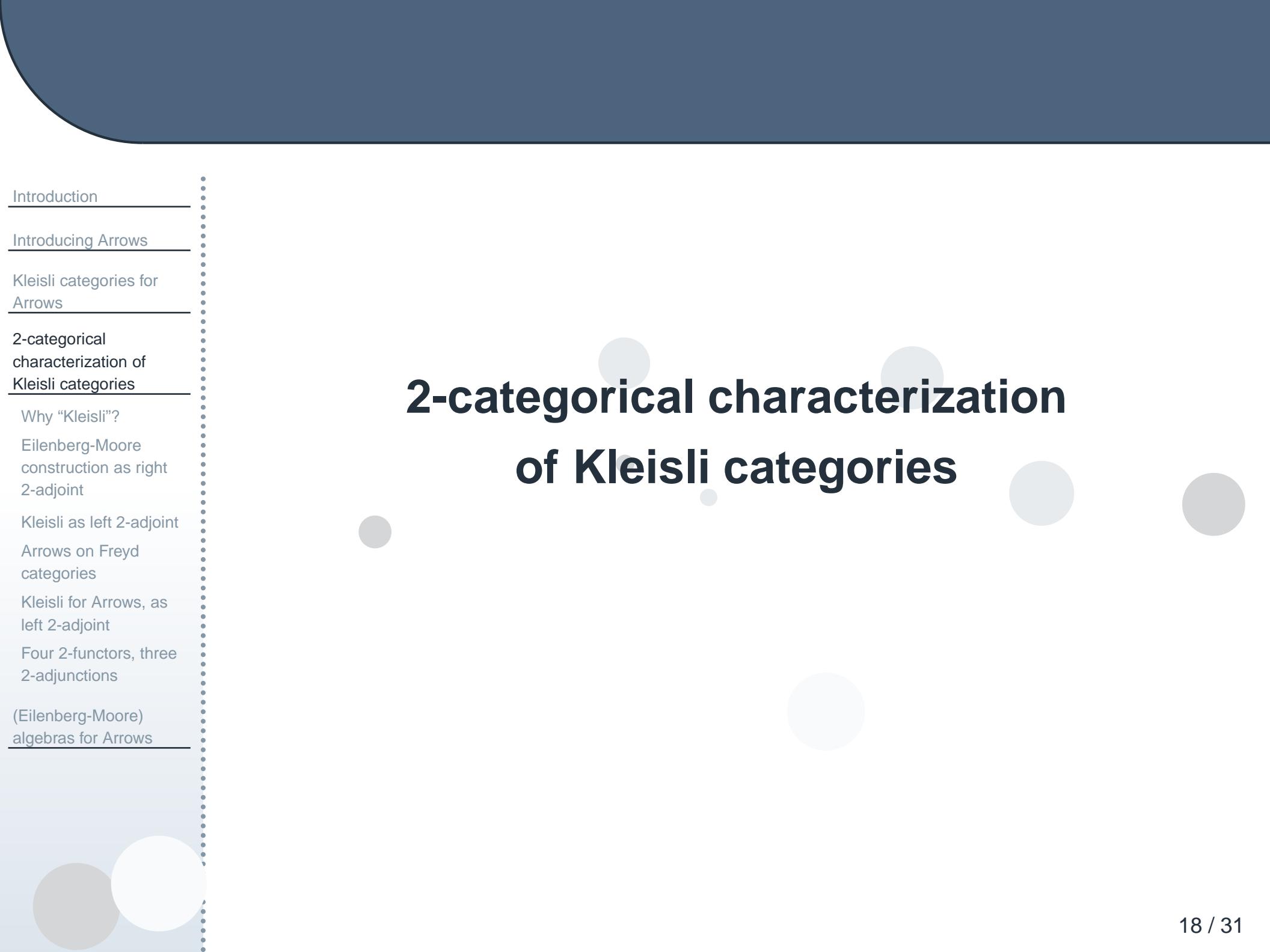
preserving appropriate structures.

- 1-1 correspondence between Arrows and Freyd categories:  
[Heunen&Jacobs, MFPS'06]

$$\{\text{Arrows on } \mathbb{C}\} \xrightarrow{\cong} \{\text{Freyd categories on } \mathbb{C}\}$$

$$A(-, +) \xrightarrow{\quad} (\mathbb{C} \rightarrow \mathbb{C}_A)$$

$$\text{Hom}_{\mathbb{K}}(J-, J+) \xleftarrow{\quad} (\mathbb{C} \xrightarrow{J} \mathbb{K})$$



# 2-categorical characterization of Kleisli categories

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# Why “Kleisli”?

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Four 2-functors, three  
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## Why is this “Kleisli”?

$$\{ \text{Arrows on } \mathbb{C} \} \xrightarrow[\text{“Kleisli”}]{\cong} \{ \text{Freyd categories on } \mathbb{C} \}$$
$$A \dashv \rightarrow (\mathbb{C} \rightarrow \mathbb{C}_A)$$

- For Arrows induced by (co)monads,  
**“Kleisli”** (for Arrows) coincides with usual Kleisli (for monads).
- 2-categorical characterization. Details now

$$\begin{array}{ccc} & \text{Ins} & \\ \text{Freyd} & \begin{array}{c} \swarrow \quad \searrow \\ T \end{array} & \text{Arr(Freyd)} , \\ & \text{“Kleisli”} & \end{array} \quad \text{just like}$$

$$\begin{array}{ccc} & \text{Ins} & \\ \text{Cat} & \begin{array}{c} \swarrow \quad \searrow \\ T \end{array} & \text{Mnd(Cat}_* \text{)} \\ & \text{Kleisli} & \end{array} \quad \text{for monads [Street'72]}$$

# Eilenberg-Moore construction as right 2-adjoint

2-categorical characterization [Street'72] of

- Eilenberg-Moore construction of algebras
- for monads

is presented as a showcase.

$$\begin{array}{ccc} \text{Cat} & \begin{array}{c} \xrightarrow{\quad Ins \quad} \\ \perp \\ \xleftarrow{\quad EMAlg \quad} \end{array} & \text{Mnd(Cat)} \end{array}$$

2-category  $\text{Mnd}(\text{Cat})$

object:  $(\mathbb{C}, T)$

$$(\mathbb{C}, T) \xrightarrow{(H, \sigma)} (\mathbb{D}, S)$$

1-cell:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\quad H \quad} & \mathbb{D} \\ T \downarrow & \Downarrow \sigma & \downarrow S \\ \mathbb{C} & \xrightarrow{\quad H \quad} & \mathbb{D} \end{array}$$

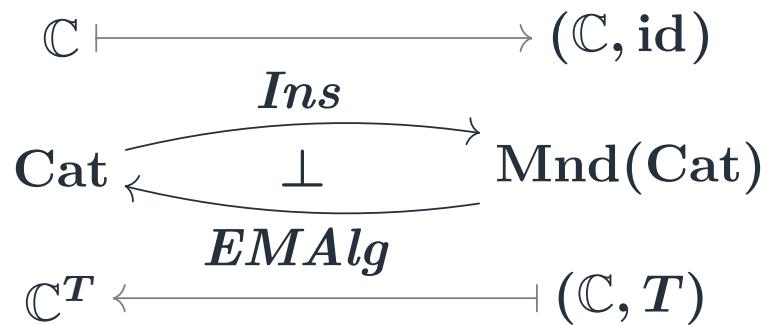
2-cell: ...

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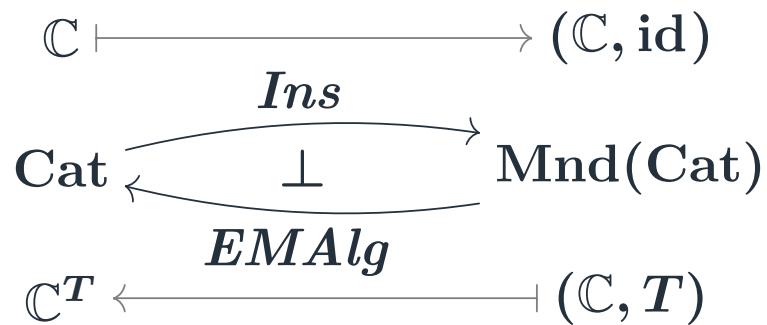
2-cell: ...

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2-category  $\mathbf{Mnd}(\mathbf{Cat})$

$$\begin{aligned}
 \text{object: } & (\mathbb{C}, T) \\
 & (\mathbb{C}, T) \xrightarrow{(H, \sigma)} (\mathbb{D}, S) \\
 \hline
 \text{1-cell: } & \begin{array}{c} \mathbb{C} \xrightarrow{H} \mathbb{D} \\ T \downarrow \quad \Downarrow \sigma \quad \downarrow S \\ \mathbb{C} \xrightarrow{H} \mathbb{D} \end{array} \\
 \text{2-cell: } & \dots
 \end{aligned}$$

obj. in  $\mathbb{C}^T \iff 1 \rightarrow \mathbb{C}^T$  in  $\mathbf{Cat}$

$\xleftrightarrow{\text{adj.}}$

$(1, \text{id}) \rightarrow (\mathbb{C}, T)$  in  $\mathbf{Mnd}(\mathbf{Cat})$

$$\begin{array}{ccc}
 1 & \xrightarrow{X} & \mathbb{C} \\
 \text{id} \downarrow & \Downarrow \sigma & \downarrow T \\
 1 & \xrightarrow{X} & \mathbb{C}
 \end{array}
 \quad \iff \quad
 \begin{array}{ccc}
 TX & & \\
 \downarrow \sigma_* & & \\
 X & &
 \end{array}$$

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left 2-adjoint

Four 2-functors, three  
2-adjunctions

(Eilenberg-Moore)  
algebras for Arrows

Similarly, [Street'72]

$$\begin{array}{ccc} & \text{Ins} & \\ \text{Cat} & \begin{array}{c} \xrightarrow{T} \\ \xleftarrow{\mathcal{K}\ell} \end{array} & \text{Mnd}(\text{Cat}_*) \end{array}$$

Can we do the same for Arrows?

$$\begin{array}{ccc} & \text{Ins} & \\ \text{FPCat} & \begin{array}{c} \xrightarrow{T} \\ \xleftarrow{\mathcal{K}\ell} \end{array} & \text{Arr(FPCat)} \end{array}$$

2-category  $\text{Arr}(\text{FPCat})$

object:  $(\mathbb{C}, A)$

1-cell: ...

2-cell: ...

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Similarly, [Street'72]

$$\begin{array}{ccc} & \text{Ins} & \\ \text{Cat} & \begin{array}{c} \xrightarrow{T} \\ \xleftarrow{\mathcal{K}\ell} \end{array} & \text{Mnd}(\text{Cat}_*) \end{array}$$

Can we do the same for Arrows? **No.**

$$\begin{array}{ccc} & \text{Ins} & \\ \text{FPCat} & \begin{array}{c} \xrightarrow{T} \\ \times \\ \xleftarrow{\mathcal{K}\ell} \end{array} & \text{Arr(FPCat)} \\ \mathbb{C}_A & \longleftarrow & (\mathbb{C}, A) \end{array}$$

pre-monoidal,  
not with finite products!

2-category  $\text{Arr}(\text{FPCat})$

object:  $(\mathbb{C}, A)$

1-cell: ...

2-cell: ...

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We extend notion of Arrows: **on FPCat  $\implies$  on Freyd**

■ Arrow  $A$  on  $\mathbb{C} \in \text{FPCat}$ :

- $A : \mathbb{C}^{\text{op}} \times \mathbb{C} \rightarrow \text{Sets}$ , with arr,  $\ggg$ , first
- with coherence conditions

■ Arrow  $A$  on  $(\mathbb{C} \xrightarrow{J} \mathbb{K}) \in \text{Freyd}$ :

- $A : \mathbb{K}^{\text{op}} \times \mathbb{K} \rightarrow \text{Sets}$ , with arr,  $\ggg$ , first
- with similar coherence conditions

Especially, first is compatible with Cartesian structure  
carried from  $\mathbb{C}$  to  $\mathbb{K}$

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Then indeed  $\mathcal{Kl} \dashv \mathcal{Ins}$  for Arrows.



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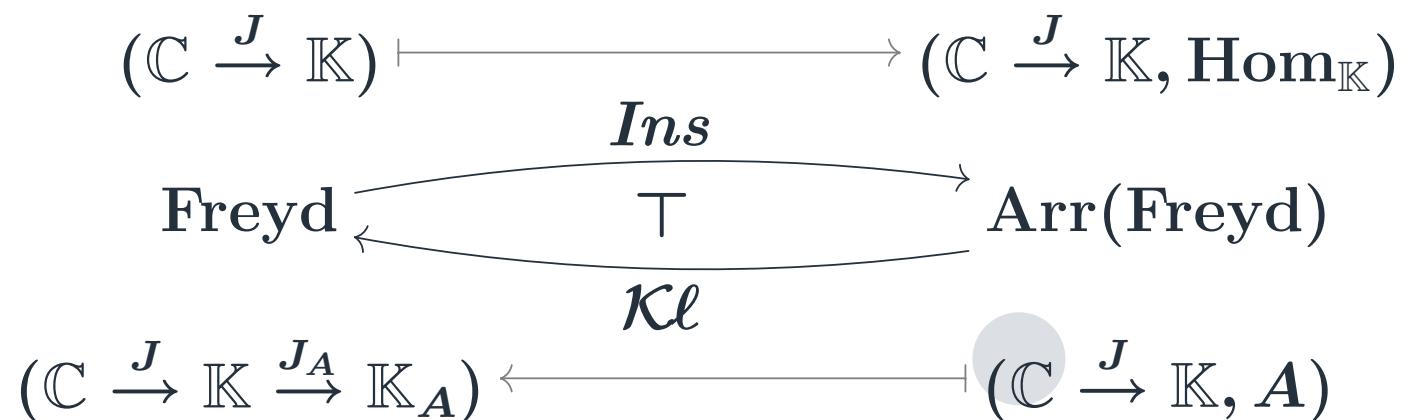
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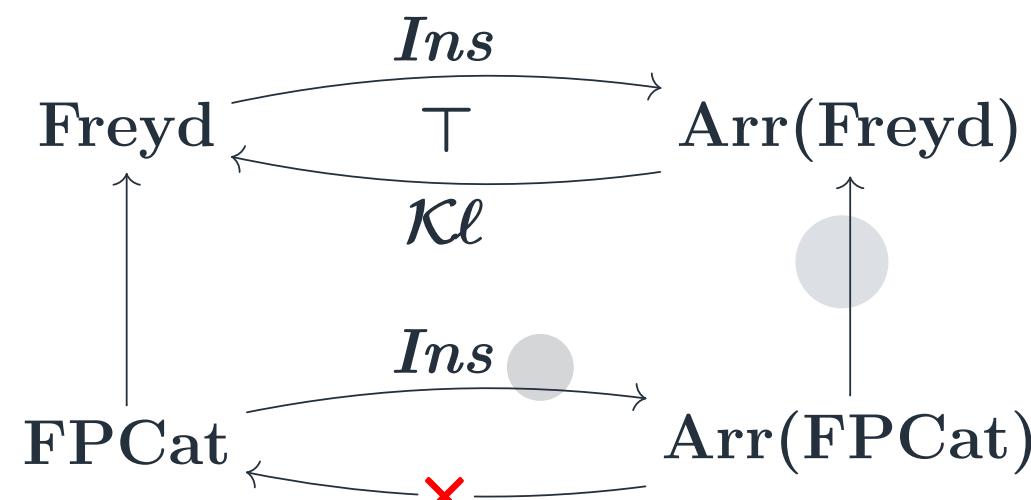
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Freyd

Arr(Freyd)

$$(\mathbb{C} \xrightarrow{J} \mathbb{K}) \begin{array}{c} \xleftarrow{\quad U \quad} \\[-1ex] \top \end{array} (\mathbb{C} \xrightarrow{J} \mathbb{K}, A)$$

$$(\mathbb{C} \xrightarrow{J} \mathbb{K}) \begin{array}{c} \vdash \\[-1ex] \xrightarrow{\quad Ins \quad} \\[-1ex] \top \end{array} (\mathbb{C} \xrightarrow{J} \mathbb{K}, \text{Hom}_{\mathbb{K}})$$

$$(\mathbb{C} \xrightarrow{J} \mathbb{K} \xrightarrow{J_A} \mathbb{K}_A) \begin{array}{c} \xleftarrow{\quad \mathcal{K}\ell \quad} \\[-1ex] \top \end{array} (\mathbb{C} \xrightarrow{J} \mathbb{K}, A)$$

$$(\mathbb{C} \xrightarrow{J} \mathbb{K}) \begin{array}{c} \vdash \\[-1ex] \xrightarrow{\quad Arr \quad} \end{array} (\mathbb{C} \xrightarrow{\text{id}} \mathbb{C}, \text{Hom}_{\mathbb{K}}(J-, J+))$$

■ *Ins* is a full embedding.

□ Freyd is a full reflective 2-subcategory of Arr(Freyd).

# Four 2-functors, three 2-adjunctions

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Freyd

Arr

Arr(Freyd)

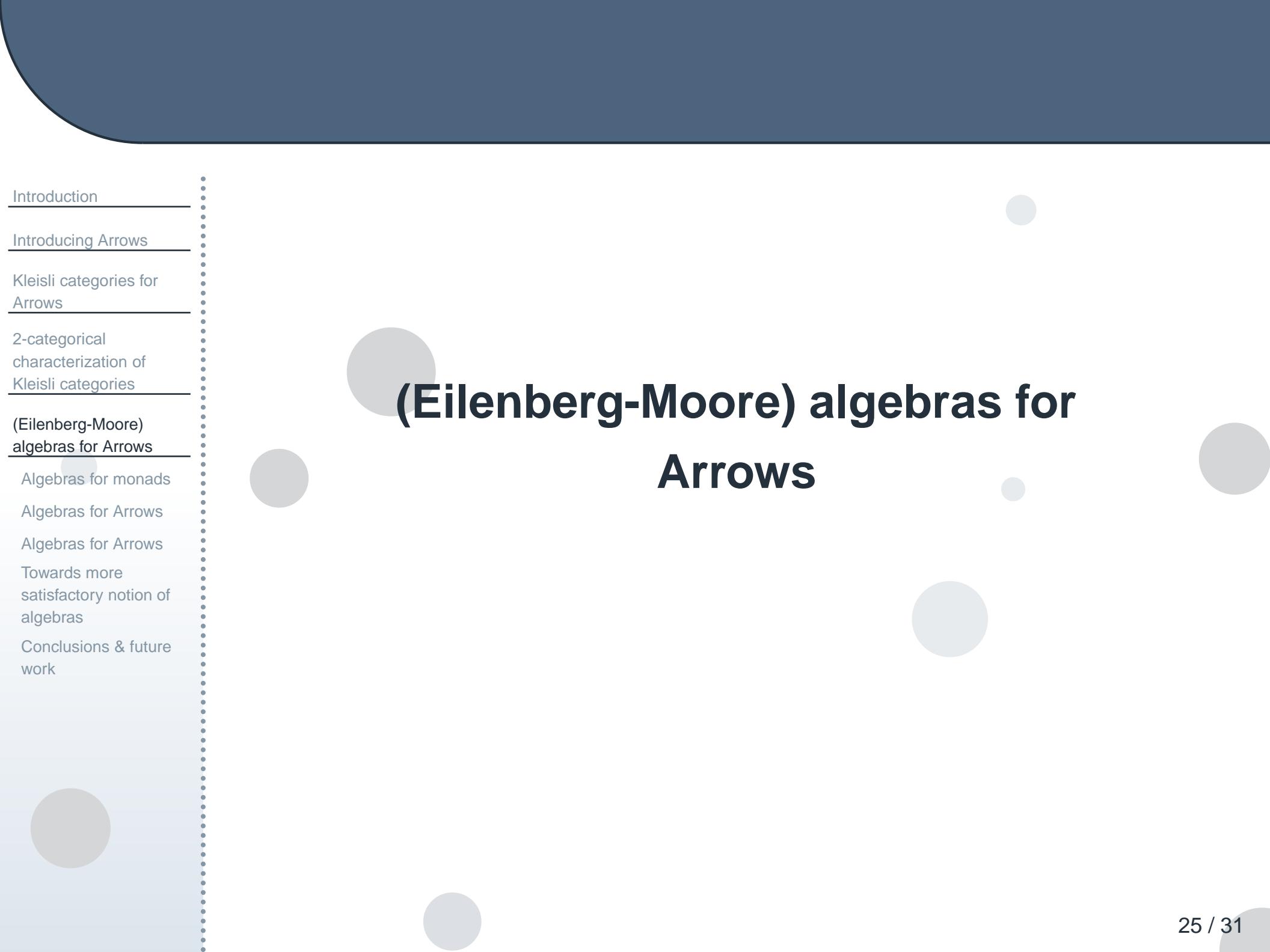
$$(\mathbb{C} \xrightarrow{J} \mathbb{K})$$

$$\begin{array}{c} \swarrow \\[-1ex] \cong \\[-1ex] \searrow \end{array}$$

Arr(FPCat)

$$(\mathbb{C}, \text{Hom}_{\mathbb{K}}(J-, J+))$$

$$\begin{array}{c} \uparrow \\[-1ex] Ins \end{array}$$



# (Eilenberg-Moore) algebras for Arrows

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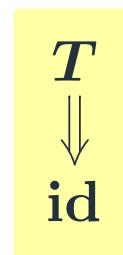
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For a monad  $T$ , consider the following notion of “algebras”...

■ **Defn.** An  $T$ -**algebra** is a natural transformation



in  $[\mathbb{C}, \mathbb{C}]$ , compatible with  $\eta, (-)^*$

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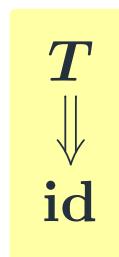
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■ **Defn.** An  $T$ -algebra is a natural transformation



in  $[\mathbb{C}, \mathbb{C}]$ , compatible with  $\eta, (-)^*$

■ We aim at generalizing this to Arrows

Recall:

Monad  
= **monoid** in  $[\mathbb{C}, \mathbb{C}]$

Arrow  
= **monoid** in  $[\mathbb{C}^{\text{op}} \times \mathbb{C}, \text{Sets}]$   
+ (compatibility with  $\times$ )

$[\mathbb{C}, \mathbb{C}]$ : monoidal  
tensor:  $F \otimes G = F \circ G$   
unit: **id**

$[\mathbb{C}^{\text{op}} \times \mathbb{C}, \text{Sets}]$ : monoidal  
tensor: like composition  
unit: **Hom** $_{\mathbb{C}}$

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- **Defn.** An *algebra* for Arrow  $A$  is a natural transformation

$$\begin{array}{ccc} A(-, +) \\ \Downarrow \\ \text{Hom}_{\mathbb{C}}(-, +) \end{array}$$

in  $[\mathbb{C}^{\text{op}} \times \mathbb{C}, \text{Sets}]$ ,

compatible with  $\text{arr}$ ,  $\ggg$ ,  $\text{first}$ .

- Justification?

- For  $A_T$  induced by a monad  $T$ ,

this is the same as algebras

$$\begin{array}{c} T \\ \Downarrow \\ \text{id} \end{array}$$

- Also for comonads, (monad + comonad + distr. law).
- Proof: non-trivial computation!

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## Justification? (ctn'd)

- Characterized as a left-inverse of Kleisli inclusion.  
[Envisaged by John Power]

$$\left\{ \begin{array}{c} A\text{-algebras} \\ \downarrow \chi \\ \text{Hom}_{\mathbb{C}}(-, +) \end{array} \right\} \cong \left\{ \begin{array}{c} \text{left inverse of} \\ \mathbb{C} \longrightarrow \mathbb{C}_A \end{array} \right\}$$

- Also the case for (co)monads. For a monad  $T$ ,

$$\left\{ \begin{array}{c} T\text{-algebras} \\ \downarrow \sigma \\ \text{id} \end{array} \right\} \cong \left\{ \begin{array}{c} \text{left inverse of} \\ \mathbb{C} \longrightarrow \mathbb{C}_T \end{array} \right\}$$

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$$\begin{array}{ccc} \mathbb{C} & \xleftarrow{\quad} & \mathbb{C}_T \\ f \nearrow & TY \downarrow \sigma_Y & \searrow f \\ X & Y & X \rightarrow Y \end{array}$$

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## Specify object(s) as a carrier?

E.g.

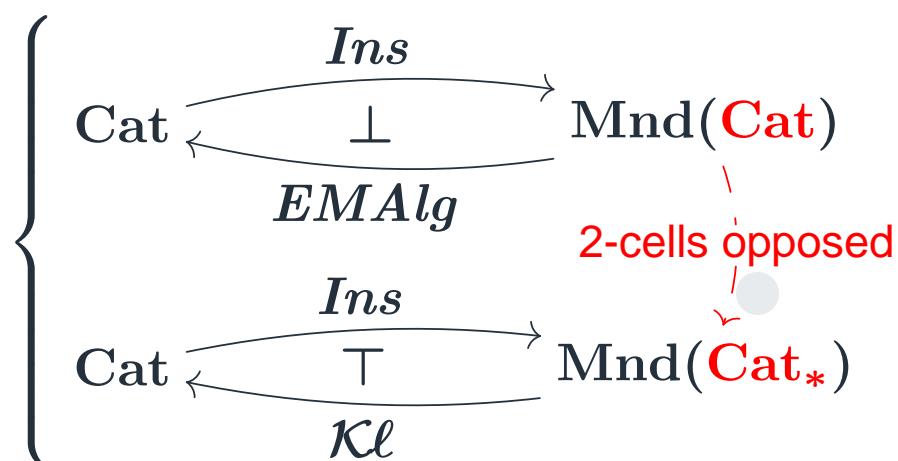
$$A(X, Y)$$

$$\downarrow$$
  
$$\text{Hom}_{\mathbb{C}}(X, Y)$$

- Doesn't work: meaning of "compatibility with  $\ggg$ " is not clear

## As a 2-categorical dual of Kleisli?

- For monads,



- Does not work for



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- Is the question “What is an algebra for Arrows?” reasonable?
  - If not, why?
  - Examples of Arrow-algebras?

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- Arrows generalize (co)monads: they are **monoids**.
- Kleisli for Arrows:

$$\{ \text{Arrows on } \mathbb{C} \} \xrightarrow[\text{Kleisli}]{\cong} \{ \text{Freyd cat. on } \mathbb{C} \}$$
$$A \dashv \dashv (\mathbb{C} \rightarrow \mathbb{C}_A)$$

- 2-categorical characterization
- Eilenberg-Moore algebras for Arrows:

$$A(-, +)$$
$$\Downarrow$$
$$\text{Hom}_{\mathbb{C}}(-, +)$$

, just like

$$T$$
$$\Downarrow$$
$$\text{id}$$

for monads .

- Examples?

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- Arrows generalize (co)monads: they are **monoids**.
- Kleisli for Arrows:

$$\begin{array}{ccc} \{\text{Arrows on } \mathbb{C}\} & \xrightarrow[\text{Kleisli}] {\cong} & \{\text{Freyd cat. on } \mathbb{C}\} \\ A & \longleftarrow & (\mathbb{C} \rightarrow \mathbb{C}_A) \end{array}$$

- 2-categorical characterization
- Eilenberg-Moore algebras for Arrows:

$$\begin{array}{c} A(-, +) \\ \Downarrow \\ \text{Hom}_{\mathbb{C}}(-, +) \end{array}$$

, just like

$$\begin{array}{c} T \\ \Downarrow \\ \text{id} \end{array}$$

for monads .

- Examples?

***Thank you for your attention!***

Contact: <http://www.cs.ru.nl/~ichiro>