Lattice-Theoretic Progress Measures and Coalgebraic Model Checking

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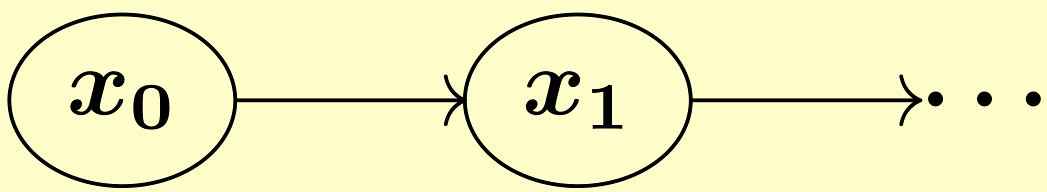
Contributions

* **Lattice-theoretic progress measure**
* **Coalgebraic model checking** as application

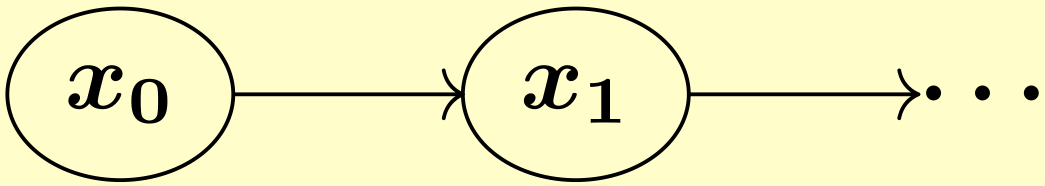
Invariant vs. Ranking Function



A linear Kripke structure:
$\mathsf{succ}\colon X \to X, \quad [\![p]\!] \subseteq X$

- **G**p (everywhere p) is a gfp $\quad \nu u.\, p \wedge \mathbf{X} u$
  - the greatest solution of $u = p \wedge \mathbf{X}\, u$
- **F**p (eventually p) is an lfp $\quad \mu u.\, p \vee \mathbf{X} u$
  - the least solution of $u = p \vee \mathbf{X}\, u$

Invariant vs. Ranking Function



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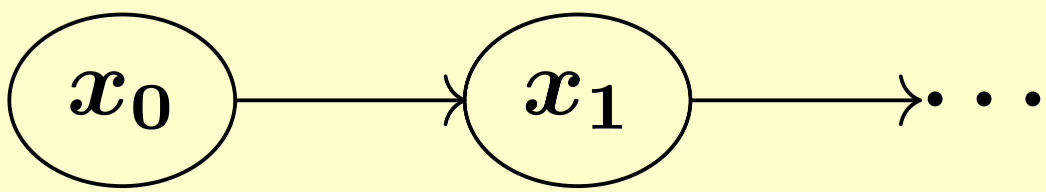

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<u>Lem.</u> (witnessing $\mathbf{G}\,p = \nu u.\,(p \wedge \mathbf{X}\,u)$)

$$\frac{I \subseteq [\![p]\!] \qquad x \in I \;\Rightarrow\; \mathsf{succ}(x) \in I}{I \subseteq [\![\mathbf{G}\,p]\!]}$$

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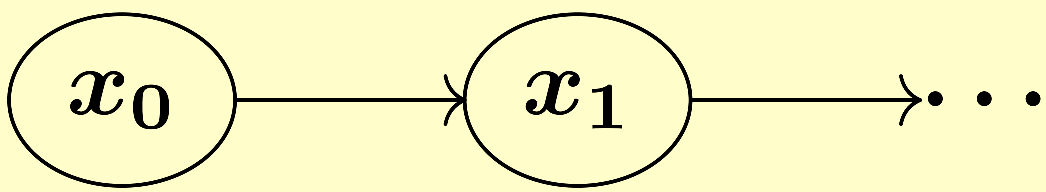

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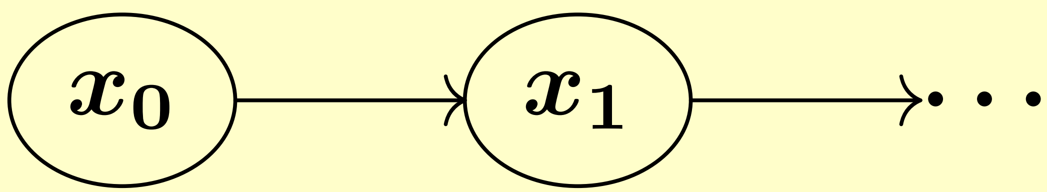

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invariant

Lem. (witnessing $\mathsf{F}\,p = \mu u.\,(p \vee \mathsf{X}\,u)$)
Let $\mathbf{rk}\colon X \to \omega \amalg \{\spadesuit\}$ be such that

$$\mathbf{rk}(x) = n \;\wedge\; x \notin [\![p]\!] \implies \mathbf{rk}(\mathsf{succ}(x)) \leq n - 1\;.$$

Then $\mathbf{rk}(x) \neq \spadesuit$ implies $x \in [\![\mathsf{F}\,p]\!]$.

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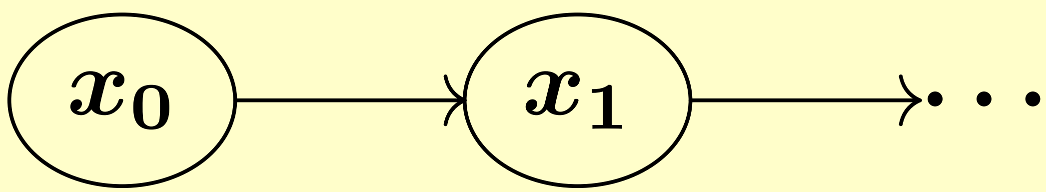

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Lem. (witnessing $\mathbf{F}\,p = \mu u.\,(p \lor \mathbf{X}\,u)$)
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ranking func.

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- How come the difference?
  ➜ Let us take a **foundational** view...

Lattice-Theoretic Foundation

L: complete lattice, $\quad f : L \to L$ monotone

Thm. (Knaster-Tarski)

- $\mu f = \min\{l \in L \mid f(l) \sqsubseteq l\}$
- $\nu f = \max\{l \in L \mid l \sqsubseteq f(l)\}$

Thm. (Cousot-Cousot)

$\bot \sqsubseteq f(\bot) \sqsubseteq \cdots \sqsubseteq f^{\omega}(\bot) \sqsubseteq \cdots$
stabilizes, and converges to μf

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Sound approx. from below

The Table

properties	witnessed by...
safety, gfp	invariants
liveness, lfp	ranking functions

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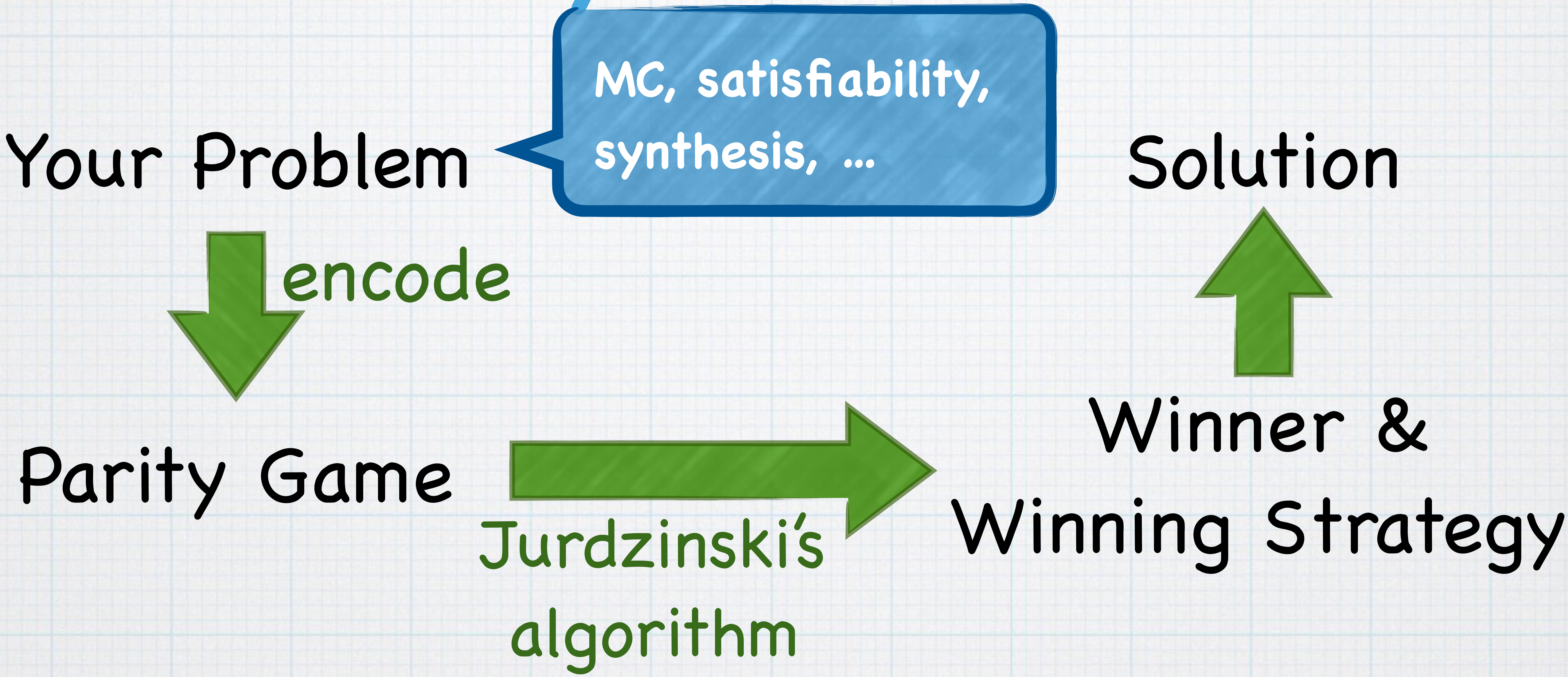
nested, alternating gfp's & lfp's	**winning strategies for a parity game** (if finitary)

The Parity-Game Workflow



The Parity-Game Workflow

Your Problem

MC, satisfiability, synthesis, …

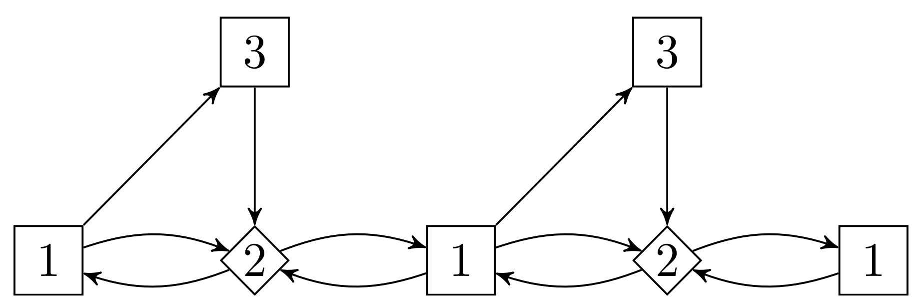
encode

Parity Game

Jurdzinski's algorithm

Winner & Winning Strategy

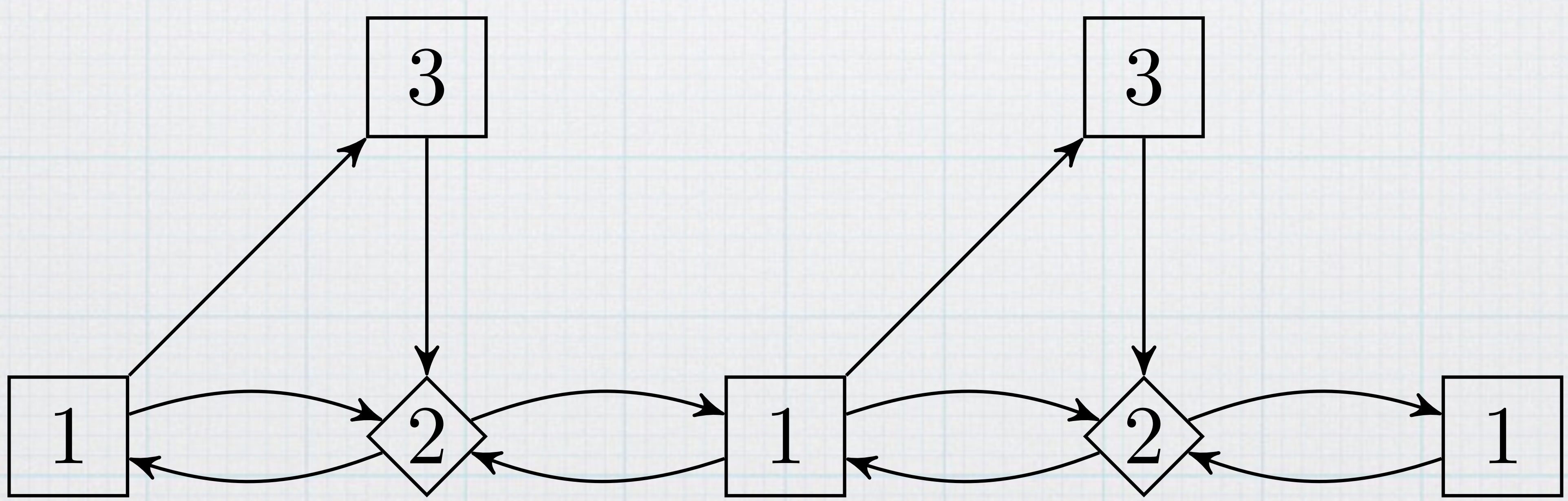
Solution



- In parity games:
  - alt. branching ($\forall$ vs $\exists$, $\wedge$ vs $\vee$)
  - parity acceptance cond.
    ➜ alt. betw. μ and ν

Jurdzinski's Progress Measure for Parity Games: Intuitions

◇: your position
□: opponent's
goal: "visit bigger even"



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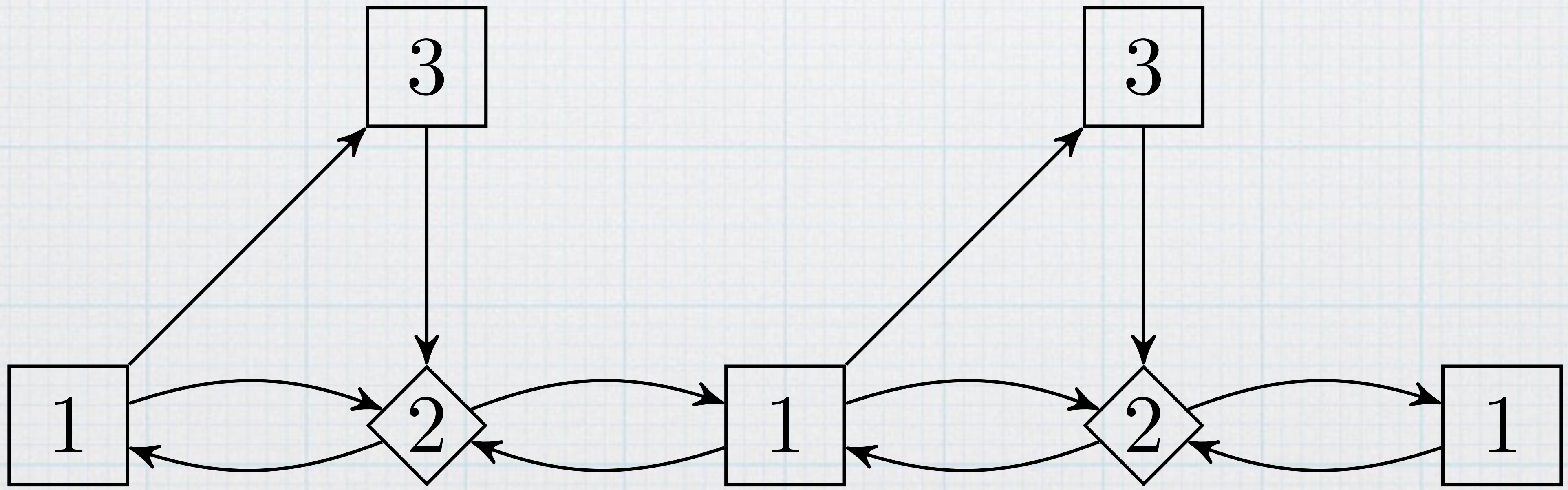
YOU WIN!



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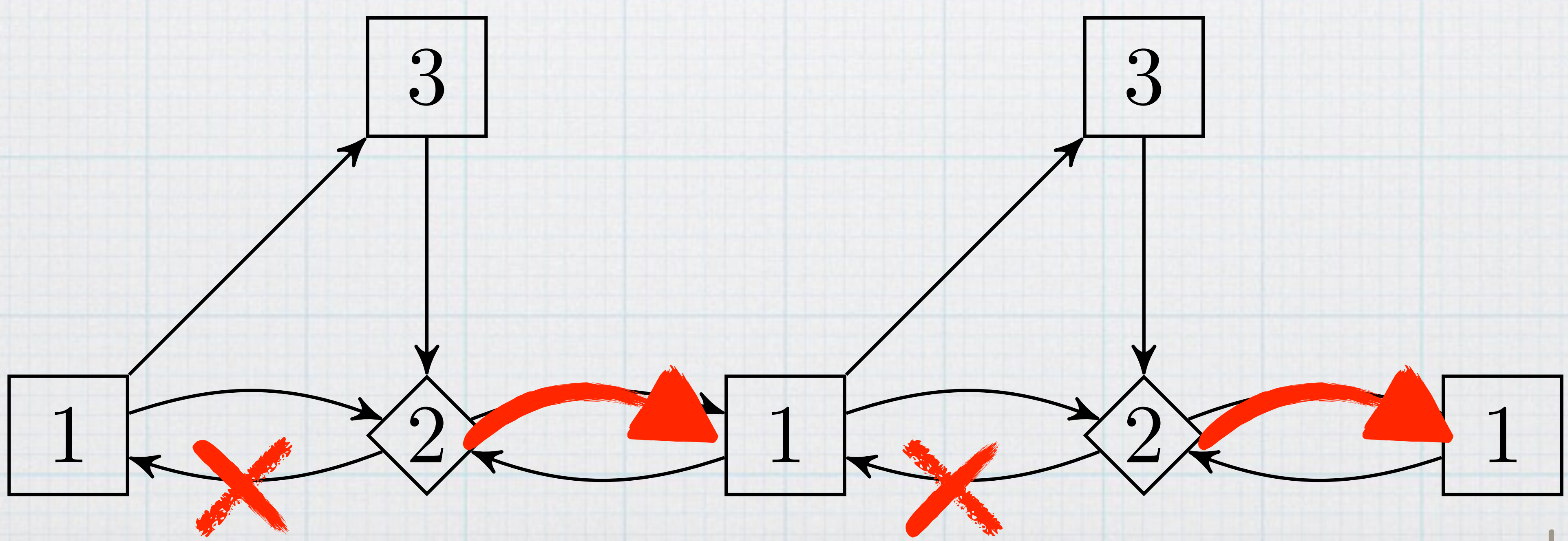

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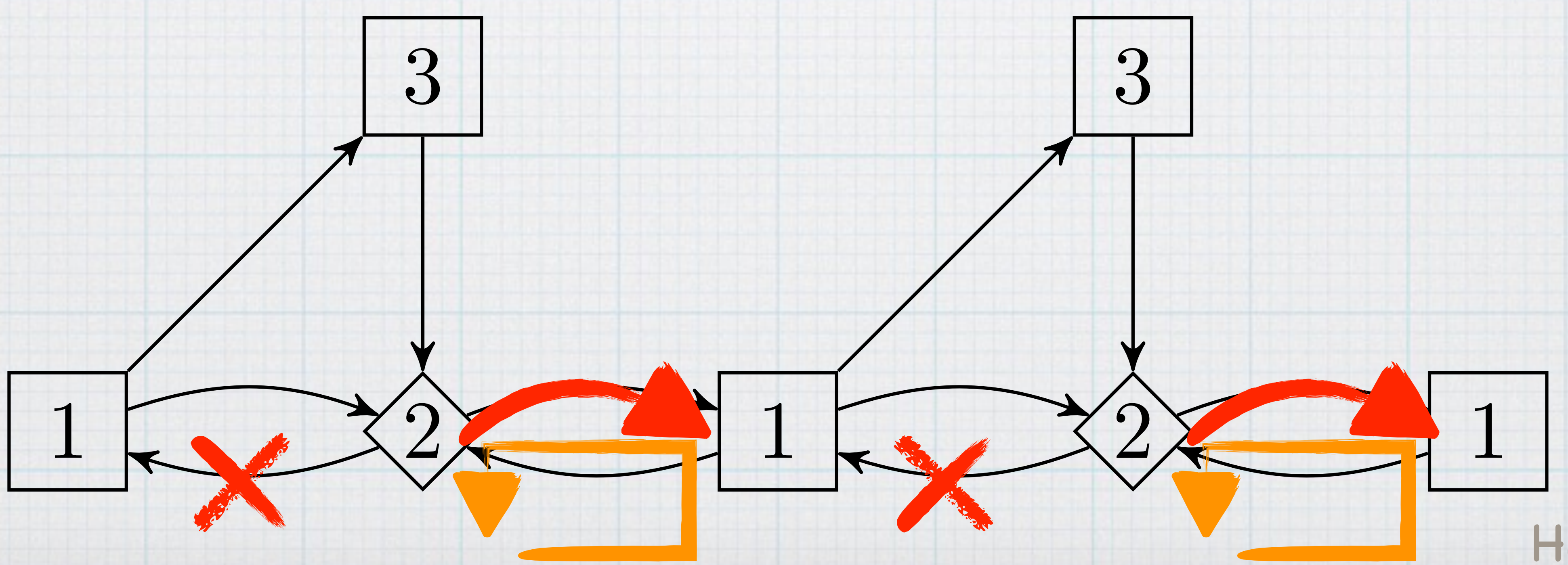

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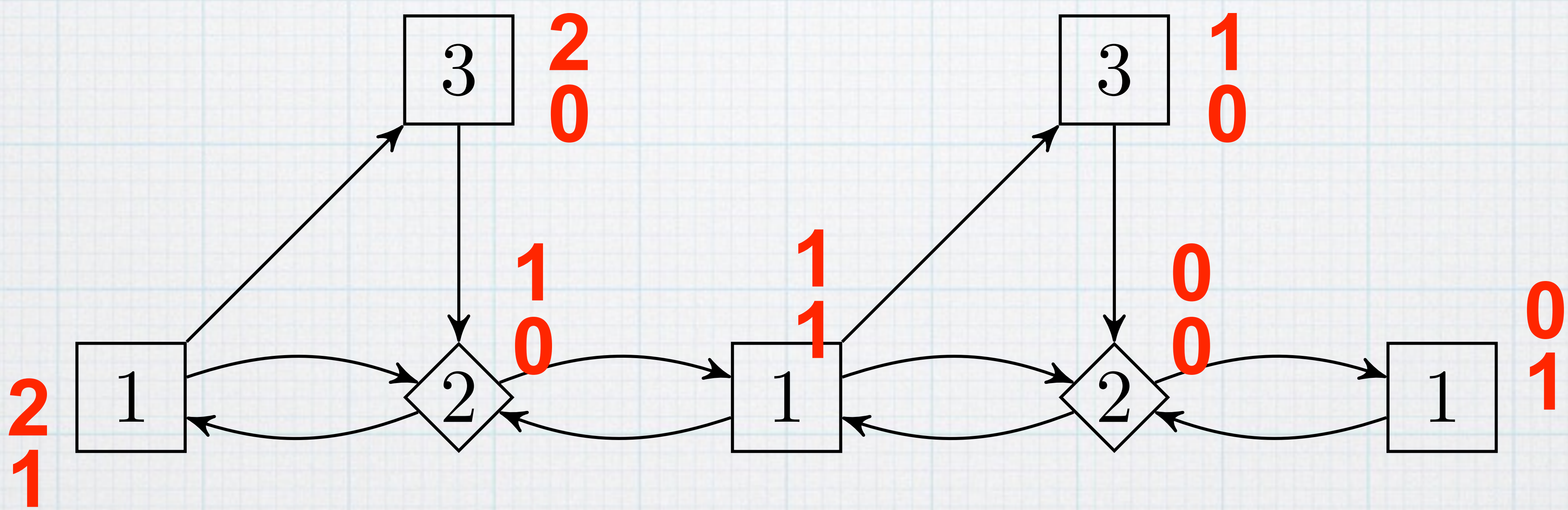

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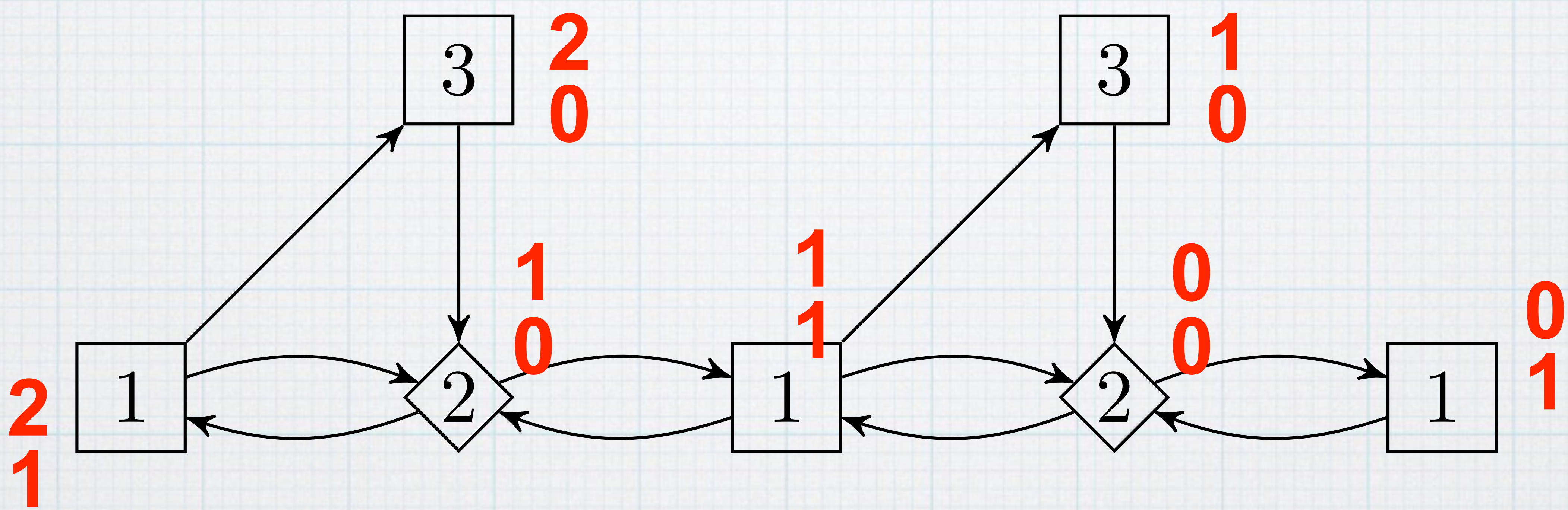

n_3 — how many 3's will be visited

n_1 — how many 1's will be visited (before visiting 2, a bigger even)

Jurdzinski's Progress Me
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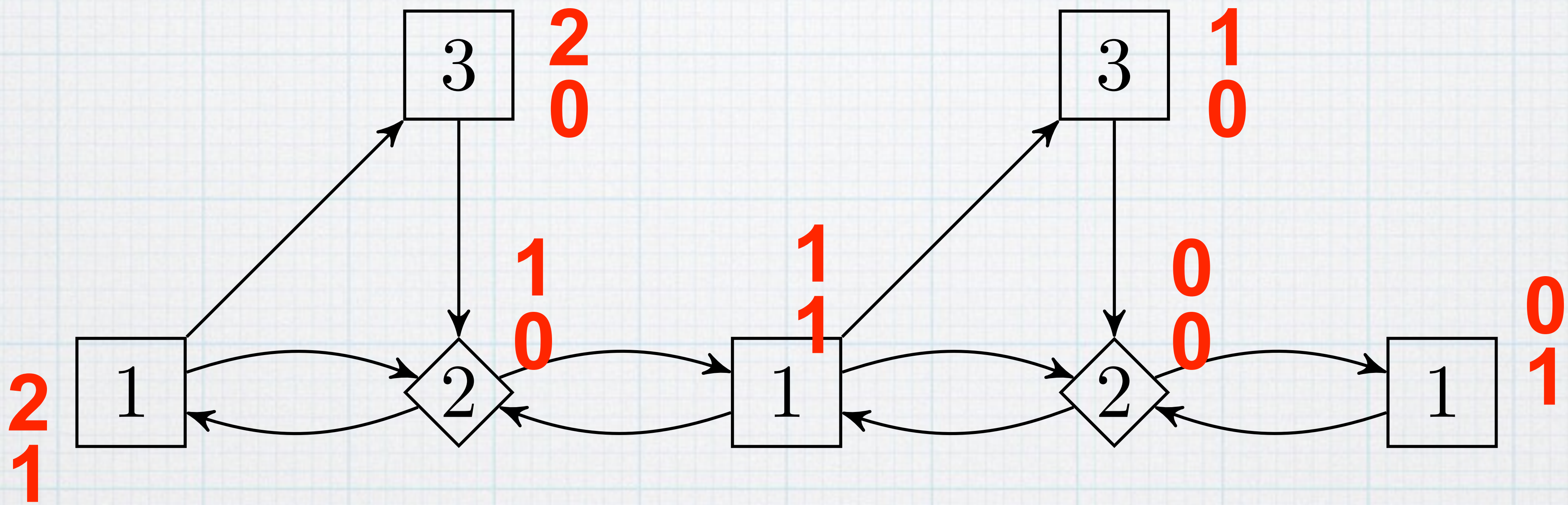

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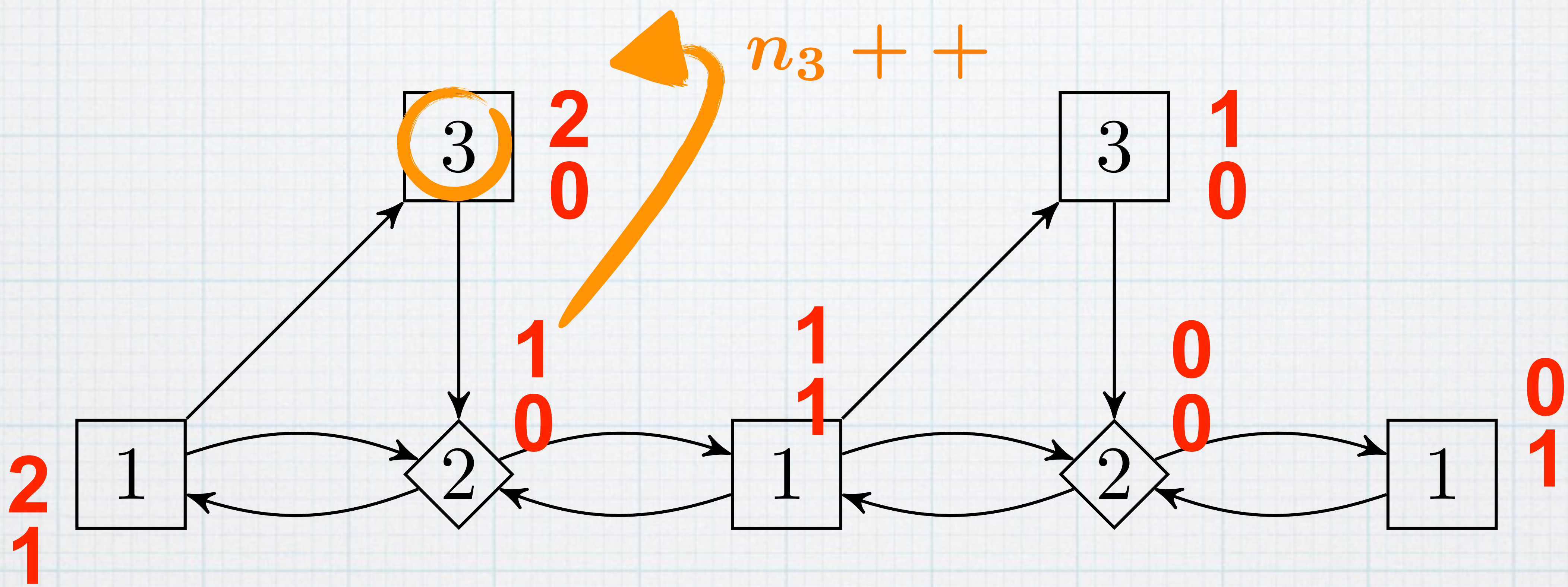

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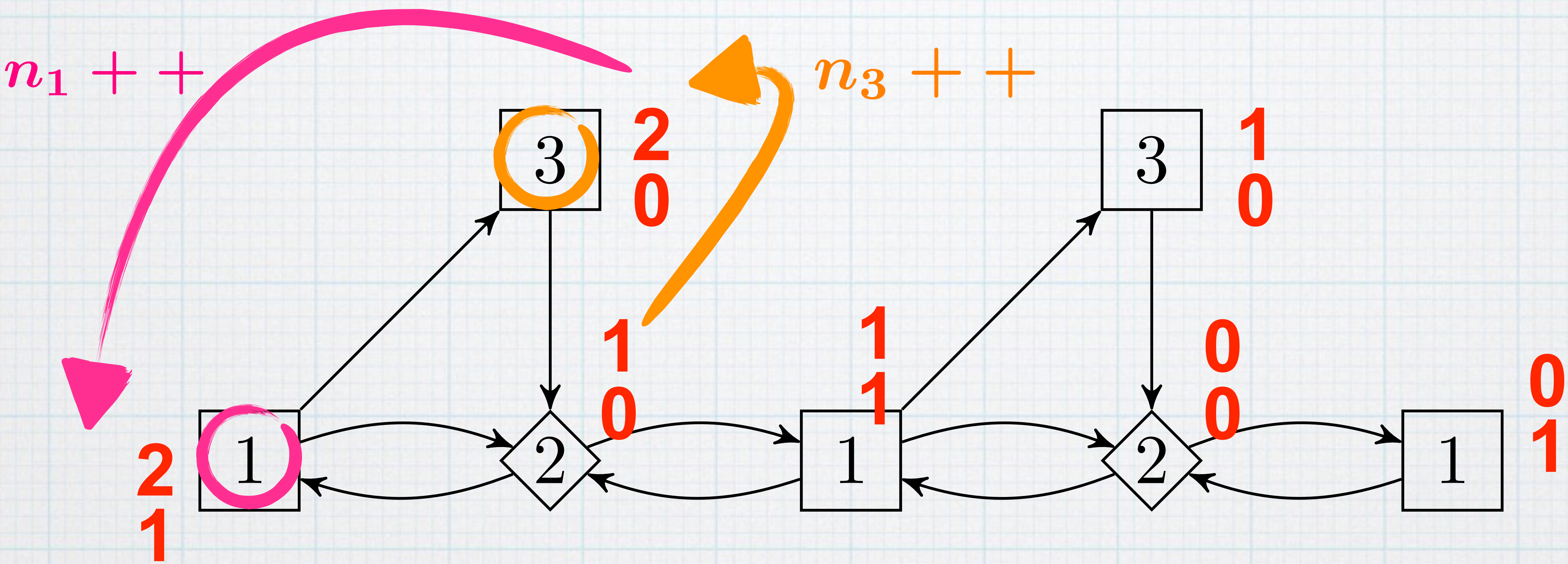

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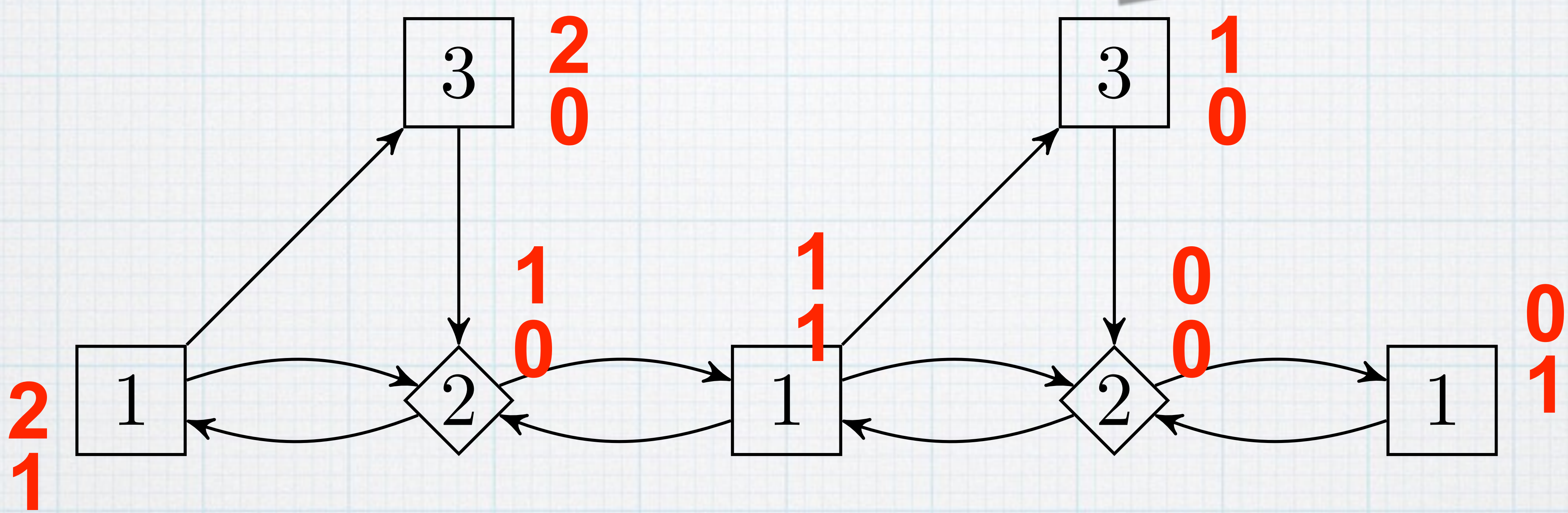

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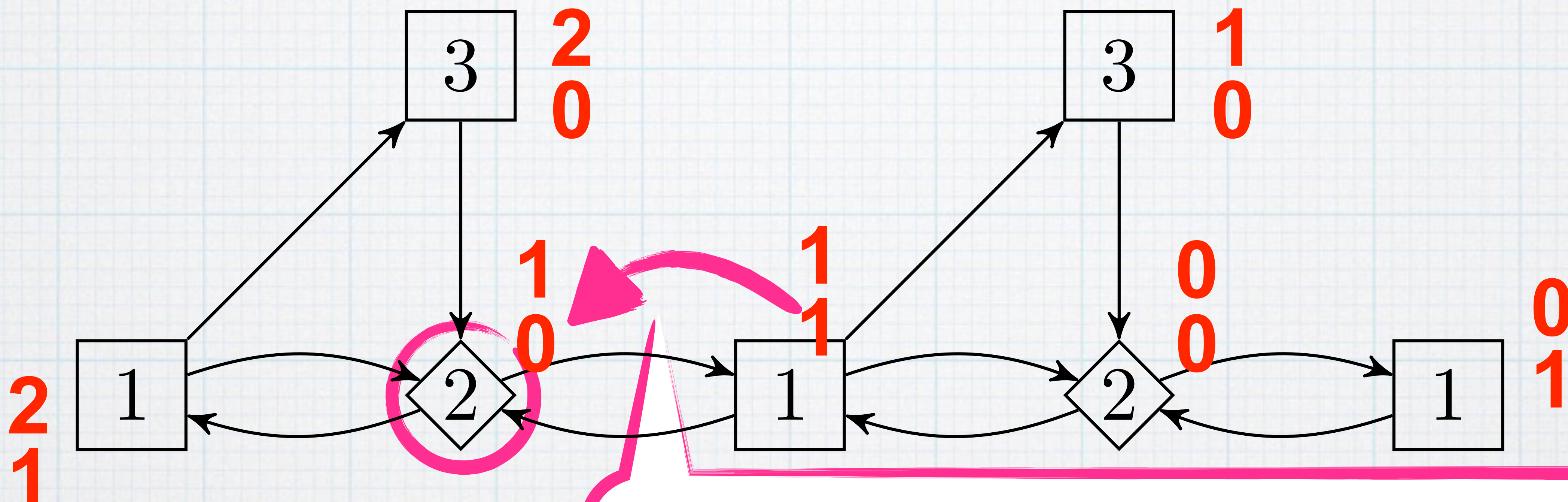

$n_1 := 0$

because visiting 2 **cancels out** visiting 1

n_3 — how many 3's will be visited

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Jurdzinski's Progress Measure
Definition

(Assuming priorities are 0, 1, ..., 6)

* A **prioritized ordinal** is α_5 α_3 α_1 (each α_j is an ordinal)

Jurdzinski's Progress Measure

Definition

(Assuming priorities are 0, 1, ..., 6)

* A **prioritized ordinal** is $\begin{matrix}\alpha_5\\\alpha_3\\\alpha_1\end{matrix}$ (each α_j is an ordinal)

* for each $i = 0, 1, \ldots, 6,$ the **i-th truncated lexicographic order**

$$\begin{matrix}\alpha_5\\\alpha_3\\\alpha_1\end{matrix} \preceq_i \begin{matrix}\beta_5\\\beta_3\\\beta_1\end{matrix}$$

is defined by

* the lexicographic order
* after truncating α_j, β_j for all $j < i$

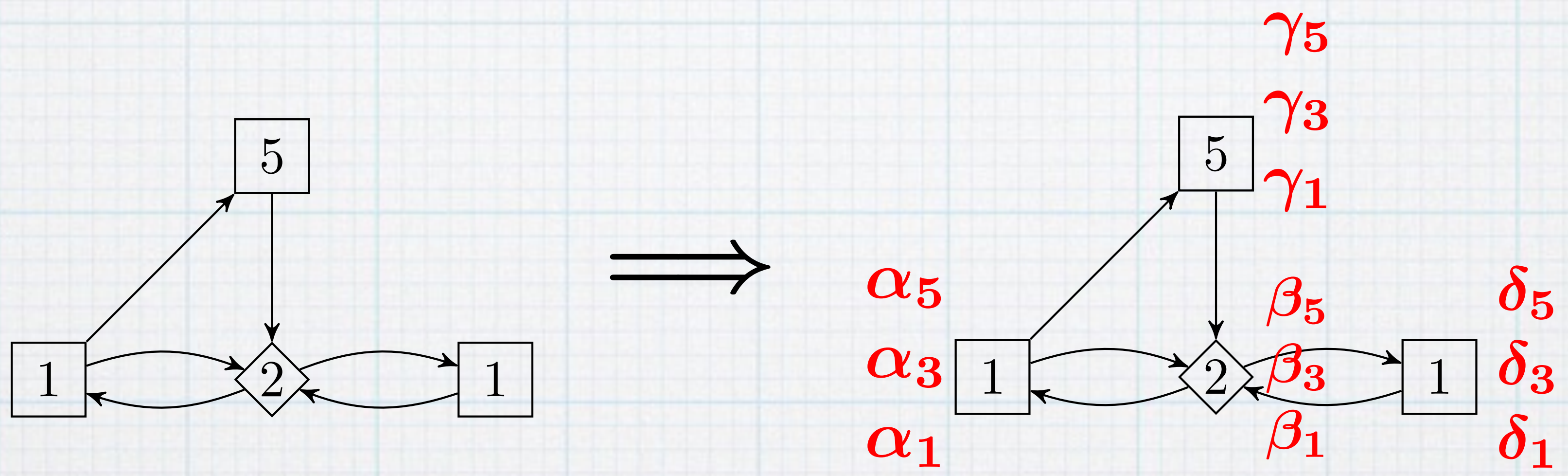
* examples: $\begin{matrix}7\\142\\63\end{matrix} \preceq_1 \begin{matrix}8\\0\\0\end{matrix}$ $\quad \begin{matrix}2\\142\\63\end{matrix} \preceq_4 \begin{matrix}2\\0\\0\end{matrix}$

truncated

Jurdzinski's Progress Measure Definition

(Assuming priorities are 0, 1, ..., 6)

* A **progress measure** is an assignment like



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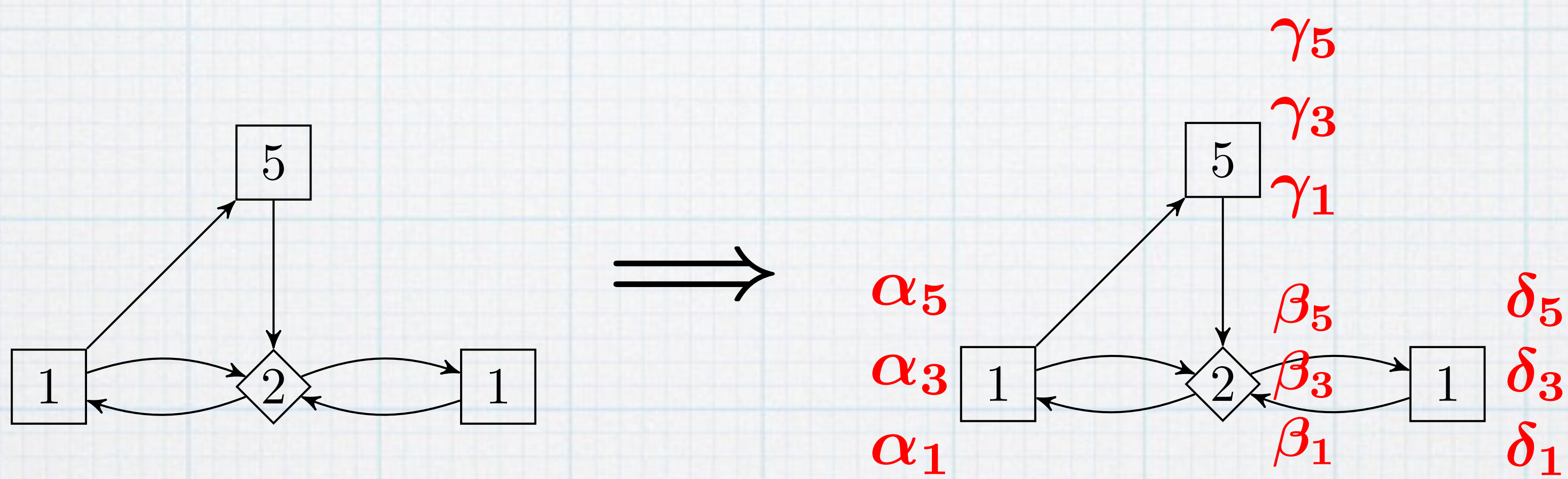

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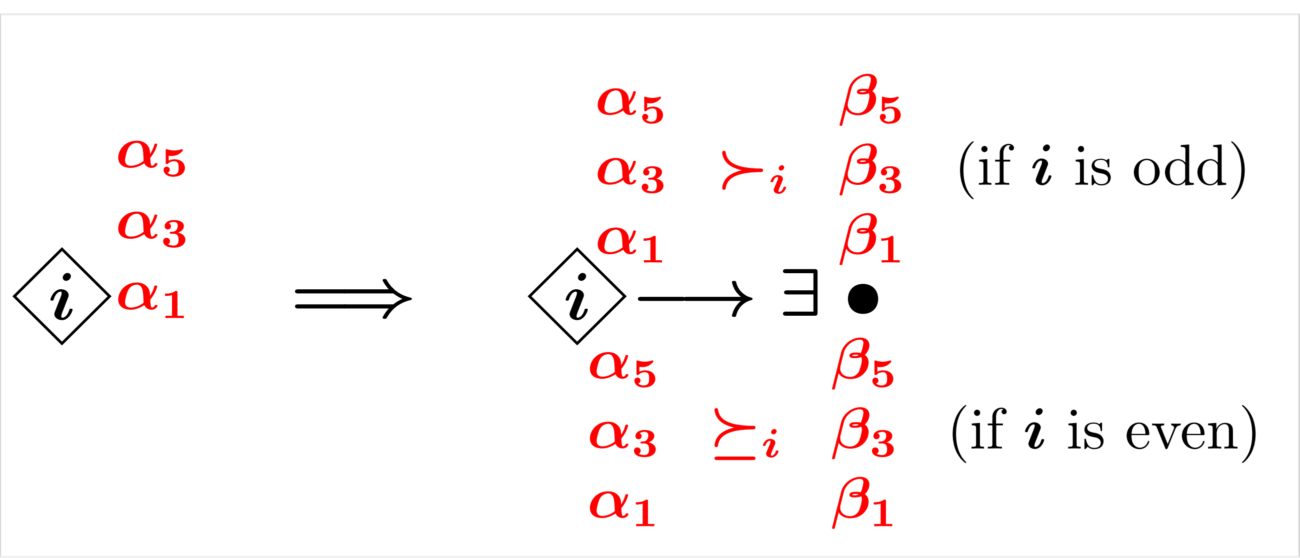

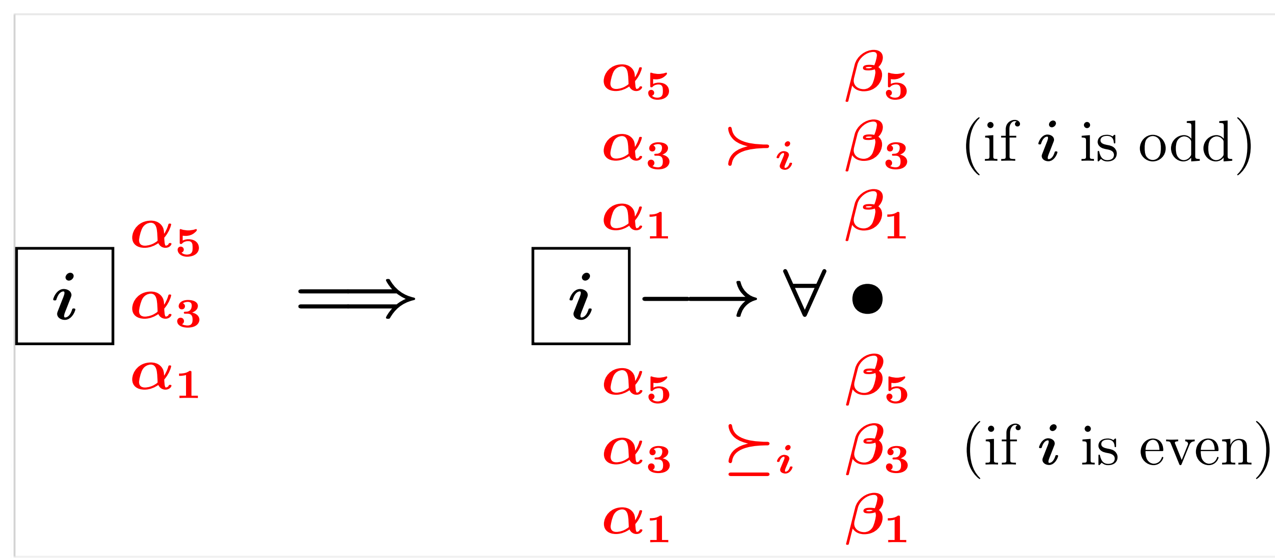

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safety, gfp	invariants
liveness, lfp	ranking functions
nested, alternating gfp's & lfp's	**progress measure for a parity game** [Jurdzinski]

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Knaster-Tarski

Cousot-Cousot

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Syntax: Equational Systems

[Arnold & Niwinski '01], [Cleaveland, Klein & Steffen, CAV'92], ...

<u>Def.</u> An *equational system* over a complete lattice L is

$$\begin{aligned} u_1 &=_{\eta_1} f_1(u_1, \dots, u_m), \\ &\vdots \\ u_m &=_{\eta_m} f_m(u_1, \dots, u_m) \end{aligned}$$

where

- $f_1, \dots, f_m \colon L^m \to L$ are monotone, and
- $\eta_1, \dots, \eta_m \in \{\mu, \nu\}$.

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$$u_1 =_\mu f_1(u_1, u_2),$$
$$u_2 =_\nu f_2(u_1, u_2)$$

//

$$\nu u_2.\, f_2(\, \mu u_1.\, f_1(u_1, u_2),\, u_2\,)$$

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solved first

$$u_1 =_\mu f_1(u_1, u_2),$$
$$u_2 =_\nu f_2(u_1, u_2)$$

$$\nu u_2.\, f_2(\,\mu u_1.\, f_1(u_1, u_2),\, u_2\,)$$

The order matters!

Definition: Progress Measure for

$$u_1 =_{\mu} f_1(\vec{u})$$
$$u_2 =_{\nu} f_2(\vec{u})$$
$$u_3 =_{\mu} f_3(\vec{u})$$
$$u_4 =_{\nu} f_4(\vec{u})$$

over L

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$$u_1 =_\mu f_1(\vec{u})$$
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over L

$$p = \begin{pmatrix} p_1(\alpha_1, \alpha_3) \\ p_2(\alpha_1, \alpha_3) \\ p_3(\alpha_1, \alpha_3) \\ p_4(\alpha_1, \alpha_3) \end{pmatrix}_{\alpha_1, \alpha_3 \in \mathbf{Ord}}$$

with $p_i(\alpha_1, \alpha_3) \in L$, $\forall i \in [1, 4]$

* "Counters" α_1, α_3 for each μ-var.
* Subject to:
  1. Monotonicity
  2. μ-var. cond.
  3. ν-var. cond.

Definition: Progress Measure for

$$\begin{array}{l} u_1 =_\mu f_1(\vec{u}) \\ u_2 =_\nu f_2(\vec{u}) \\ u_3 =_\mu f_3(\vec{u}) \\ u_4 =_\nu f_4(\vec{u}) \end{array}$$

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1. (Monotonicity) For $i \in [1,4]$,

$$(\alpha_1, \alpha_3) \preceq_i (\beta_1, \beta_3) \implies p_i(\alpha_1, \alpha_3) \sqsubseteq p_i(\beta_1, \beta_3)$$

Definition: Progress Measure for

$$
\begin{aligned}
u_1 &=_\mu f_1(\vec{u}) \\
u_2 &=_\nu f_2(\vec{u}) \\
u_3 &=_\mu f_3(\vec{u}) \\
u_4 &=_\nu f_4(\vec{u})
\end{aligned}
$$

over L

$$
p = \begin{pmatrix} p_1(\alpha_1, \alpha_3) \\ p_2(\alpha_1, \alpha_3) \\ p_3(\alpha_1, \alpha_3) \\ p_4(\alpha_1, \alpha_3) \end{pmatrix}_{\alpha_1, \alpha_3 \in \mathbf{Ord}}
$$

with $p_i(\alpha_1, \alpha_3) \in L, \quad \forall i \in [1, 4]$

2. (μ-var. cond.)

- (base) $p_1(0, \alpha_3) = \bot, \quad p_3(\alpha_1, 0) = \bot$
- (step)

$$
p_1(\alpha_1 + 1, \quad \alpha_3 \quad) \sqsubseteq f_1 \begin{pmatrix} p_1(\alpha_1, \alpha_3) \\ p_3(\alpha_1, \alpha_3) \end{pmatrix}
$$

$$
p_3(\quad \alpha_1 \quad, \alpha_3 + 1) \sqsubseteq f_3 \begin{pmatrix} p_1(\beta_1, \alpha_3) \\ p_3(\beta_1, \alpha_3) \end{pmatrix} \qquad (\exists \beta_1)
$$

- (limit)

$$
p_1(\alpha_1, \alpha_3) \sqsubseteq \bigsqcup_{\beta_1 < \alpha_1} p_1(\beta_1, \alpha_3) \qquad (\alpha_1 \text{: a limit ord.})
$$

(same for α_3)

Defi...

Thm. (Cousot-Cousot)

$\bot \sqsubseteq f(\bot) \sqsubseteq \cdots \sqsubseteq f^{\omega}(\bot) \sqsubseteq \cdots$

stabilizes, and converges to μf

for

$$\begin{aligned} u_1 &=_{\mu} f_1(\vec{u}) \\ u_2 &=_{\nu} f_2(\vec{u}) \\ u_3 &=_{\mu} f_3(\vec{u}) \\ u_4 &=_{\nu} f_4(\vec{u}) \end{aligned}$$

over L

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Definition: Progress Measure for

$$\begin{aligned} u_1 &=_{\mu} f_1(\vec{u}) \\ u_2 &=_{\nu} f_2(\vec{u}) \\ u_3 &=_{\mu} f_3(\vec{u}) \\ u_4 &=_{\nu} f_4(\vec{u}) \end{aligned}$$

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$$p_2(\alpha_1, \alpha_3) \sqsubseteq f_2\left(\overrightarrow{p}(\beta_1, \alpha_3)\right) \qquad (\exists \beta_1)$$

$$p_4(\alpha_1, \alpha_3) \sqsubseteq f_4\left(\overrightarrow{p}(\beta_1, \beta_3)\right) \qquad (\exists \beta_1, \beta_3)$$

Thm. (Knaster-Tarski)

- $\nu f = \max\{l \in L \mid l \sqsubseteq f(l)\}$

$$\Longrightarrow \quad \frac{l \sqsubseteq f(l)}{l \sqsubseteq \nu f}$$

for

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over L

$$\begin{pmatrix} \ldots \\ p_3(\alpha_1, \alpha_3) \\ p_4(\alpha_1, \alpha_3) \end{pmatrix}_{\alpha_1, \alpha_3 \in \mathbf{Ord}}$$

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1. Monotonicity
2. μ-var. cond. (base, step, limit)
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$$p_1(\alpha_1 + 1, \quad \alpha_3 \quad) \sqsubseteq f_1 \begin{pmatrix} p_1(\alpha_1, \alpha_3) \\ p_3(\alpha_1, \alpha_3) \end{pmatrix}$$

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with $p_i(\alpha_1, \alpha_3) \in L$, $\forall i \in [\;\; 4]$

$$\begin{matrix} 2 \\ 142 \\ 63 \end{matrix} \preceq_4 \begin{matrix} 2 \\ 0 \\ 0 \end{matrix}$$

1. Monotonicity
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$$p_1(\alpha_1 + 1, \quad \alpha_3 \quad) \sqsubseteq f_1 \begin{pmatrix} p_1(\alpha_1, \alpha_3) \\ p_3(\alpha_1, \alpha_3) \end{pmatrix}$$

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Correctness

* (soundness)

  Let $p = \begin{pmatrix} p_1(\alpha_1, \alpha_3) \\ p_2(\alpha_1, \alpha_3) \\ p_3(\alpha_1, \alpha_3) \\ p_4(\alpha_1, \alpha_3) \end{pmatrix}_{\alpha_1, \alpha_3 \in \mathbf{Ord}}$ be a prog. meas. for $\begin{array}{l} u_1 =_\mu f_1(\vec{u}) \\ u_2 =_\nu f_2(\vec{u}) \\ u_3 =_\mu f_3(\vec{u}) \\ u_4 =_\nu f_4(\vec{u}) \end{array}$

  Then p underapproximates the solution:

$$p_i(\alpha_1, \alpha_3) \sqsubseteq (\text{the solution for } u_i),$$

$$\text{for each } \alpha_1, \alpha_3 \in \mathbf{Ord} \text{ and } i \in [1, 4]$$

* **(completeness)** There is a prog. meas. that achieves equalities

The Table

properties	witnessed by...
safety, gfp	invariants
liveness, lfp	ranking functions
nested, alternating gfp's & lfp's	**progress measure for a parity game** (if finitary); **(lattice-theoretic) progress measure** (in general)

(Potential) applications:

- **Theorem proving**, proof rules
- Program verification: "**synthesis of symbolic progress measures**"
- In metatheories
- Generic "coalgebraic" model checking

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Lattice-Theoretic Progress Measures in Coalgebraic Model Checking

(in 2 min.)

See e.g. [Rutten, TCS'00] [Jacobs, book draft]

Coalgebra

* Categorical abstraction of **state-based dynamics**

$$\begin{array}{c} FX \\ c\uparrow \\ X \end{array}$$

$F = 2 \times (\_)^{\Sigma}$	DFA
$F = (\mathcal{P}\_)^{\Sigma}$	LTS
$F = \mathcal{D}$	Markov chain
$F = \mathcal{P}\mathcal{P}$	nbd. frames, games
other F's	graded sys., game frames, $\cdots$

Coalgebraic Modal Logic

[Moss, Pattinson, Kurz, Cirstea, Kupke, Venema, Schroeder,...]

$$\frac{\text{modal logic}}{\text{Kripke frame}} = \frac{\textbf{coalg. modal logic}}{\text{coalgebra}}$$

Coalgebraic Modal Logic

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$$\frac{\text{modal logic}}{\text{Kripke frame}} = \frac{\text{coalg. modal logic}}{\text{coalgebra}}$$

- Modalities by **predicate liftings**

$$\lambda\colon \Omega^X \longrightarrow \Omega^{FX}, \quad \text{natural in } X$$

where $\Omega = \{\text{truth values}\}$ (e.g. $\{\mathbf{t}, \mathbf{f}\}$ or $[0,1]$)

- Hennessy-Milner logic, neighborhood logic, graded logic ("in more than k successors"), game logic ("a coalition C forces..."), Lukasiewicz logic [Mio, Simpson, ...], logics w/ future discounting [Almagor, Boker, Kupfermann, ...], ...

- Fixed points by eq. sys.

$$u_1 =_\mu (p \wedge u_2) \vee \mathsf{X} u_1$$
$$u_2 =_\nu u_2$$

Contributions I: Branching Time

* **Progress measure**, as a witness in model checking
  * X can be infinite
  * $\mathbf{\Omega}$ can be $[0,1]$
* **MC algorithm**, if finitary
  * X: finite, $\mathbf{\Omega} = \{\mathrm{t}, \mathrm{f}\}$
  * Generic algorithm, works for a variety of logics
  * Complexity exponential only in alternation depth

Def.

$$p = \begin{pmatrix} p_1(\alpha_1, \dots, \alpha_k) \\ \vdots \\ p_m(\alpha_1, \dots, \alpha_k) \end{pmatrix}_{\vec{\alpha} \in \mathbf{Ord}}$$

- with $p_i(\vec{\alpha}) \in \mathbf{\Omega}^X$,
- subject to
  1. monotonicity
  2. μ-var. cond.
  3. ν-var. cond.

Thm.

$$\begin{pmatrix} p_1(\alpha_1, \dots, \alpha_k) \\ \vdots \\ p_m(\alpha_1, \dots, \alpha_k) \end{pmatrix}_{\vec{\alpha} \in \mathbf{Ord}} \sqsubseteq \left[\!\left[\begin{matrix} u_1 =_{\eta_1} \varphi_1(\vec{u}) \\ \vdots \\ u_m =_{\eta_m} \varphi_m(\vec{u}) \end{matrix} \right]\!\right]_{\substack{FX \\ \uparrow \\ X}}$$

Contributions II: Linear Time

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- Like **LTL** (as opp. to CTL)
- More challenging for coalgebra
  ➜
  in a **Kleisli** category
- We focus on **nondet.** branching

Contributions II: Linear Time

* **Progress measure**, for linear-time model checking
  * $\left(\begin{array}{c} \mathcal{P}FX \\ \uparrow \\ X \end{array}, x \right) \models \left(\begin{array}{ccc} u_1 & =_{\eta_1} & \varphi_1(\vec{u}) \\ & \vdots & \\ u_m & =_{\eta_m} & \varphi_m(\vec{u}) \end{array} \right)$

    is witnessed by
    * a **runtree** $\begin{array}{c} FY \\ \uparrow \\ Y \end{array}$, and
    * data like $\left(\begin{array}{c} p_1(\alpha_1, \dots, \alpha_k) \\ \vdots \\ p_m(\alpha_1, \dots, \alpha_k) \end{array} \right)_{\vec{\alpha} \in \mathbf{Ord}}$

      with $p_i(\vec{\alpha}) \in \Omega^Y$

* **Decision procedure**, if finitary
  * Exploits the **small runtree** theorem

- Like **LTL** (as opp. to CTL)
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Conclusions

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(Potential) applications:

- **[done]** Generic "coalgebraic" model checking
- **[Forthcoming]** Coalgebraic modeling of Buechi automata & simulations
- **Theorem proving**, proof rules
- Program verification: "**synthesis of symbolic progress measures**"
- In metatheories, e.g. for higher-order model checking [Ong, Kobayashi, Tsukada, ...]

Conclusions

Thank you for your attention!

Ichiro Hasuo (Dept. CS, U. Tokyo)
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