Semantics of Higher-Order Quantum Computation via Geometry of Interaction

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東京大学
THE UNIVERSITY OF TOKYO

京都大学
KYOTO UNIVERSITY

Contribution

Denotational semantics of a **functional quantum programming language**

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Denotational semantics of a **functional quantum programming language**

Linear λ-calculus + quantum primitives

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One of the first to cover the full features!

- !-modality for duplicable data
- recursion

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Denotational semantics of a **functional quantum programming language**

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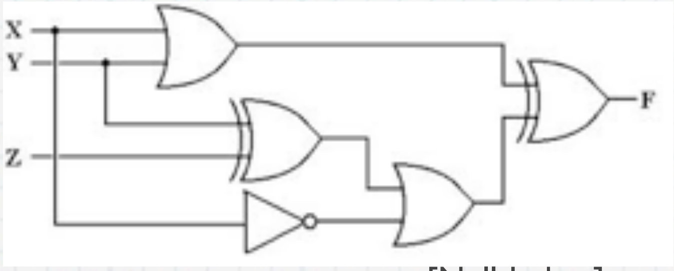
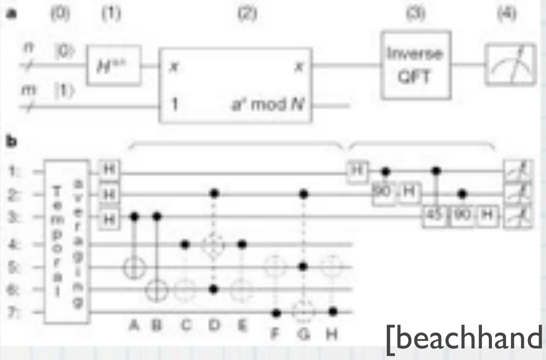
... via **GoI** (Geometry of Interaction)

Part 1

Functional QPL: Some Contexts

Quantum Programming Language

Classical		Quantum	
(Boolean) circuit	 [Null-Lobur]	Quantum circuit	 [beachhandball.es]
Programming language	see code below		

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int i,j;
int factorial(int k)
{
    j=1;
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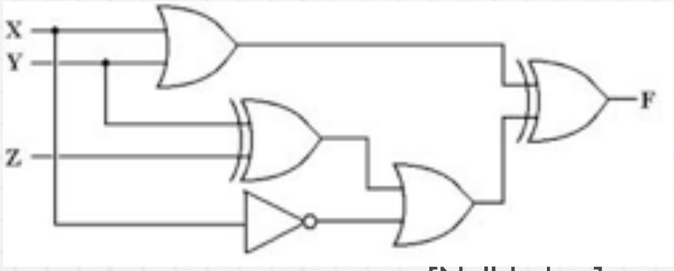
Quantum Programming Language

Classical

(Boolean) circuit



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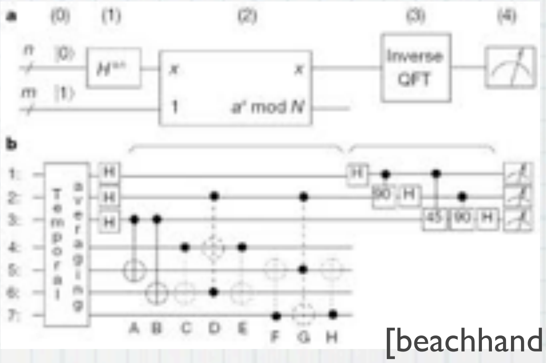

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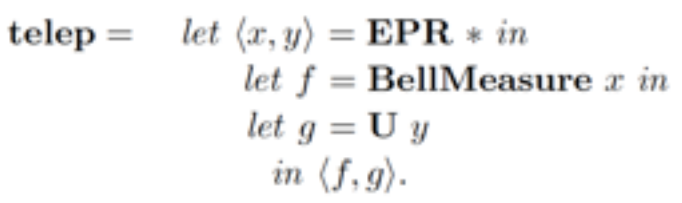

$$\begin{aligned}\mathbf{telep} = \quad & \mathit{let}\ \langle x, y\rangle = \mathbf{EPR} * \mathit{in} \\ & \mathit{let}\ f = \mathbf{BellMeasure}\ x\ \mathit{in} \\ & \mathit{let}\ g = \mathbf{U}\ y \\ & \mathit{in}\ \langle f, g\rangle.\end{aligned}$$

[Selinger-Valiron]

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Quantum Programming Language

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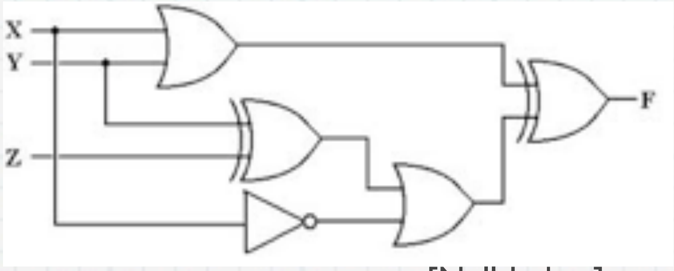

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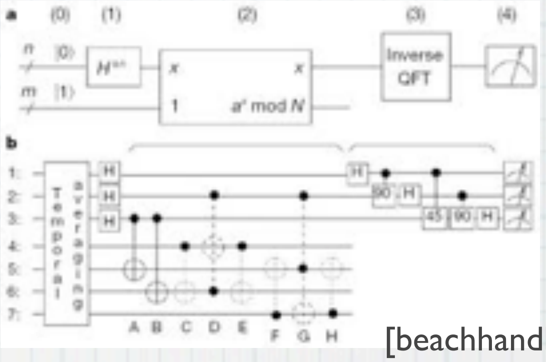

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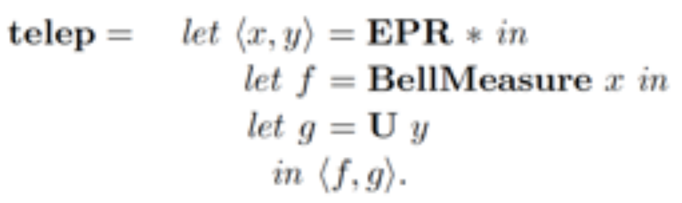
Quantum programming language



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[Selinger-Valiron]

- For discovery of **algorithms**
- For **reasoning, verification**

Functional Quantum Programming Language

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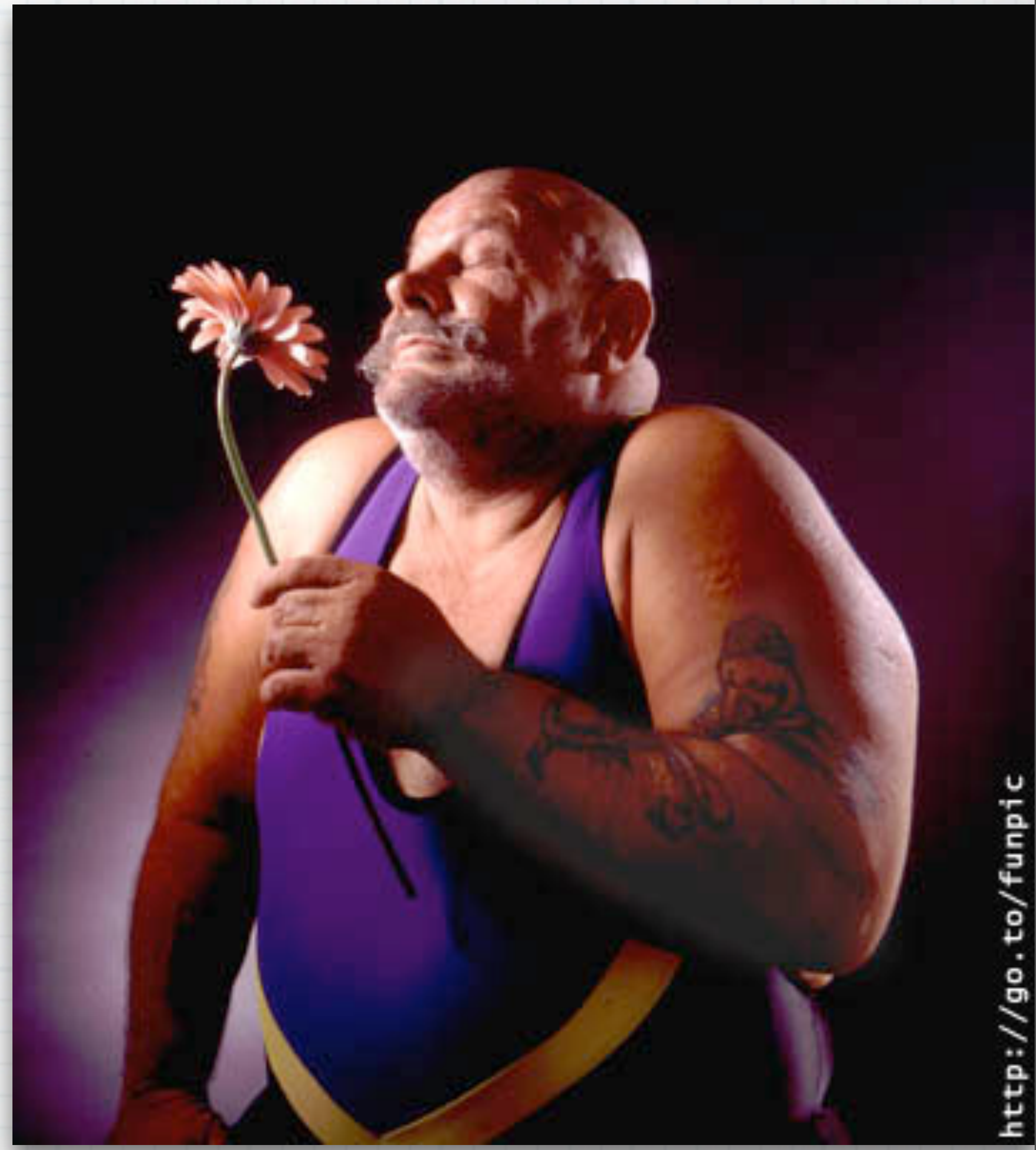
Functional Quantum Programming Language

- A **real man's** programming style



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- Heavily used in the **financial sector**
- ...

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SOLUTIONS
JANE STREET
galois
IIJ
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NII
National Institute of Informatics
Standard Chartered
TSURU CAPITAL
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Functional Quantum Programming Language

- A **real man's** programming style
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- **Mathematically nice and clean**
  - Aids semantical study
  - **Transfer** from classical to quantum

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Functional QPL: Syntax

- **Linear λ-calculus**
  + quantum primitives [van Tonder, Selinger, Valiron, ...]
- Linearity for **no-cloning**
  - "Input can be used only once"
  - Not allowed/typable:
  - Duplicable (classical) data: by the **!-modality**

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“arbitrary many copies”

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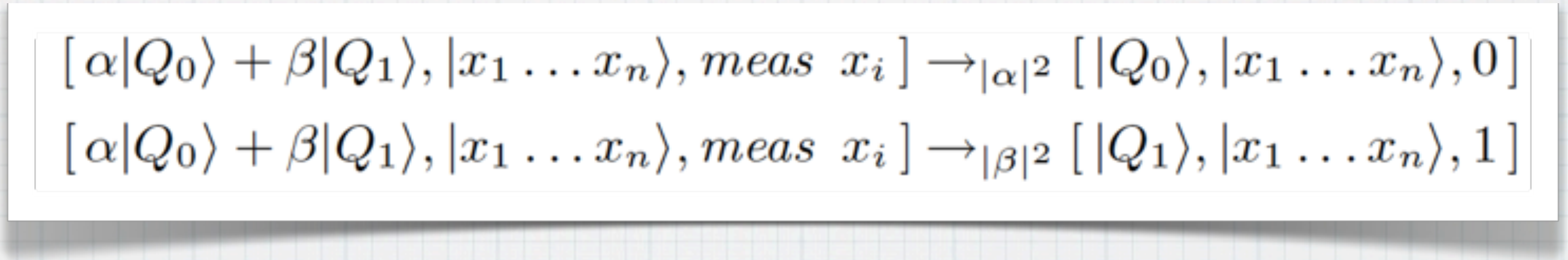
- Operational semantics: [Selinger & Valiron, '09]
  - "Quantum closure," reduction with probabilistic branching



$$[\,\alpha|Q_0\rangle + \beta|Q_1\rangle, |x_1 \dots x_n\rangle, meas\ \ x_i\,] \rightarrow_{|\alpha|^2} [\,|Q_0\rangle, |x_1 \dots x_n\rangle, 0\,]$$

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  - Allows to identify **linear logic** $\otimes$ and **quantum** $\otimes$ (feature of the Selinger-Valiron language; not in ours)

Functional QPL: Semantics

$[\![M]\!]$

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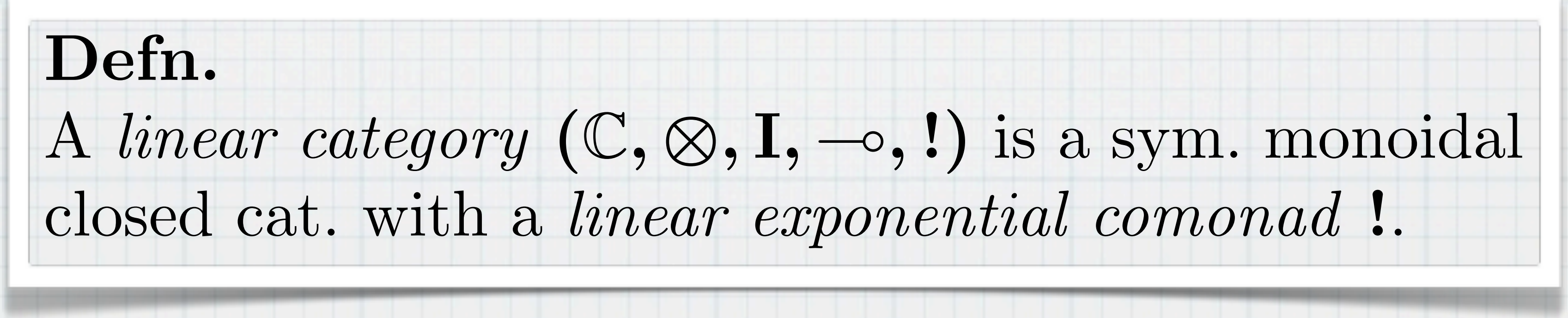

Defn.
A *linear category* $(\mathbb{C}, \otimes, \mathbf{I}, \multimap, \mathbf{!})$ is a sym. monoidal closed cat. with a *linear exponential comonad* $\mathbf{!}$.

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  - For functional QPL? Is **Hilb** (or alike) a linear cat.?

Functional QPL: Semantics

- **Hilb** (or alike) is **not** a linear category

- **Challenge**: coexistence of **quantum** and **classical** data
- Only partial results
  - [Selinger & Valiron, '08]: for strictly linear fragment (w/o !)

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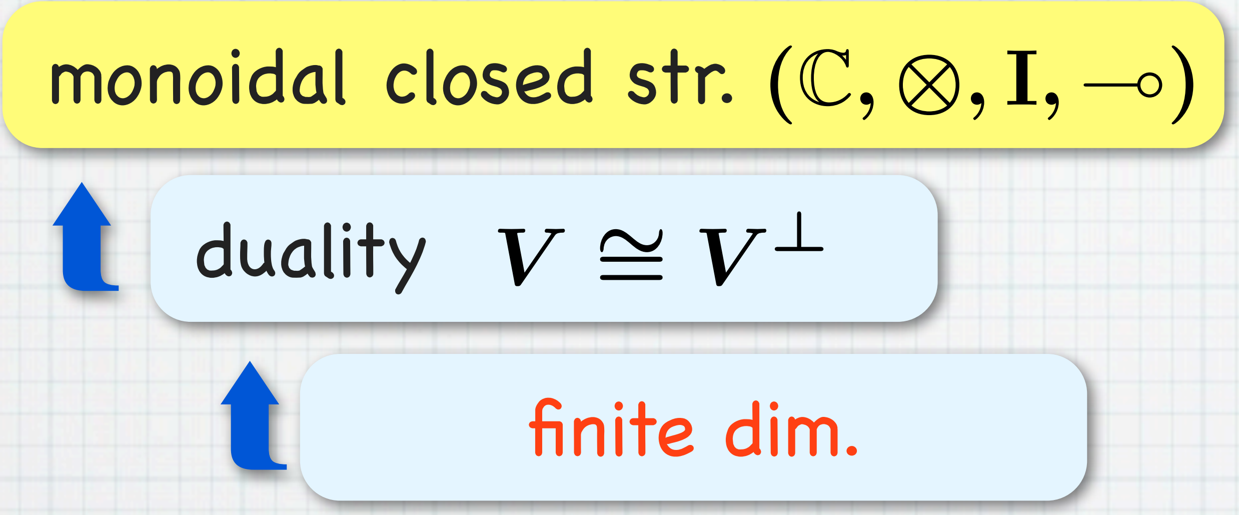

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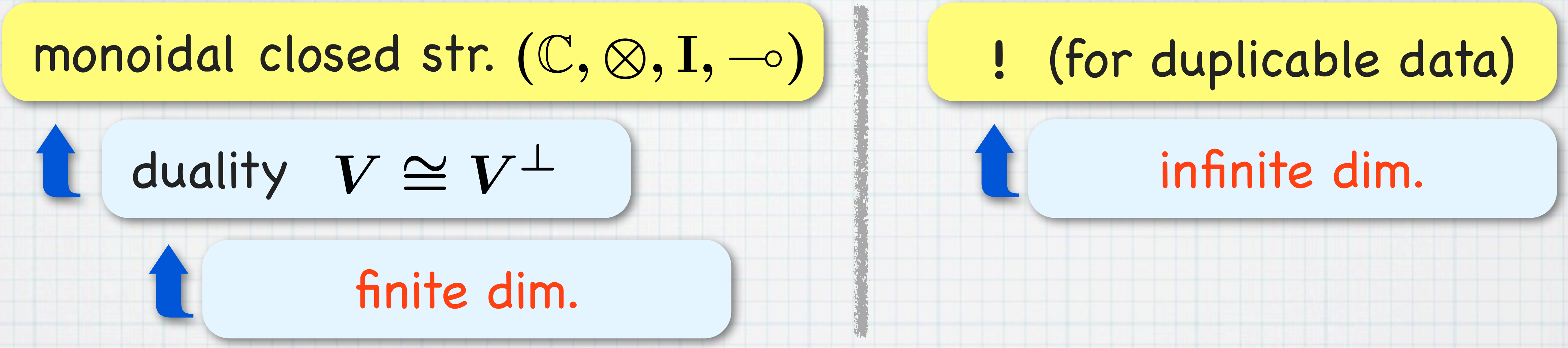

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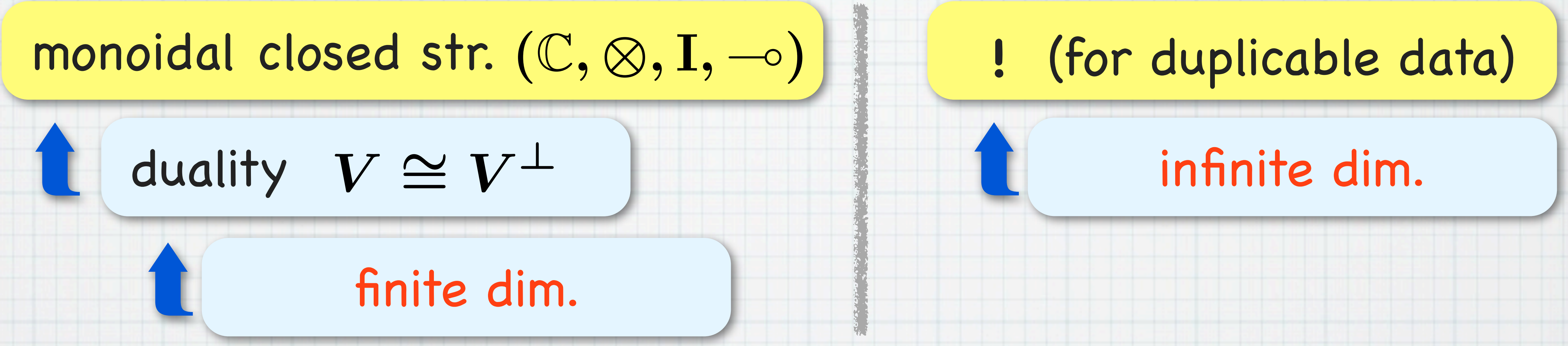

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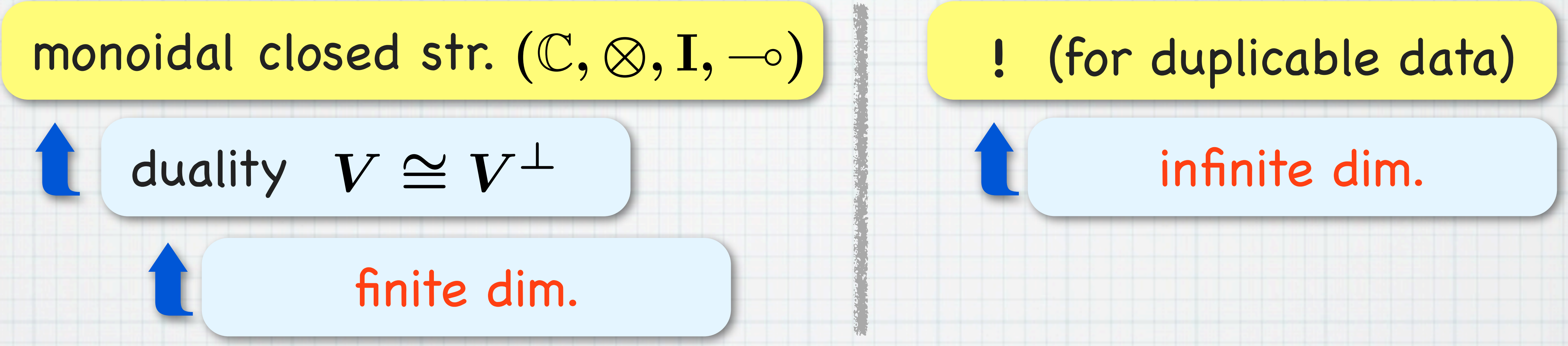

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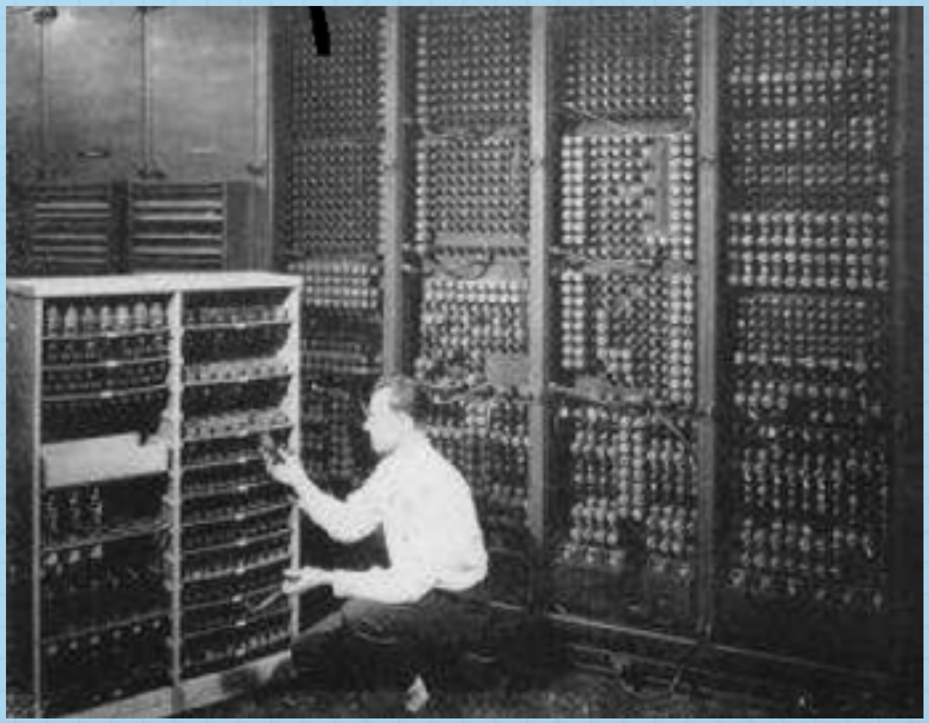
“Quantum Data, Classical Control”

Illustration by N. Hoshino

Quantum data

Classical control



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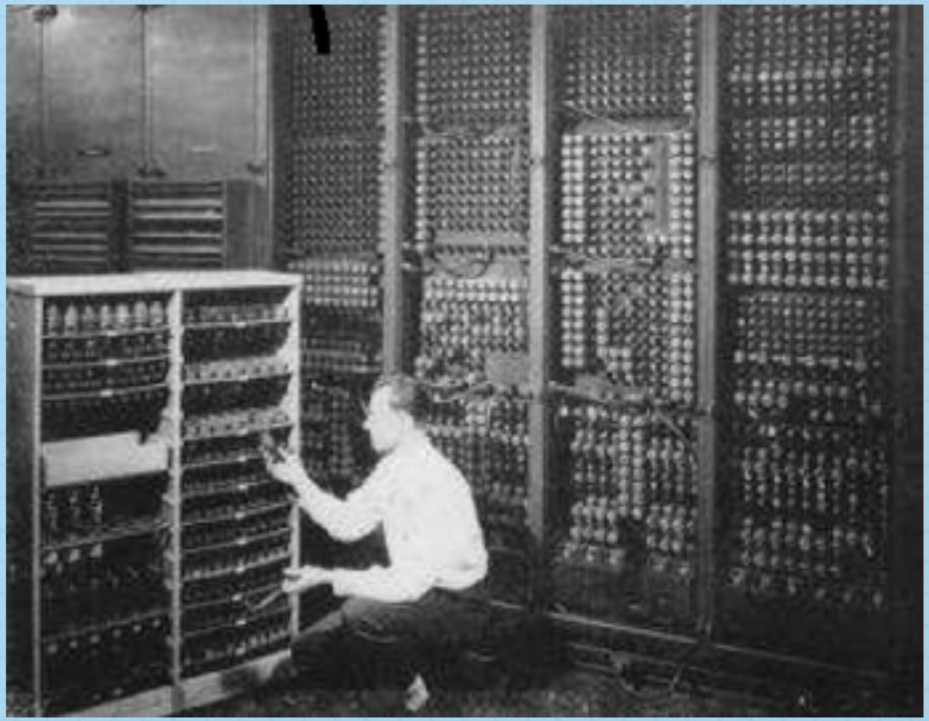

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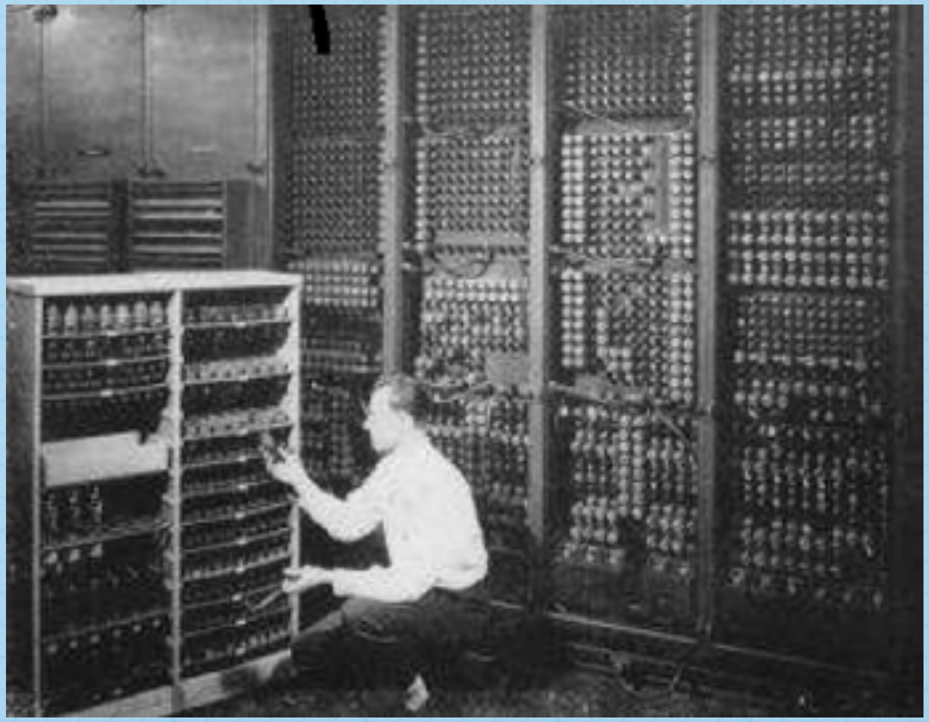

What We Do

- **GoI (Geometry of Interaction)** [Girard '89]

  An "implementation" of **classical control**

$$\mathbf{tr}(f) =$$

$$f_{XY} \sqcup \left(\coprod_{n \in \mathbb{N}} f_{ZY} \circ (f_{ZZ})^{n} \circ f_{XZ} \right)$$

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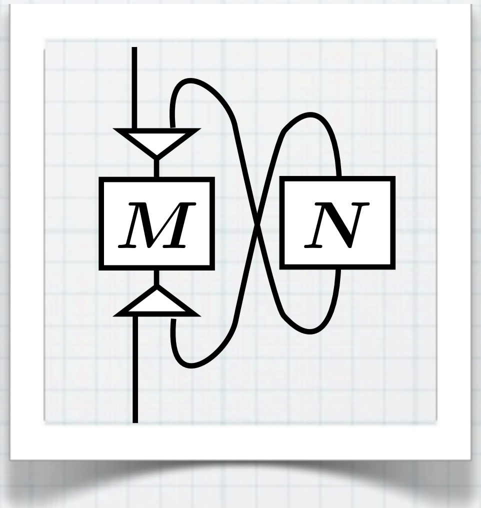
  Its categorical axiomatics



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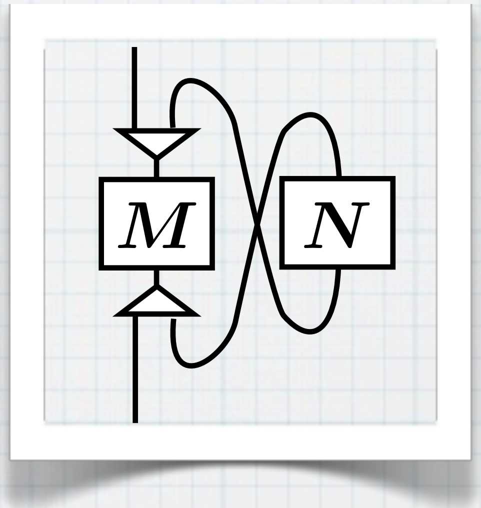
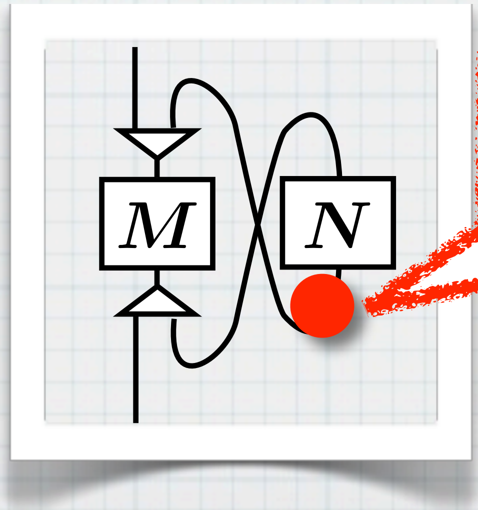


- We add a **quantum layer** to GoI
  - ➜ "Quantum data, classical control"
  - Used: theory of coalgebra

    [Hasuo, Jacobs, Sokolova '07] [Jacobs '10]



quantum state

Part 2

The Categorical GoI Workflow

GoI:
Geometry of Interaction

- J.-Y. Girard, at Logic Colloquium '88

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- **<u>But</u>** I'm no linear logician!

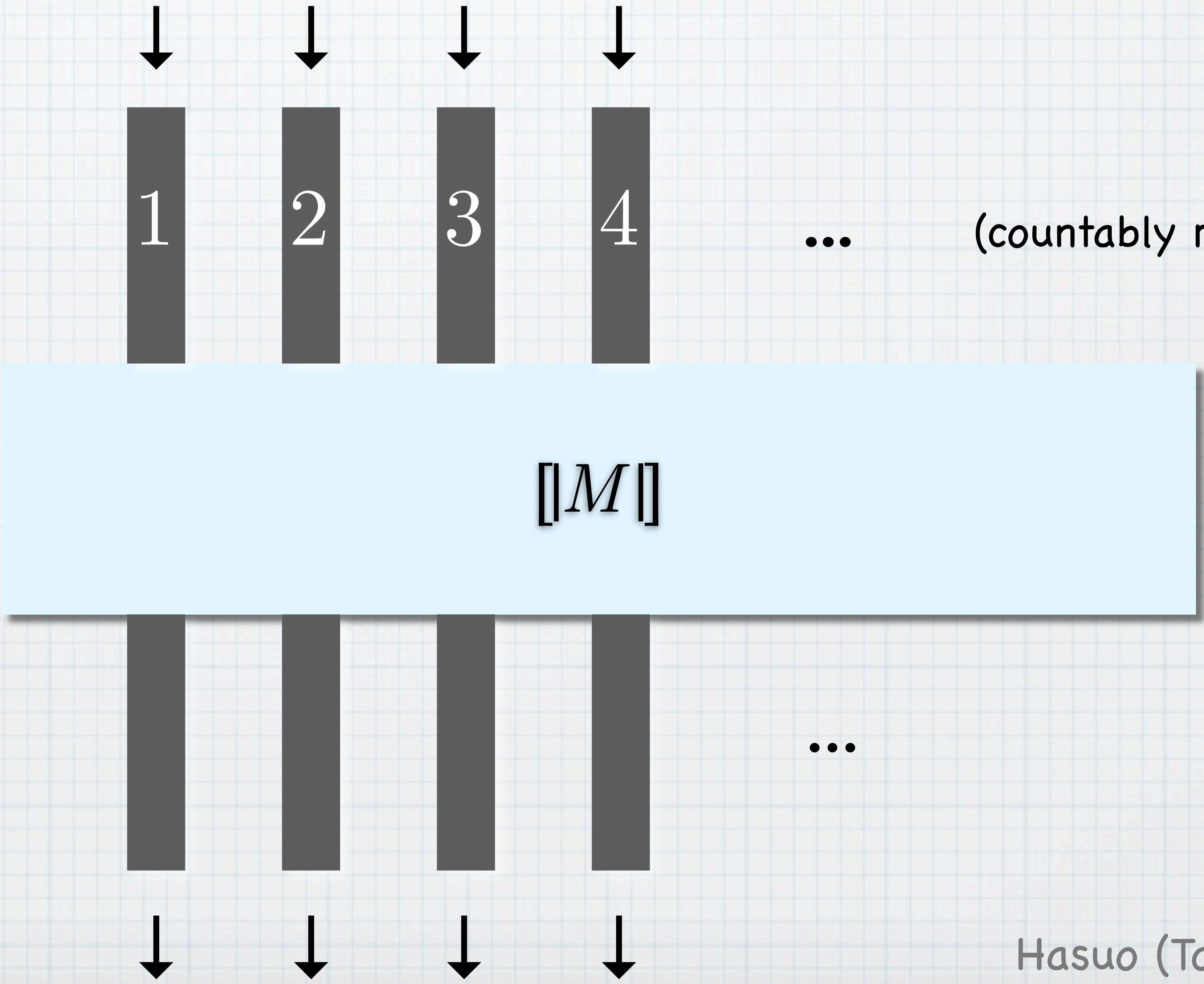
GoI:
Geometry of Interaction

- J.-Y. Girard, at Logic Colloquium '88
- **But** I'm no linear logician!
- In this talk:
  - Its categorical formulation
    [Abramsky, Haghverdi, Scott '02]
  - "The GoI Animation"

The GoI Animation

$$[\![\boldsymbol{M}]\!] \;=\; (\,\mathbb{N} \rightharpoonup \mathbb{N},\ \text{a partial function}\,)$$

$$=\ \text{“piping”}$$



(countably many)

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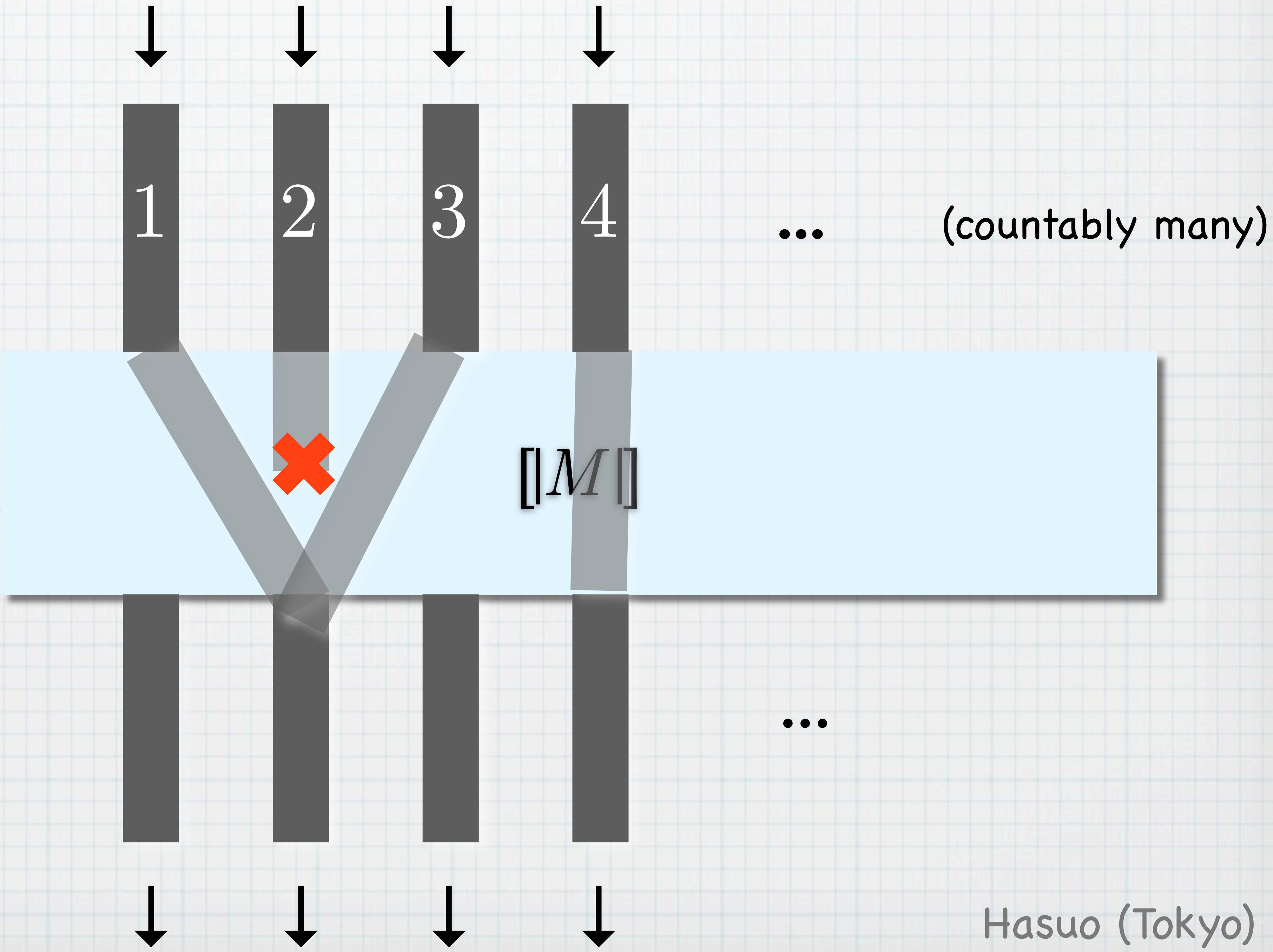

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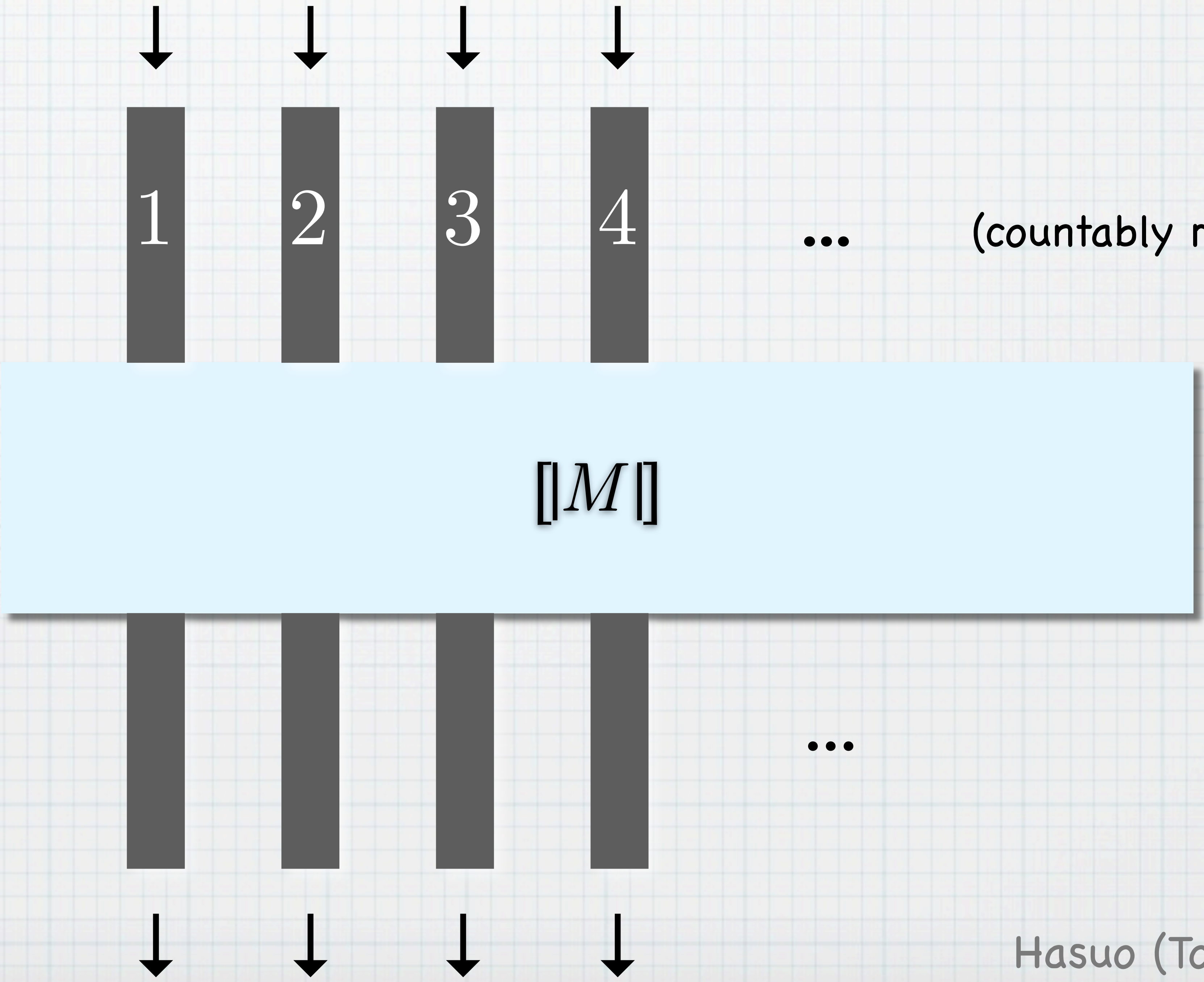

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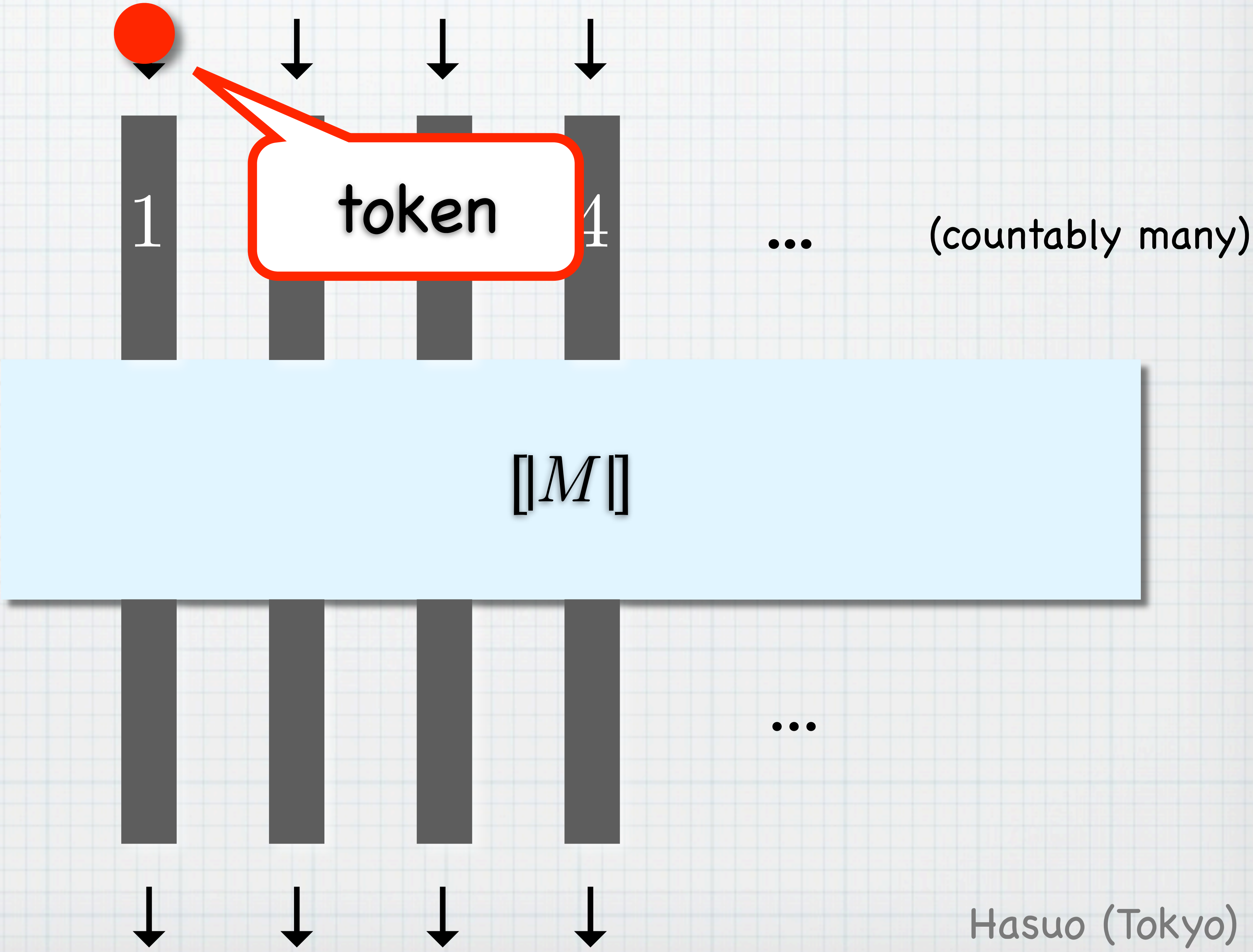

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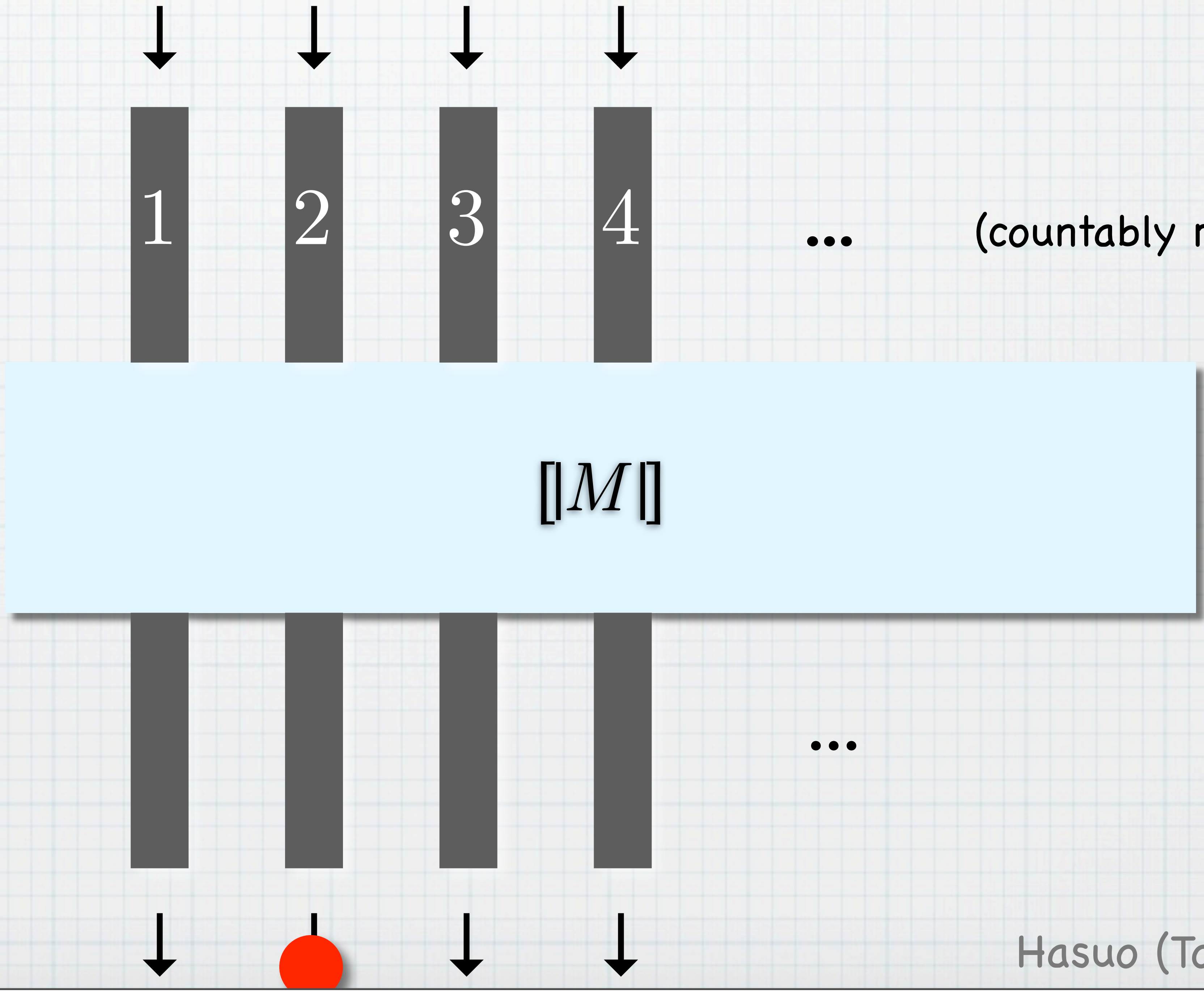

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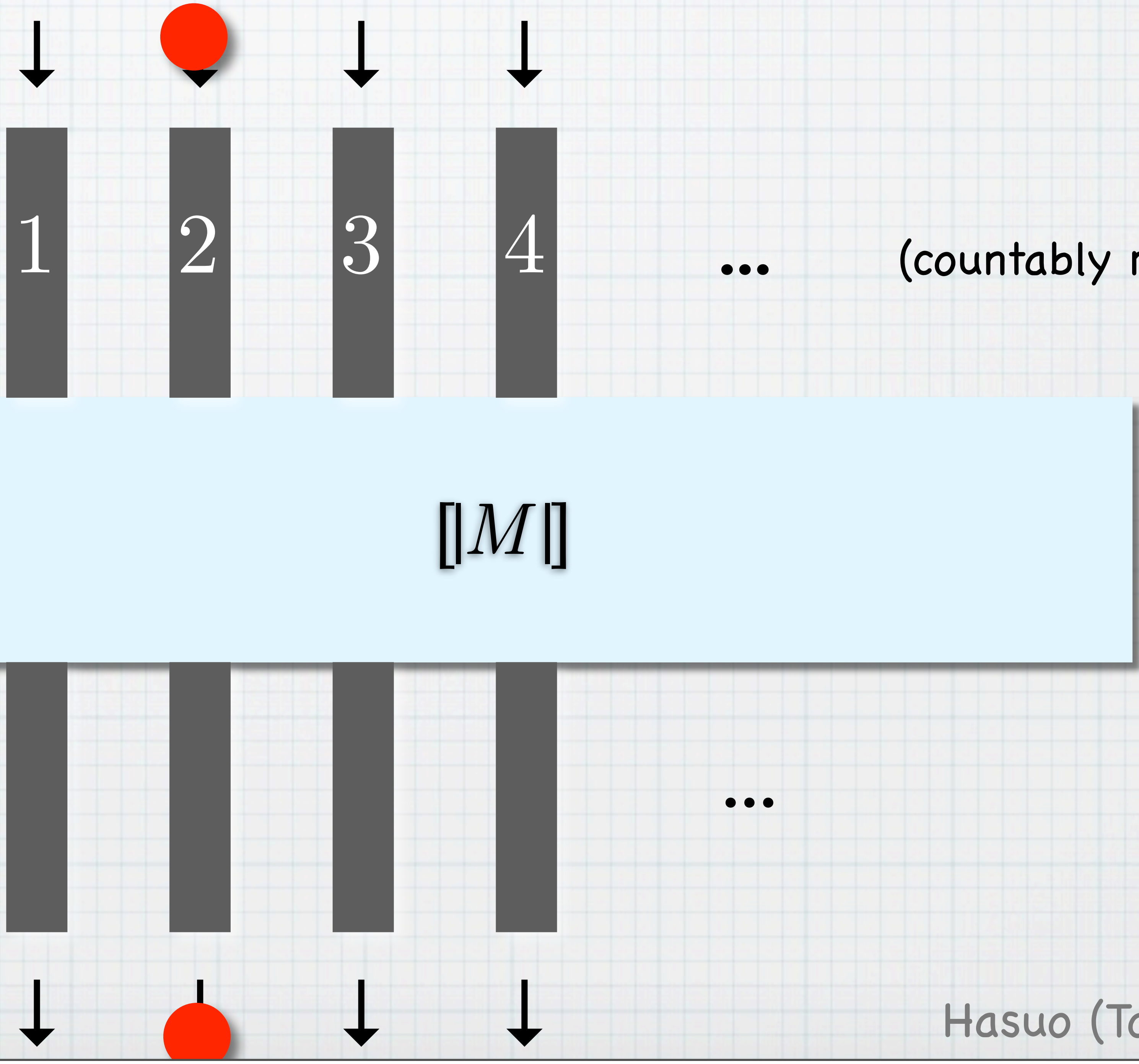

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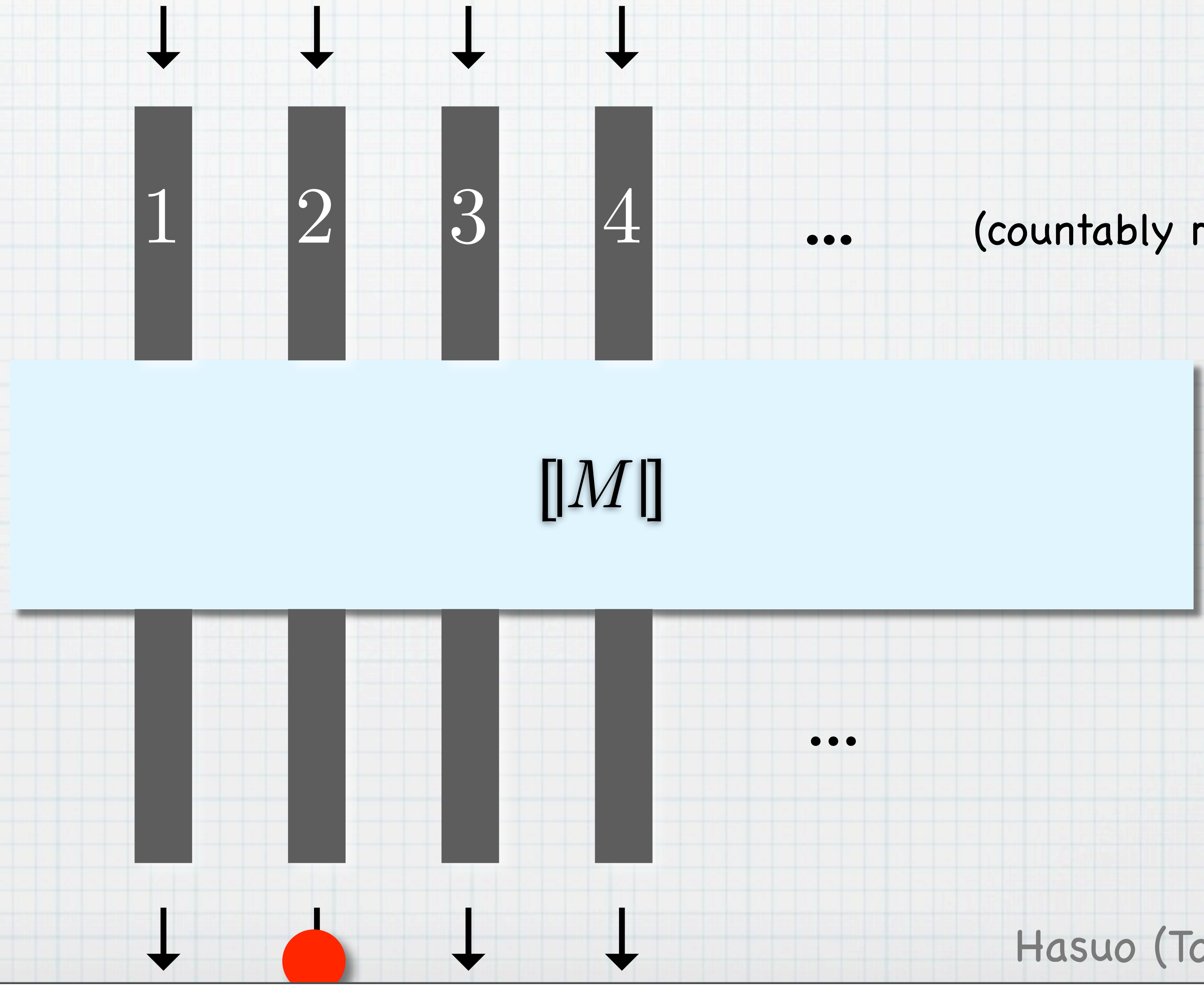

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The GoI Animation

- Function application $[\![M N]\!]$
  - by “parallel composition + hiding”

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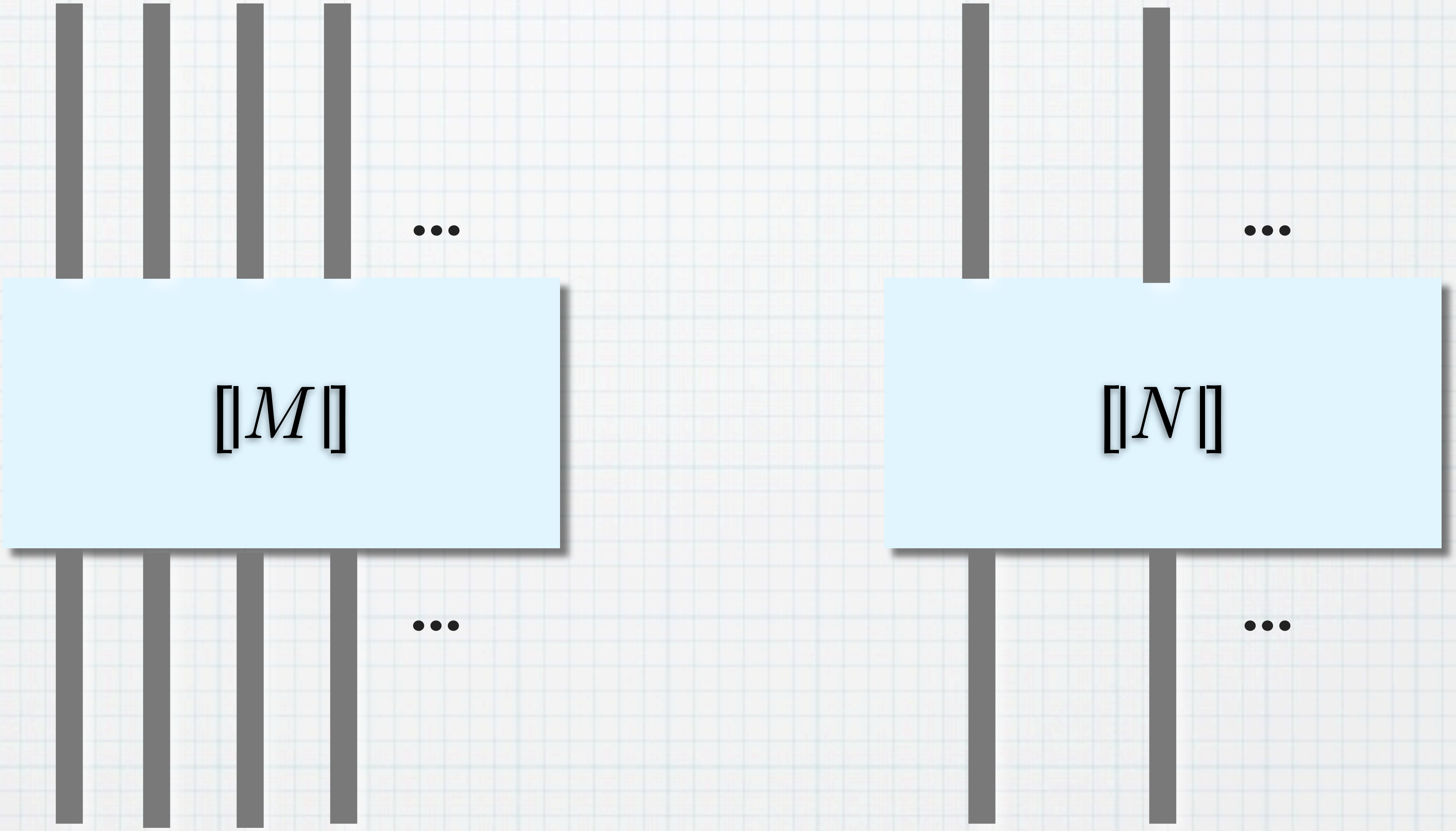
$$[\![\boldsymbol{M N}]\!] =$$



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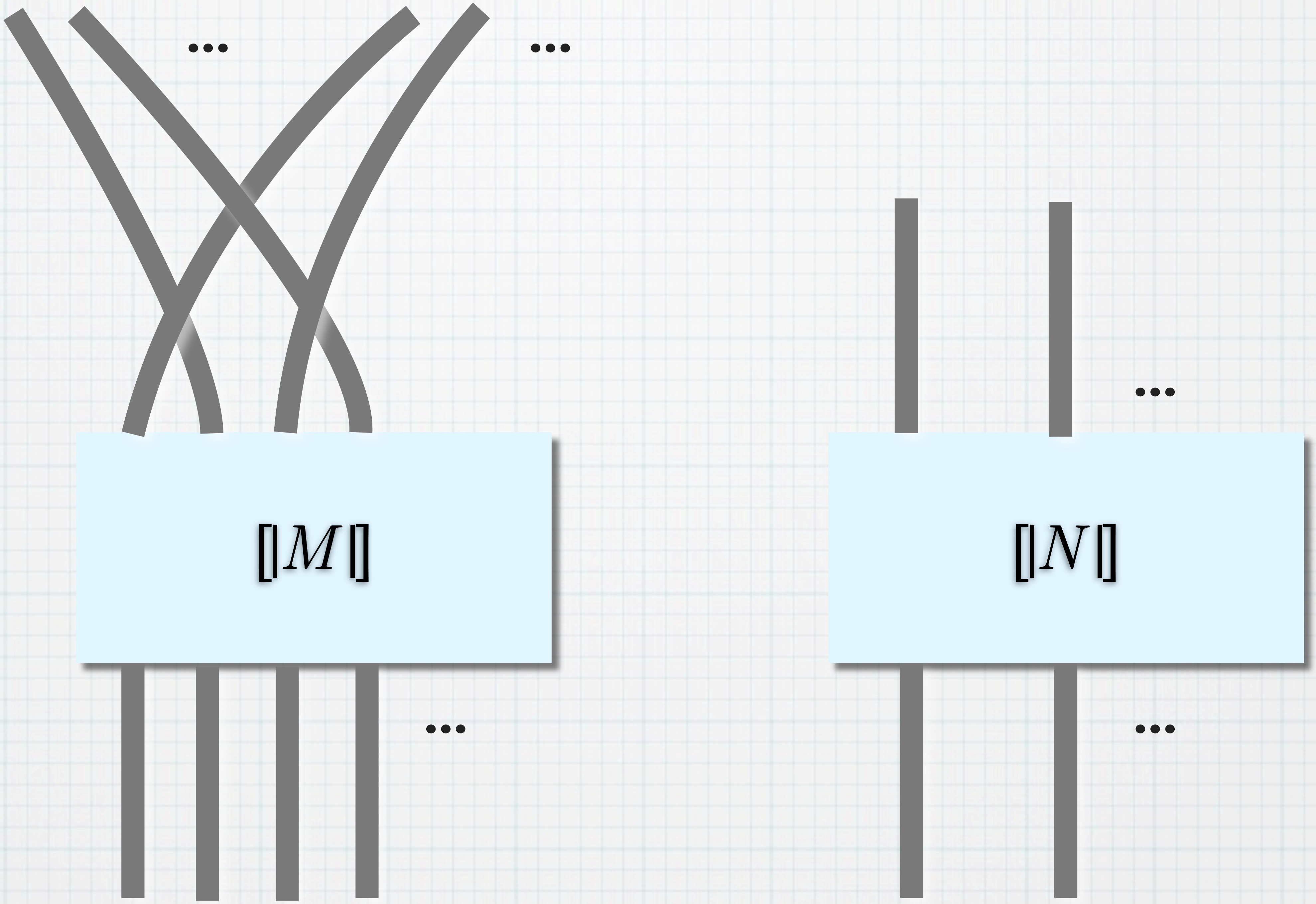

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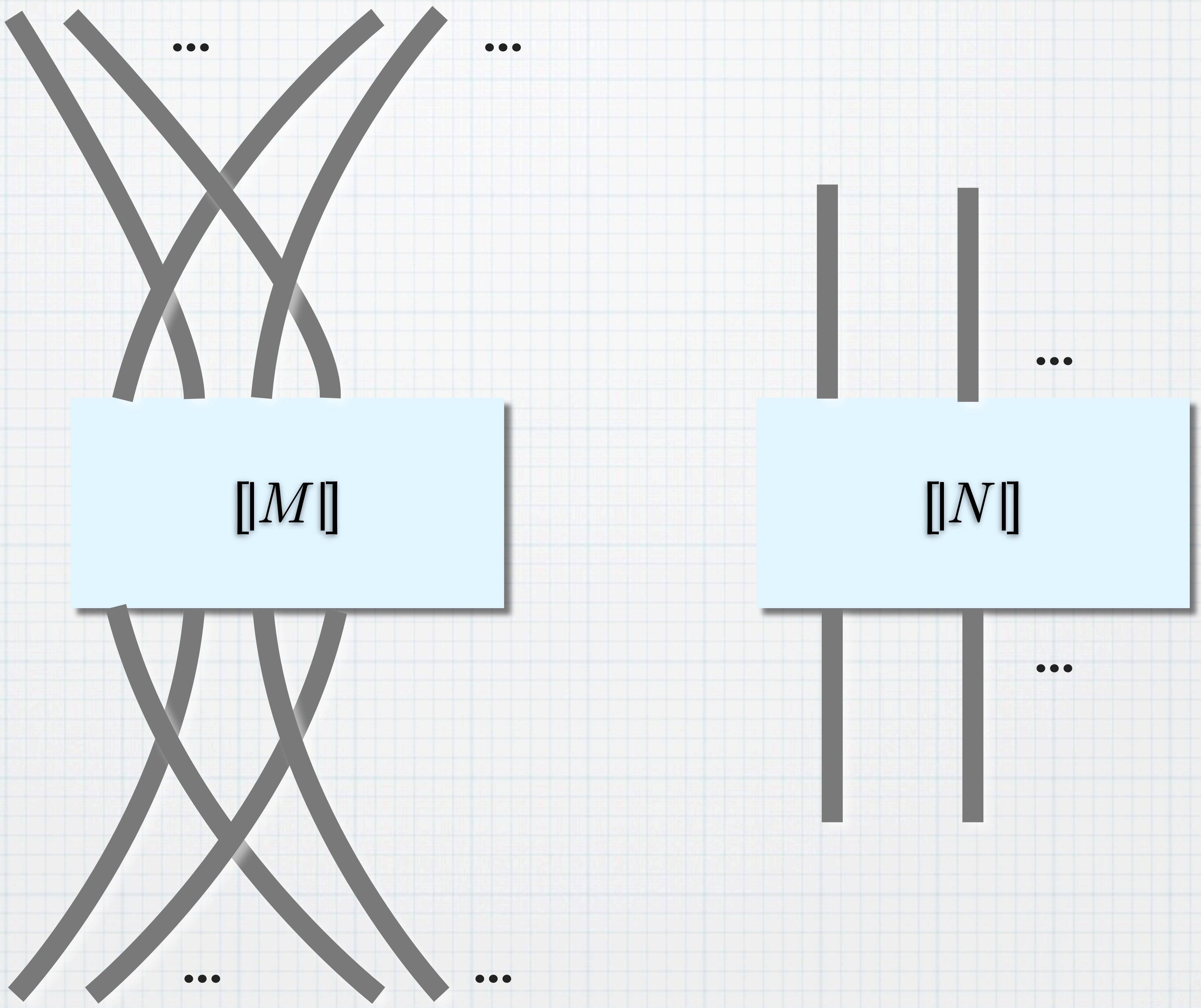
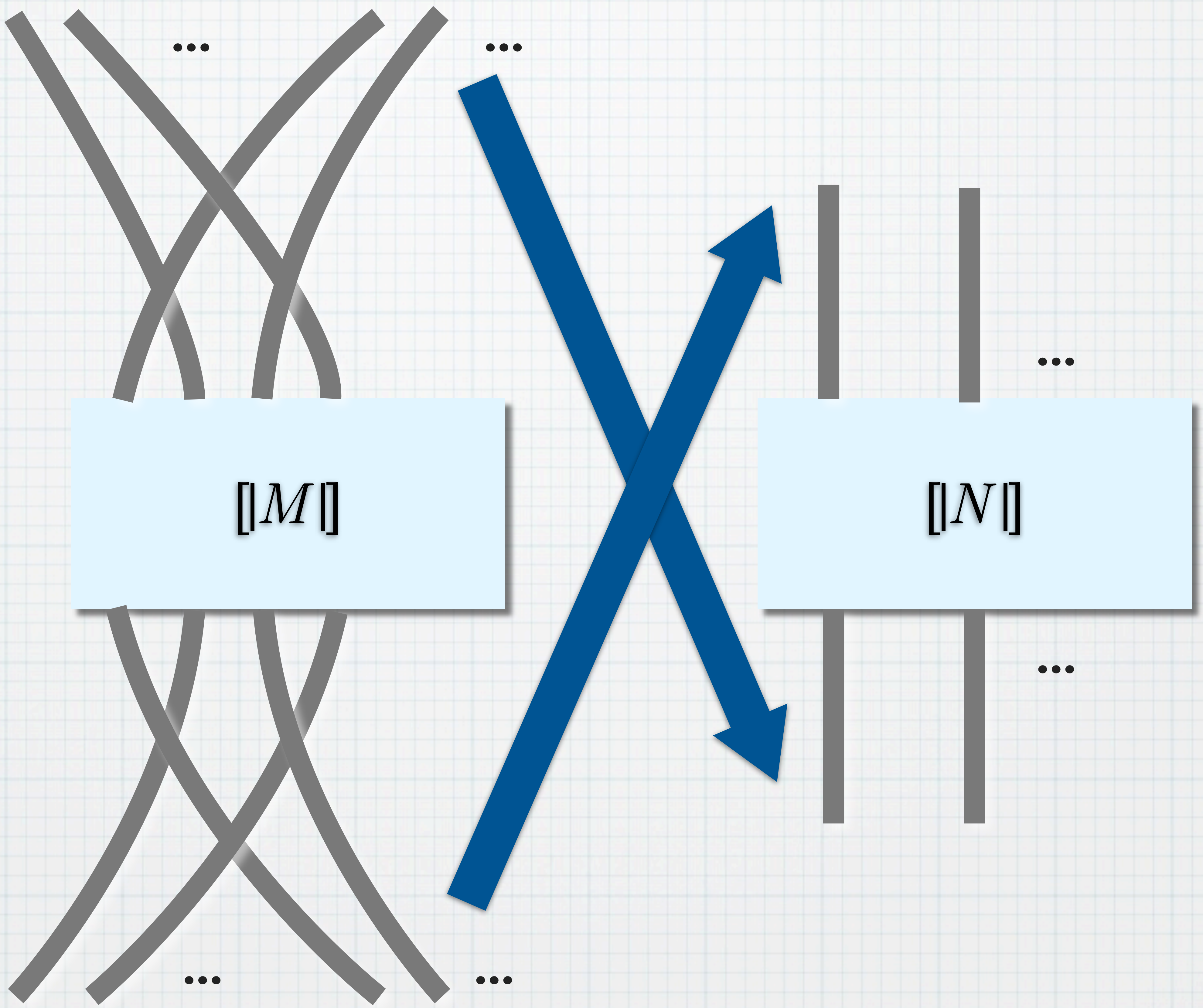
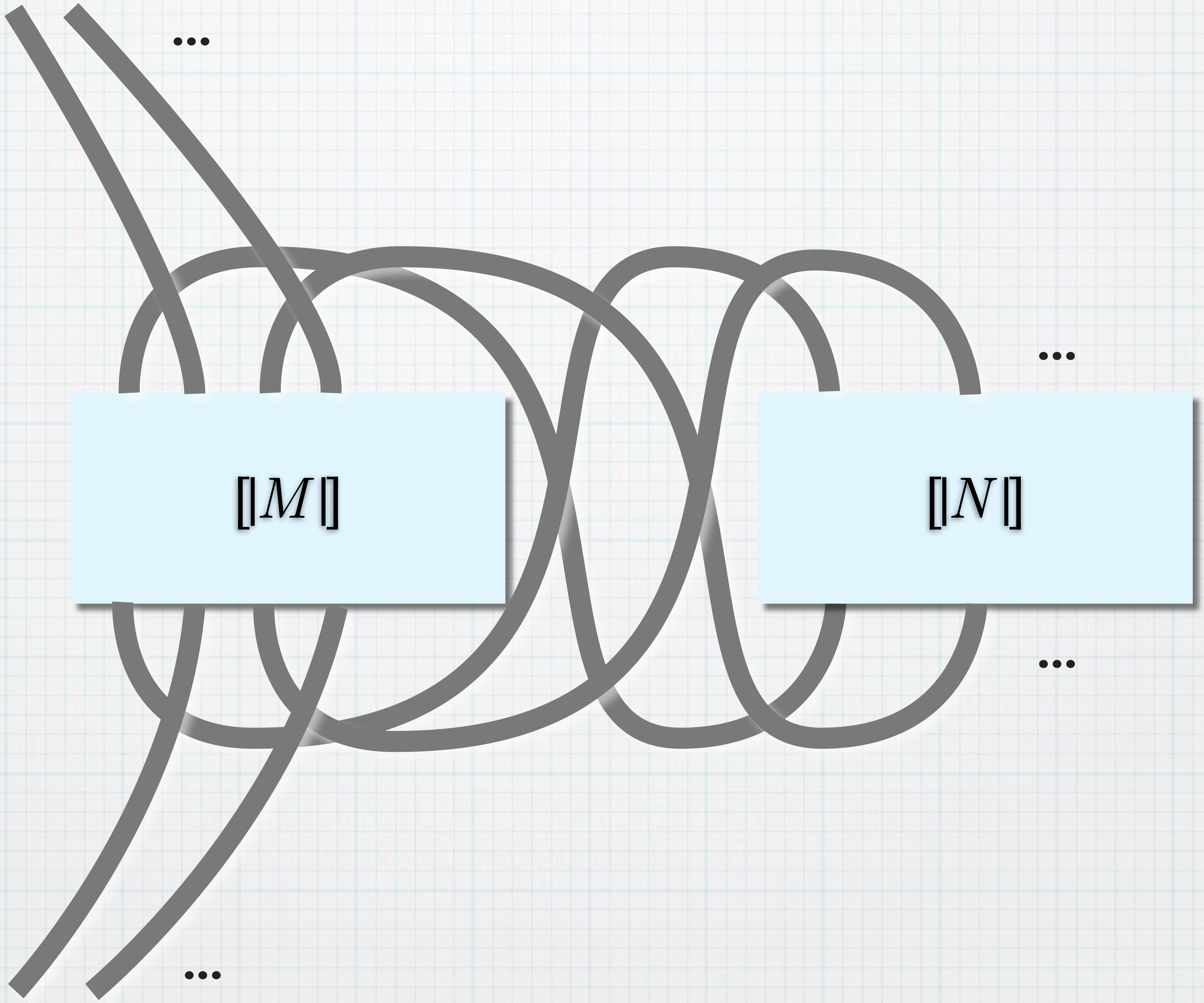


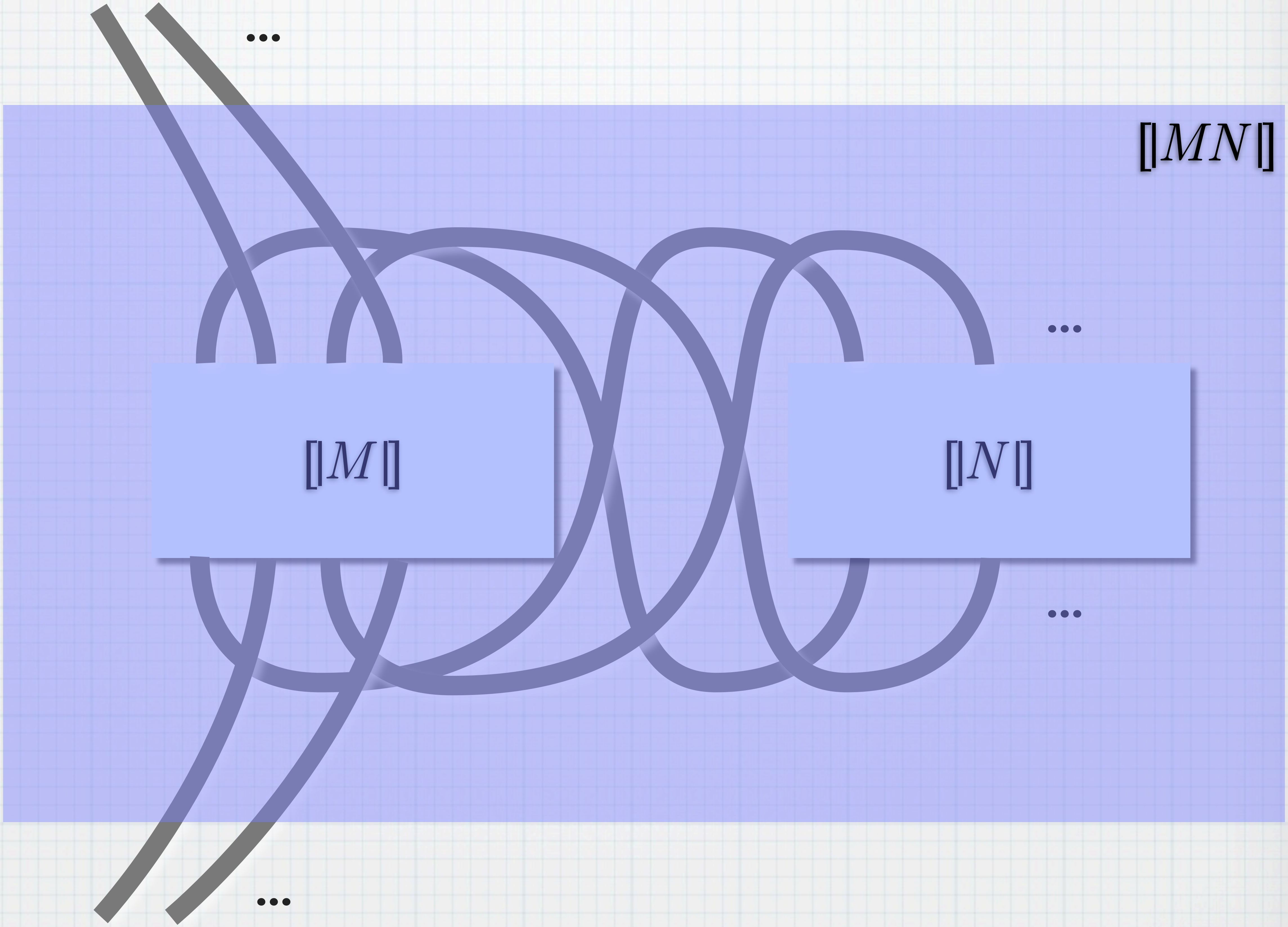
$$[\![MN]\!] =$$



$$[\![\boldsymbol{M} \boldsymbol{N}]\!] =$$

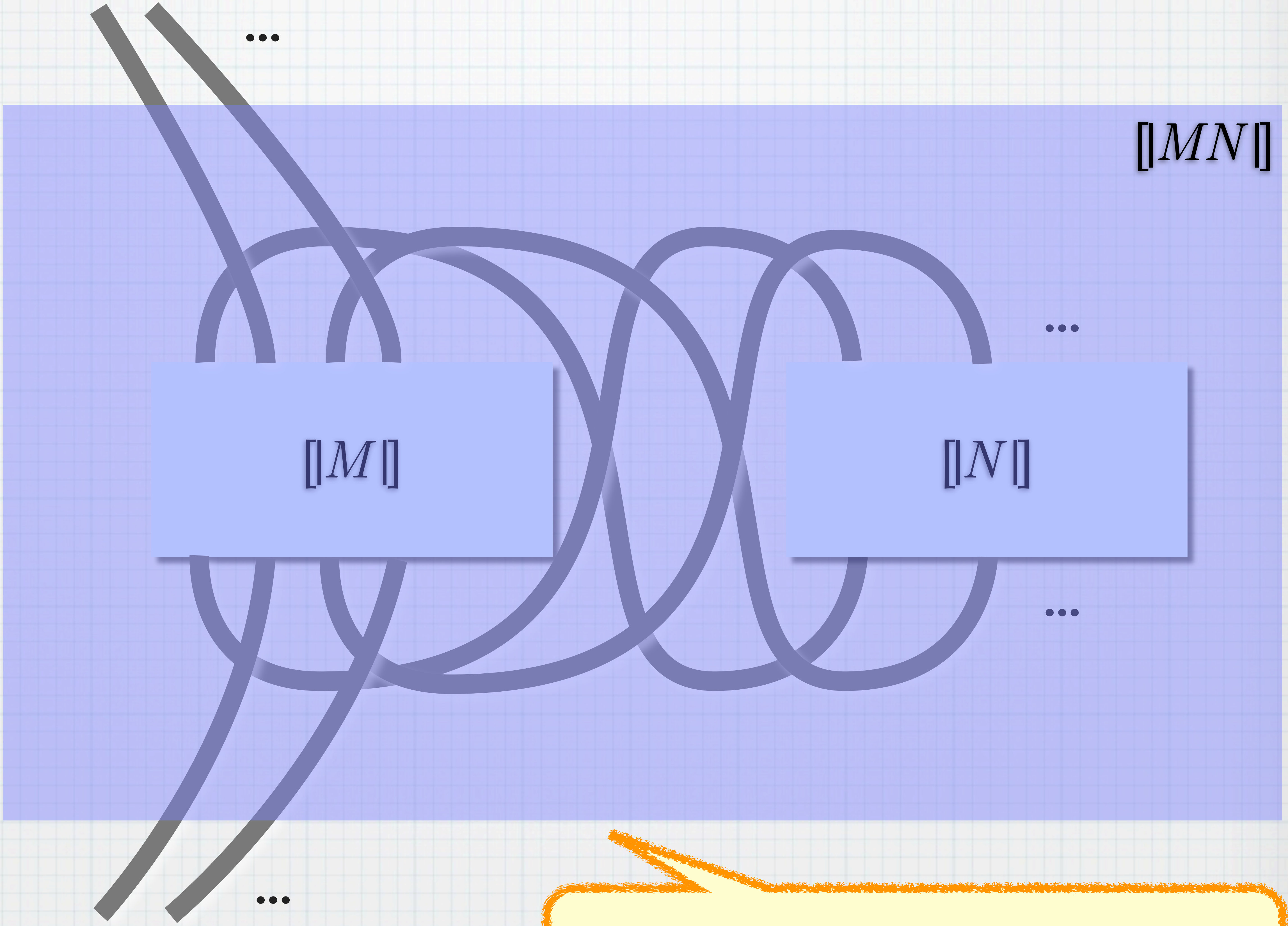


$$\llbracket \boldsymbol{M N} \rrbracket =$$


...
$\llbracket MN \rrbracket$
...
$\llbracket M \rrbracket$
$\llbracket N \rrbracket$
...
...

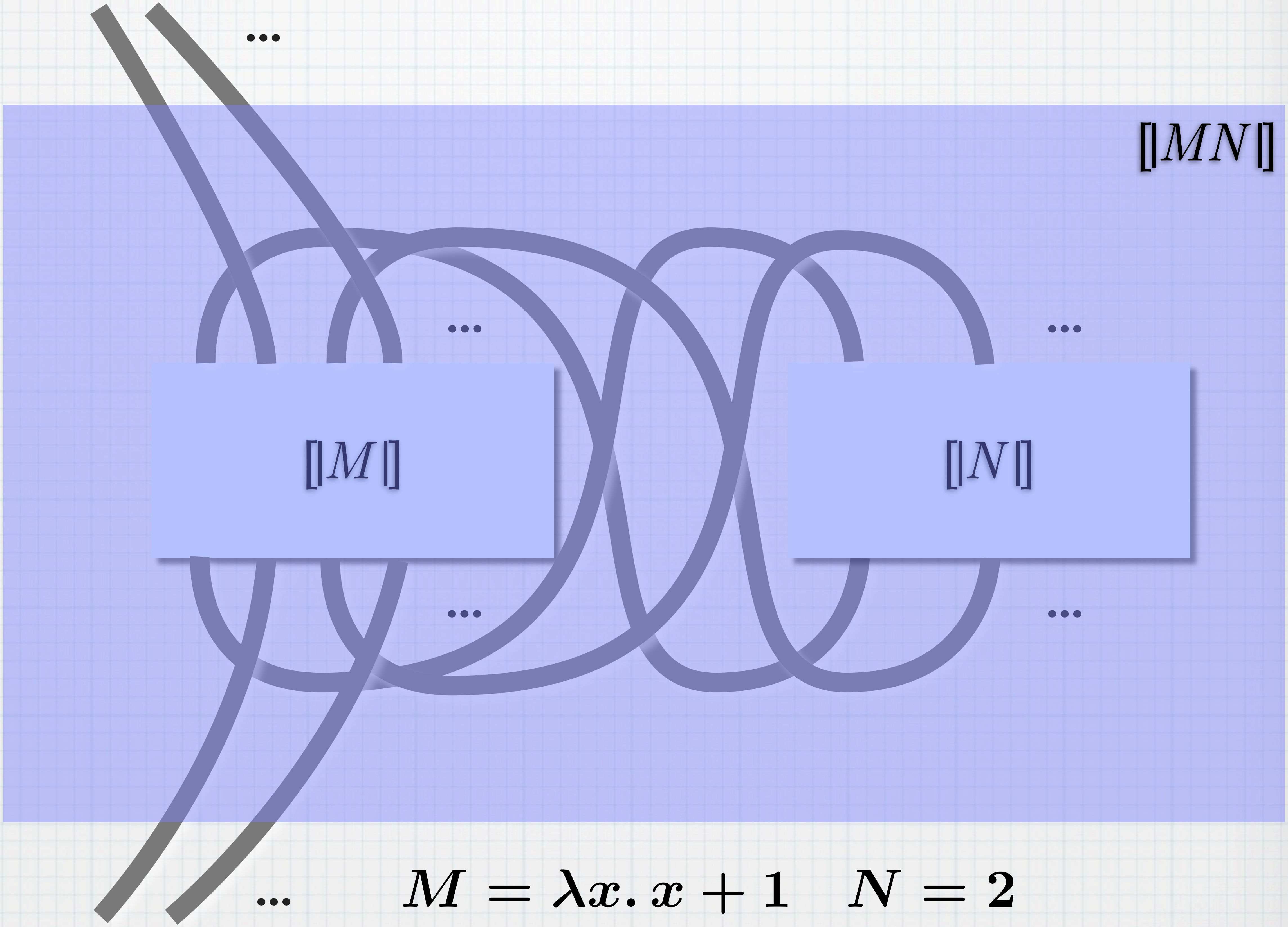
$[\![MN]\!]$

$=$



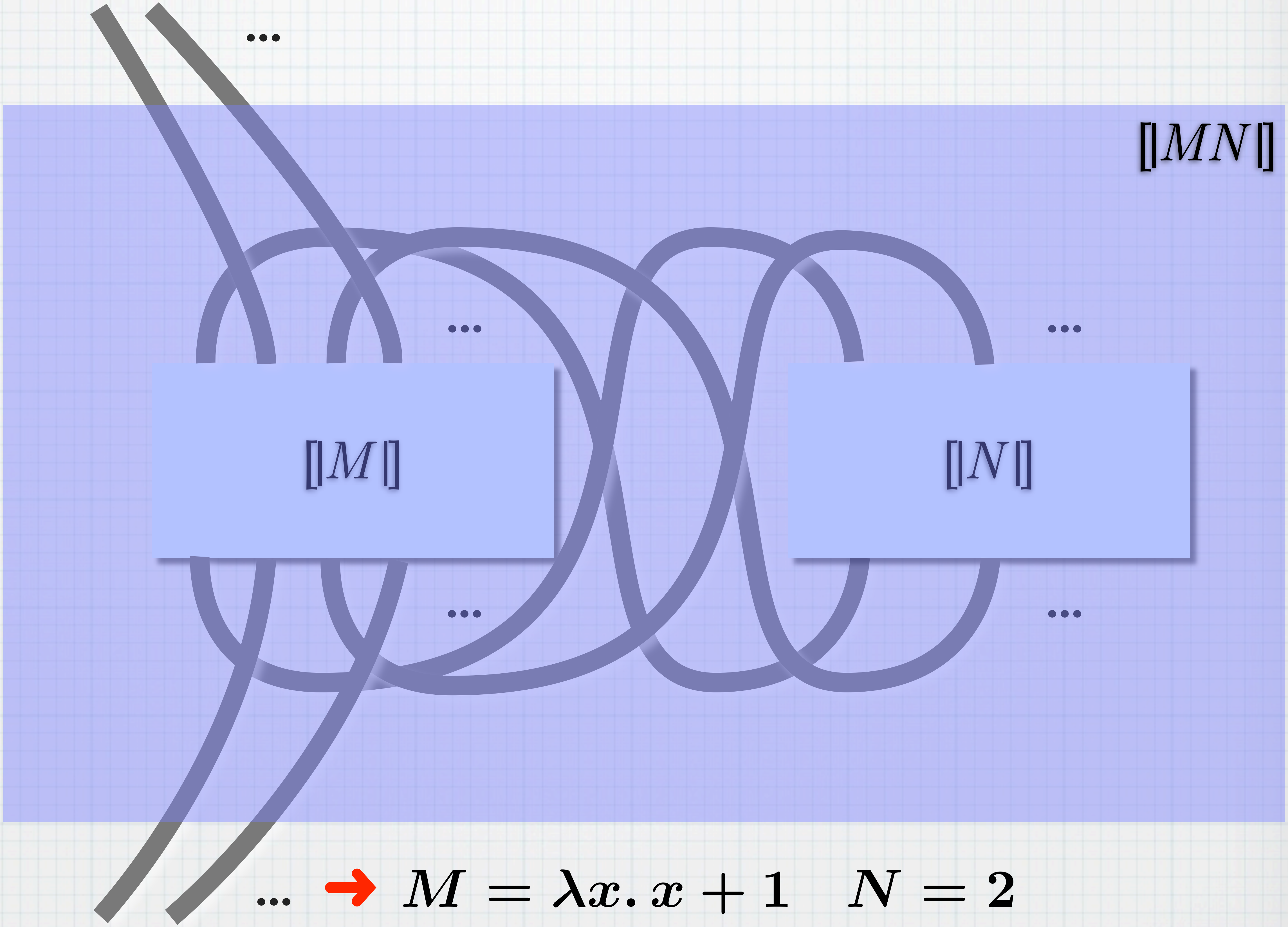
"parallel composition + hiding"
(cf. games)

$$[\![\boldsymbol{MN}]\!] =$$



$$\begin{array}{ll} \boldsymbol{M = \lambda x.\, x + 1} & \boldsymbol{N = 2} \\ \boldsymbol{M = \lambda x.\, 1} & \boldsymbol{N = 2} \\ \boldsymbol{M = \lambda f.\, f 1} & \boldsymbol{N = \lambda x.\, (x + 1)} \end{array}$$

$$[\![\boldsymbol{MN}]\!] =$$



➜ $\boldsymbol{M = \lambda x.\, x + 1 \qquad N = 2}$

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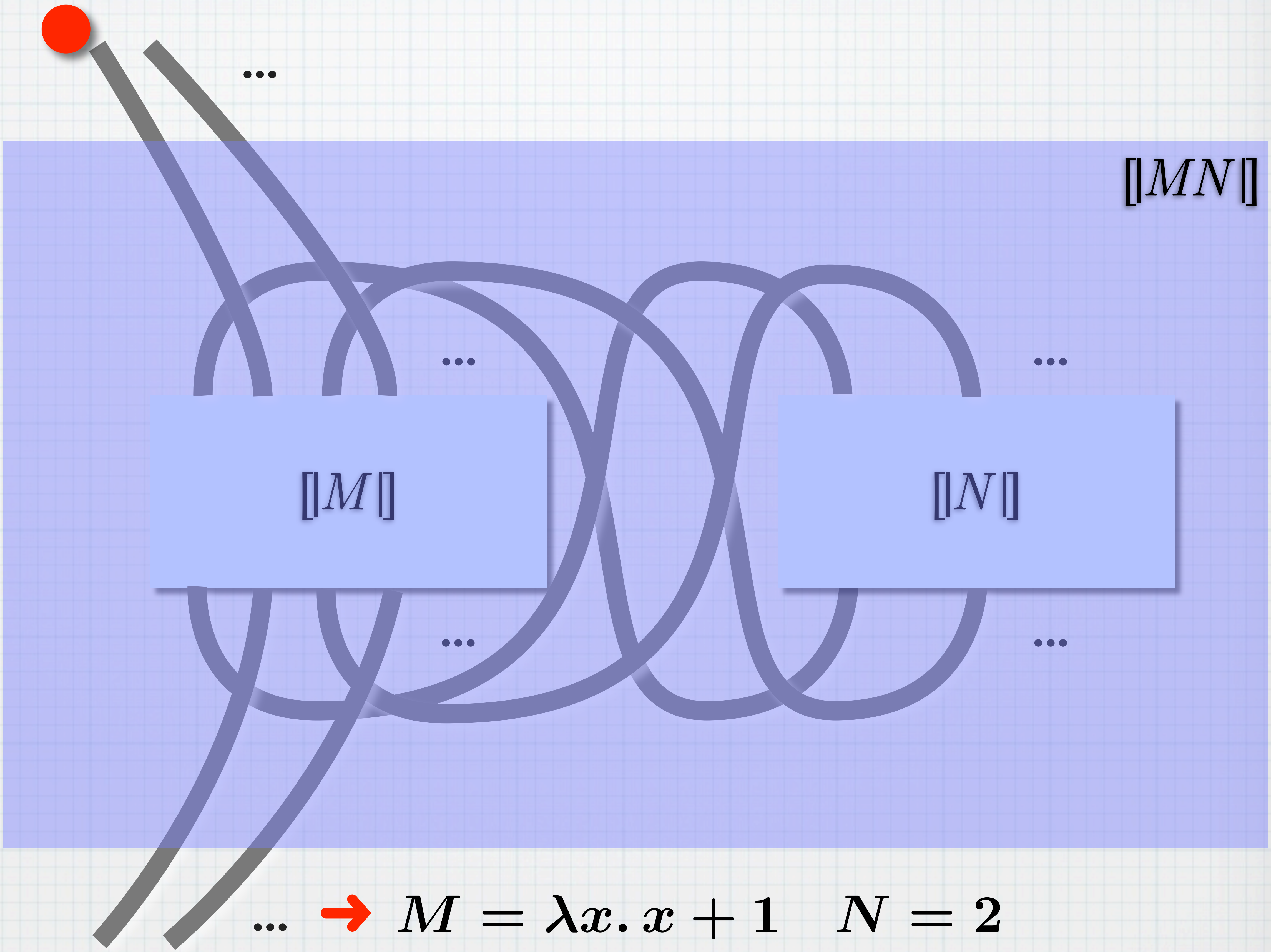

$$\begin{array}{llll} \rightarrow & M = \lambda x.\, x+1 & & N = 2 \\ & M = \lambda x.\, 1 & & N = 2 \\ & M = \lambda f.\, f 1 & & N = \lambda x.\, (x+1) \end{array}$$

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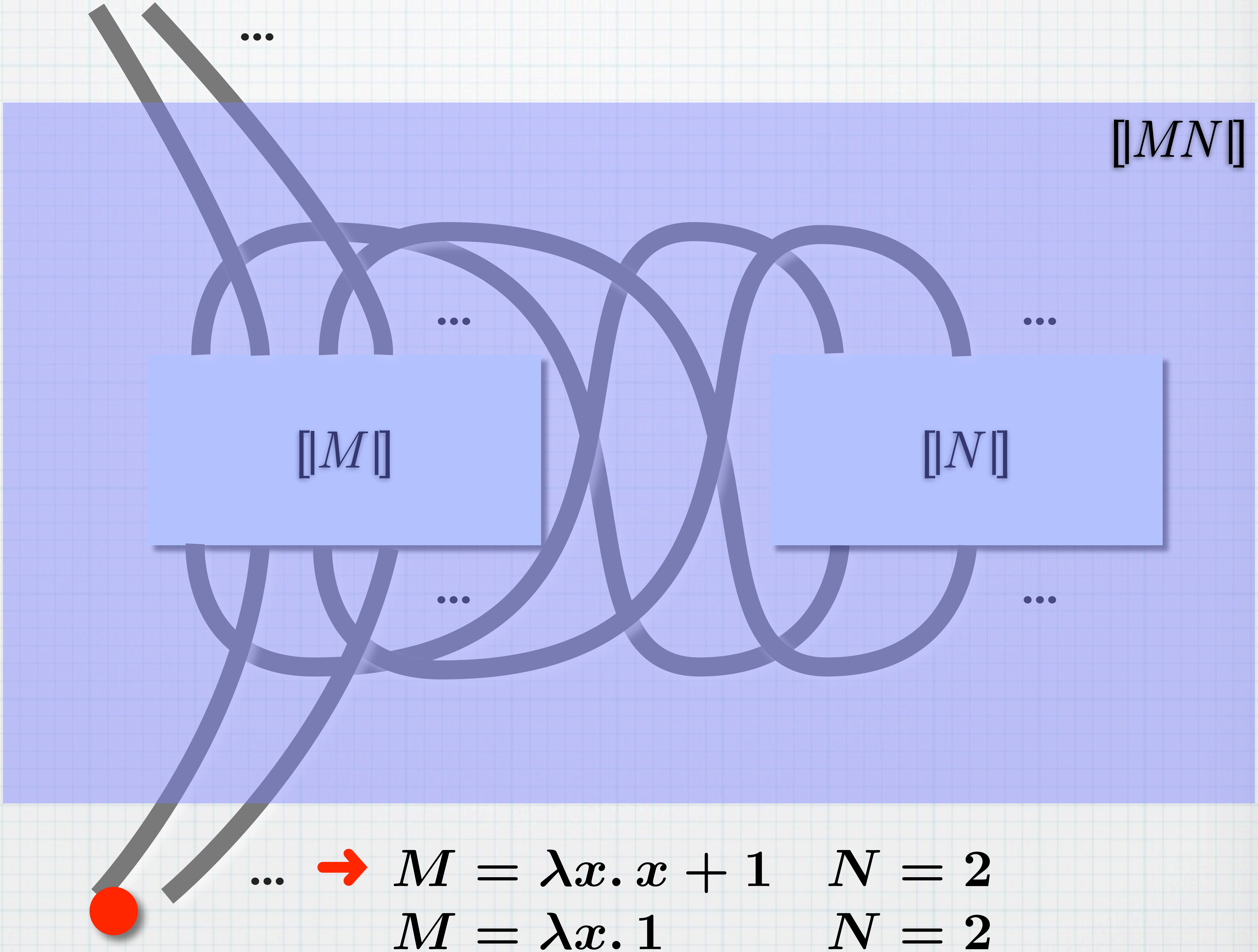
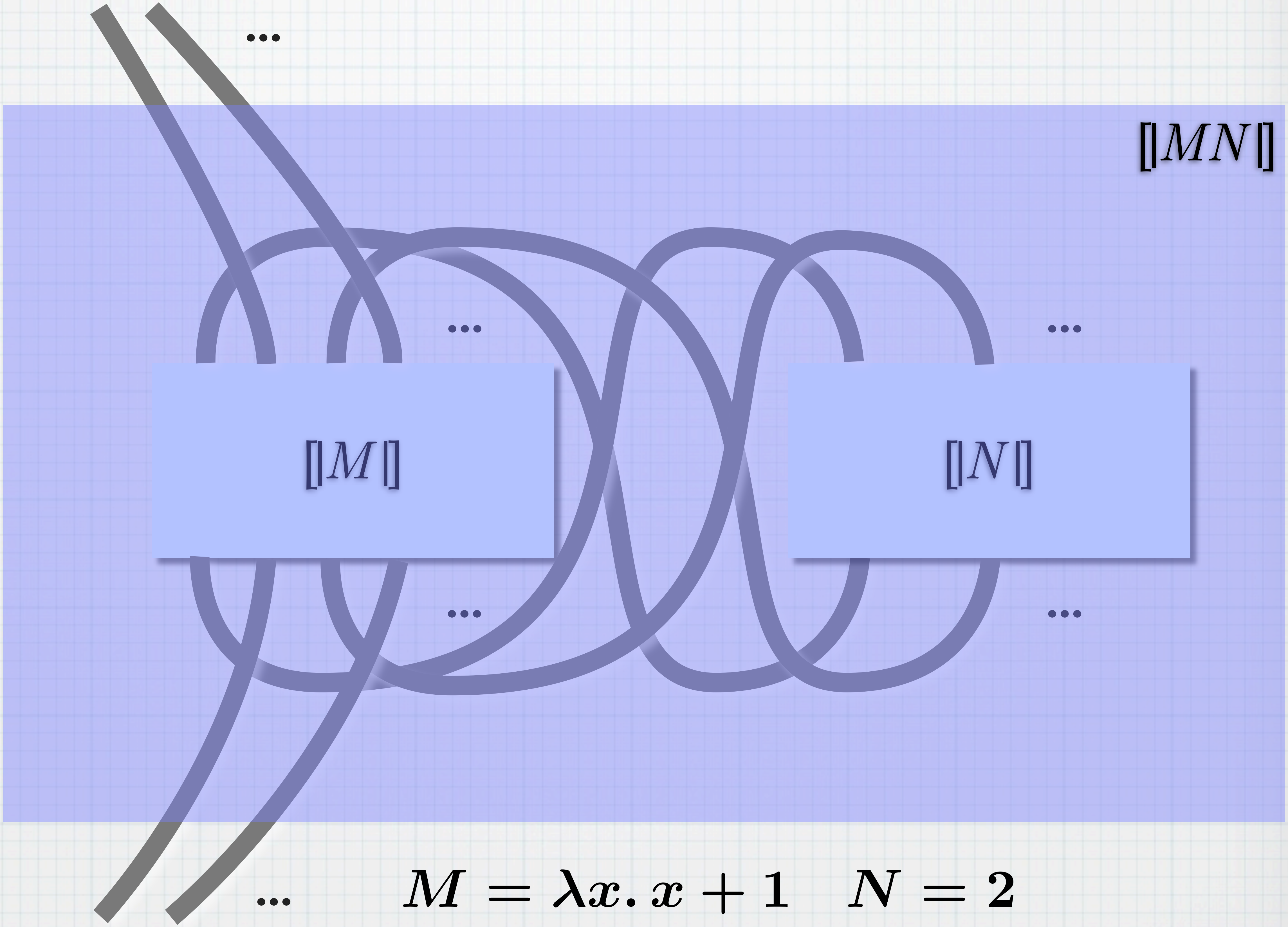


$\rightarrow$ $M = \lambda x.\, x + 1$ $N = 2$

$M = \lambda x.\, 1$ $N = 2$

$M = \lambda f.\, f 1$ $N = \lambda x.\, (x + 1)$

$$[\![\boldsymbol{MN}]\!] =$$



$$\begin{aligned}
\boldsymbol{M} &= \boldsymbol{\lambda x.\, x + 1} & \boldsymbol{N} &= \boldsymbol{2} \\
\boldsymbol{M} &= \boldsymbol{\lambda x.\, 1} & \boldsymbol{N} &= \boldsymbol{2} \\
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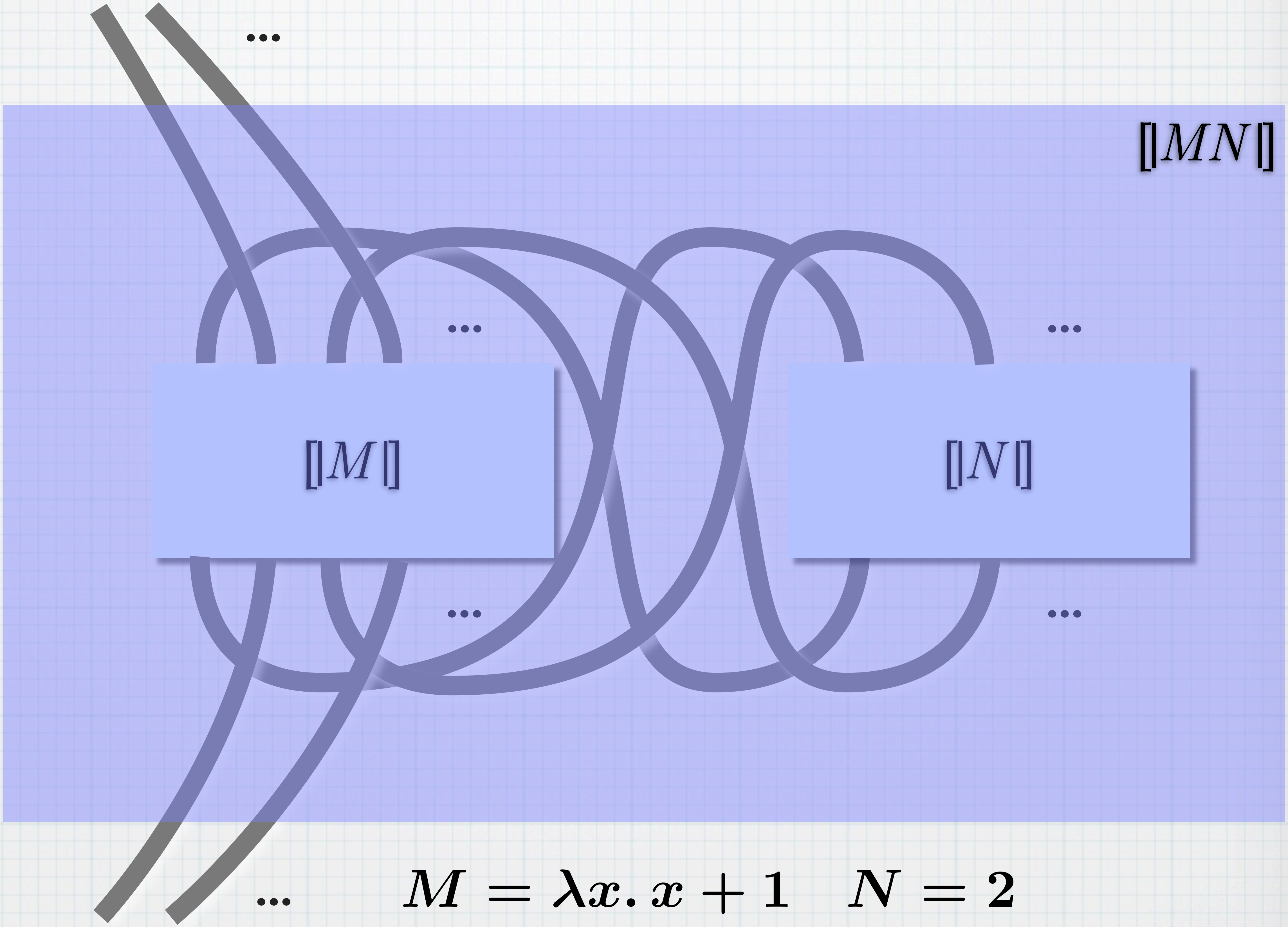
\end{aligned}$$

$[\![MN]\!] =$



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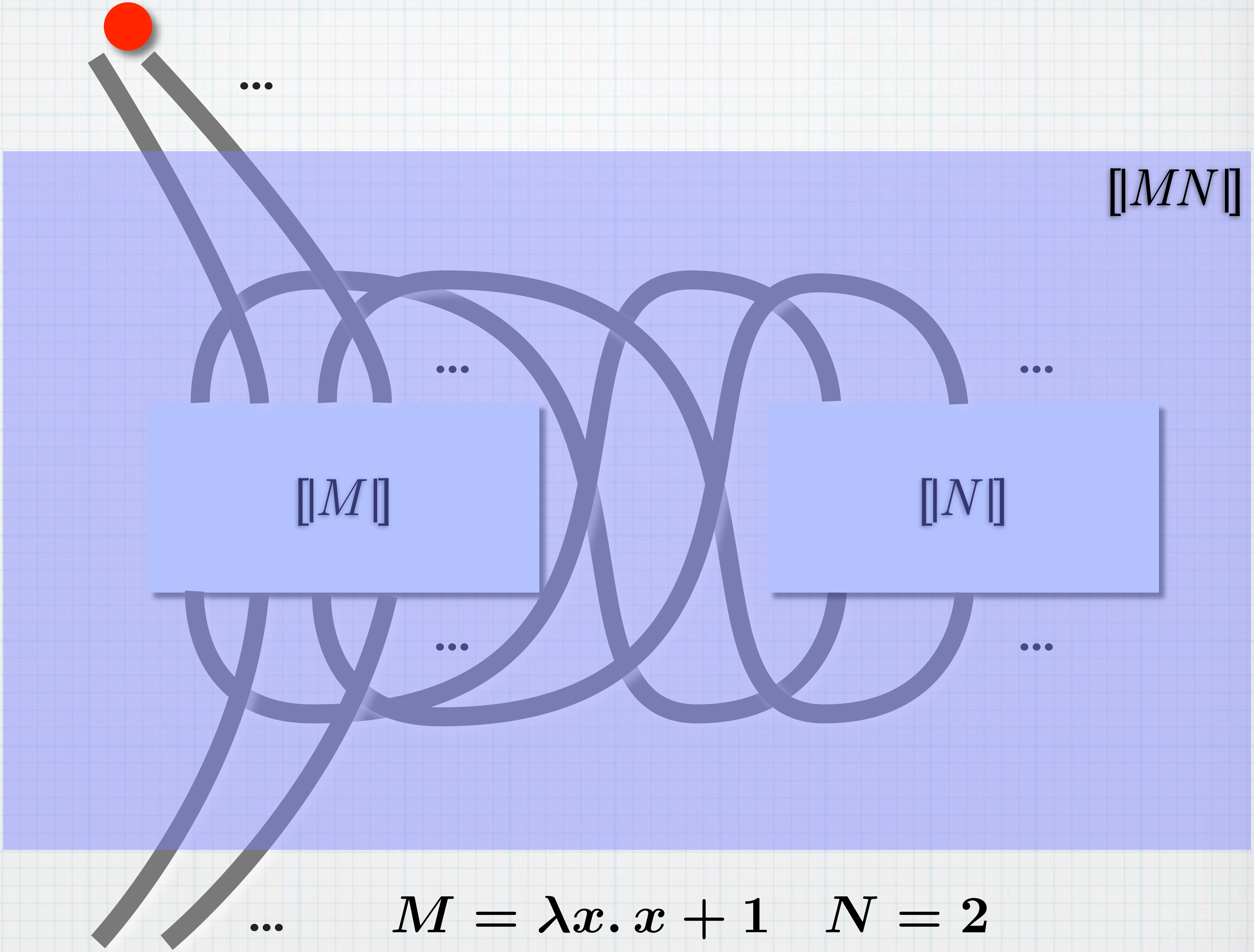

$\llbracket MN \rrbracket$
$\llbracket M \rrbracket$
$\llbracket N \rrbracket$
...
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...

$$\boldsymbol{M = \lambda x.\, x + 1 \qquad N = 2}$$

→ $\boldsymbol{M = \lambda x.\, 1 \qquad N = 2}$

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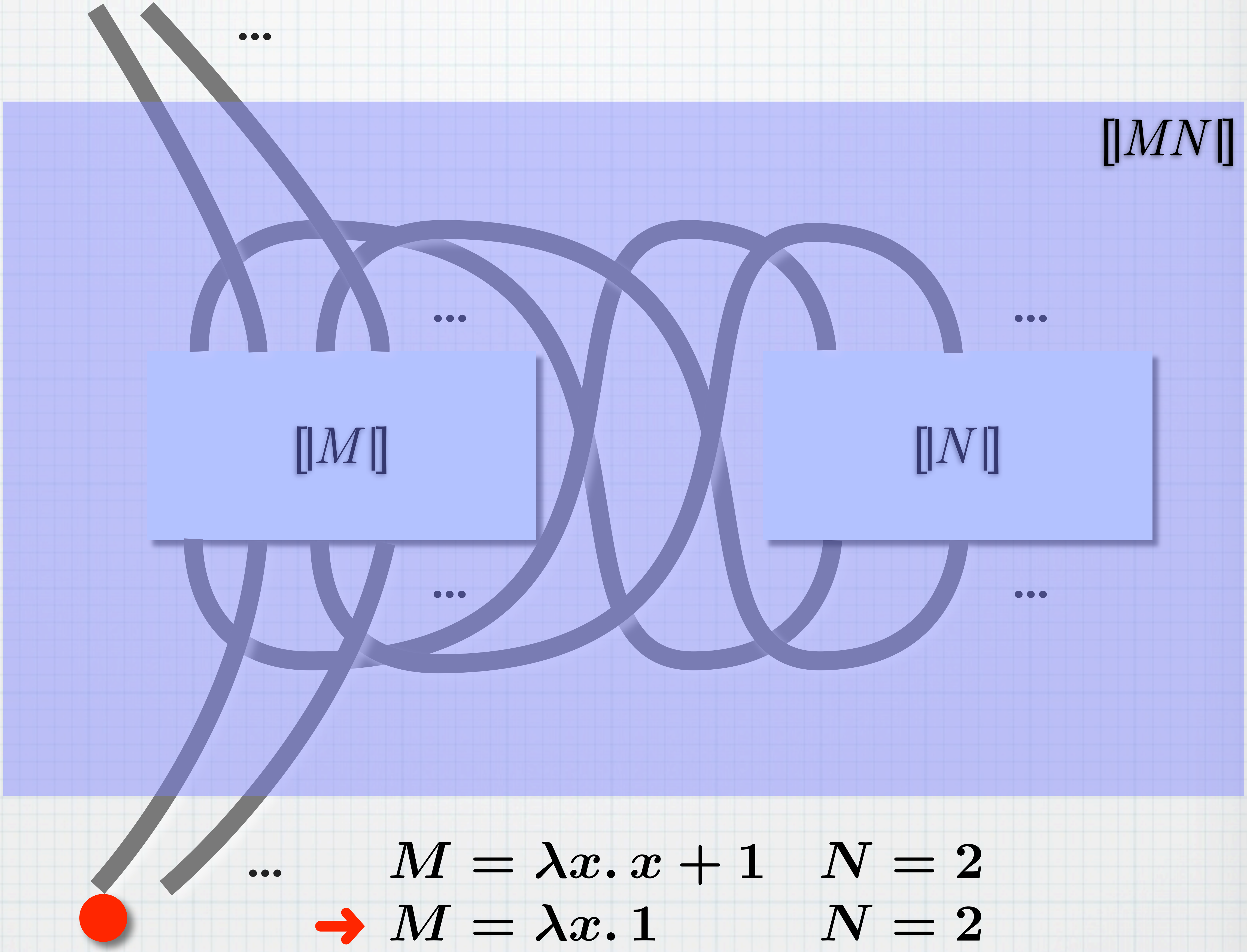
$$[\![MN]\!] =$$



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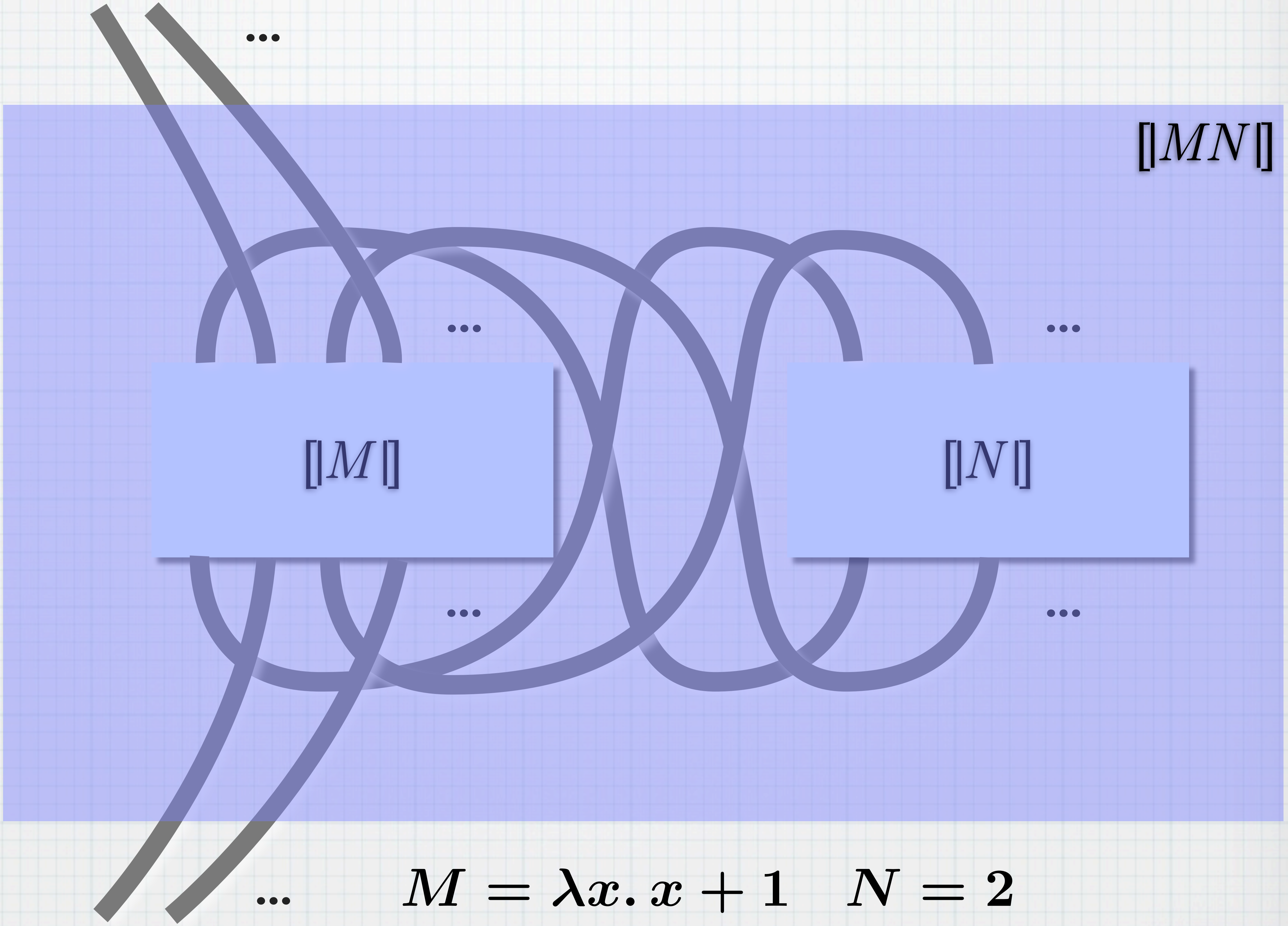
$[\![MN]\!]$

$=$



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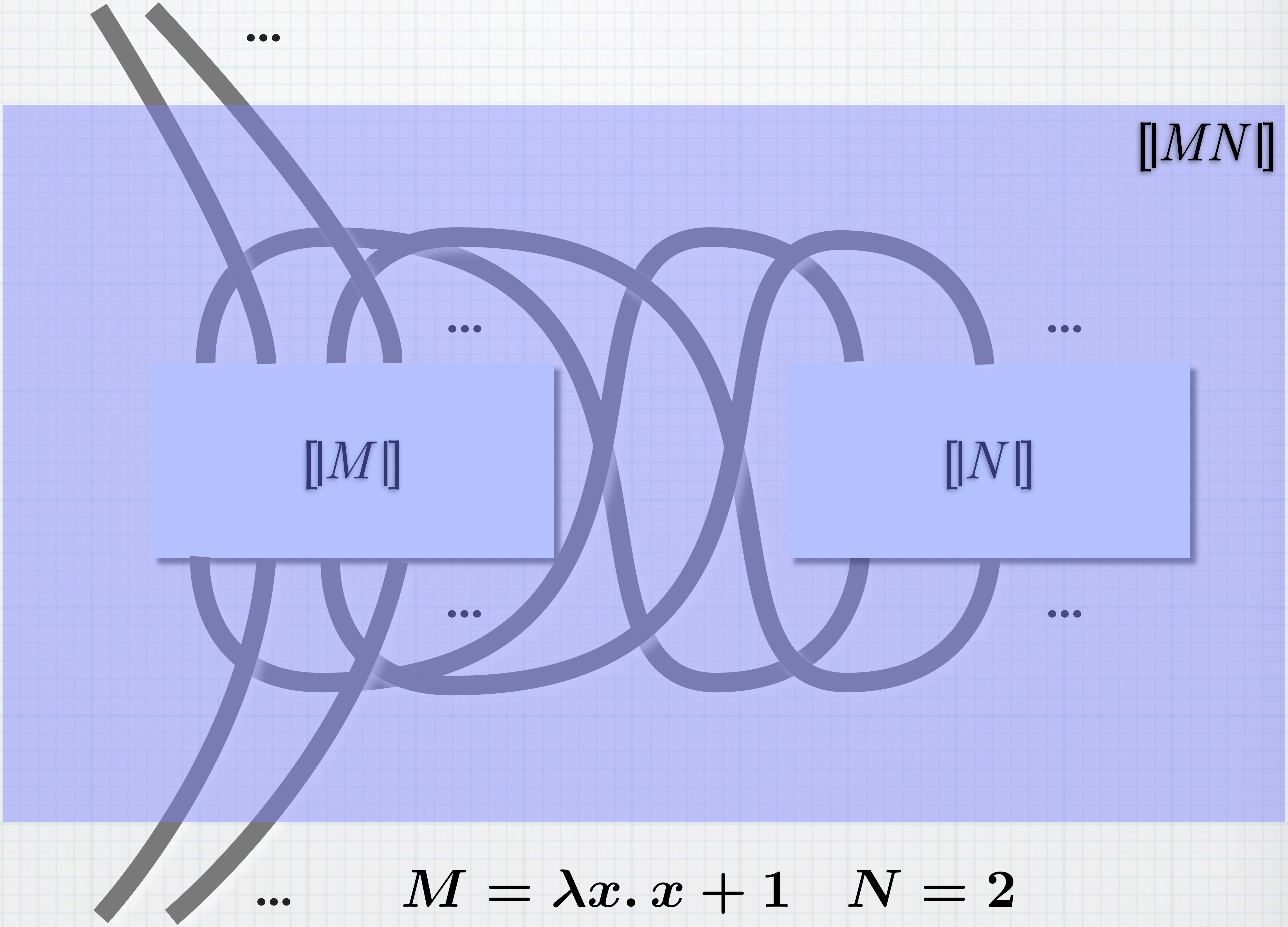
$\llbracket MN \rrbracket$

$=$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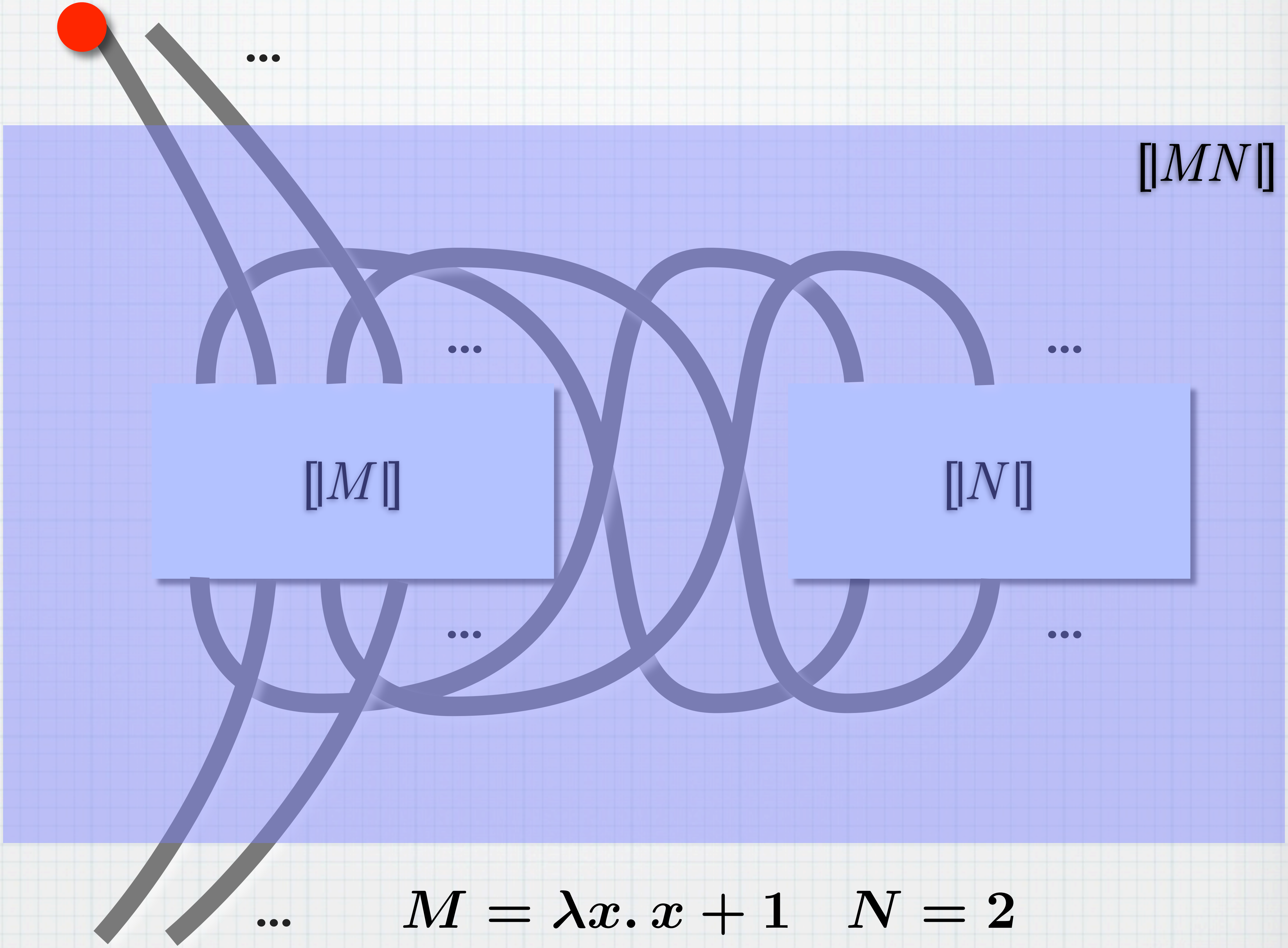

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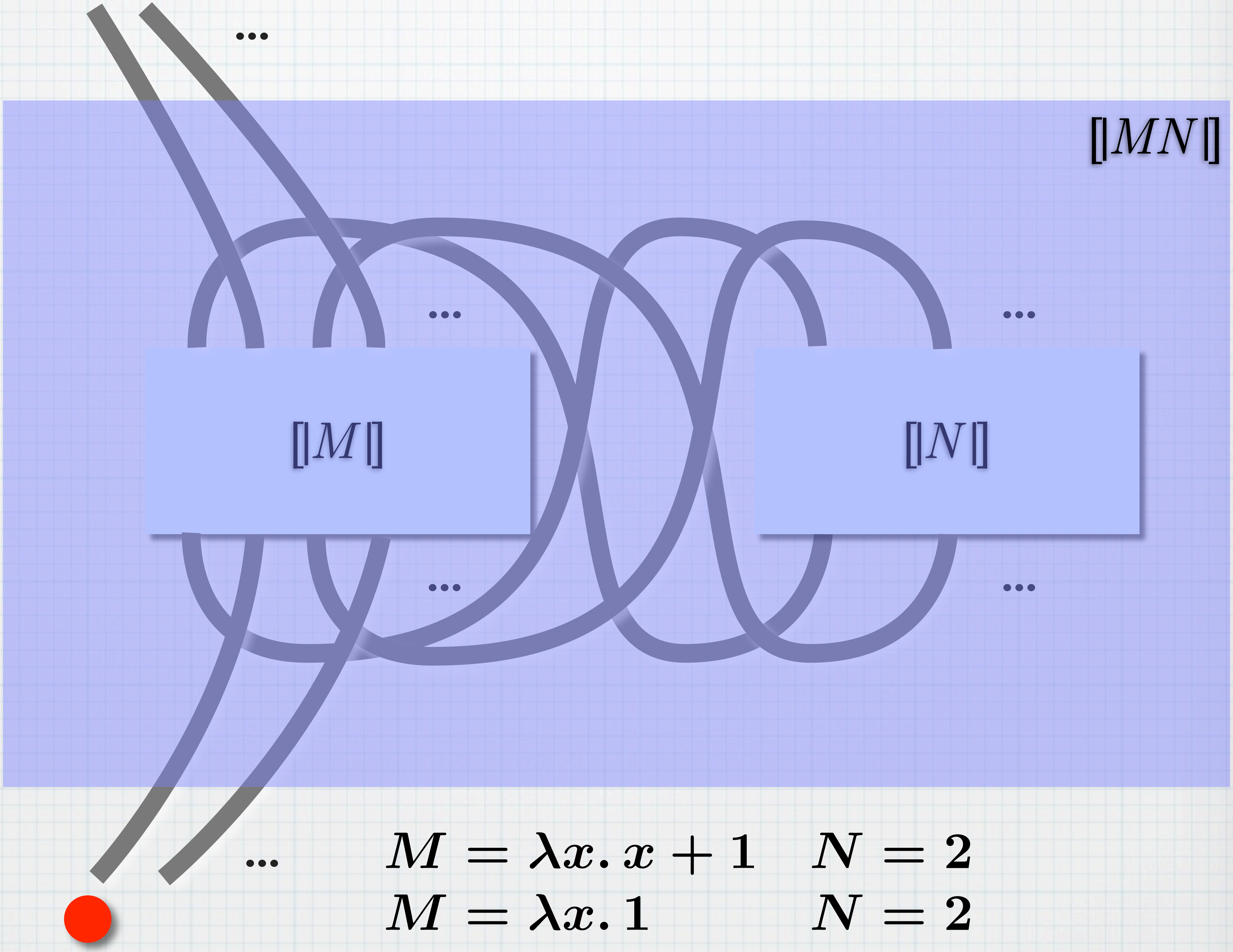
$=$



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→ $M = \lambda f.\, f1 \quad N = \lambda x.\, (x + 1)$

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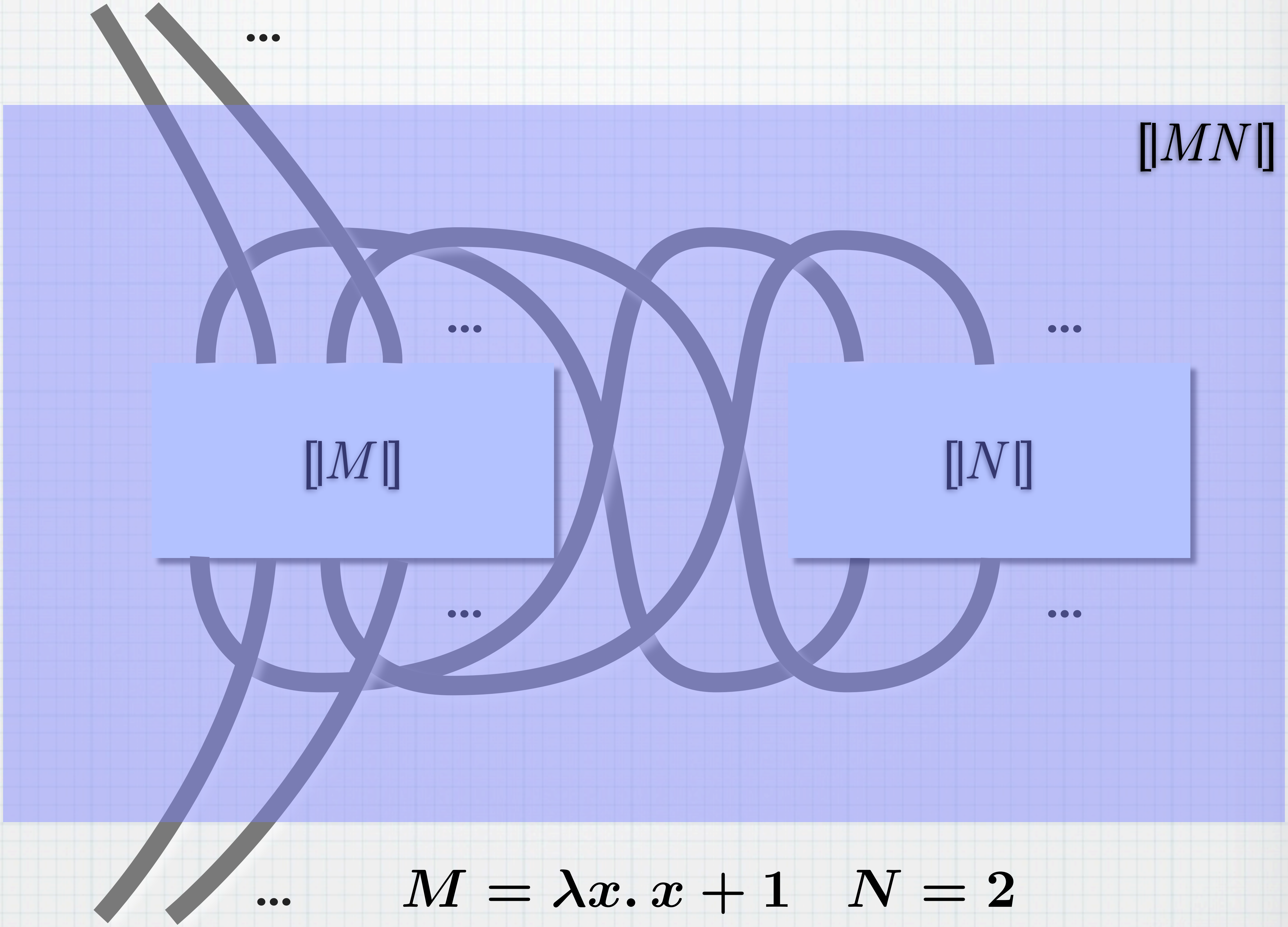
$=$



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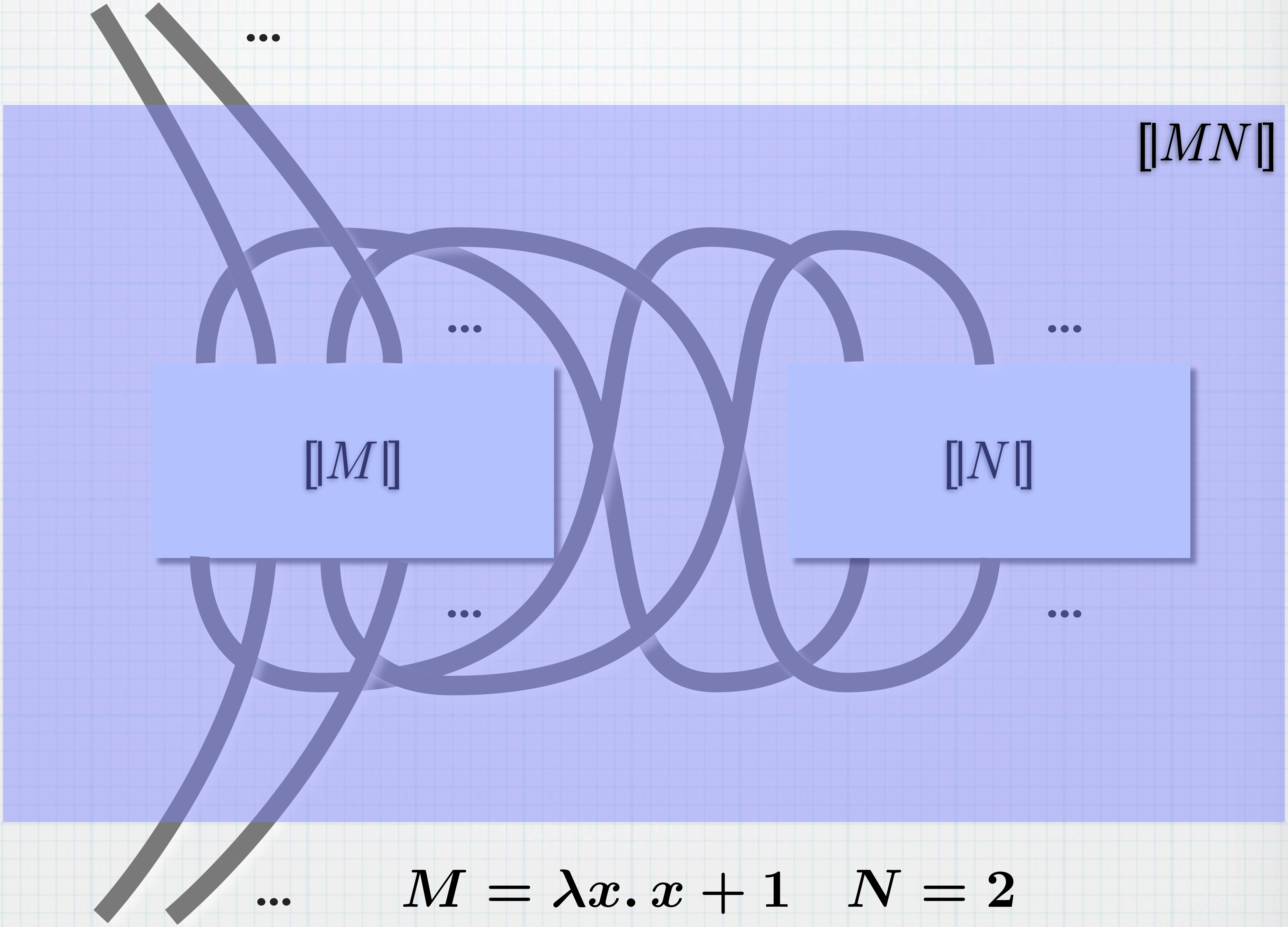

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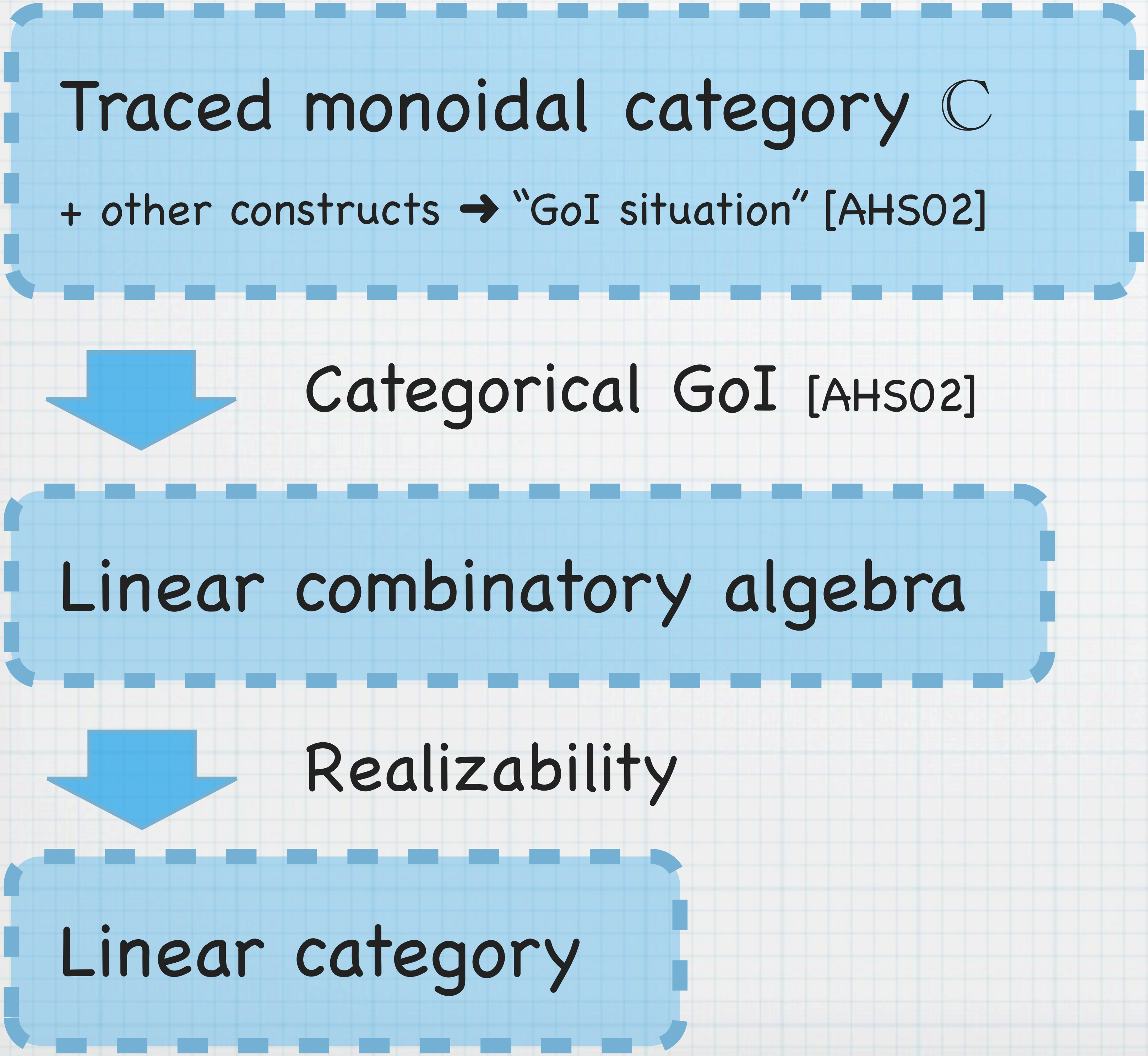
Categorical GoI

- Axiomatics of GoI in the categorical language

- Our main reference:
  - [AHS02] S. Abramsky, E. Haghverdi, and P. Scott, "Geometry of interaction and linear combinatory algebras," MSCS 2002
  - Especially its technical report version (Oxford CL), since it's a bit more detailed

The Categorical GoI Workflow



Traced monoidal category $\mathbb{C}$

+ other constructs ➜ "GoI situation" [AHS02]

Categorical GoI [AHS02]

Linear combinatory algebra

Realizability

Linear category

The Categorical GoI Workflow

Traced monoidal category $\mathbb{C}$

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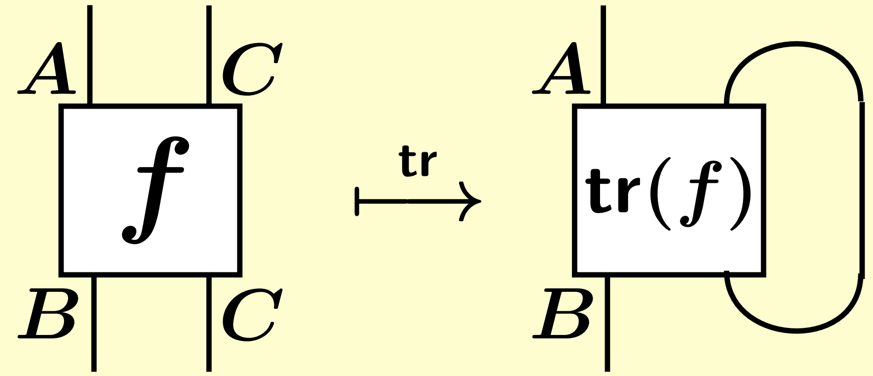

Categorical GoI [AHS02]

Linear combinatory algebra

Realizability

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The Categorical GoI Workflow

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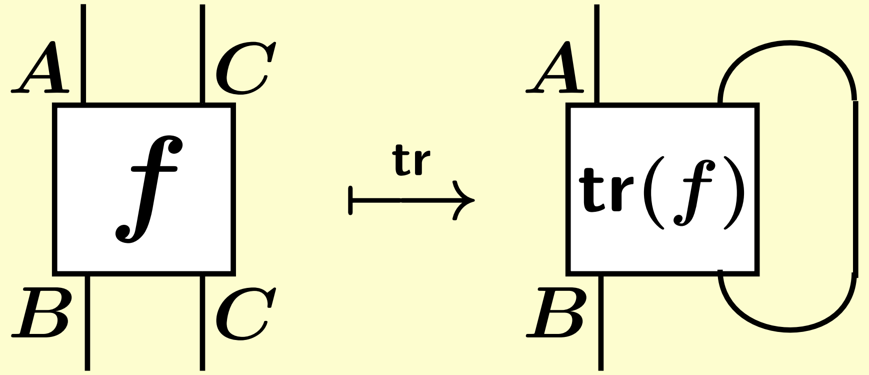

Categorical GoI [AHS02]

- Applicative str. + combinators
- Model of untyped calculus

Linear combinatory algebra

Realizability

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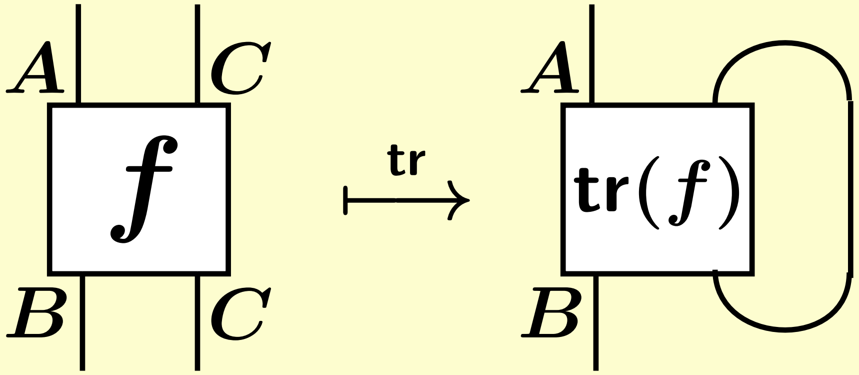

Categorical GoI [AHS02]

Linear combinatory algebra

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Model of typed calculus

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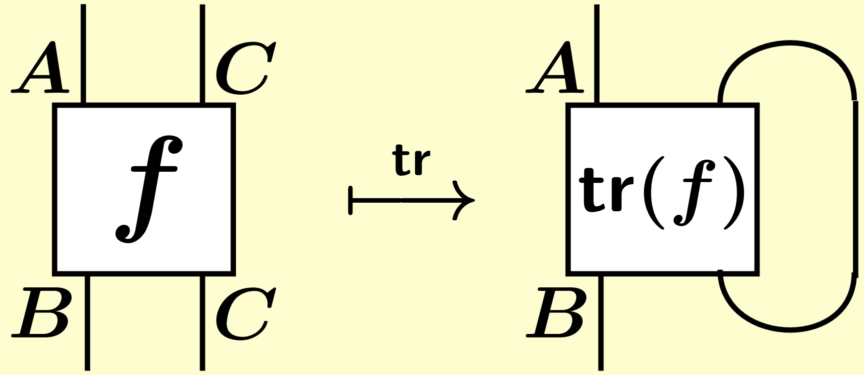

Categorical GoI [AHS02]

- Applicative str. + combinators
- Model of untyped calculus

Linear combinatory algebra

Realizability

- PER, ω-set, assembly, ...
- "Programming in untyped λ"

Linear category

Model of typed calculus

The Categorical GoI Workflow

Traced monoidal category $\mathbb{C}$

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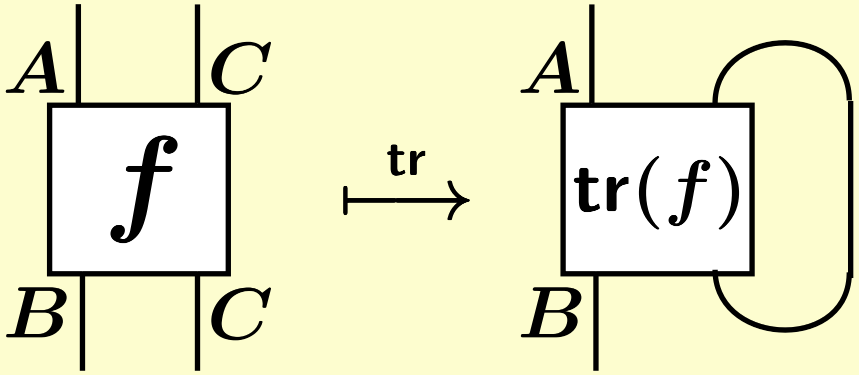

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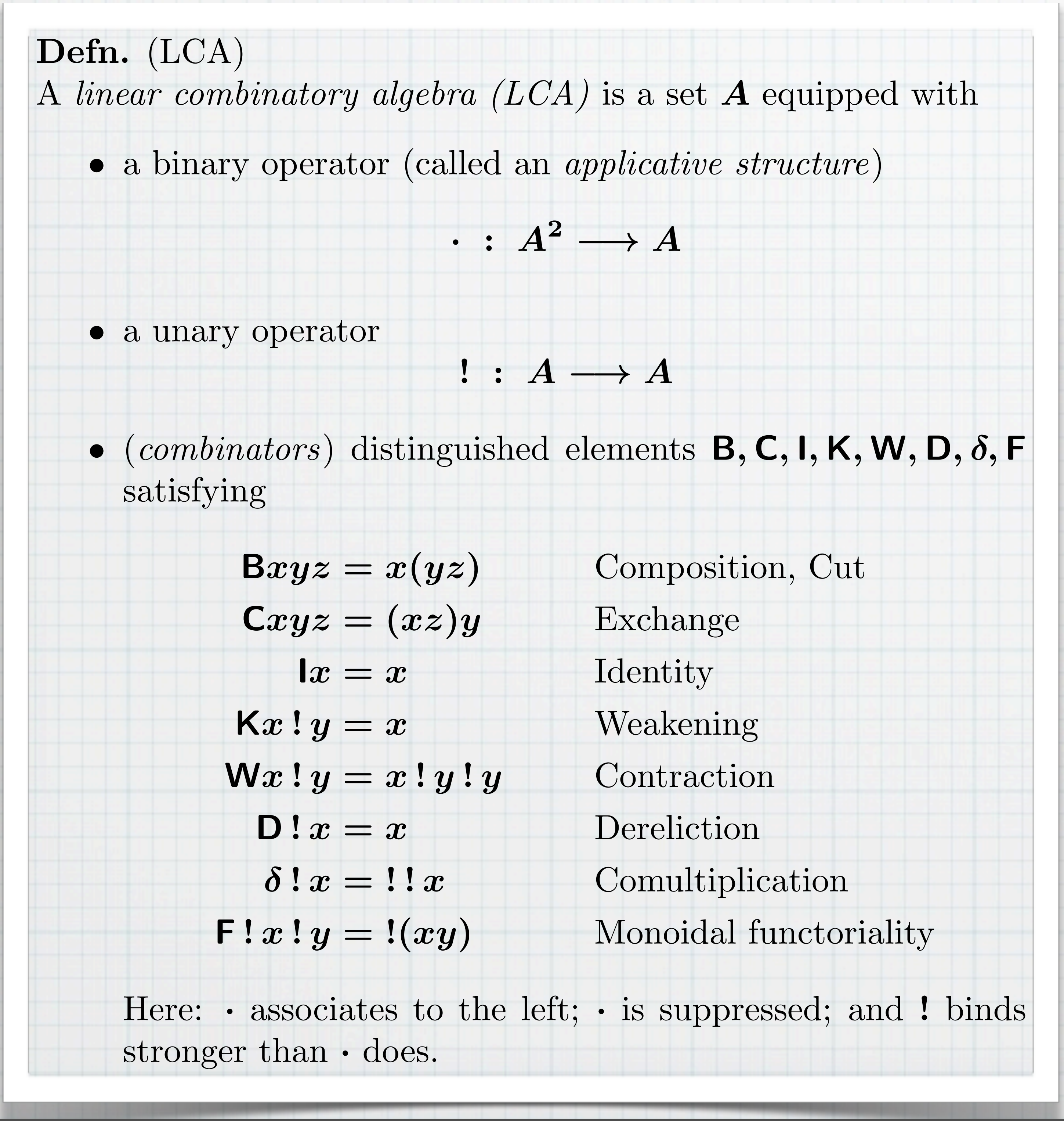
Linear Combinatory Algebra (LCA)



Defn. (LCA)
A *linear combinatory algebra (LCA)* is a set $\boldsymbol{A}$ equipped with

- a binary operator (called an *applicative structure*)

$$\cdot \; : \; \boldsymbol{A}^2 \longrightarrow \boldsymbol{A}$$

- a unary operator

$$! \; : \; \boldsymbol{A} \longrightarrow \boldsymbol{A}$$

- (*combinators*) distinguished elements $\mathbf{B}, \mathbf{C}, \mathbf{I}, \mathbf{K}, \mathbf{W}, \mathbf{D}, \boldsymbol{\delta}, \mathbf{F}$ satisfying

$\mathbf{B}xyz = x(yz)$	Composition, Cut
$\mathbf{C}xyz = (xz)y$	Exchange
$\mathbf{I}x = x$	Identity
$\mathbf{K}x\,!\,y = x$	Weakening
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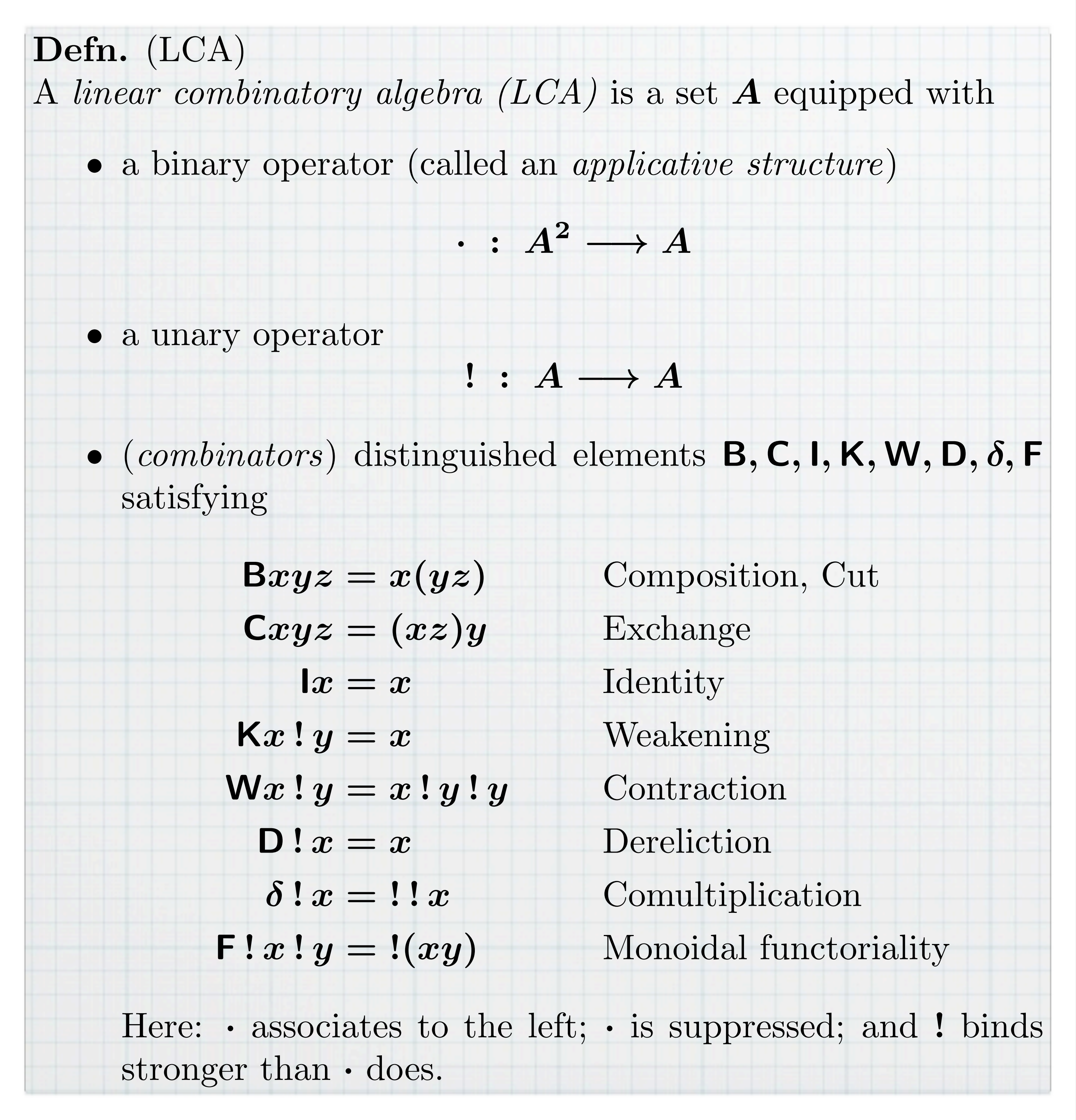
$\boldsymbol{\delta}\,!\,x = !\,!\,x$	Comultiplication
$\mathbf{F}\,!\,x\,!\,y = !(xy)$	Monoidal functoriality

Here: $\cdot$ associates to the left; $\cdot$ is suppressed; and $!$ binds stronger than $\cdot$ does.

Linear Combinatory Algebra (LCA)

What
we want (outcome)



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Linear Combinatory Algebra (LCA)

What
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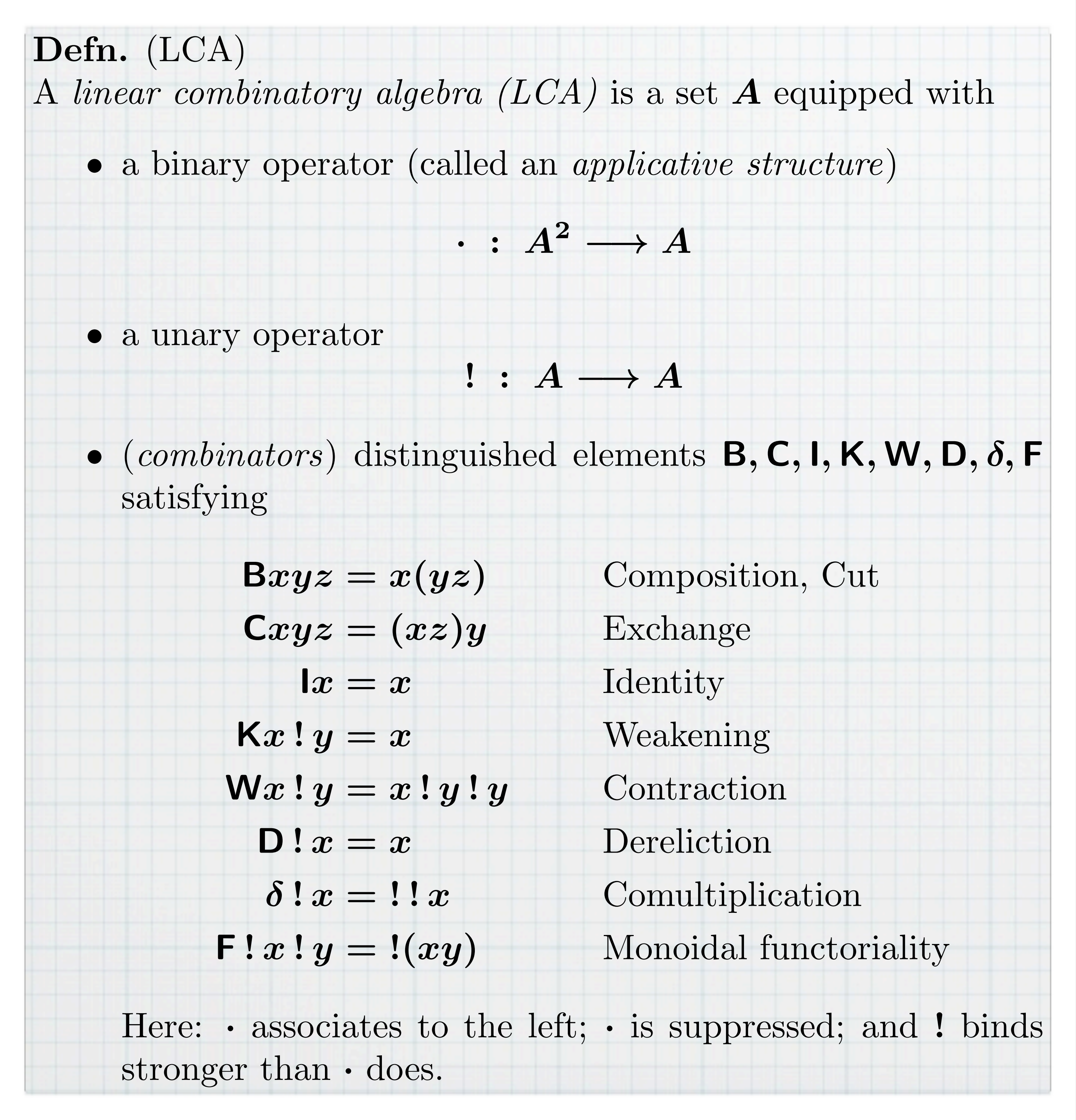
- Model of untyped linear λ



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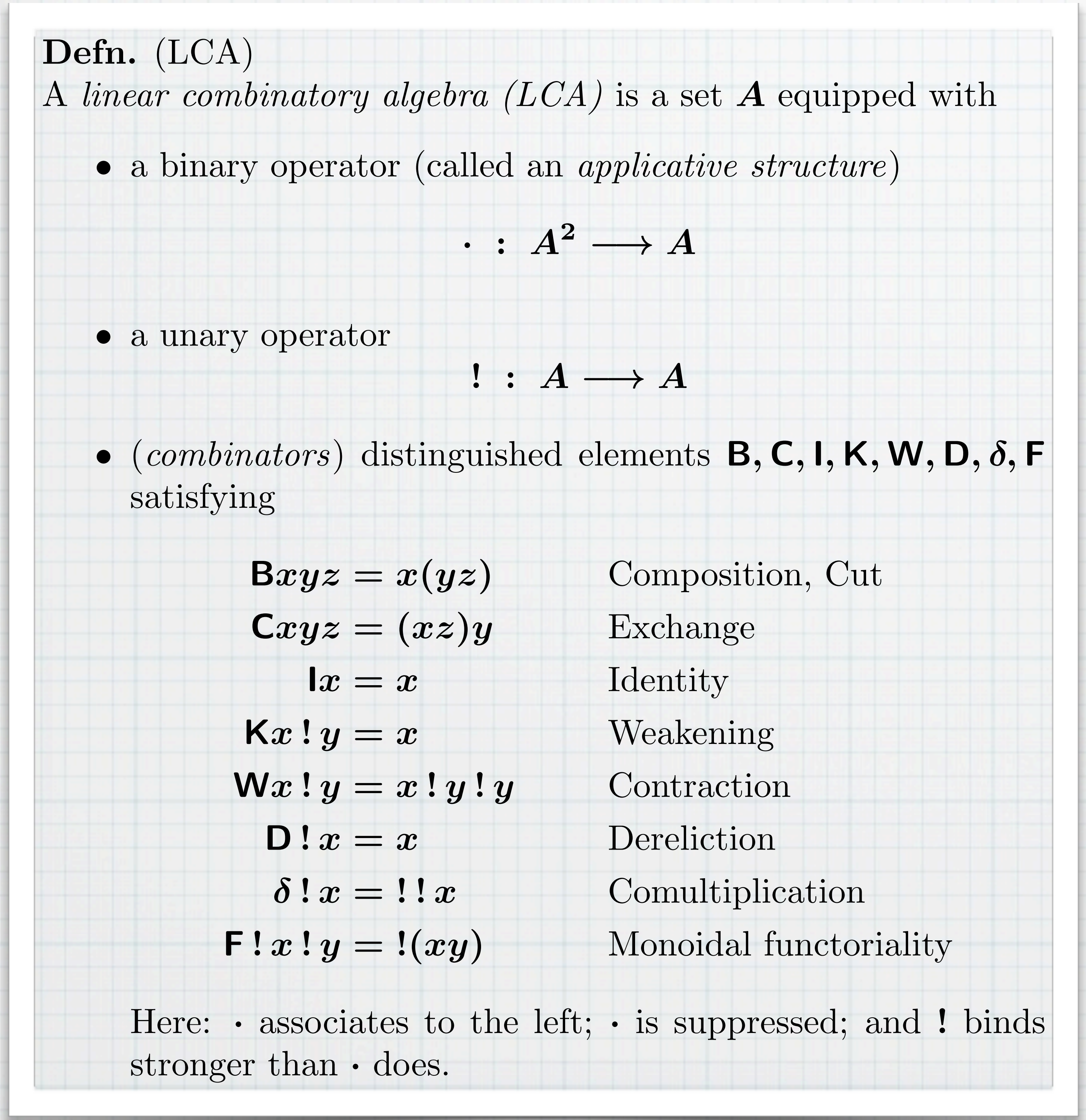

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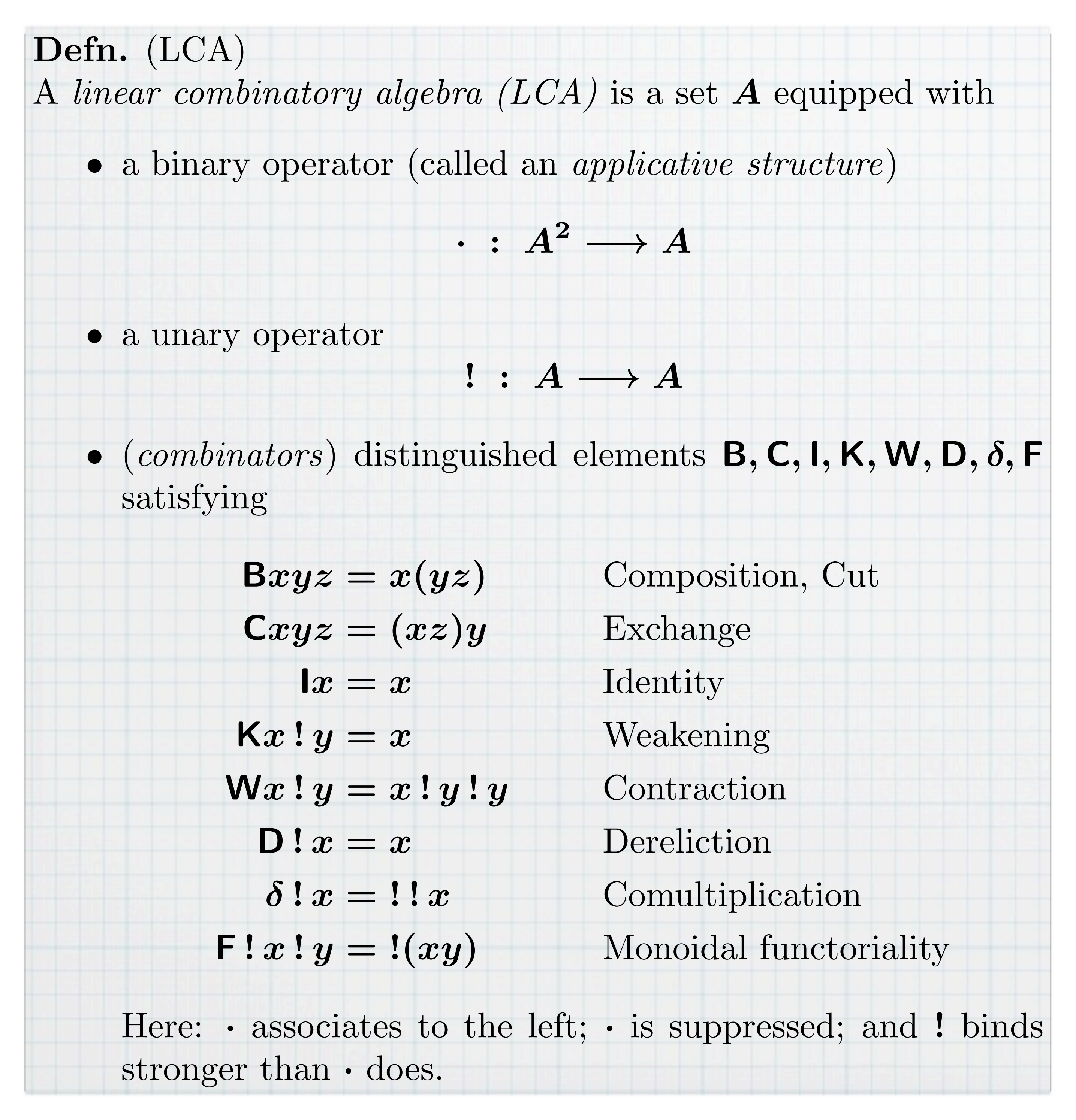
What
we want (outcome)

- ✱ Model of untyped linear λ
- ✱ a ∈ A ≈ closed linear λ-term

Linear Combinatory Algebra (LCA)



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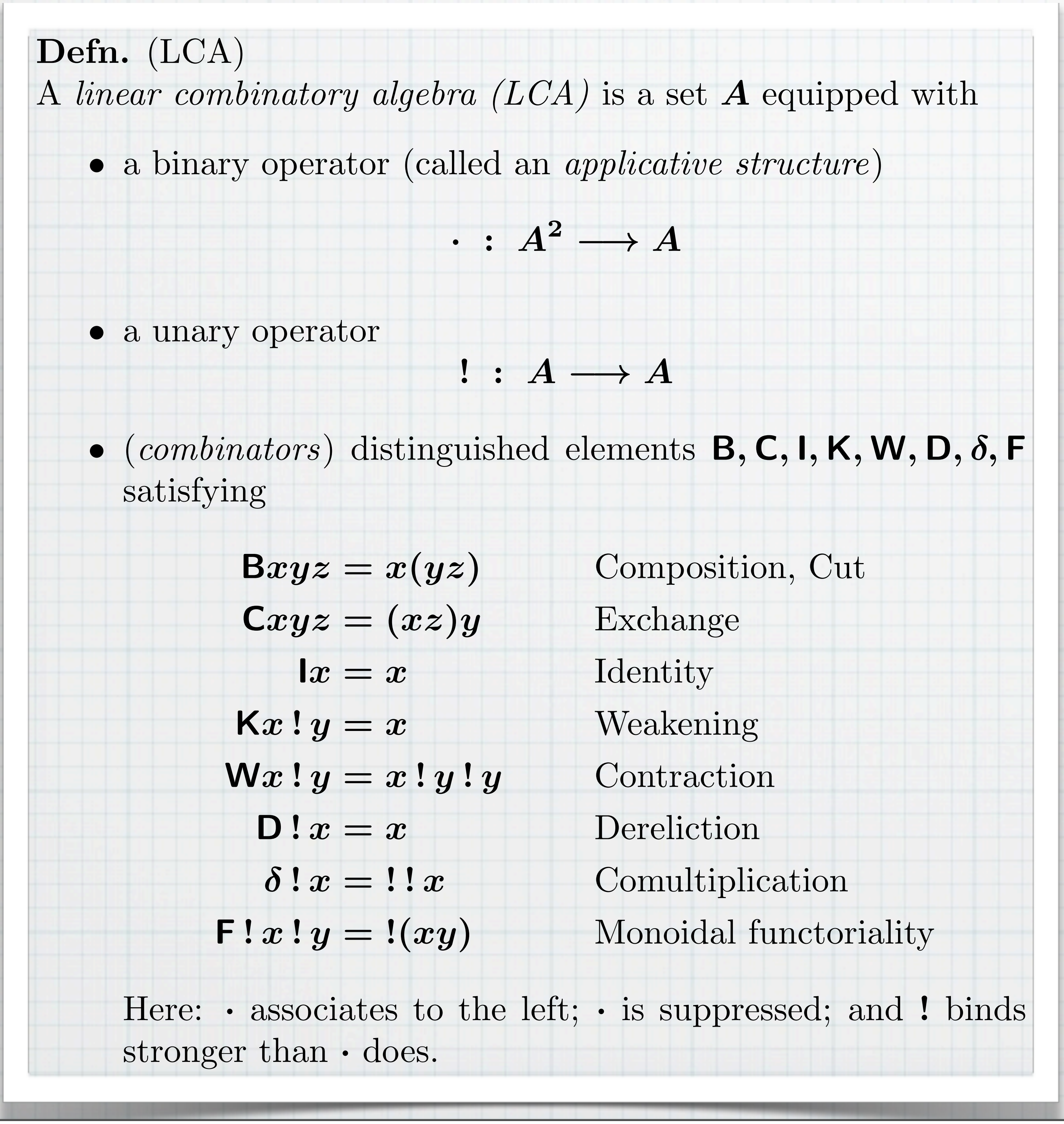
What
we want (outcome)

- Model of untyped linear λ
- a ∈ A ≈ closed linear λ-term
- No **S** or **K** (linear!)

Hasuo (Tokyo)

Linear Combinatory Algebra (LCA)



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\end{array}$$

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What
we want (outcome)

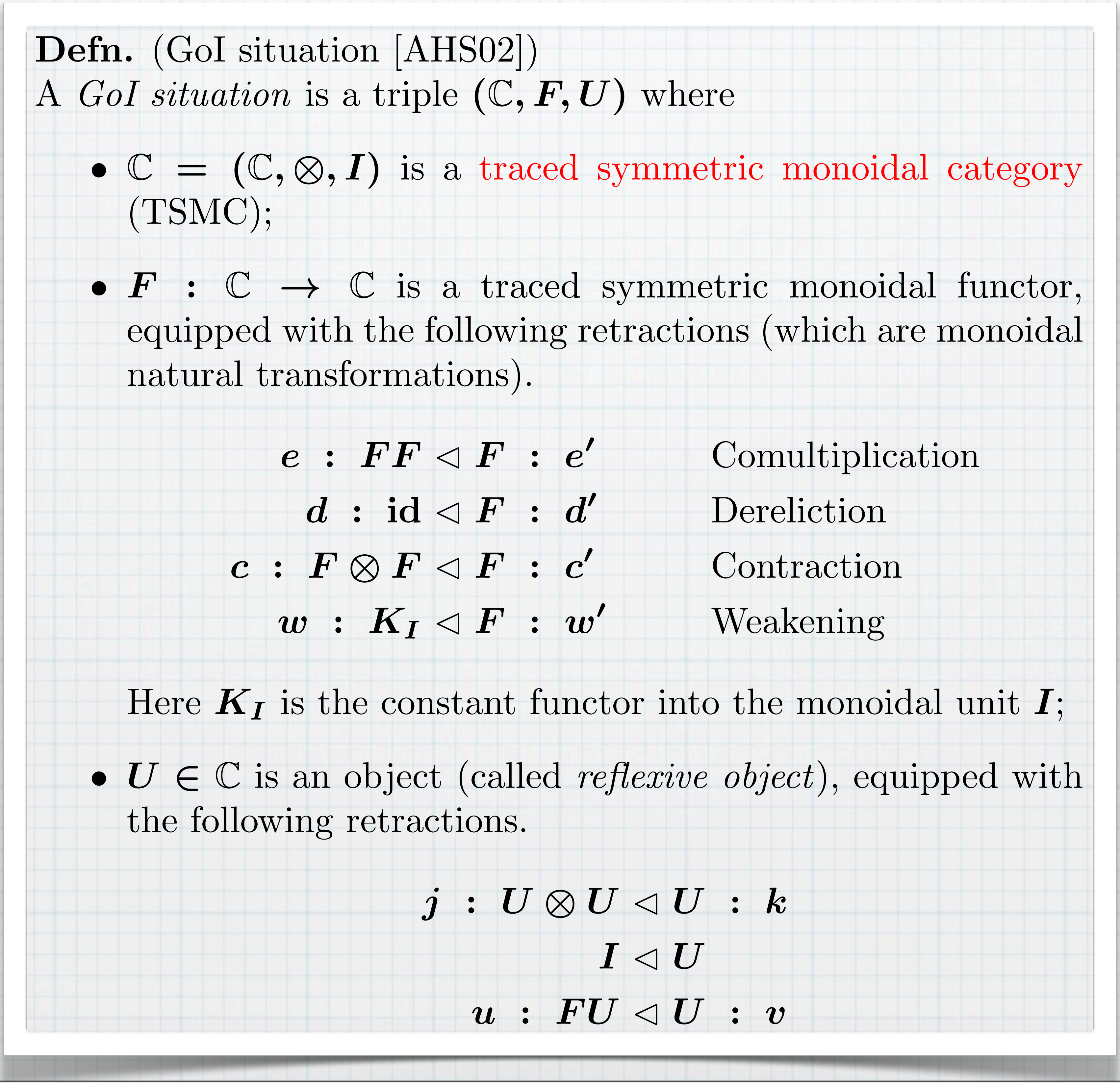
- Model of untyped linear λ
- a ∈ A ≈ closed linear λ-term
- No **S** or **K** (linear!)
- Combinatory completeness: e.g.

$$\lambda xyz.\, zxy$$

designates an elem. of A

What we use (ingredient)

GoI situation



Defn. (GoI situation [AHS02])
A *GoI situation* is a triple $(\mathbb{C}, F, U)$ where

- $\mathbb{C} = (\mathbb{C}, \otimes, I)$ is a traced symmetric monoidal category (TSMC);
- $F : \mathbb{C} \to \mathbb{C}$ is a traced symmetric monoidal functor, equipped with the following retractions (which are monoidal natural transformations).

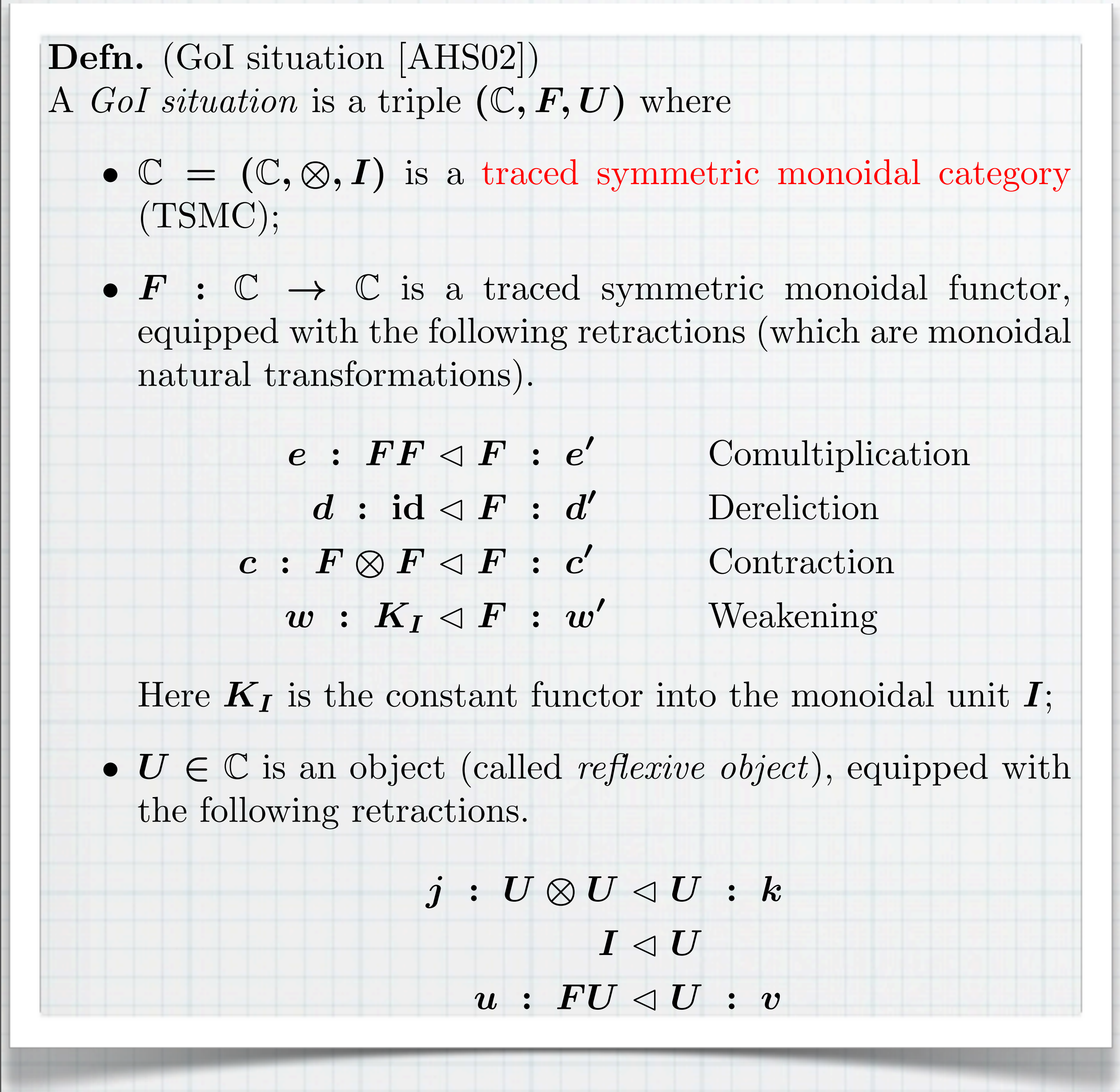
  $$\begin{aligned} e &: FF \lhd F : e' && \text{Comultiplication} \\ d &: \mathrm{id} \lhd F : d' && \text{Dereliction} \\ c &: F \otimes F \lhd F : c' && \text{Contraction} \\ w &: K_I \lhd F : w' && \text{Weakening} \end{aligned}$$

  Here K_I is the constant functor into the monoidal unit I;
- $U \in \mathbb{C}$ is an object (called *reflexive object*), equipped with the following retractions.

  $$\begin{aligned} j &: U \otimes U \lhd U : k \\ & I \lhd U \\ u &: FU \lhd U : v \end{aligned}$$

Hasuo (Tokyo)

GoI situation



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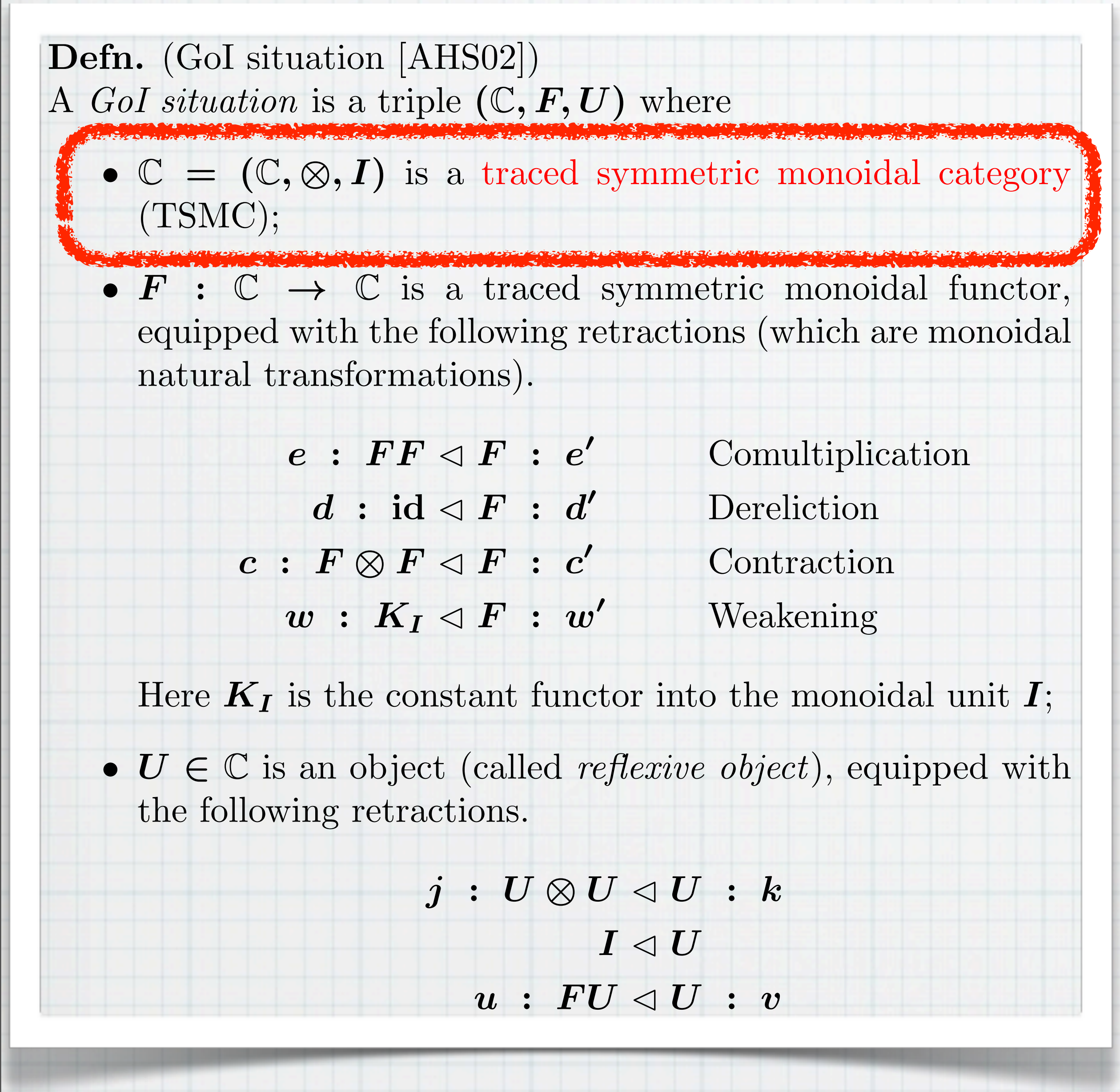
- Monoidal category $(\mathbb{C}, \otimes, \boldsymbol{I})$
  - String diagrams

GoI situation



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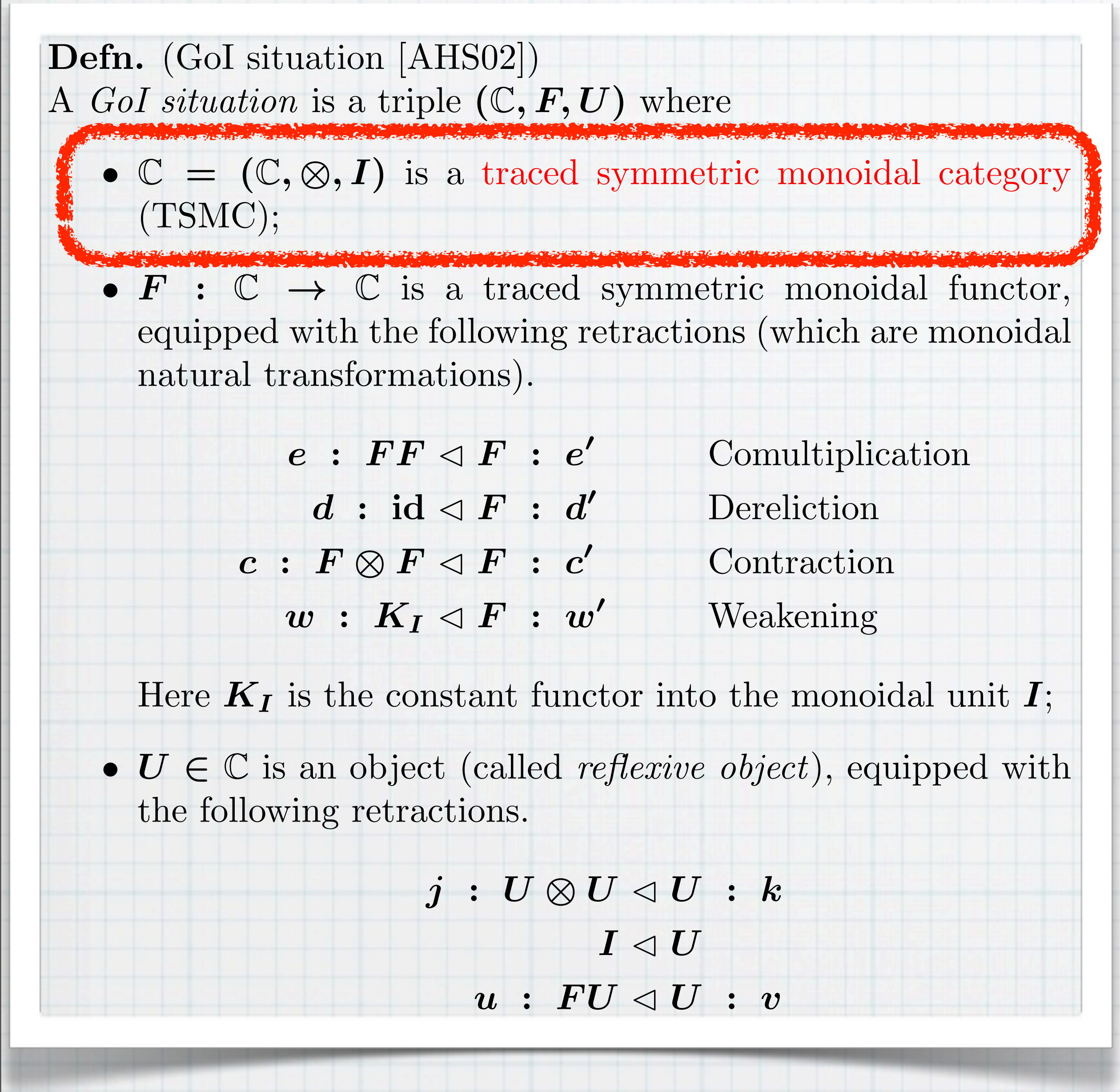

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* Monoidal category $(\mathbb{C}, \otimes, \boldsymbol{I})$

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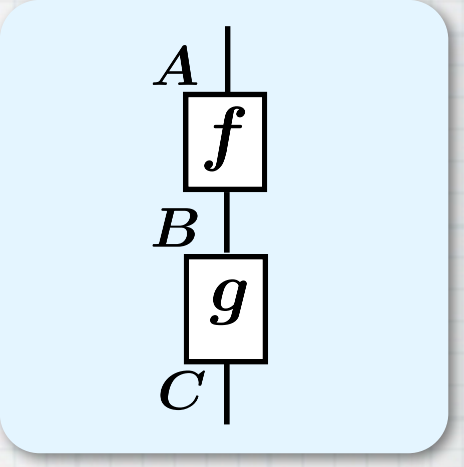
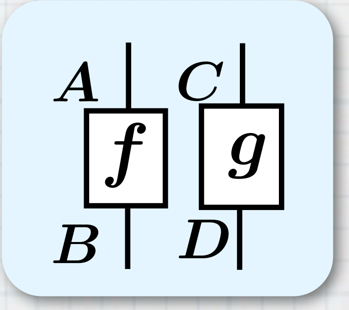
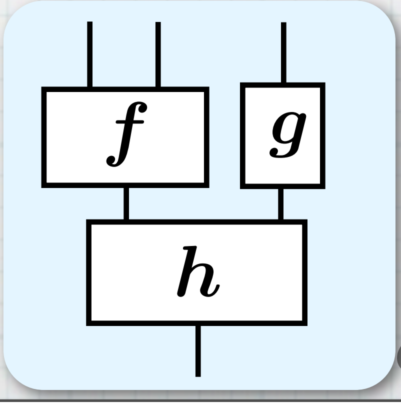
$$\frac{A \xrightarrow{f} B \quad B \xrightarrow{g} C}{A \xrightarrow{g \circ f} C}$$



$$\frac{A \xrightarrow{f} B \quad C \xrightarrow{g} D}{A \otimes C \xrightarrow{f \otimes g} B \otimes D}$$

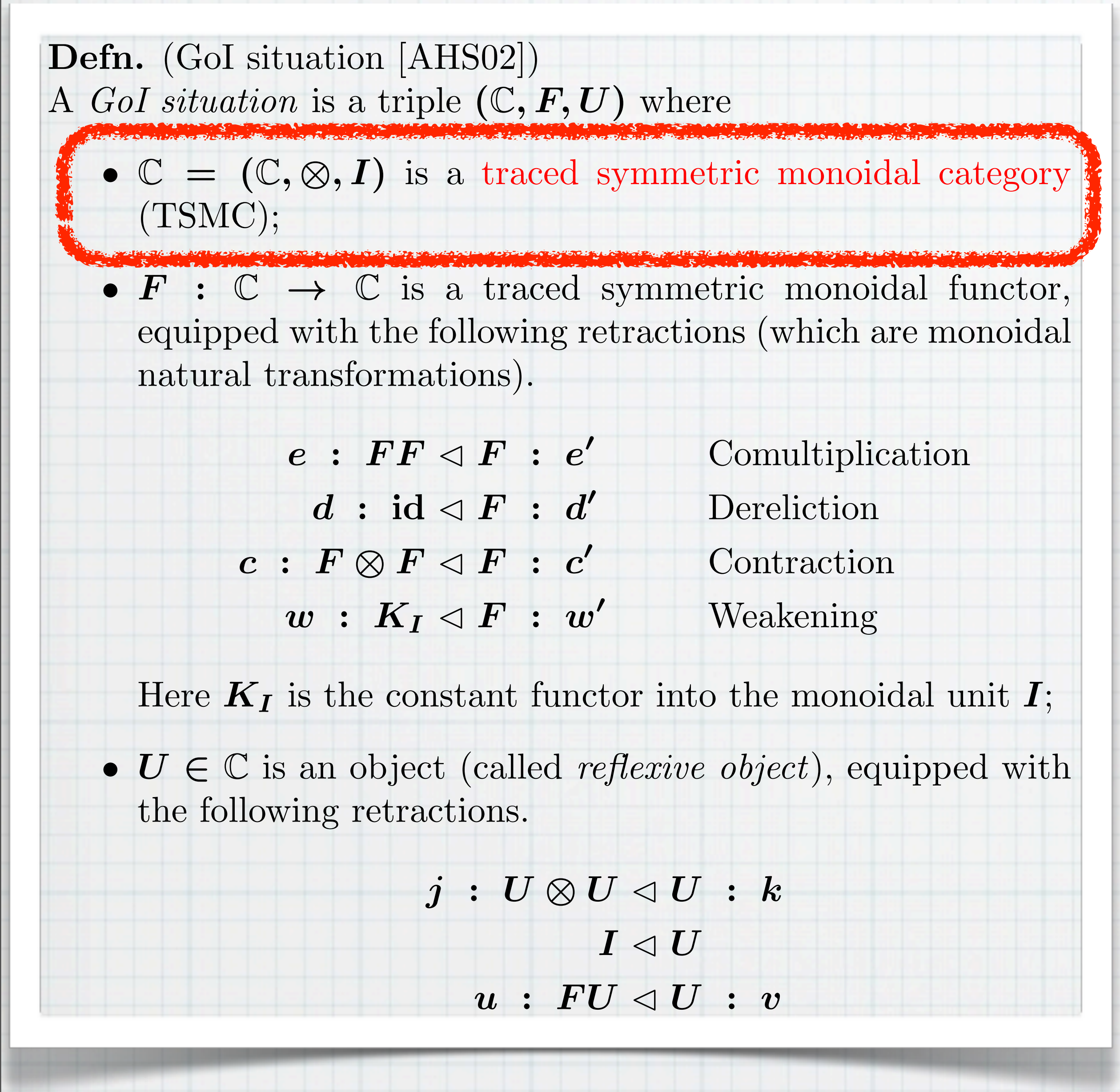


$$h \circ (f \otimes g)$$



o)

GoI situation



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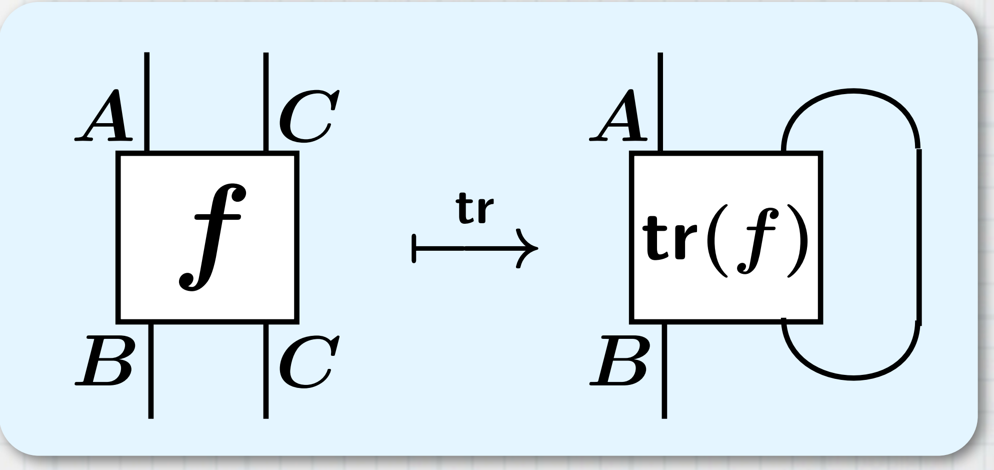
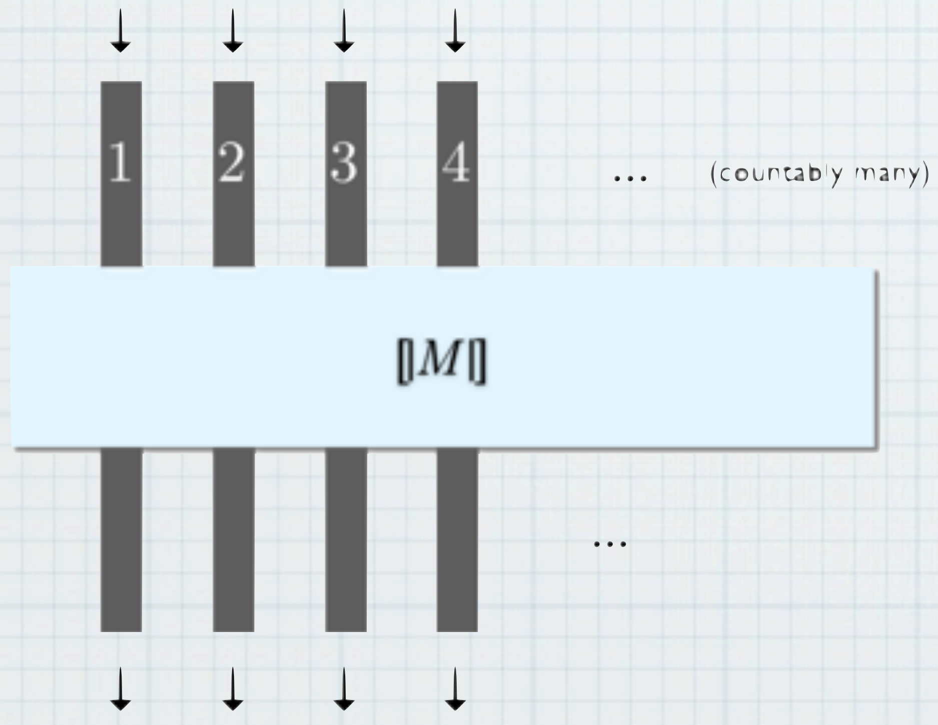
$$\frac{A \otimes C \xrightarrow{f} B \otimes C}{A \xrightarrow{\mathrm{tr}(f)} B}$$

that is

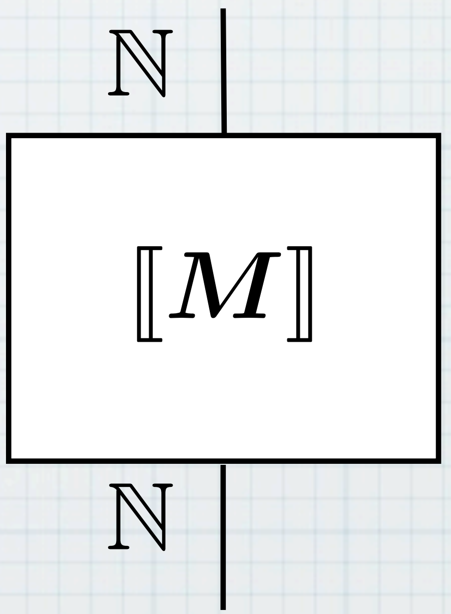


String Diagram vs. “Pipe Diagram”

- I use two ways of depicting partial functions $\mathbb{N} \rightharpoonup \mathbb{N}$



Pipe diagram



String diagram

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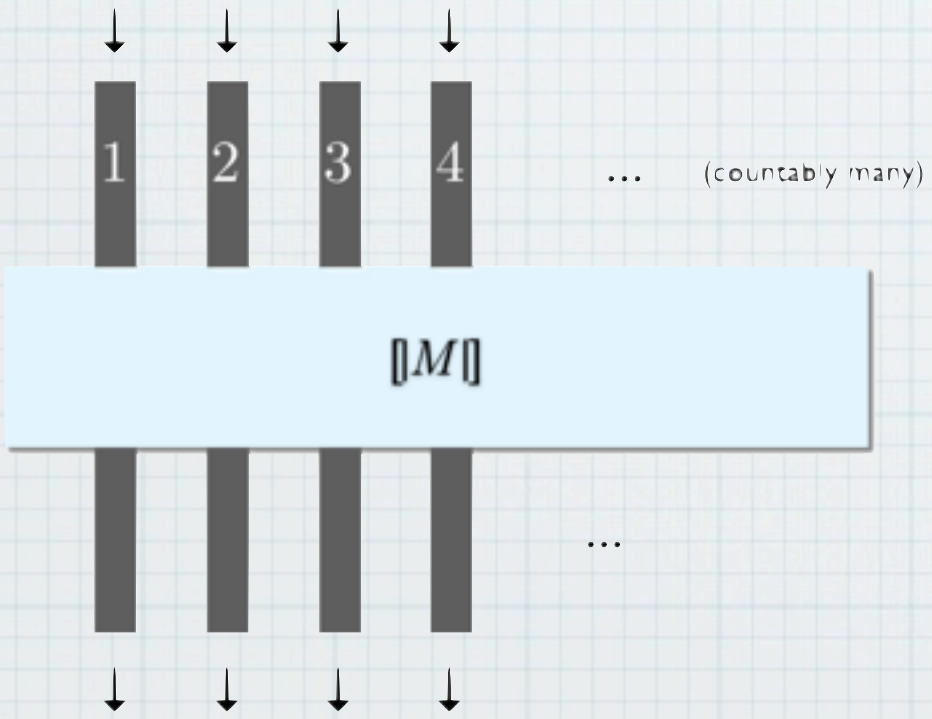
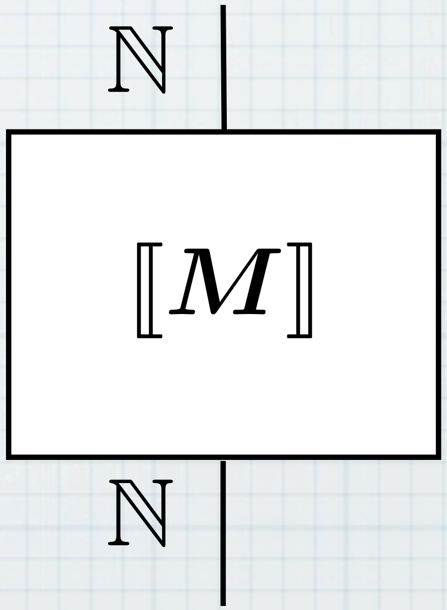
In the monoidal category $(\mathbf{Pfn}, +, 0)$



Pipe diagram

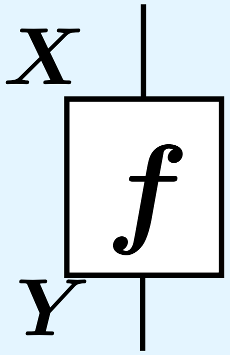


String diagram

Traced Sym. Monoidal Category $(\mathbf{Pfn}, +, 0)$

- Category **Pfn** of partial functions
  - Obj. A set X
  - Arr. A partial function

$$\begin{array}{c} X \to Y \ \text{ in } \mathbf{Pfn} \\ \hline\hline X \rightharpoonup Y, \text{ partial function} \end{array}$$



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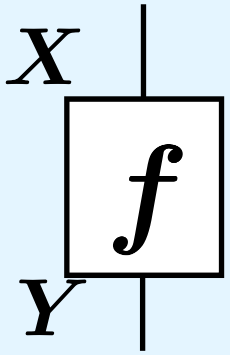

- is traced symmetric monoidal

Traced Sym. Monoidal Category

$(\mathbf{Pfn}, +, 0)$

- $$\frac{X + Z \xrightarrow{f} Y + Z \qquad \text{in } \mathbf{Pfn}}{X \xrightarrow{\mathsf{tr}(f)} Y \qquad \text{in } \mathbf{Pfn}}$$

  How?

-

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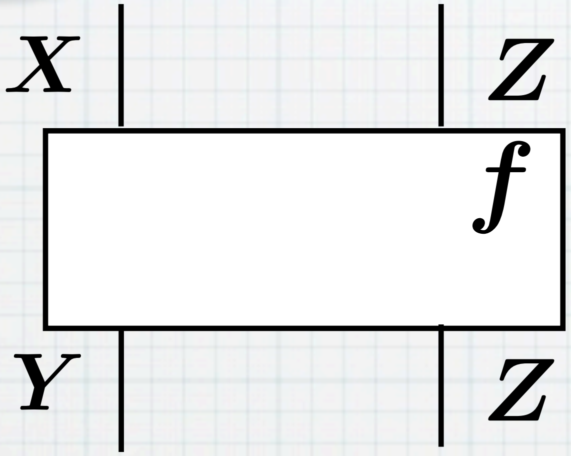
- 

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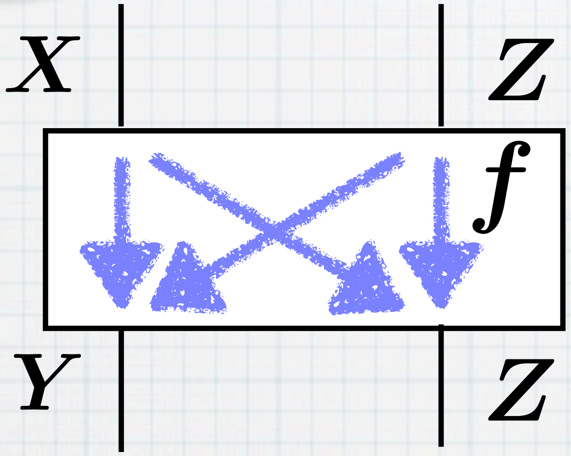
- 

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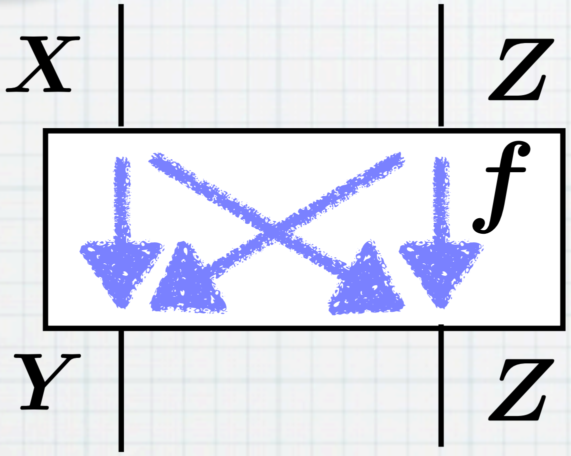
- 

$$f_{XY}(x) := \begin{cases} f(x) & \text{if } f(x) \in Y \\ \bot & \text{o.w.} \end{cases}$$

Similar for f_{XZ}, f_{ZY}, f_{ZZ}

Hasuo (Tokyo)

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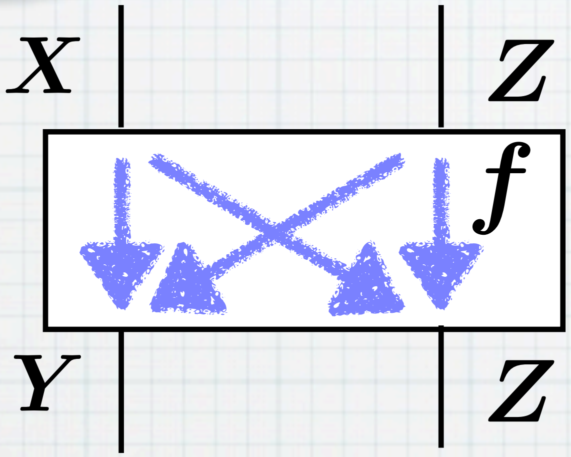
- 

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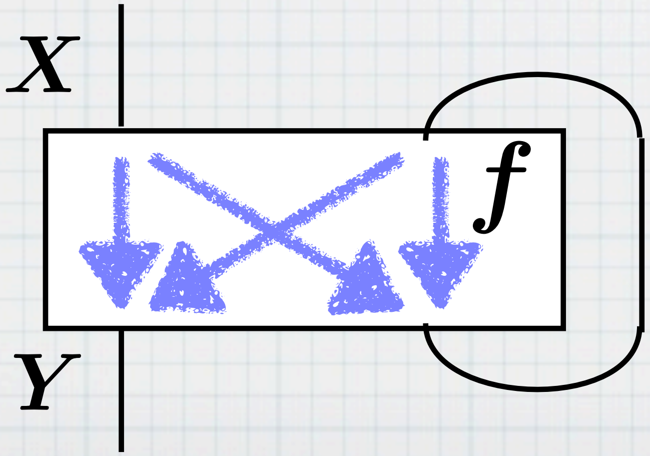
- Trace operator:



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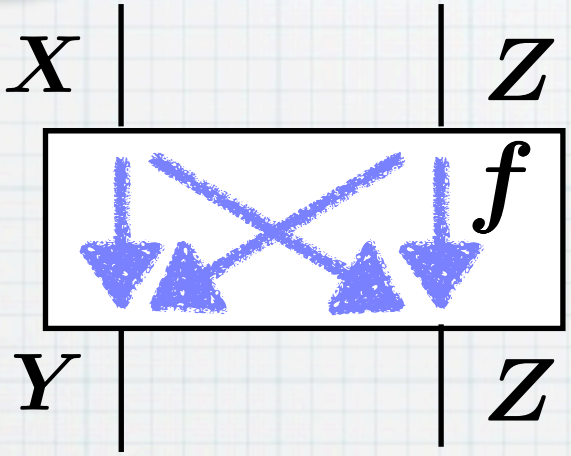
- 

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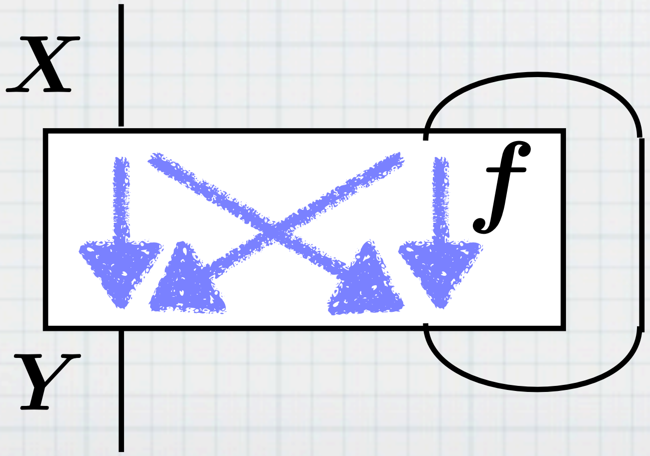

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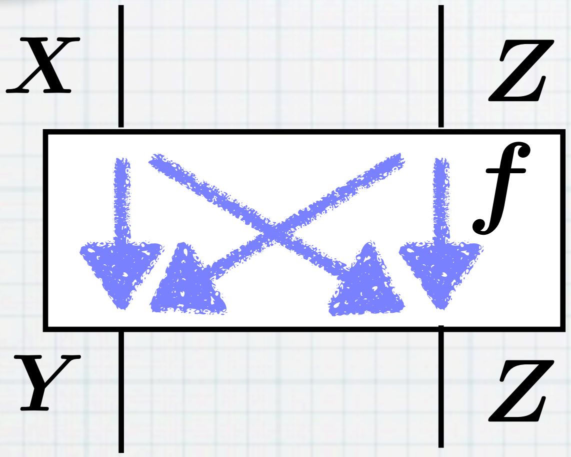
- 

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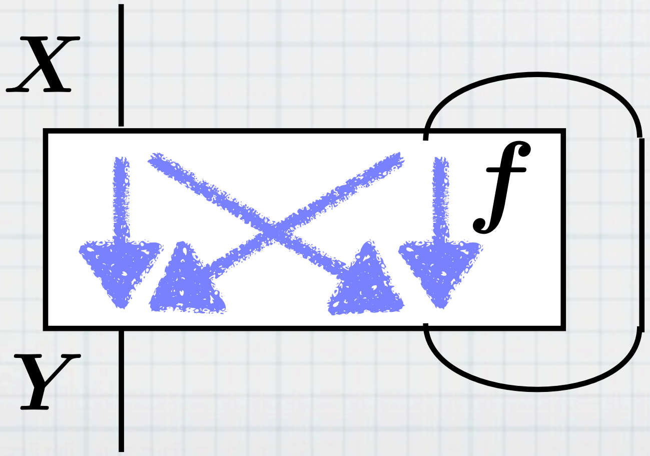
- **Execution formula** (Girard)
- Partiality is essential (infinite loop)

- Trace operator:



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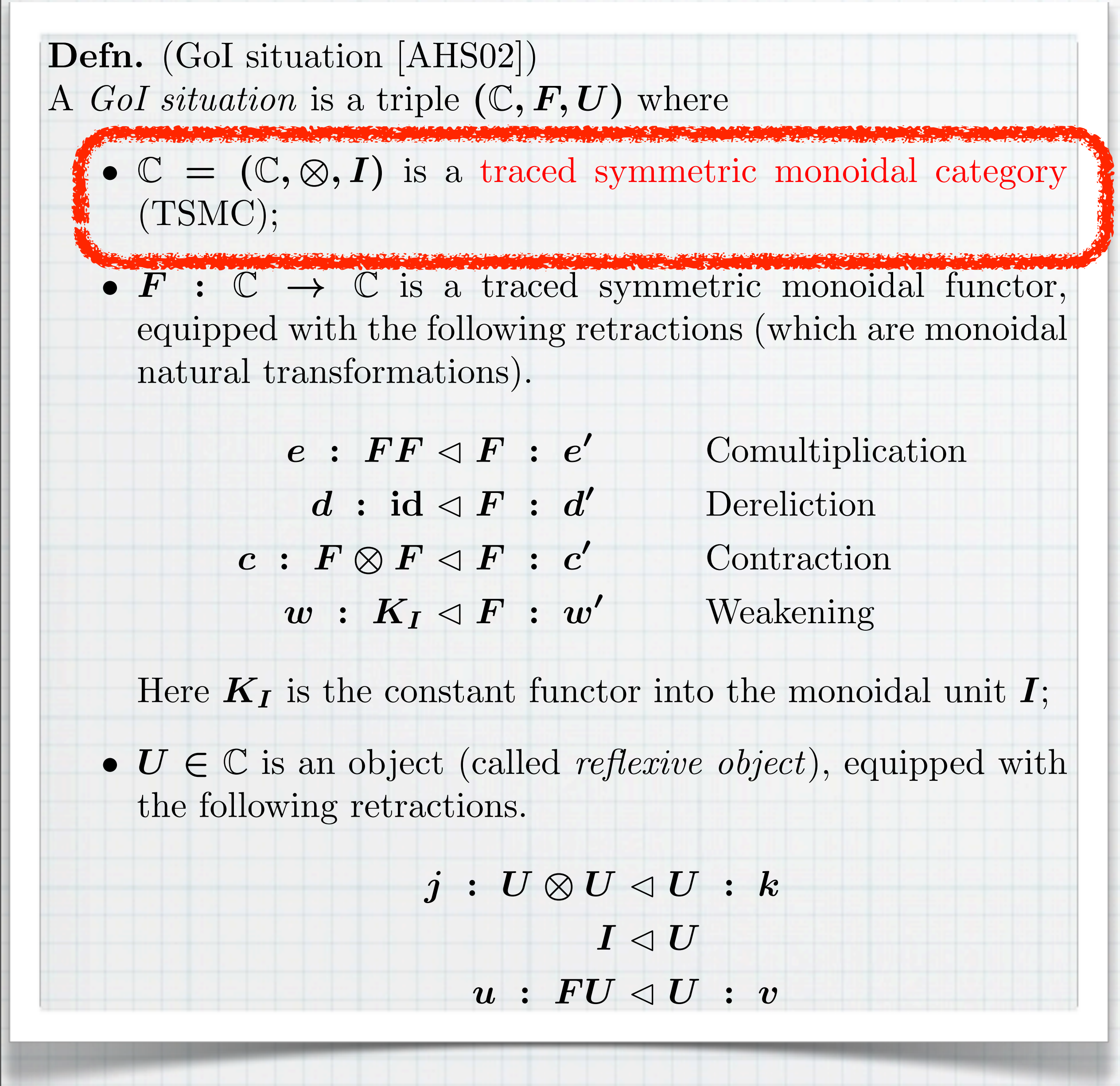
GoI situation



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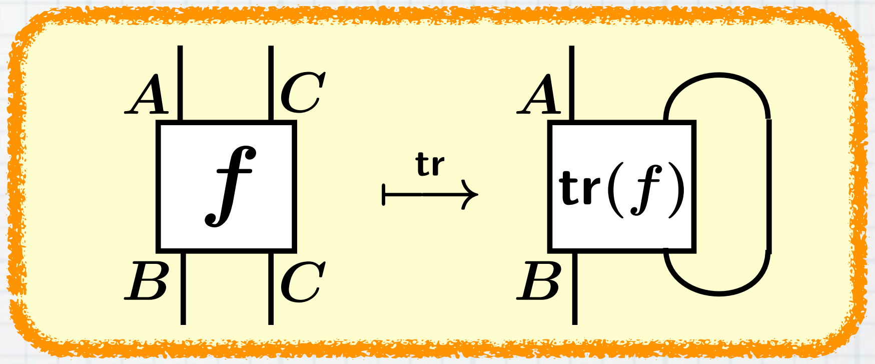
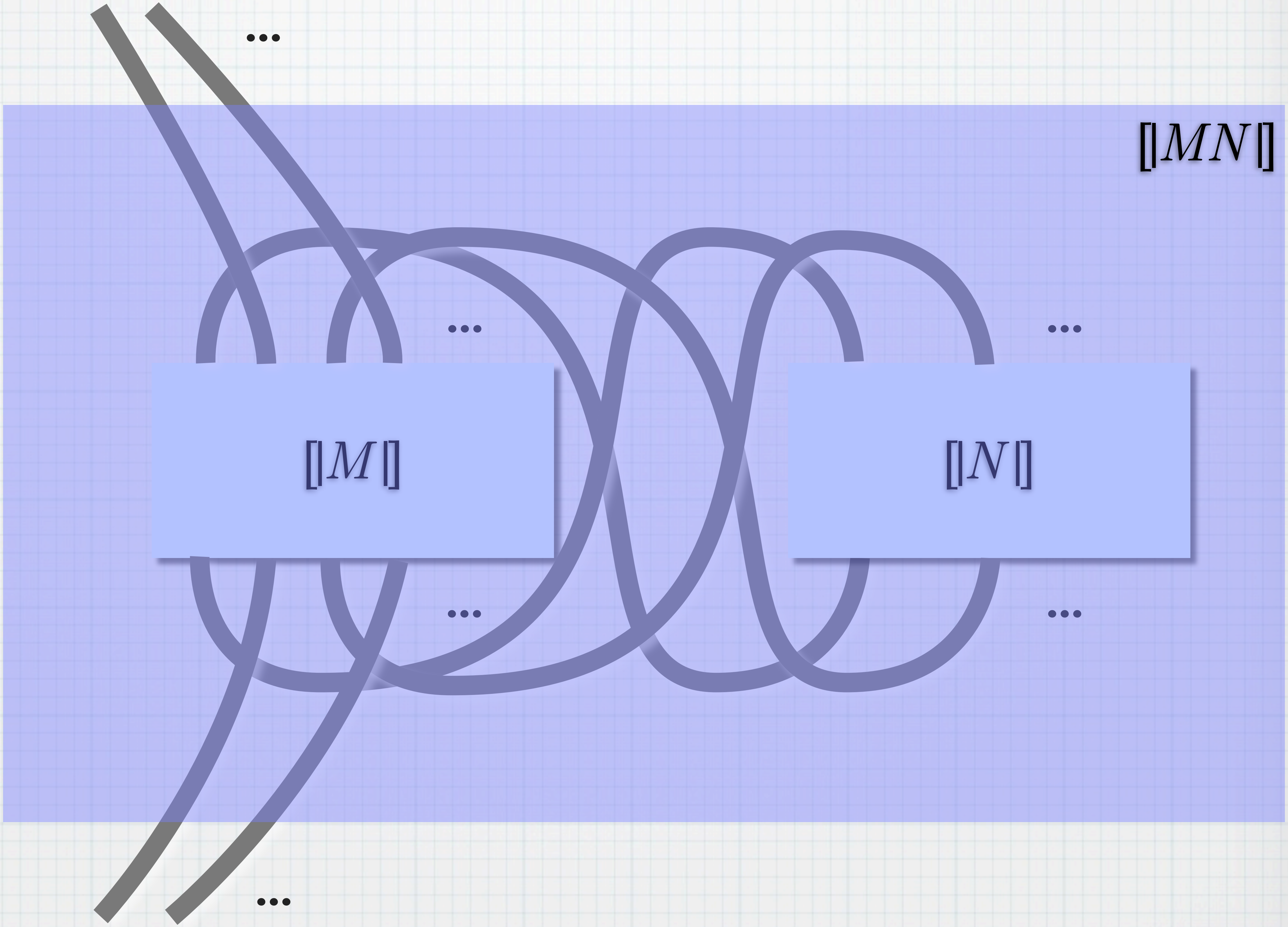
- Traced sym. monoidal cat.
  - Where one can "feedback"



$$f \overset{\mathrm{tr}}{\longmapsto} \mathrm{tr}(f)$$

  - Why for GoI?

$$[\![\boldsymbol{MN}]\!] =$$



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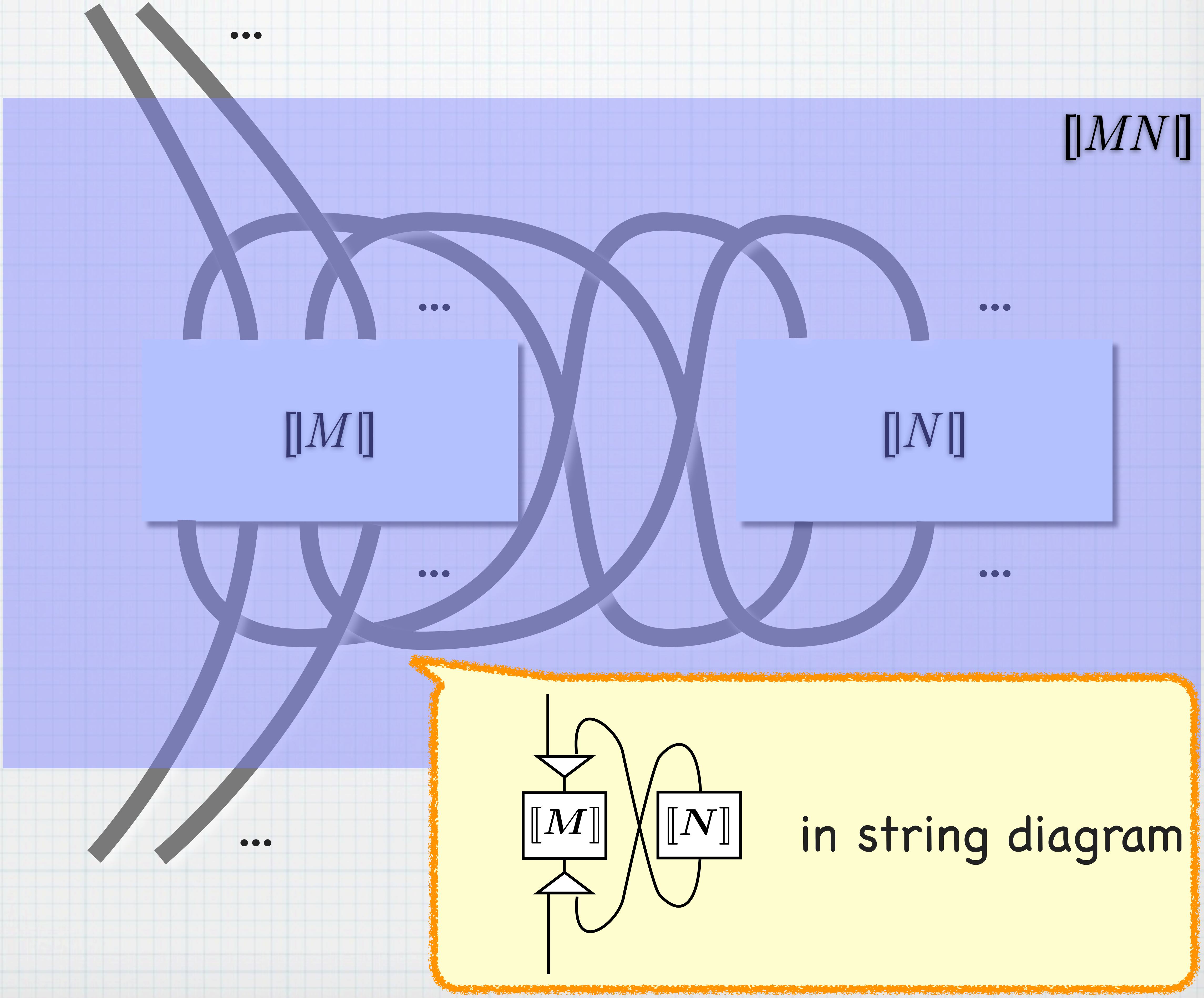
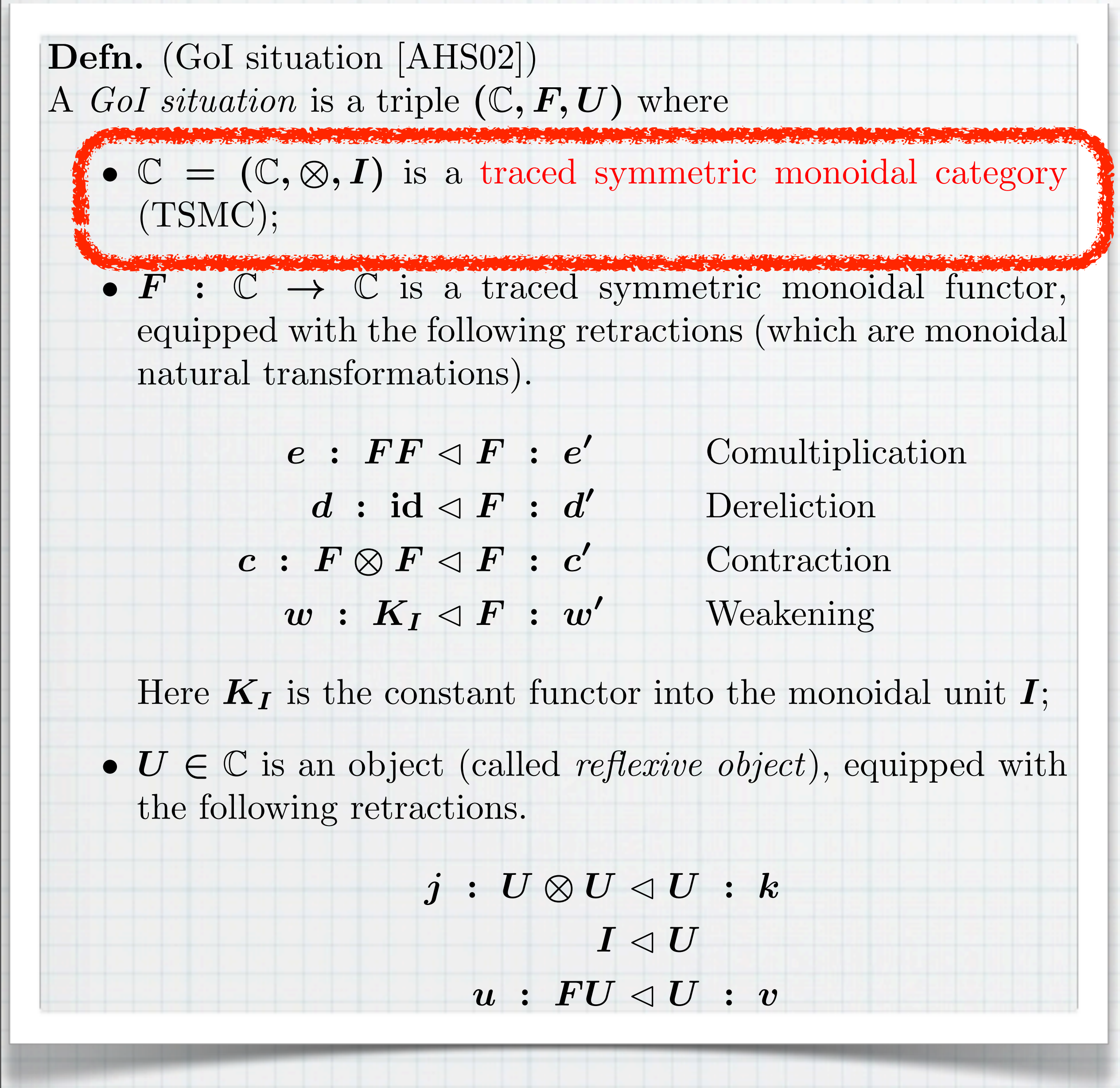


in string diagram

GoI situation



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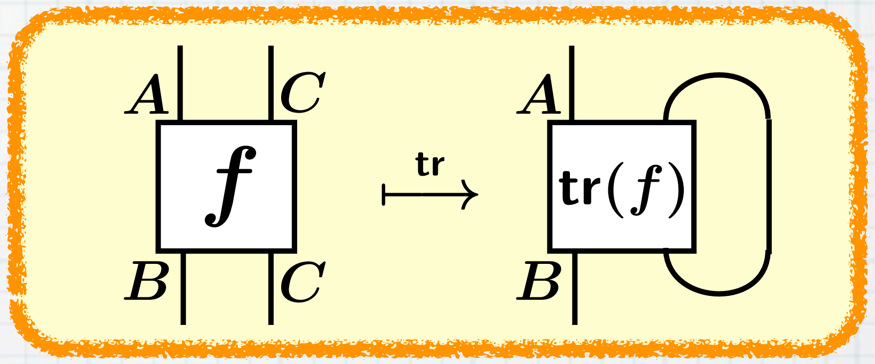
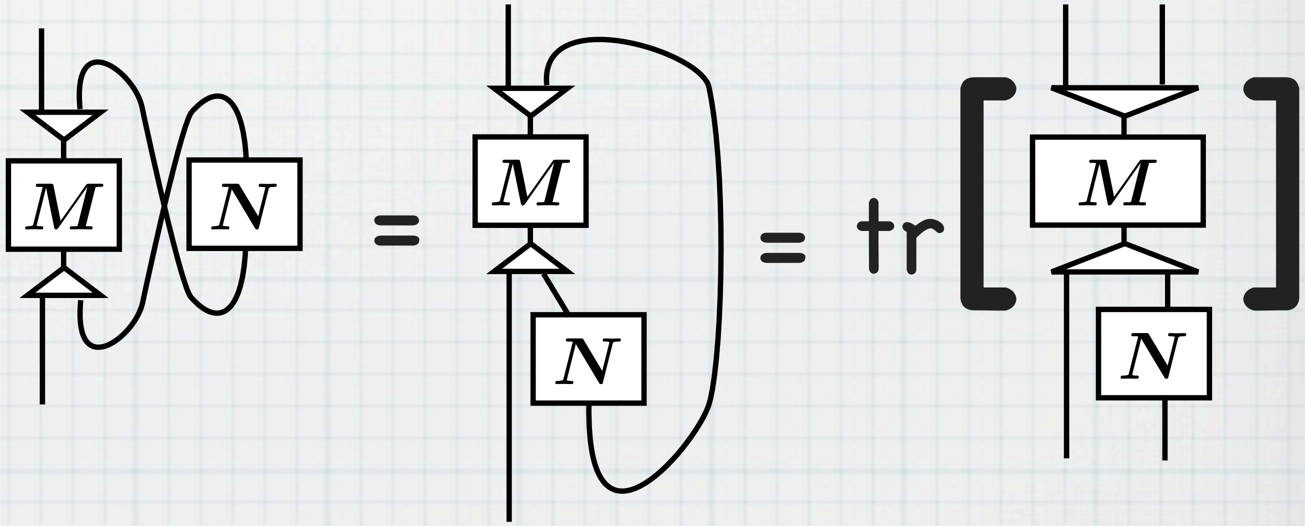
- Traced sym. monoidal cat.
  - Where one can "feedback"



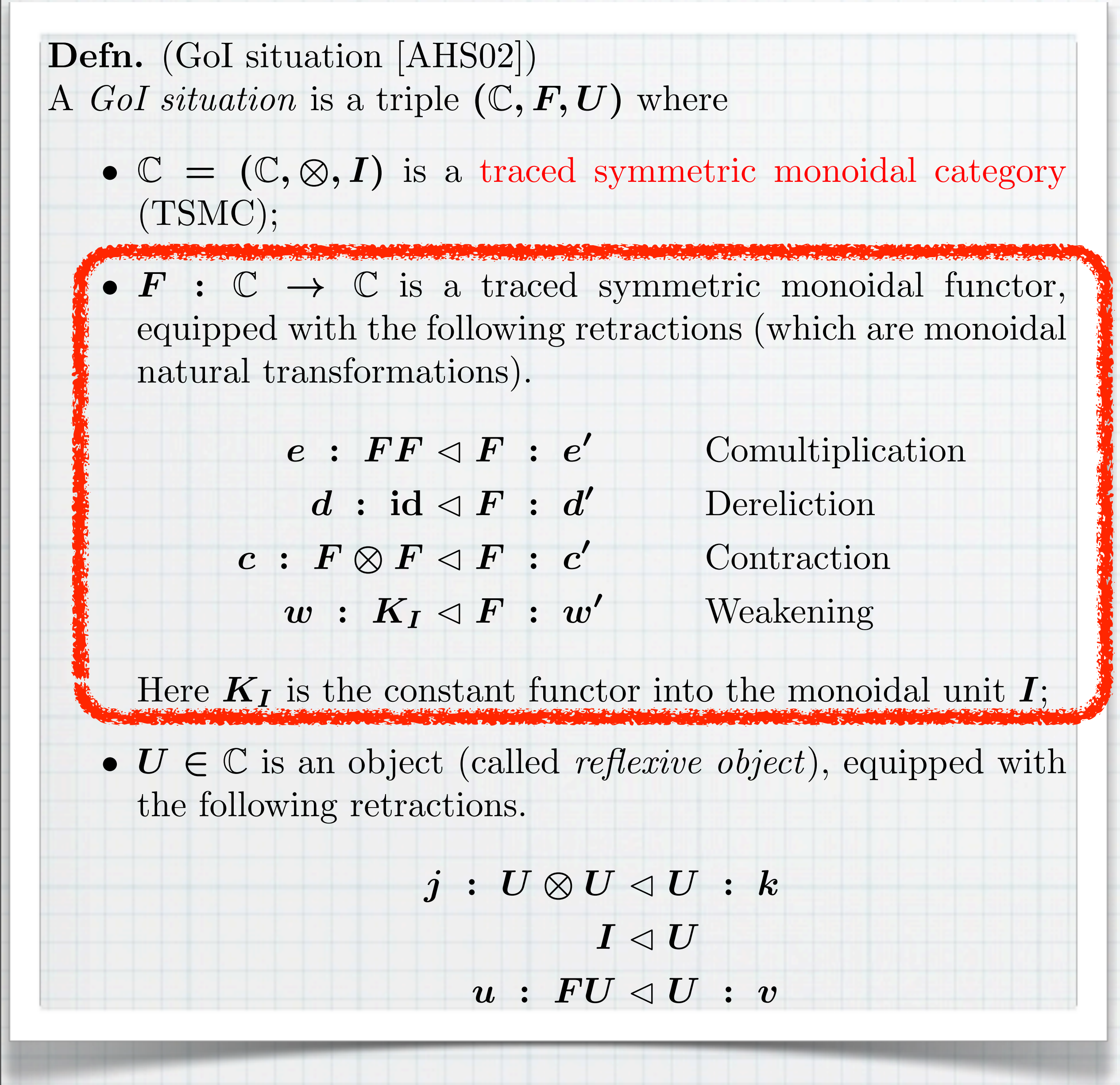
  - Why for GoI?



- Leading example: **Pfn**

Hasuo (Tokyo)

GoI situation



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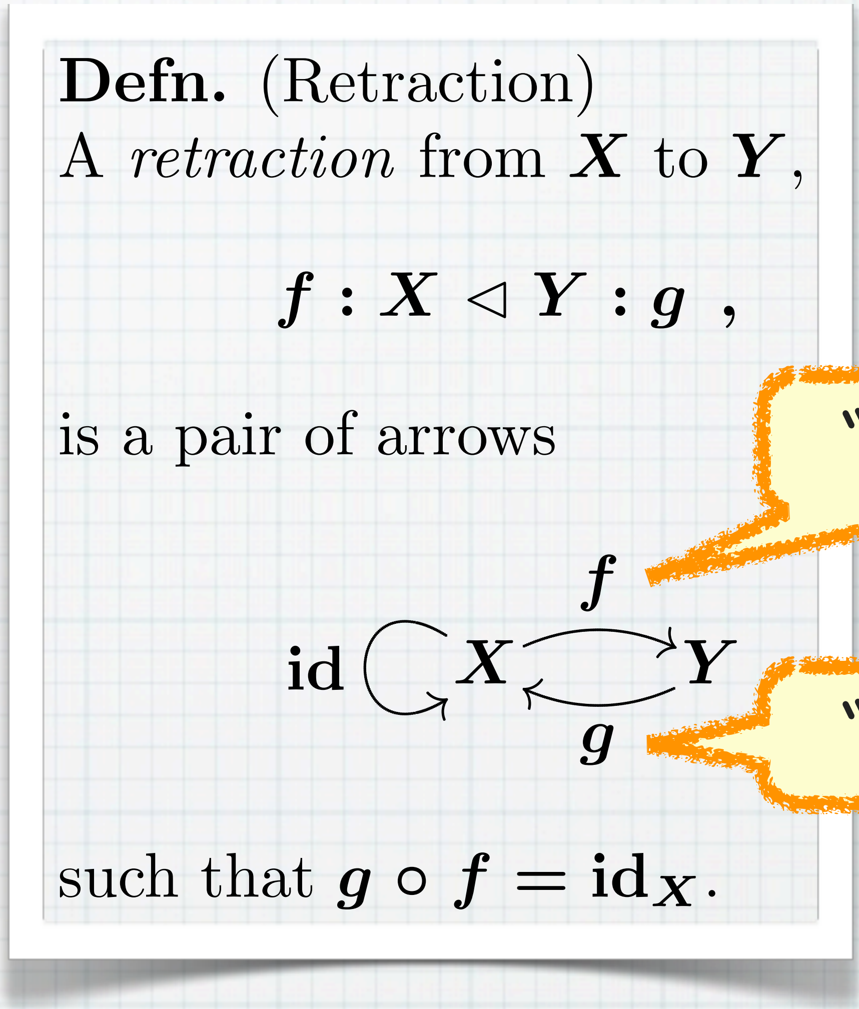


Defn. (Retraction)
A *retraction* from $\boldsymbol{X}$ to $\boldsymbol{Y}$,

$$\boldsymbol{f} : \boldsymbol{X} \lhd \boldsymbol{Y} : \boldsymbol{g} \ ,$$

is a pair of arrows

$$\begin{tikzcd} \boldsymbol{X} \arrow[loop left, "\mathbf{id}"] \arrow[r, bend left, "\boldsymbol{f}"] & \boldsymbol{Y} \arrow[l, bend left, "\boldsymbol{g}"] \end{tikzcd}$$

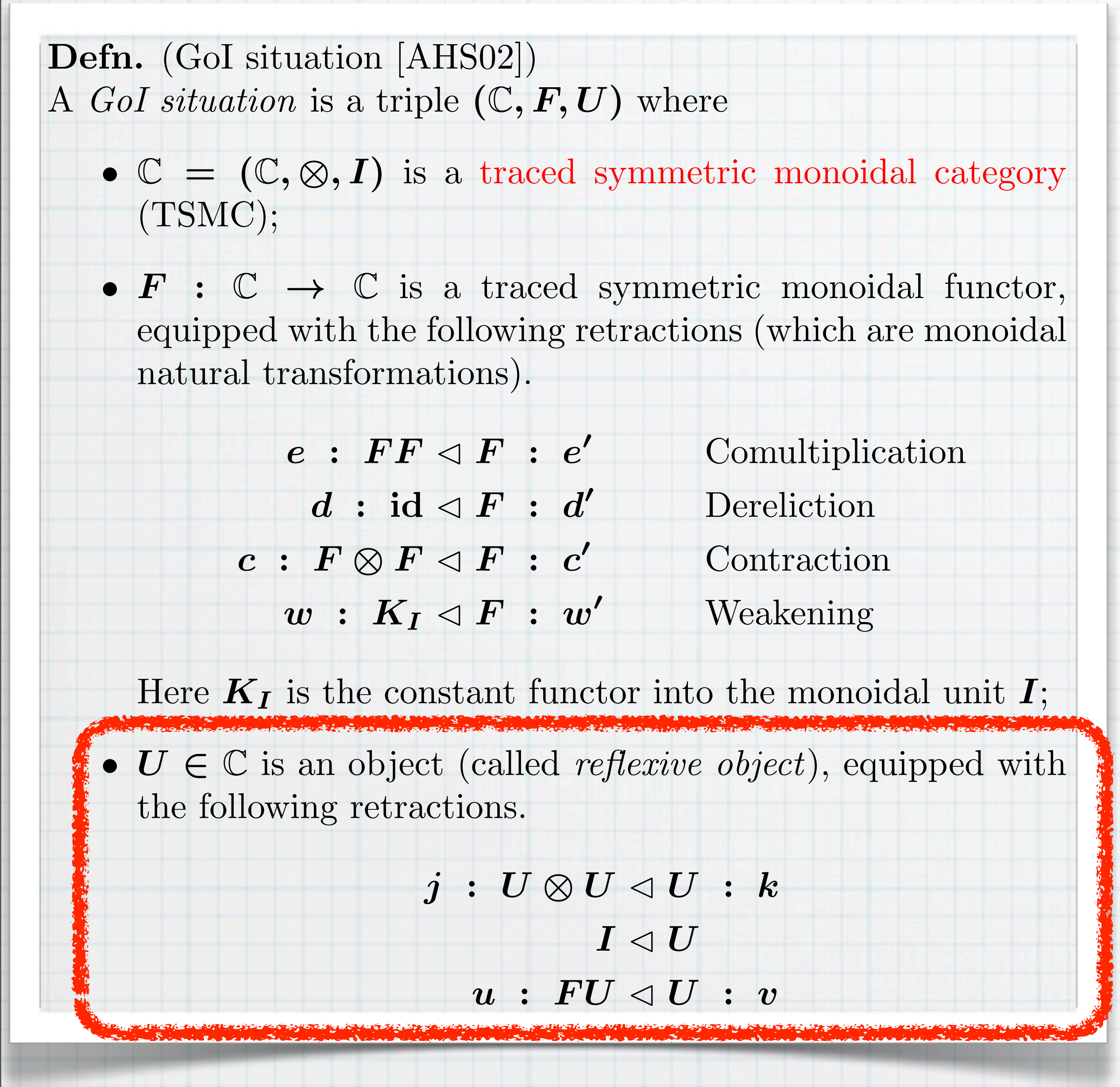
such that $\boldsymbol{g} \circ \boldsymbol{f} = \mathbf{id}_{\boldsymbol{X}}$.

"embedding"

"projection"

- ✱ Functor F
  - ✱ For obtaining $! : A \to A$

GoI situation



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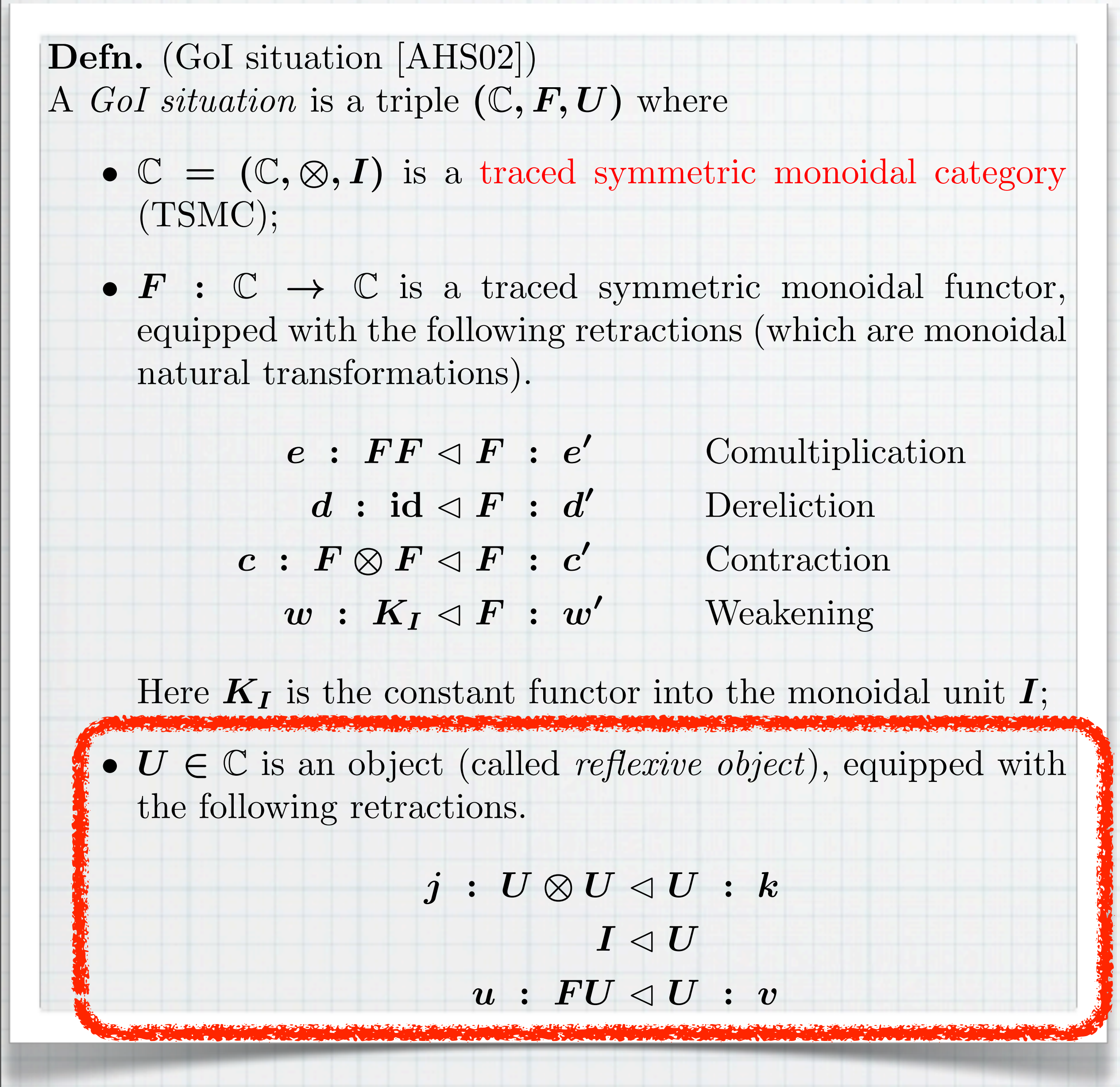
$$\begin{aligned} \boldsymbol{j} : \boldsymbol{U} \otimes \boldsymbol{U} \lhd \boldsymbol{U} &: \boldsymbol{k} \\ \boldsymbol{I} \lhd \boldsymbol{U} & \\ \boldsymbol{u} : \boldsymbol{F}\boldsymbol{U} \lhd \boldsymbol{U} &: \boldsymbol{v} \end{aligned}$$

- The reflexive object U
  - Retr. $$\begin{tikzcd} \boldsymbol{U} \otimes \boldsymbol{U} \arrow[r, bend left, "\boldsymbol{j}"] & \boldsymbol{U} \arrow[l, bend left, "\boldsymbol{k}"] \end{tikzcd}$$

GoI situation



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$$\begin{array}{rcll} e : FF \lhd F : e' & & \text{Comultiplication} \\ d : \mathrm{id} \lhd F : d' & & \text{Dereliction} \\ c : F \otimes F \lhd F : c' & & \text{Contraction} \\ w : K_I \lhd F : w' & & \text{Weakening} \end{array}$$

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- The reflexive object U
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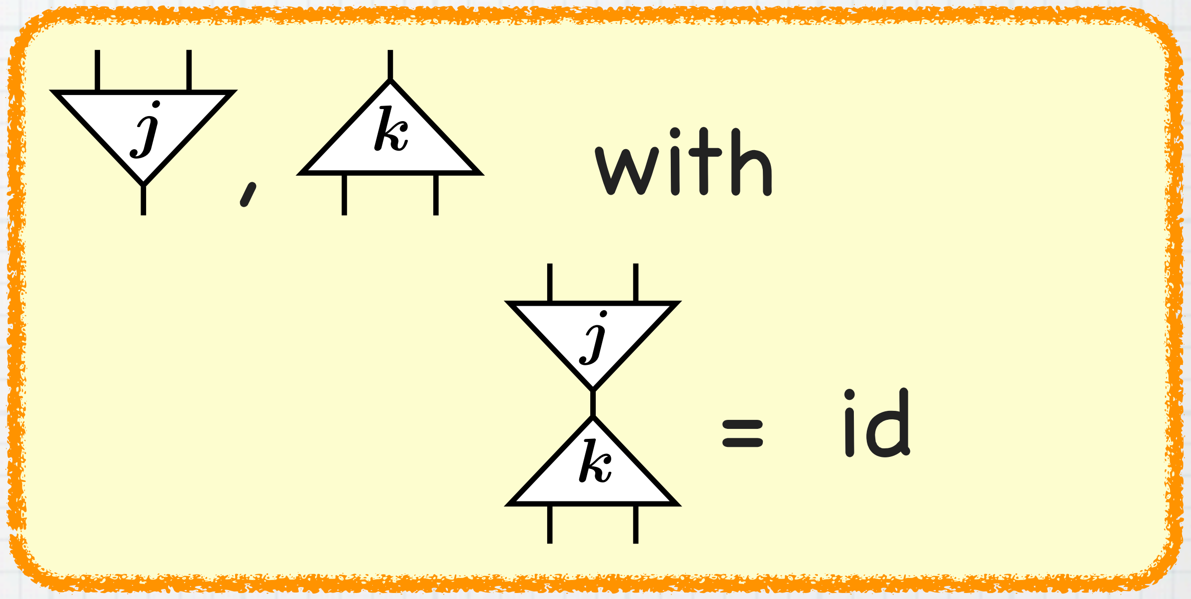
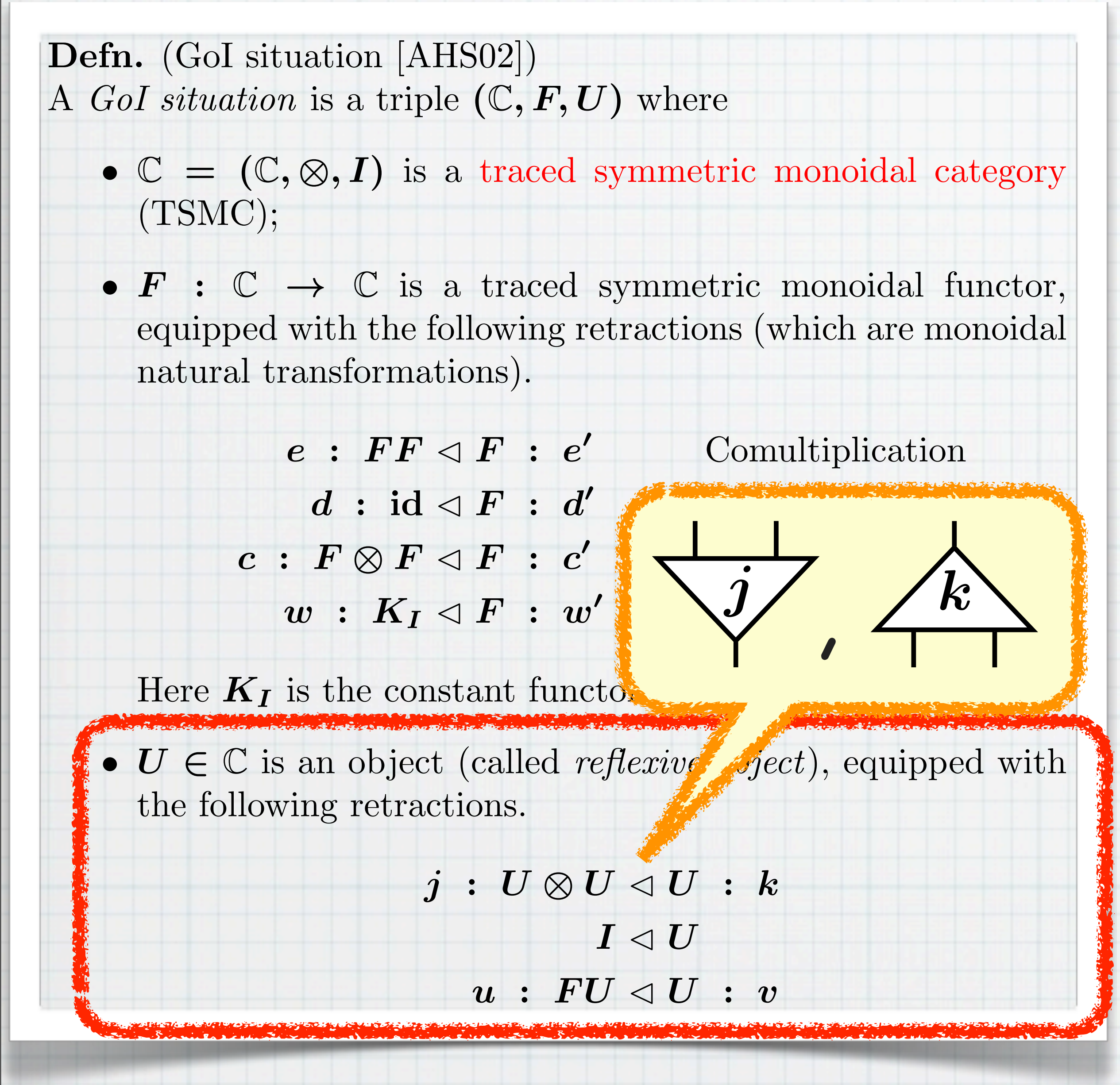


with

= id

GoI situation



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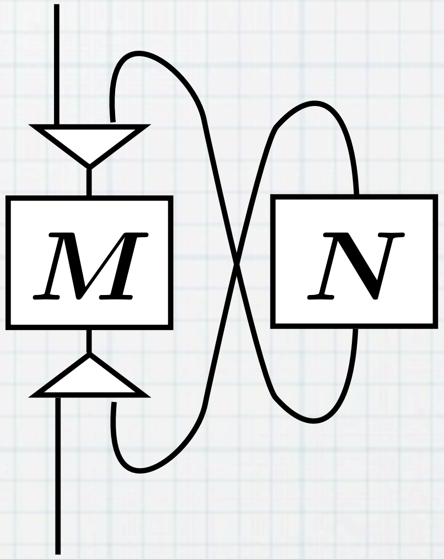
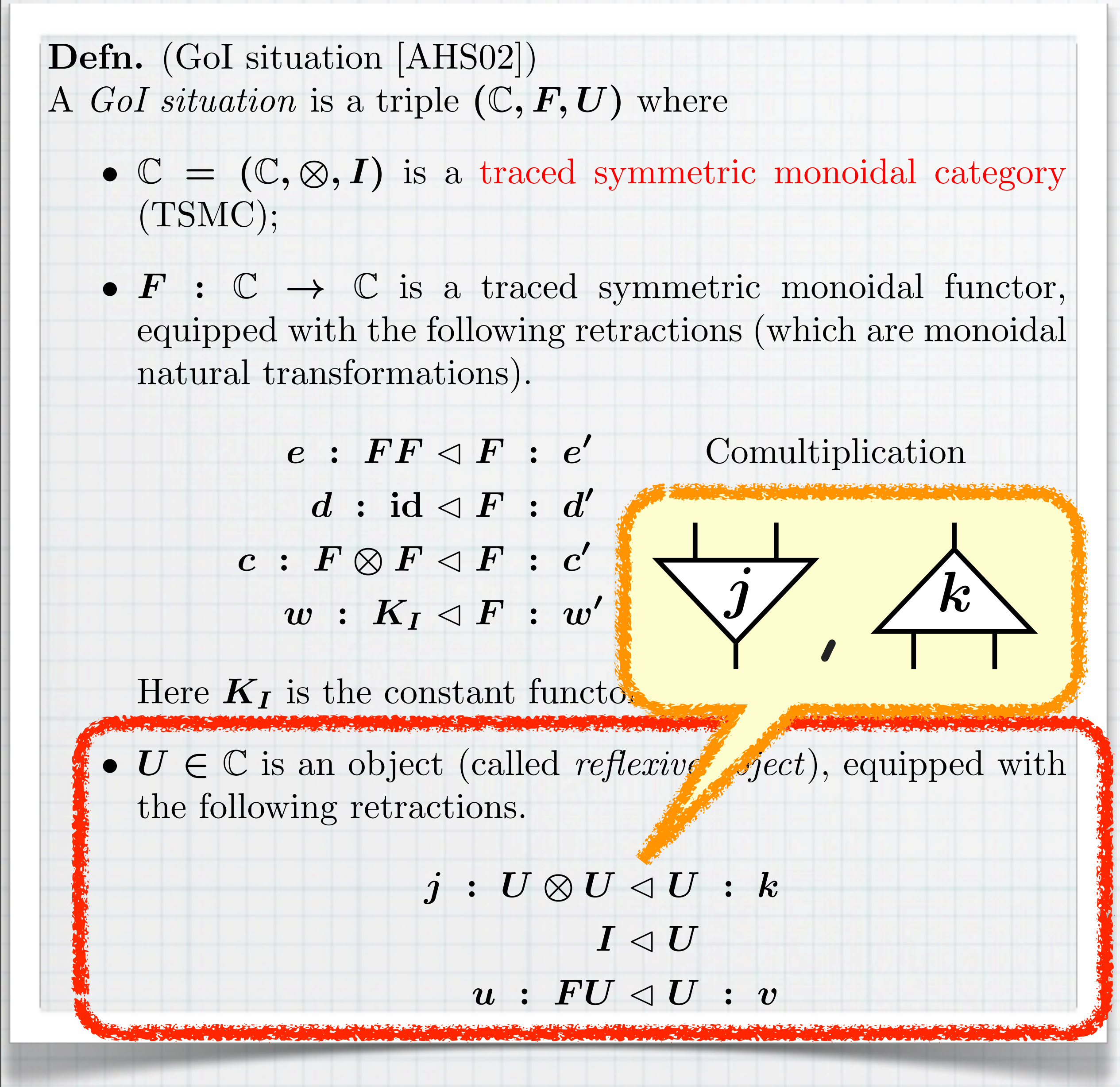
- The reflexive object U
  - Why for GoI?



  - Example in **Pfn**:

GoI situation



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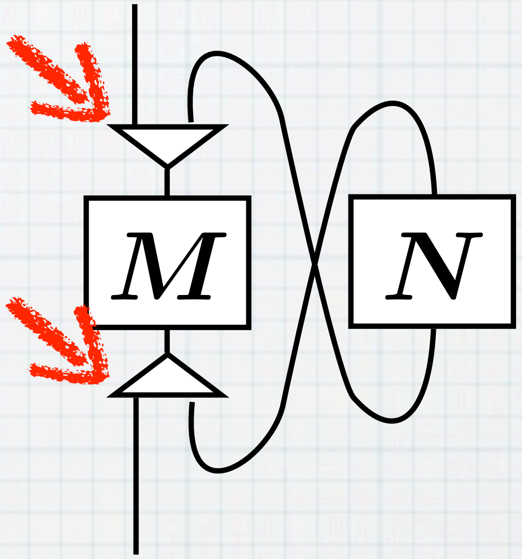

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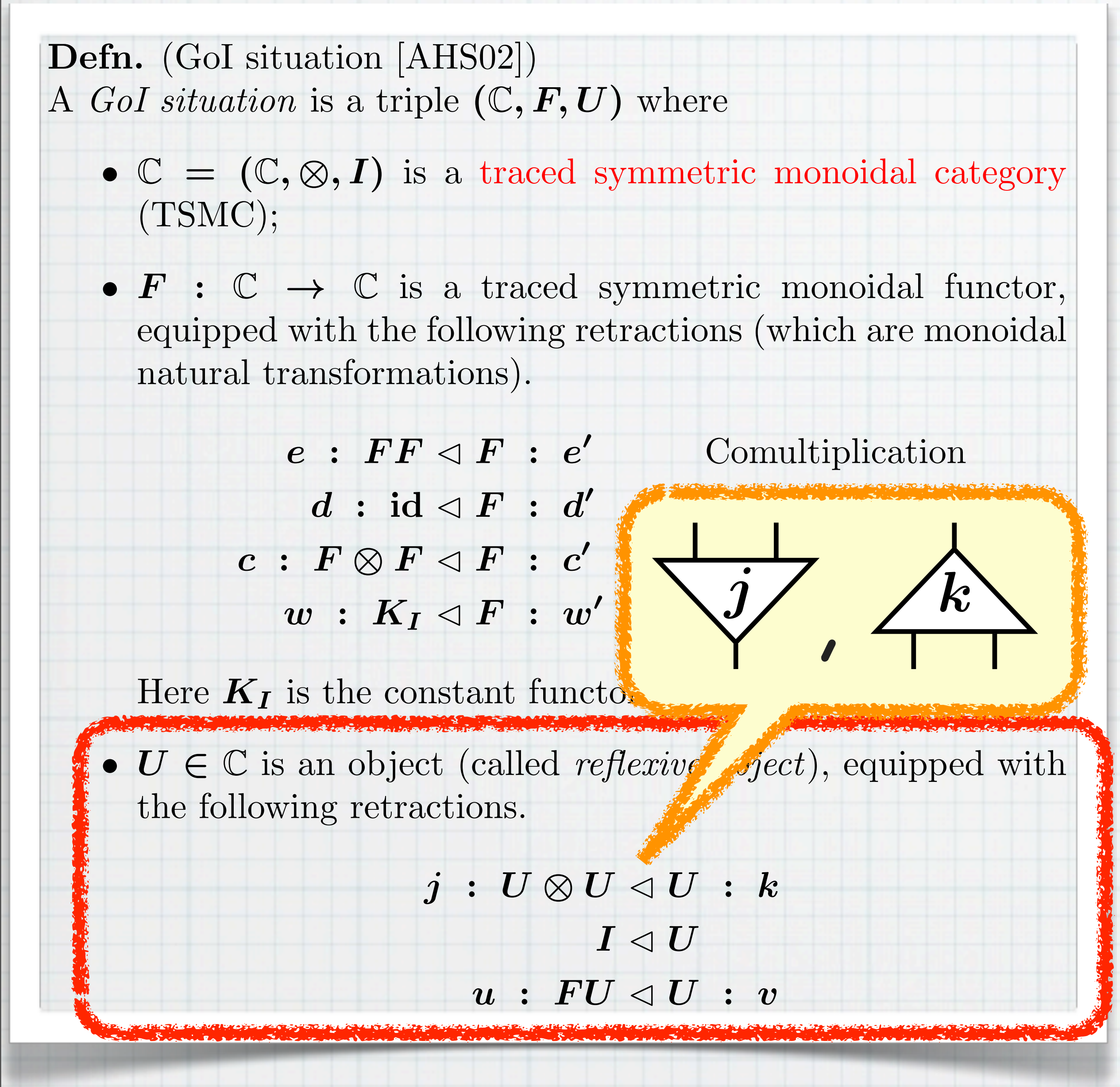

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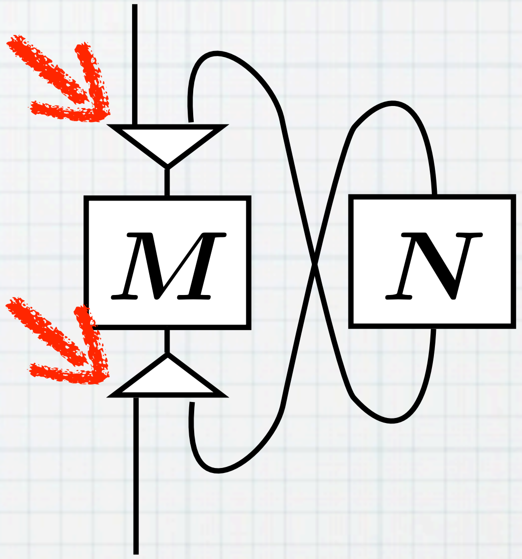
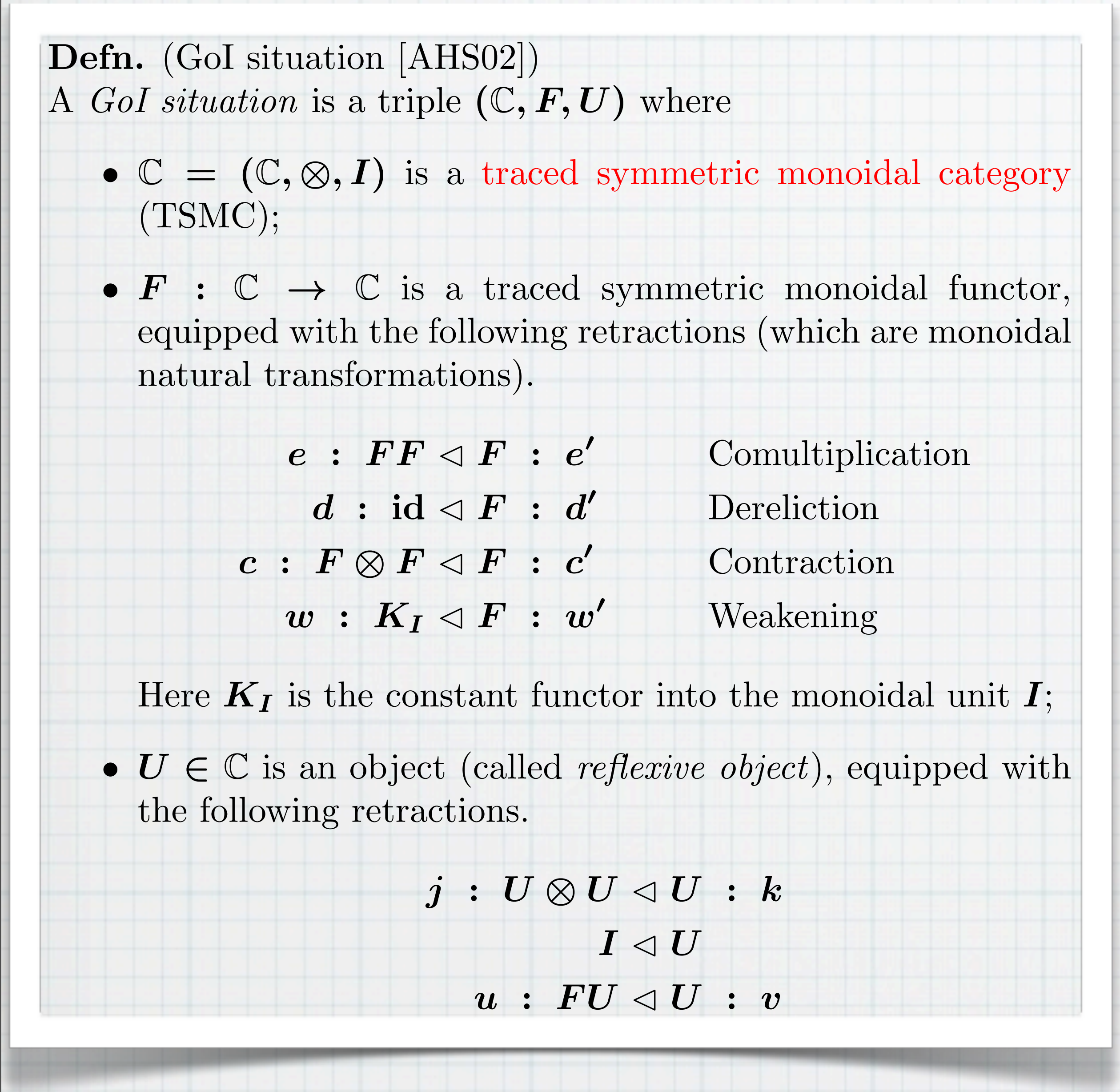


- Example in **Pfn**:

$\mathbb{N} \in \mathbf{Pfn}$, with

$$\mathbb{N} + \mathbb{N} \cong \mathbb{N},$$
$$\mathbb{N} \cdot \mathbb{N} \cong \mathbb{N}$$

GoI Situation: Summary



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$\boldsymbol{c} : \boldsymbol{F} \otimes \boldsymbol{F} \lhd \boldsymbol{F} : \boldsymbol{c}'$	Contraction
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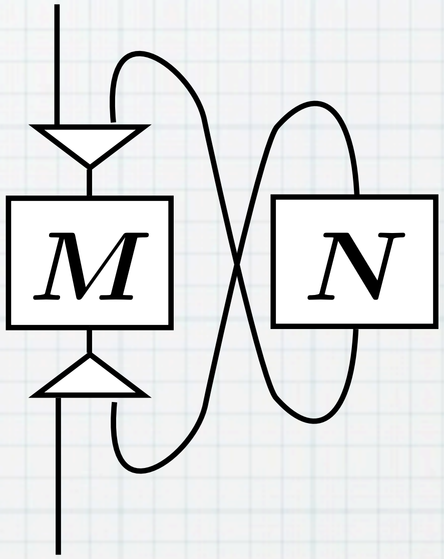
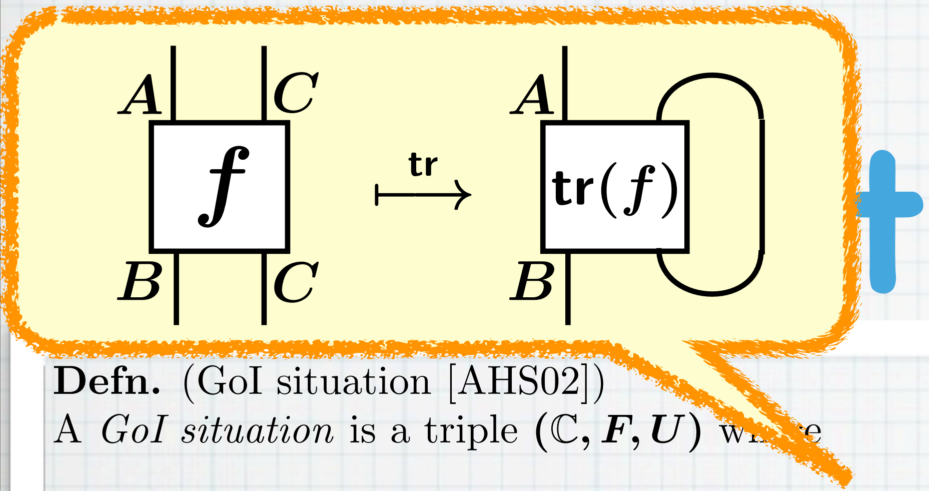
- Categorical axiomatics of the "GoI animation"



- Example:

$$(\mathbf{Pfn}, \mathbb{N} \cdot \_\,, \mathbb{N})$$



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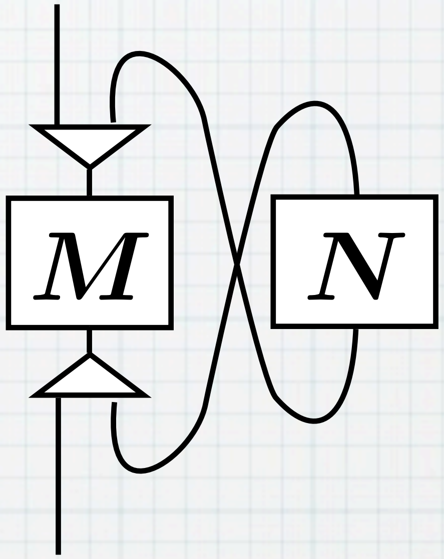

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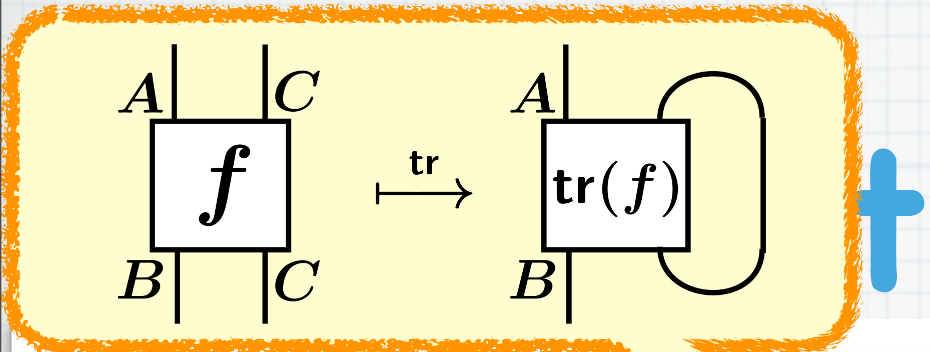

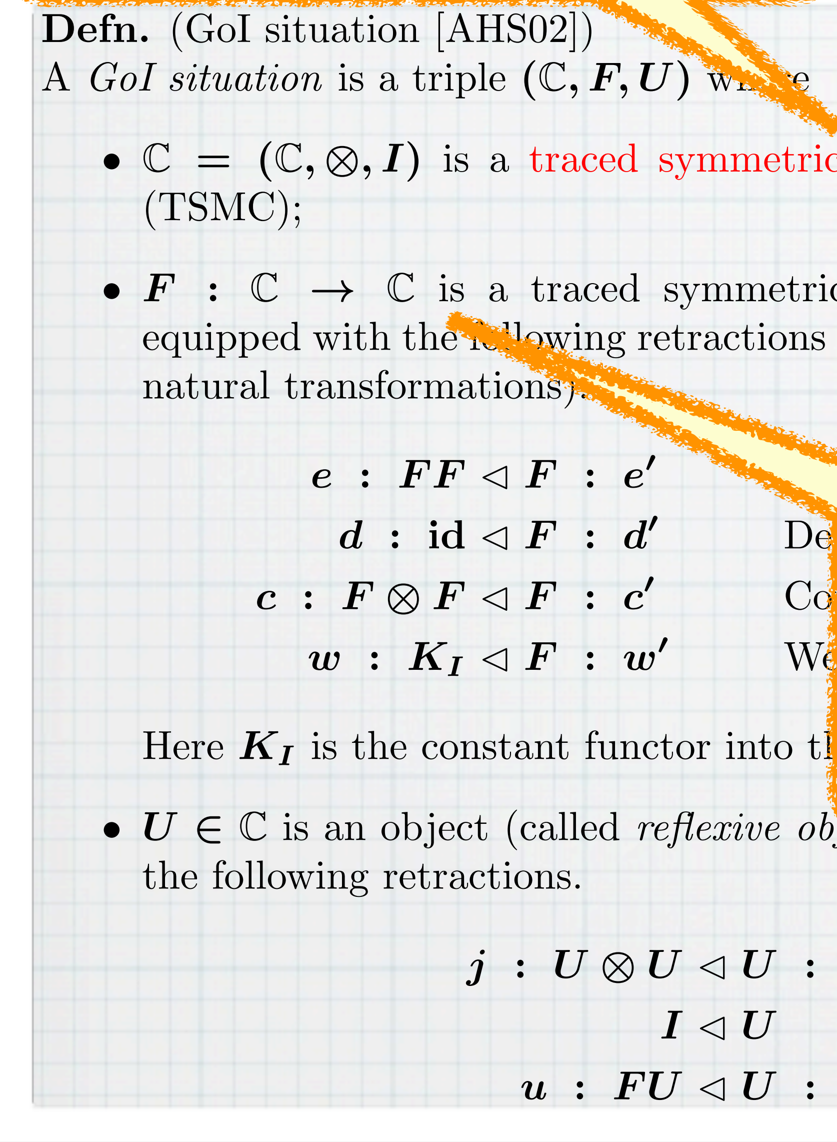
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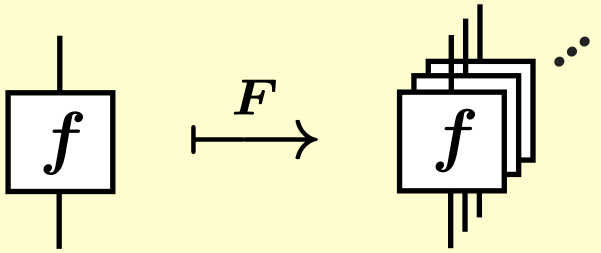

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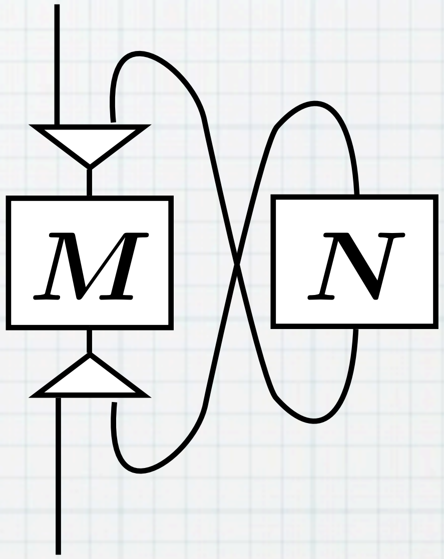

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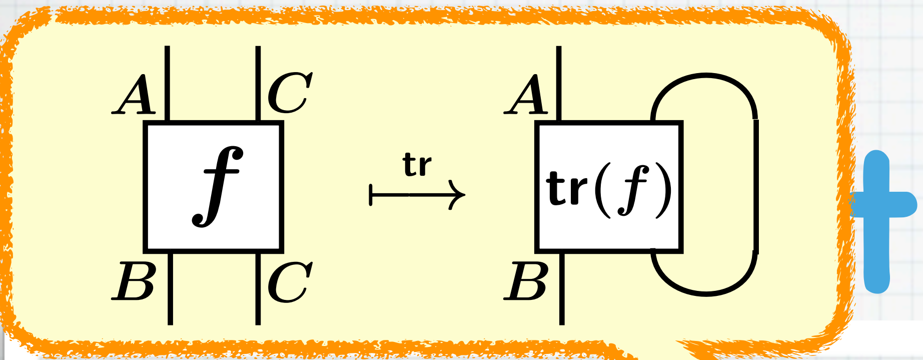

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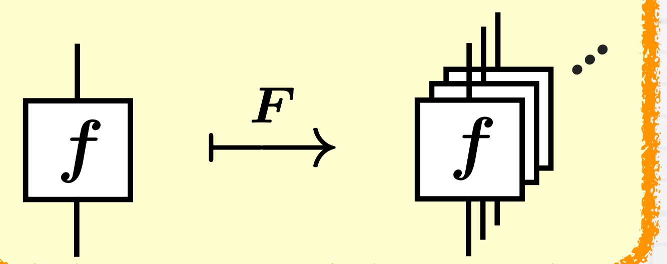

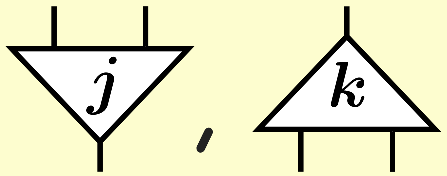

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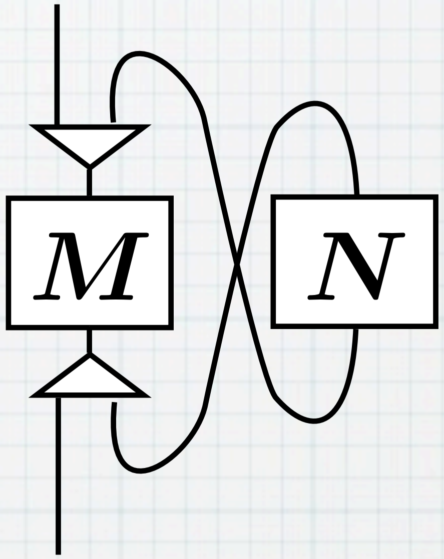

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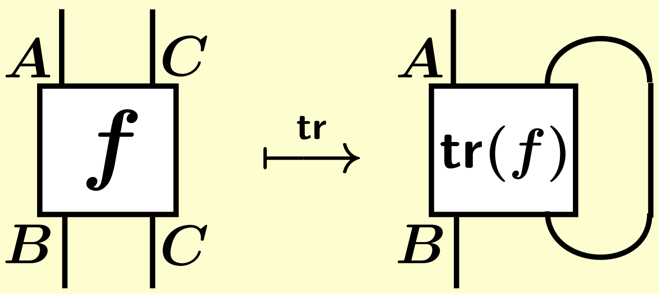

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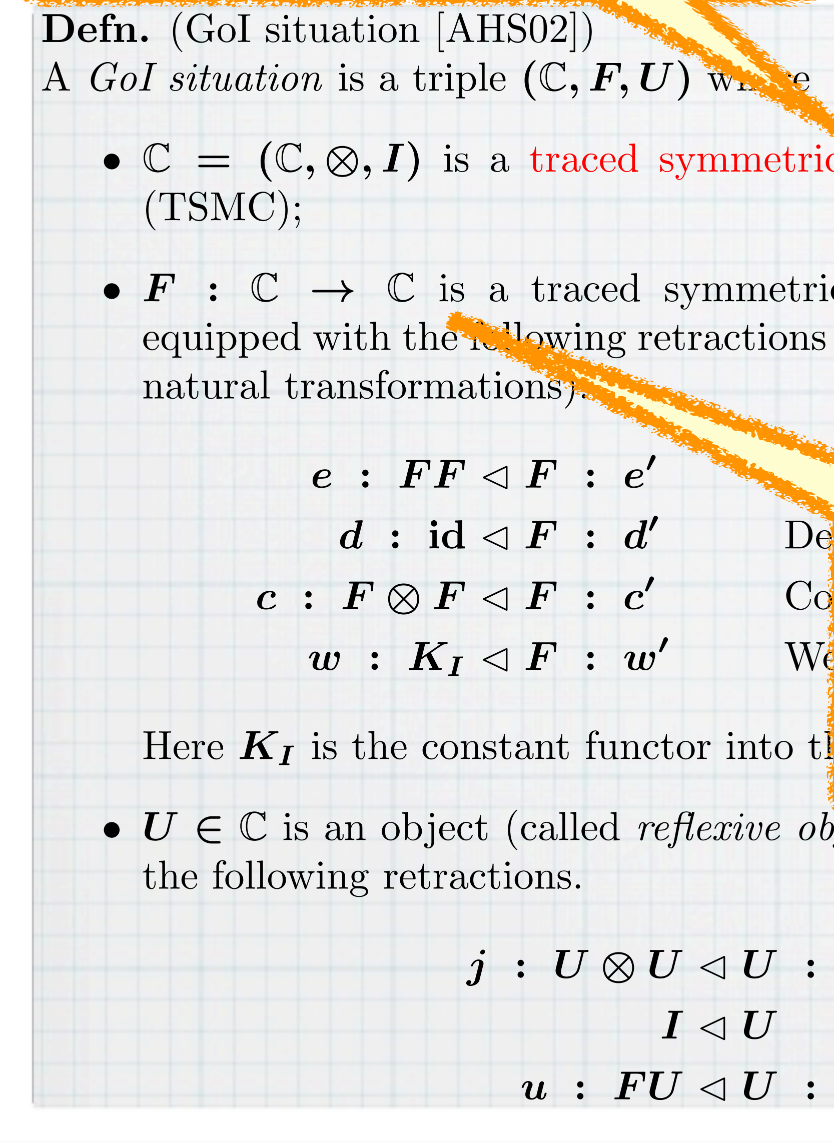

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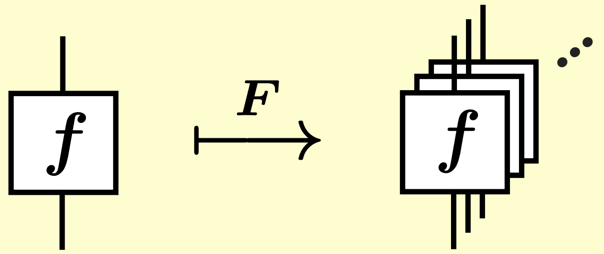

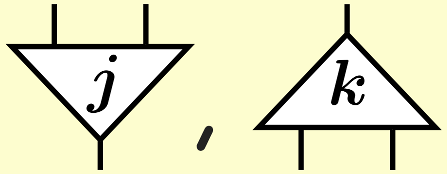

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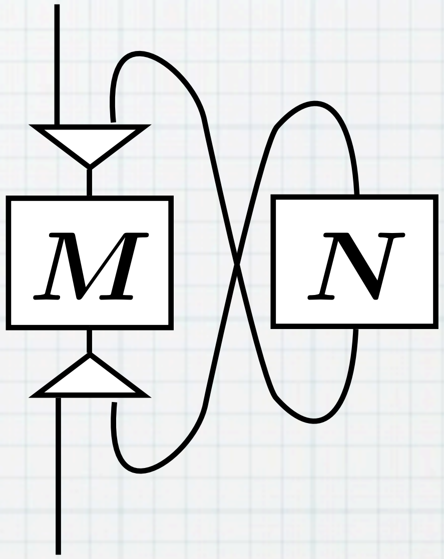

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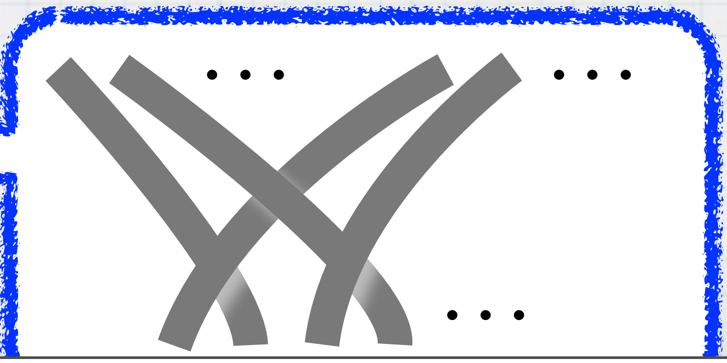

Categorical GoI: Constr. of an LCA

Thm. ([AHS02])
Given a GoI situation $(\mathbb{C}, \boldsymbol{F}, \boldsymbol{U})$, the homset

$$\mathbb{C}(\boldsymbol{U}, \boldsymbol{U})$$

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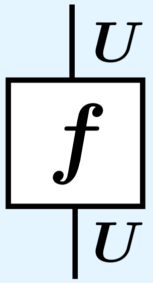

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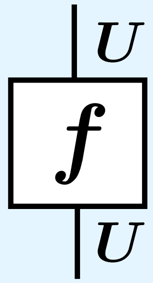
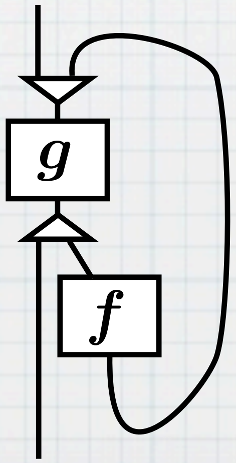


$\in \mathbb{C}(\boldsymbol{U}, \boldsymbol{U})$

- $\boldsymbol{g} \cdot \boldsymbol{f}$

$$:= \mathbf{tr}\Big((\boldsymbol{U} \otimes \boldsymbol{f}) \circ \boldsymbol{k} \circ \boldsymbol{g} \circ \boldsymbol{j} \Big)$$

=



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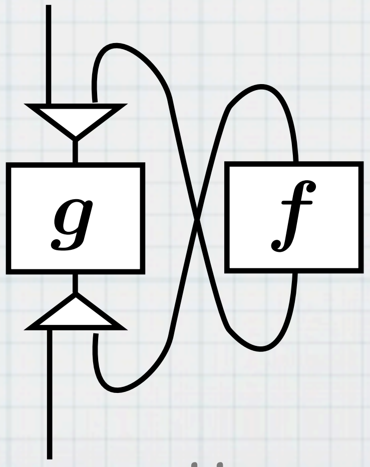

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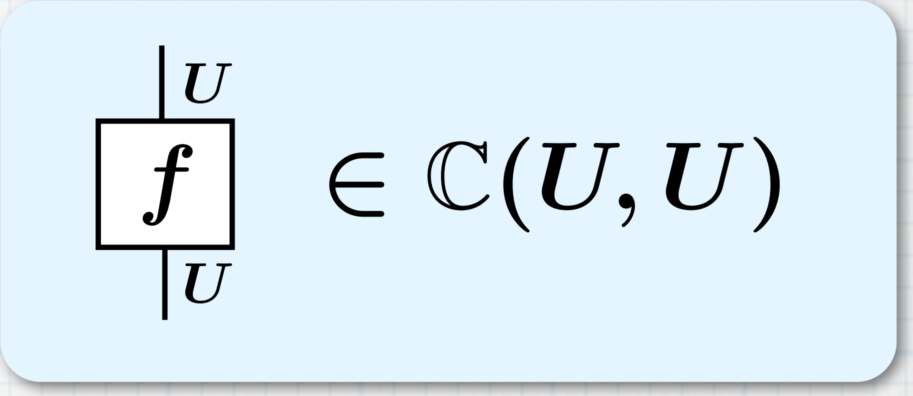
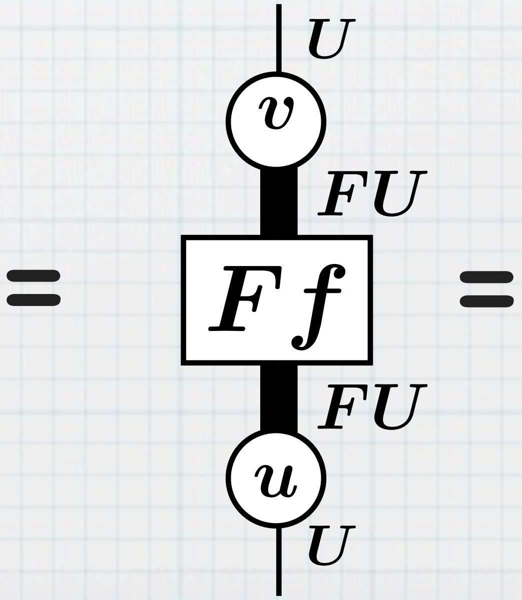
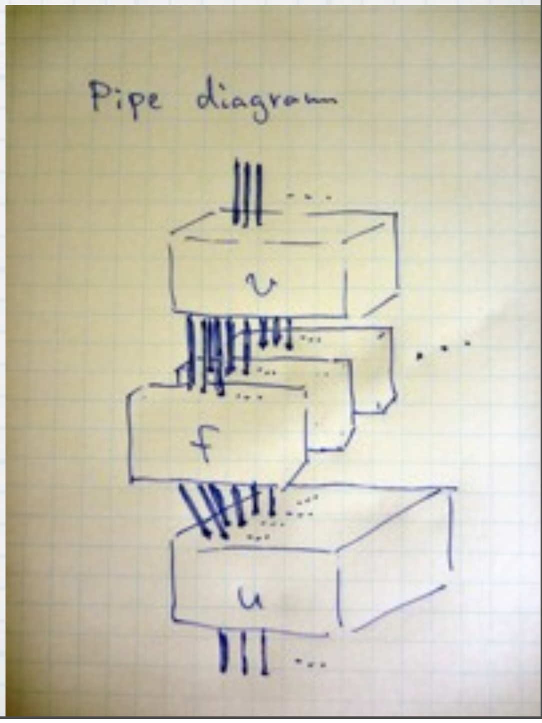


$\in \mathbb{C}(\boldsymbol{U}, \boldsymbol{U})$

- Applicative str. ·
- ! operator
- Combinators B, C, I, ...

- $!\boldsymbol{f} \;:=\; \boldsymbol{u} \circ \boldsymbol{F}\boldsymbol{f} \circ \boldsymbol{v}$





Categorical GoI: Constr. of an LCA

- Combinator $Bxyz = x(yz)$

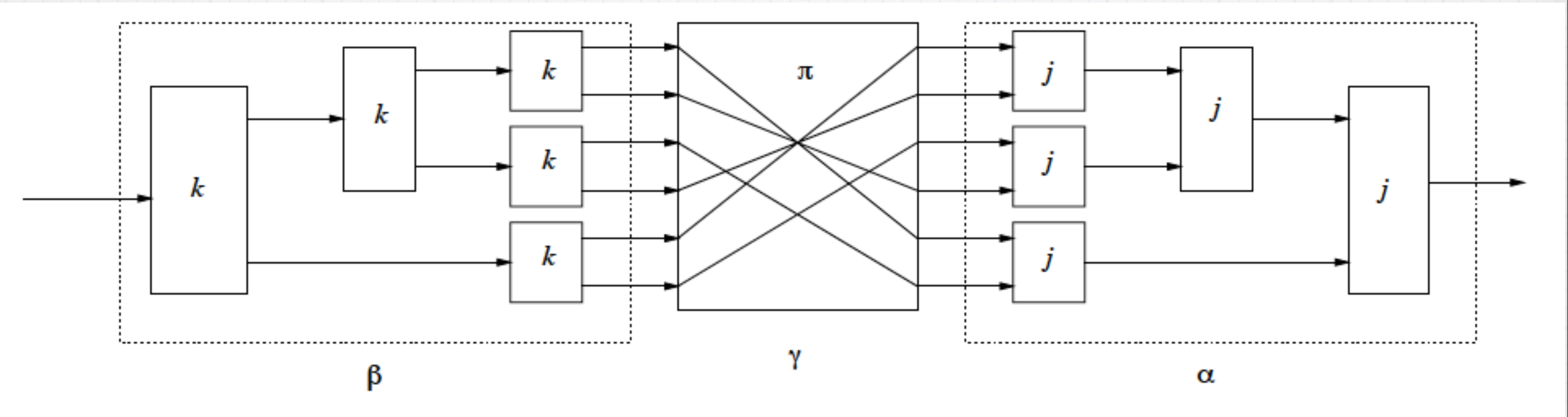


Figure 7: Composition Combinator B

from [AHS02]

Categorical GoI: Constr. of an LCA

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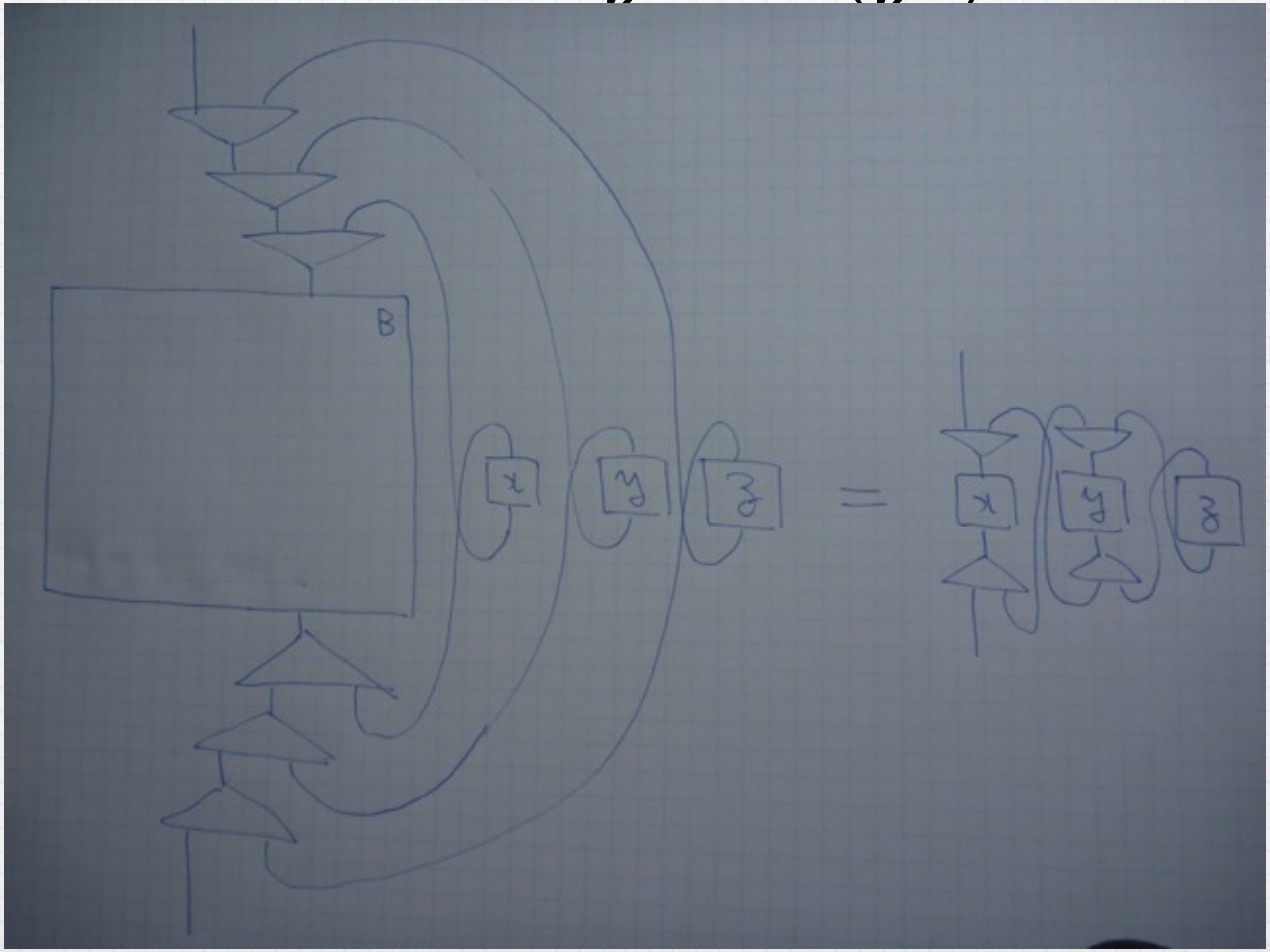
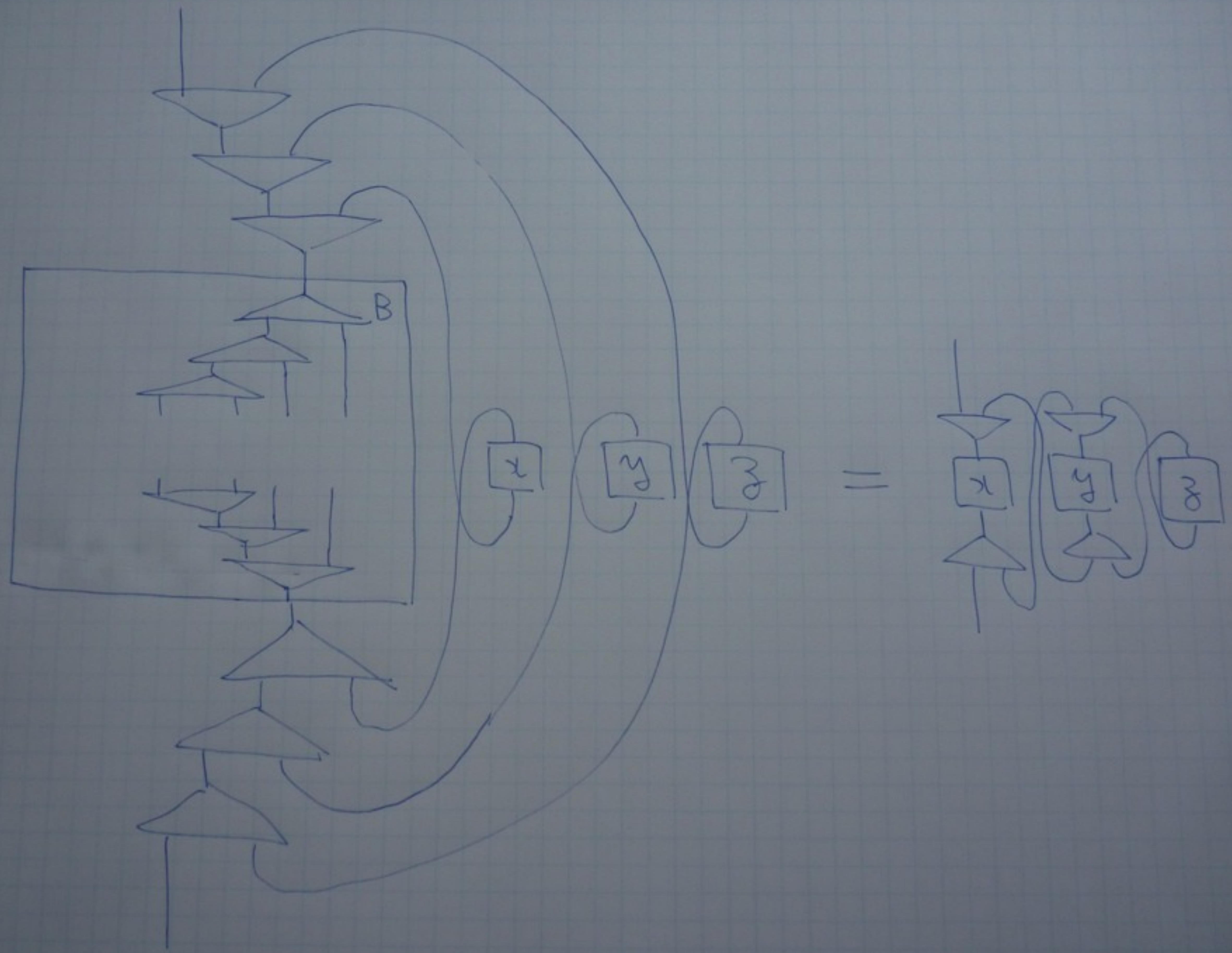
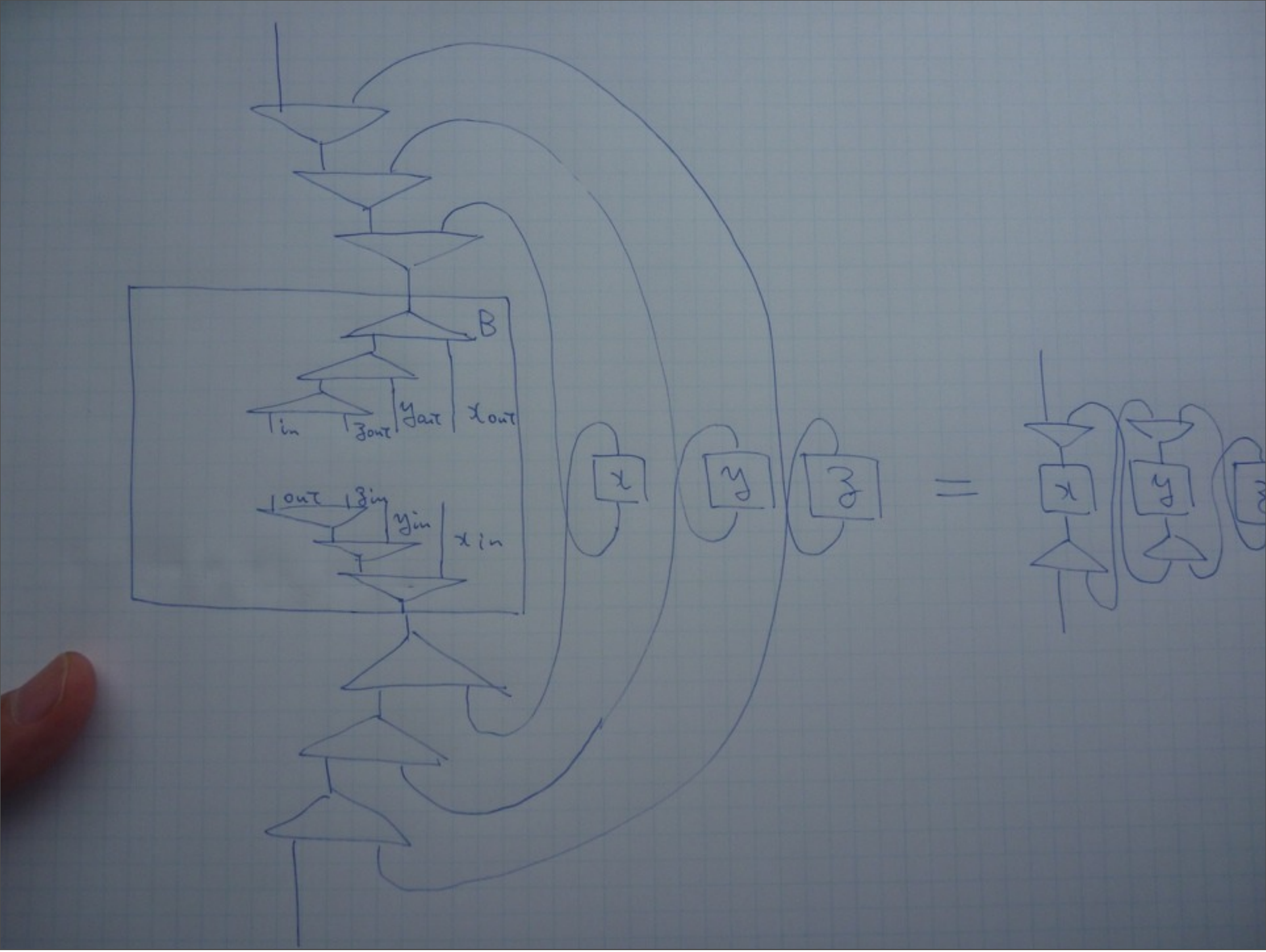



B
x
y
z
=
x
y
z


B
in
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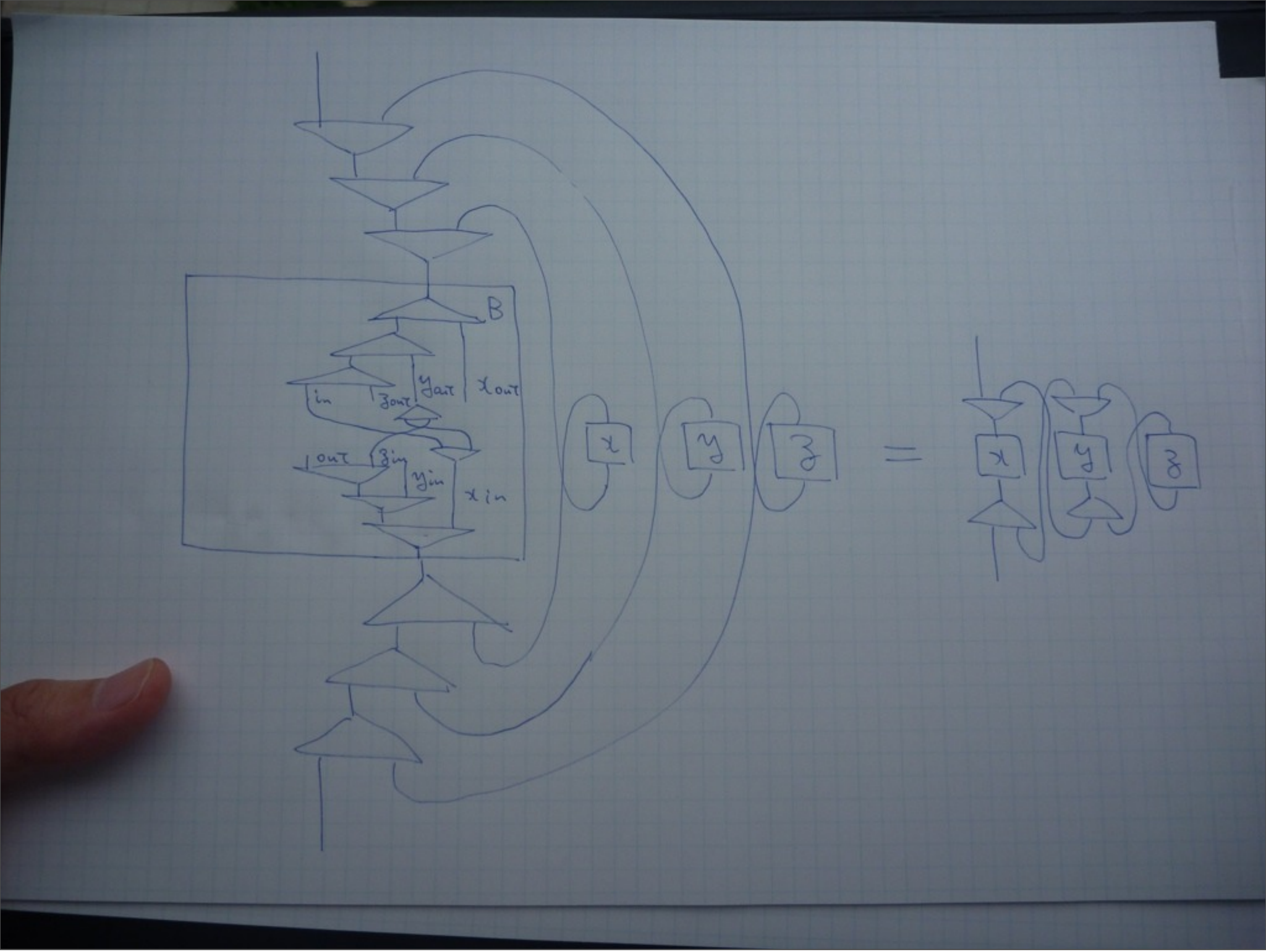
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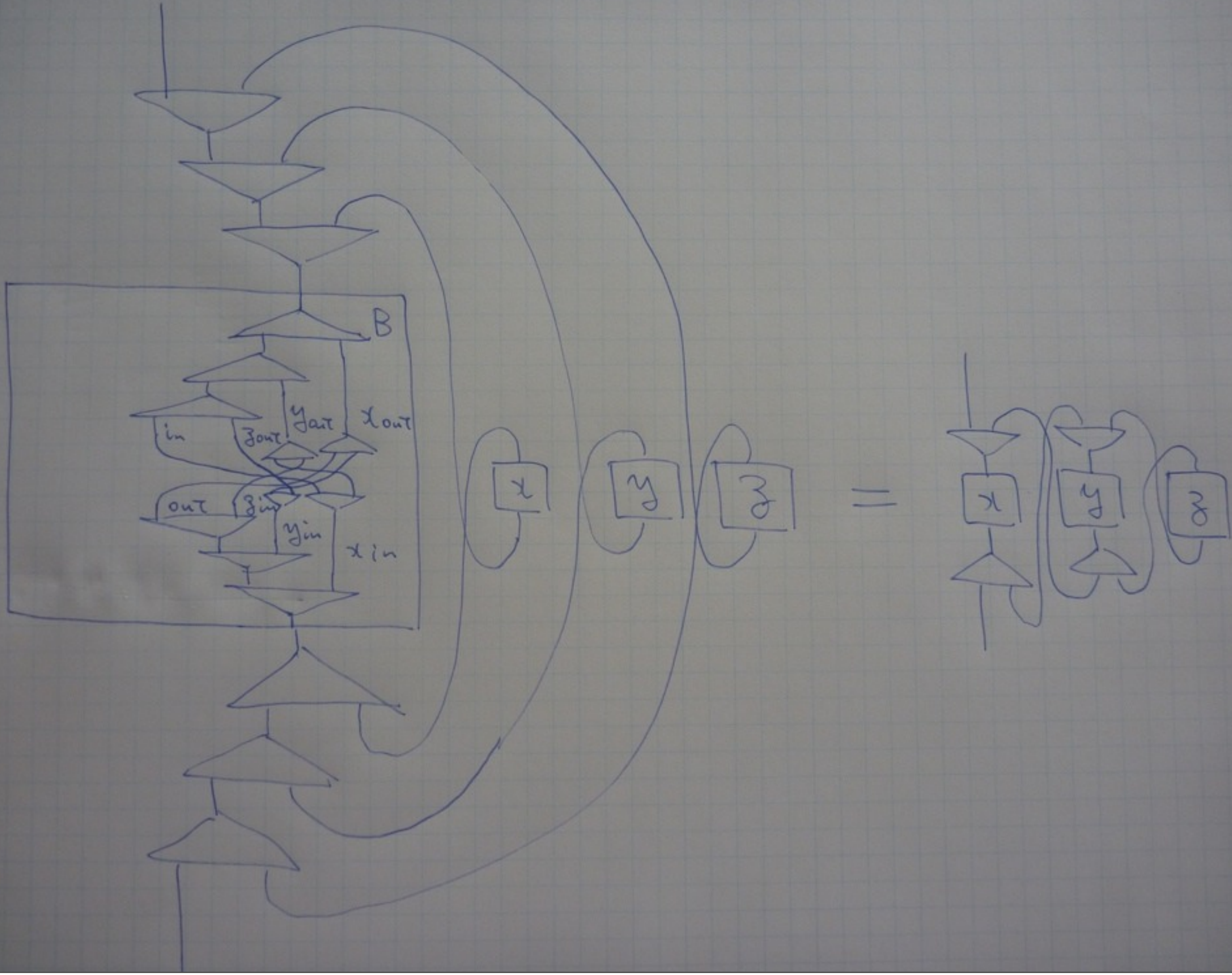
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Categorical GoI: Constr. of an LCA

- Combinator $Bxyz = x(yz)$

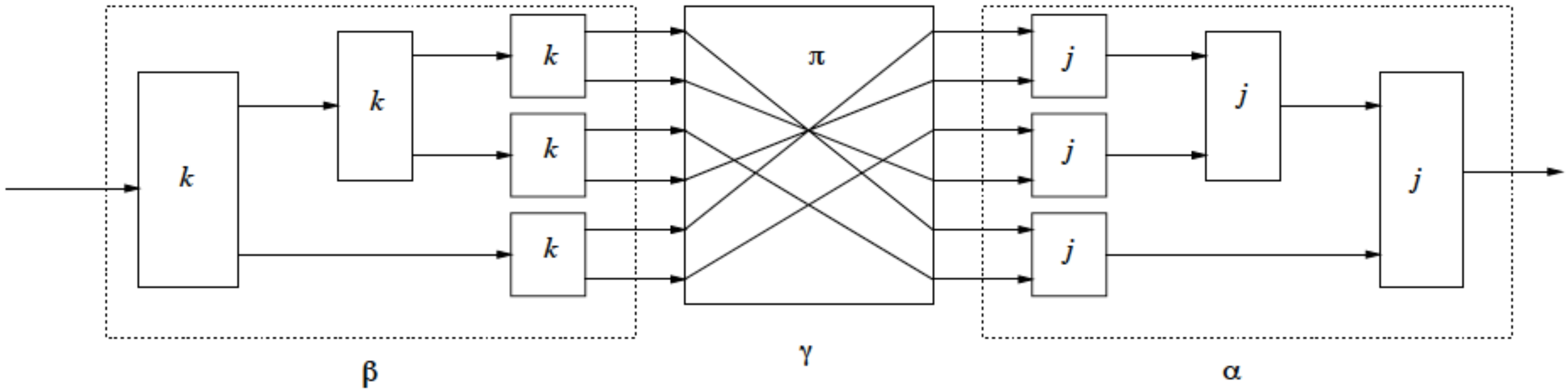

Figure 7: Composition Combinator B

from [AHS02]

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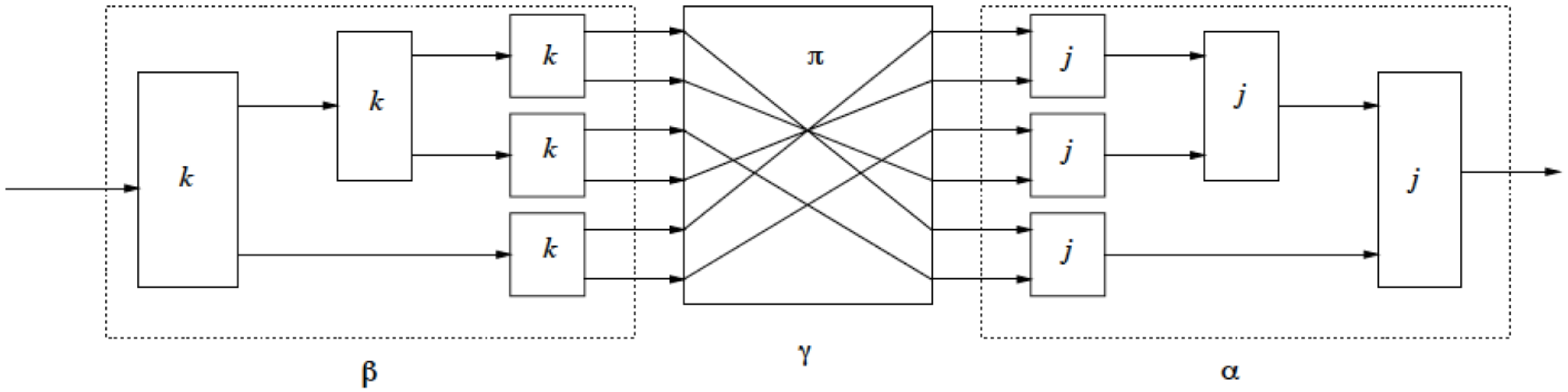


Figure 7: Composition Combinator B

from [AHS02]

Nice dynamic interpretation of (linear) computation!!

Summary: Categorical GoI

Defn. (GoI situation [AHS02])
A *GoI situation* is a triple $(\mathbb{C}, \boldsymbol{F}, \boldsymbol{U})$ where

- $\mathbb{C} = (\mathbb{C}, \otimes, \boldsymbol{I})$ is a traced symmetric monoidal category (TSMC);
- $\boldsymbol{F} : \mathbb{C} \to \mathbb{C}$ is a traced symmetric monoidal functor, equipped with the following retractions (which are monoidal natural transformations).

$$\begin{array}{rcccl}
\boldsymbol{e} & : & \boldsymbol{FF} \lhd \boldsymbol{F} & : & \boldsymbol{e'} \qquad \text{Comultiplication} \\
\boldsymbol{d} & : & \mathbf{id} \lhd \boldsymbol{F} & : & \boldsymbol{d'} \qquad \text{Dereliction} \\
\boldsymbol{c} & : & \boldsymbol{F} \otimes \boldsymbol{F} \lhd \boldsymbol{F} & : & \boldsymbol{c'} \qquad \text{Contraction} \\
\boldsymbol{w} & : & \boldsymbol{K_I} \lhd \boldsymbol{F} & : & \boldsymbol{w'} \qquad \text{Weakening}
\end{array}$$

  Here $\boldsymbol{K_I}$ is the constant functor into the monoidal unit $\boldsymbol{I}$;
- $\boldsymbol{U} \in \mathbb{C}$ is an object (called *reflexive object*), equipped with the following retractions.

$$\begin{array}{rcccl}
\boldsymbol{j} & : & \boldsymbol{U} \otimes \boldsymbol{U} \lhd \boldsymbol{U} & : & \boldsymbol{k} \\
 & & \boldsymbol{I} \lhd \boldsymbol{U} & & \\
\boldsymbol{u} & : & \boldsymbol{FU} \lhd \boldsymbol{U} & : & \boldsymbol{v}
\end{array}$$

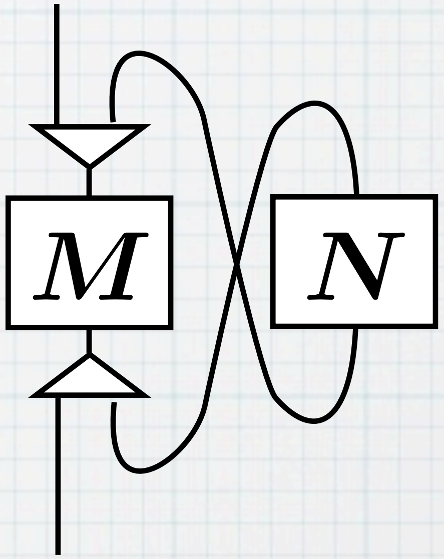
Thm. ([AHS02])
Given a GoI situation $(\mathbb{C}, \boldsymbol{F}, \boldsymbol{U})$, the homset

$$\mathbb{C}(\boldsymbol{U}, \boldsymbol{U})$$

carries a canonical LCA structure.

Why Categorical Generalization?: Examples Other Than Pfn [AHS02]

- Strategy: find a TSMC!



- "Wave-style" examples
  - ⊗ is Cartesian product(-like)
    - in which case,

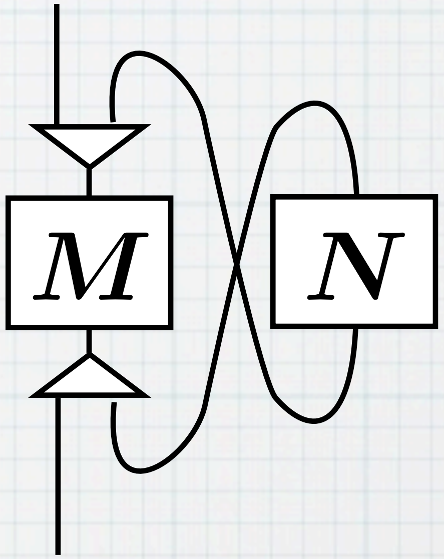
      trace ≈ fixed point operator [Hasegawa/Hyland]
  - An example: $\big(\,(\omega\text{-}\mathbf{Cpo}, \times, 1),\ (\_)^{\mathbb{N}},\ A^{\mathbb{N}}\,\big)$
  - (... less of a dynamic flavor)

Hasuo (Tokyo)

Monday, November 7, 2011

Why Categorical Generalization?: Examples Other Than Pfn [AHS02]

- "Particle-style" examples
  - Obj. X∈C is set-like; ⊗ is coproduct-like
    - The GoI animation is valid



  - Examples:
    - Partial functions $\big(\, (\mathbf{Pfn}, +, \mathbf{0}),\; \mathbb{N}\cdot\_\,,\; \mathbb{N} \,\big)$
    - Binary relations $\big(\, (\mathbf{Rel}, +, \mathbf{0}),\; \mathbb{N}\cdot\_\,,\; \mathbb{N} \,\big)$
    - "Discrete stochastic relations" $\big(\, (\mathbf{DSRel}, +, \mathbf{0}),\; \mathbb{N}\cdot\_\,,\; \mathbb{N} \,\big)$

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$$\begin{array}{c} X \to Y \text{ in } \mathbf{Pfn} \\ \hline\hline X \rightharpoonup Y, \text{ partial function} \\ \hline\hline X \to \mathcal{L}Y \text{ in } \mathbf{Sets} \end{array} \quad \text{where } \mathcal{L}Y = \{\bot\} + Y$$

- **Rel** (relations)

$$\begin{array}{c} X \to Y \text{ in } \mathbf{Rel} \\ \hline\hline R \subseteq X \times Y, \text{ relation} \\ \hline\hline X \to \mathcal{P}Y \text{ in } \mathbf{Sets} \end{array} \quad \text{where } \mathcal{P} \text{ is the powerset monad}$$

- **DSRel**

$$\begin{array}{c} X \to Y \text{ in } \mathbf{DSRel} \\ \hline\hline X \to \mathcal{D}Y \text{ in } \mathbf{Sets} \end{array}$$

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Why Catego
Examples Other Than Pfn [AHS02]

Categories of sets and
(functions with different branching/partiality)

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Why Catego
Examples Other Than Fun [AHS02]

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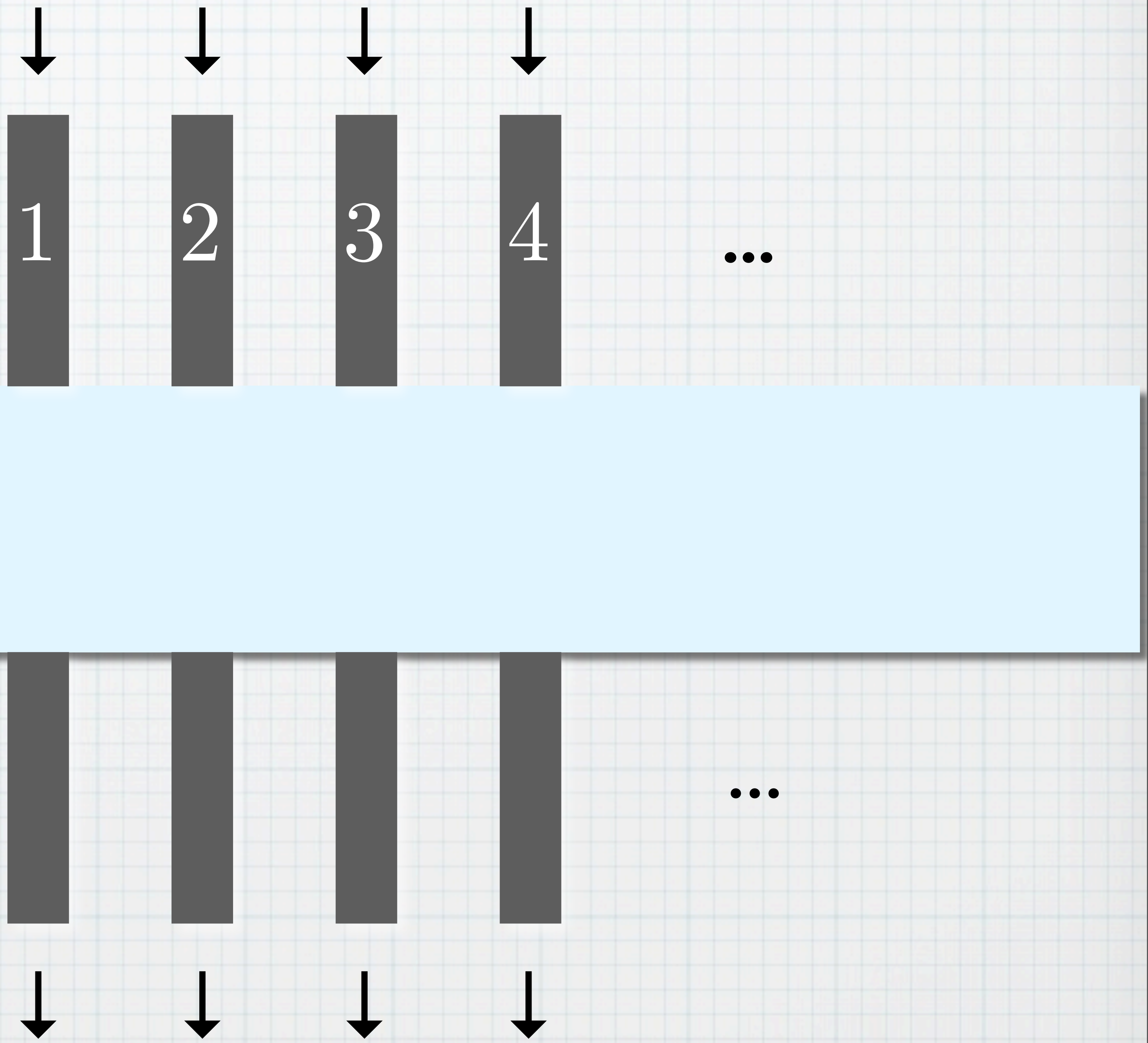
Probabilistic branching

Different Branching in The GoI Animation

- **Pfn** (partial functions)
  - Pipes can be stuck
- **Rel** (relations)
  - Pipes can branch
- **DSRel**
  - Pipes can branch probabilistically



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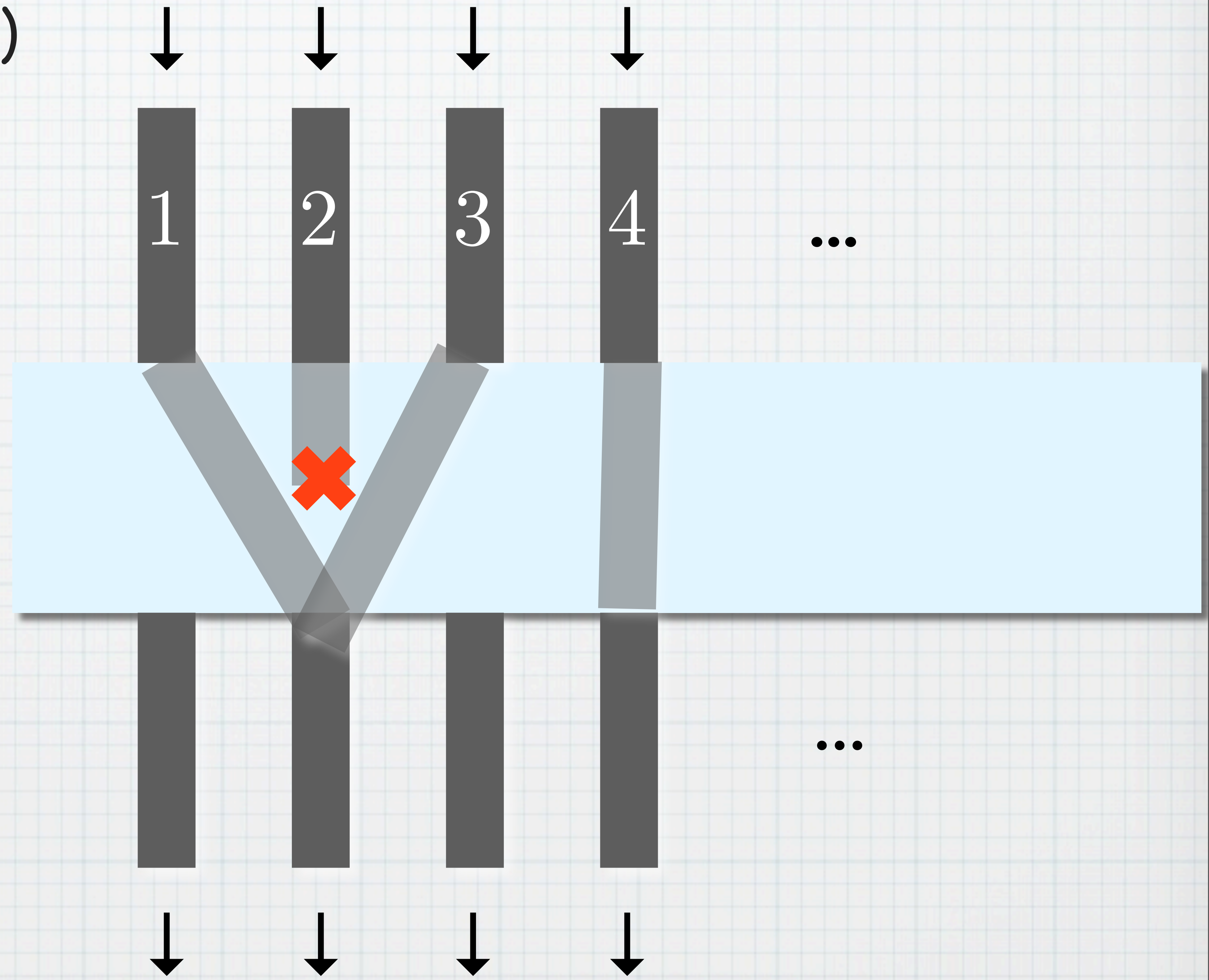

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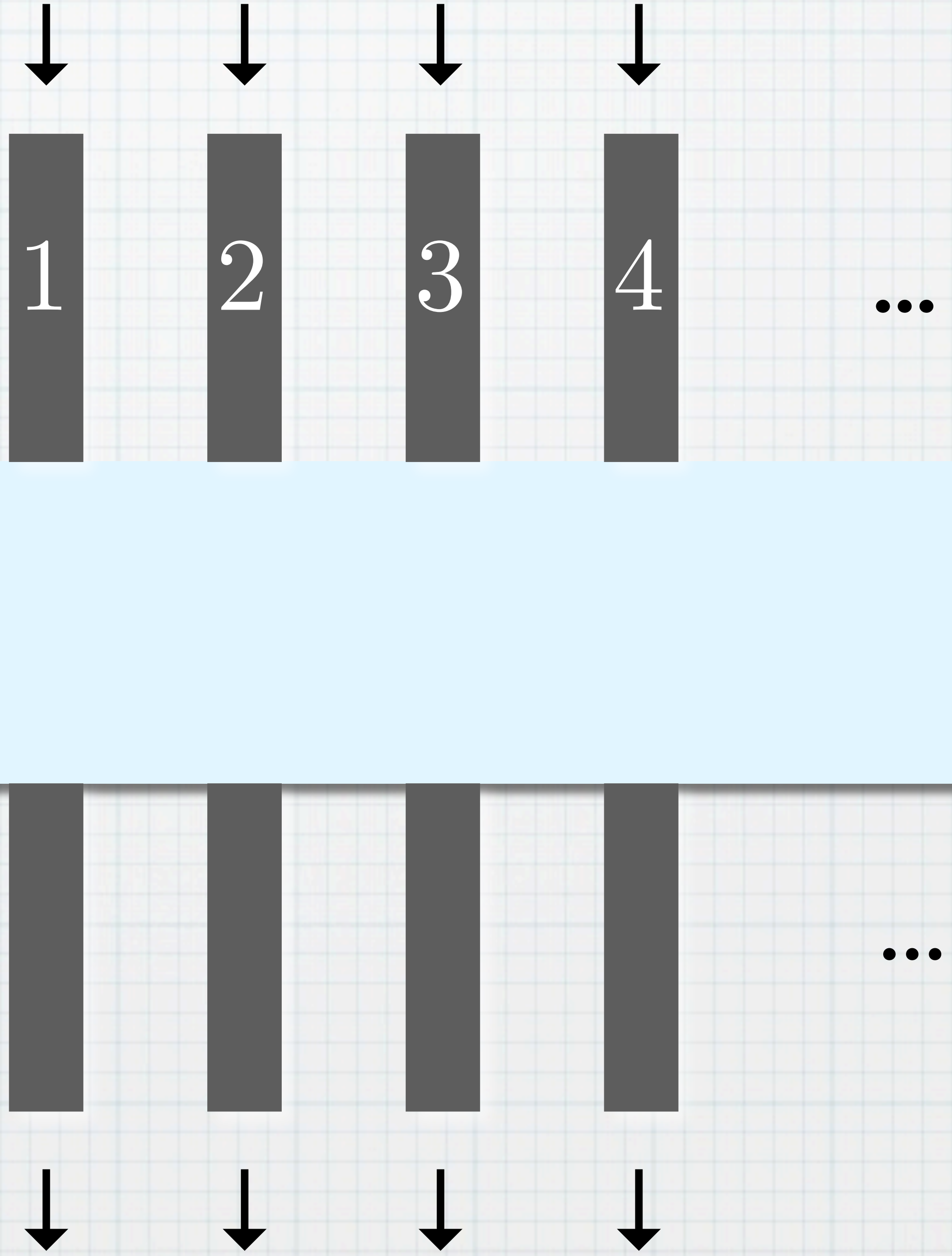

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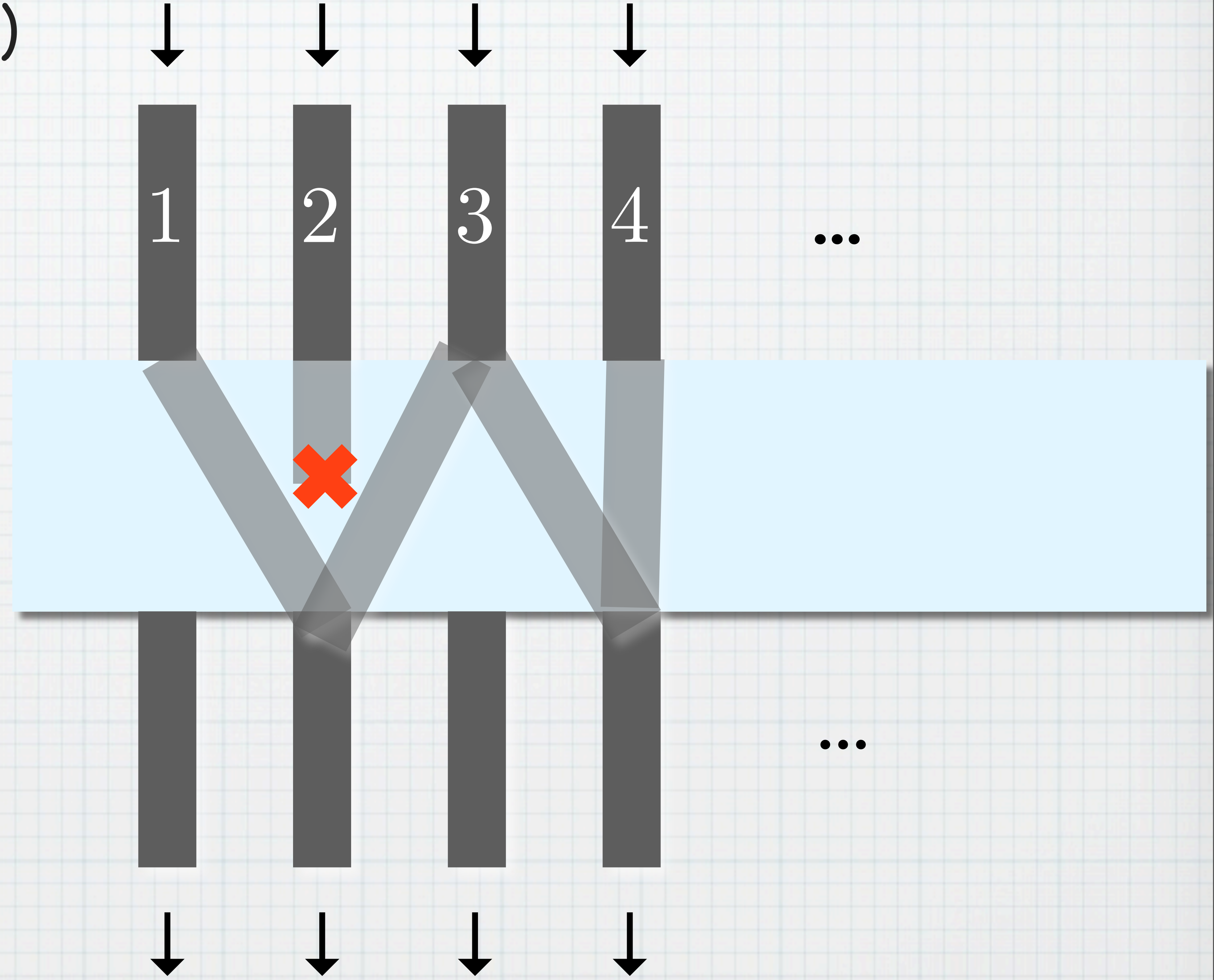

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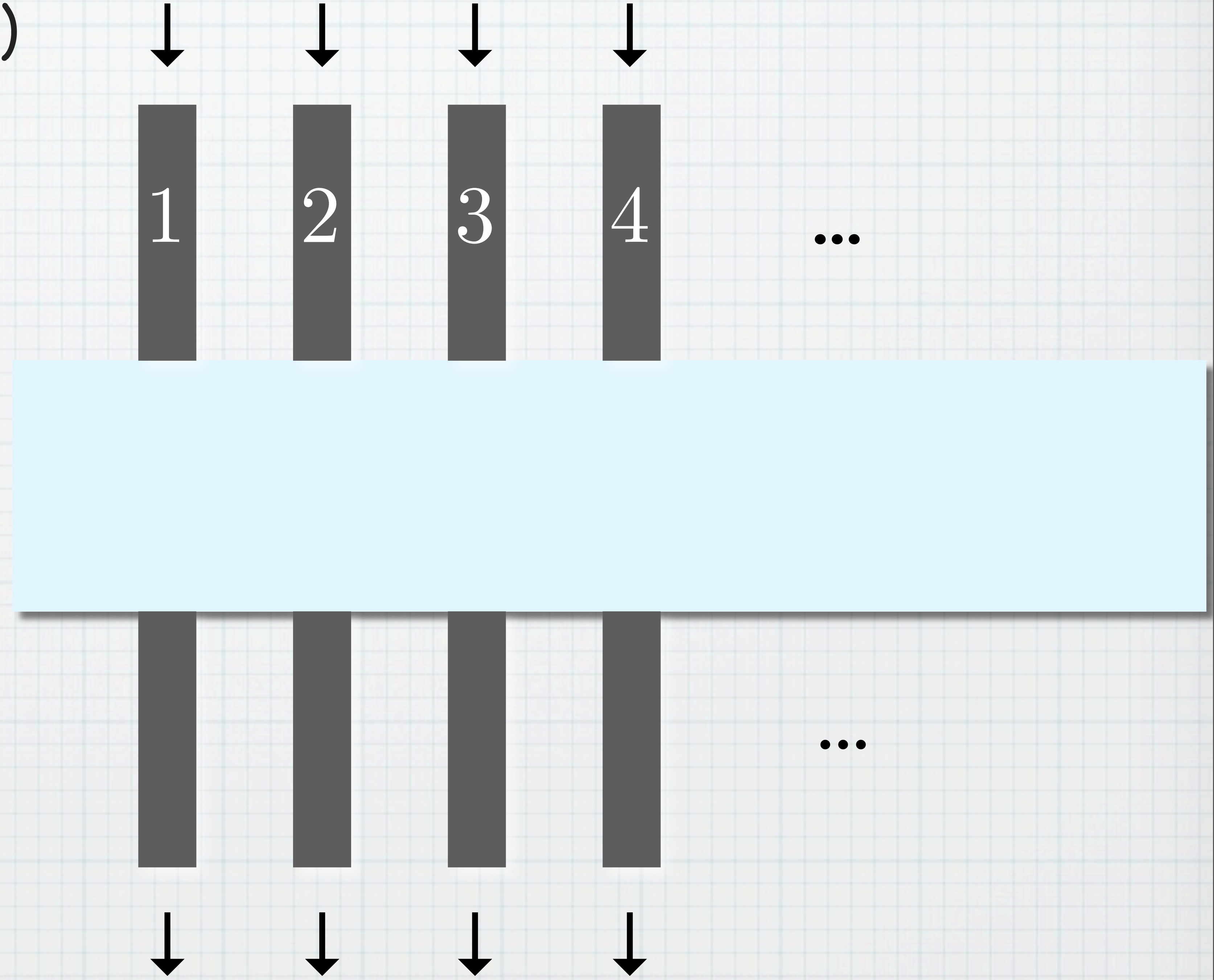

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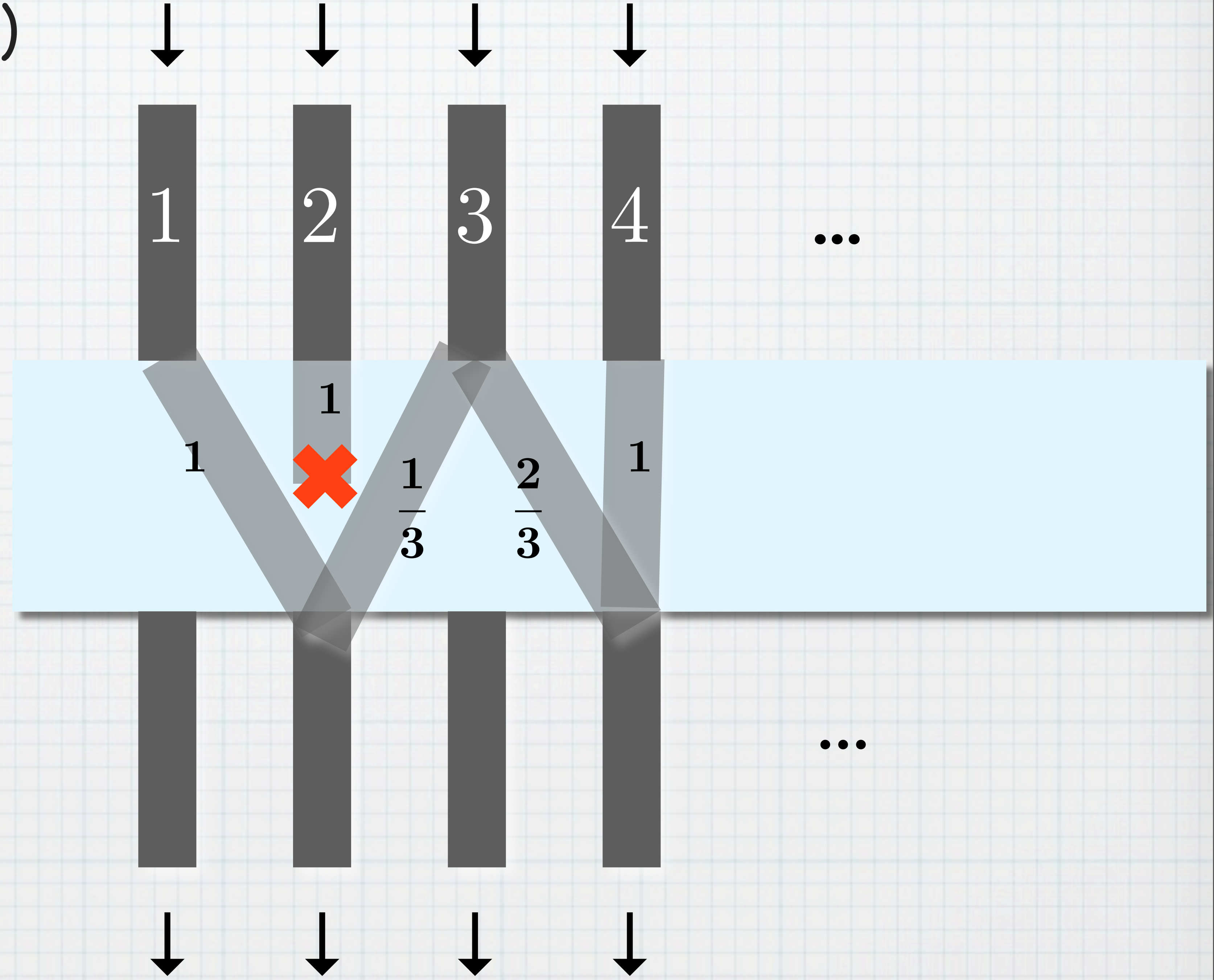

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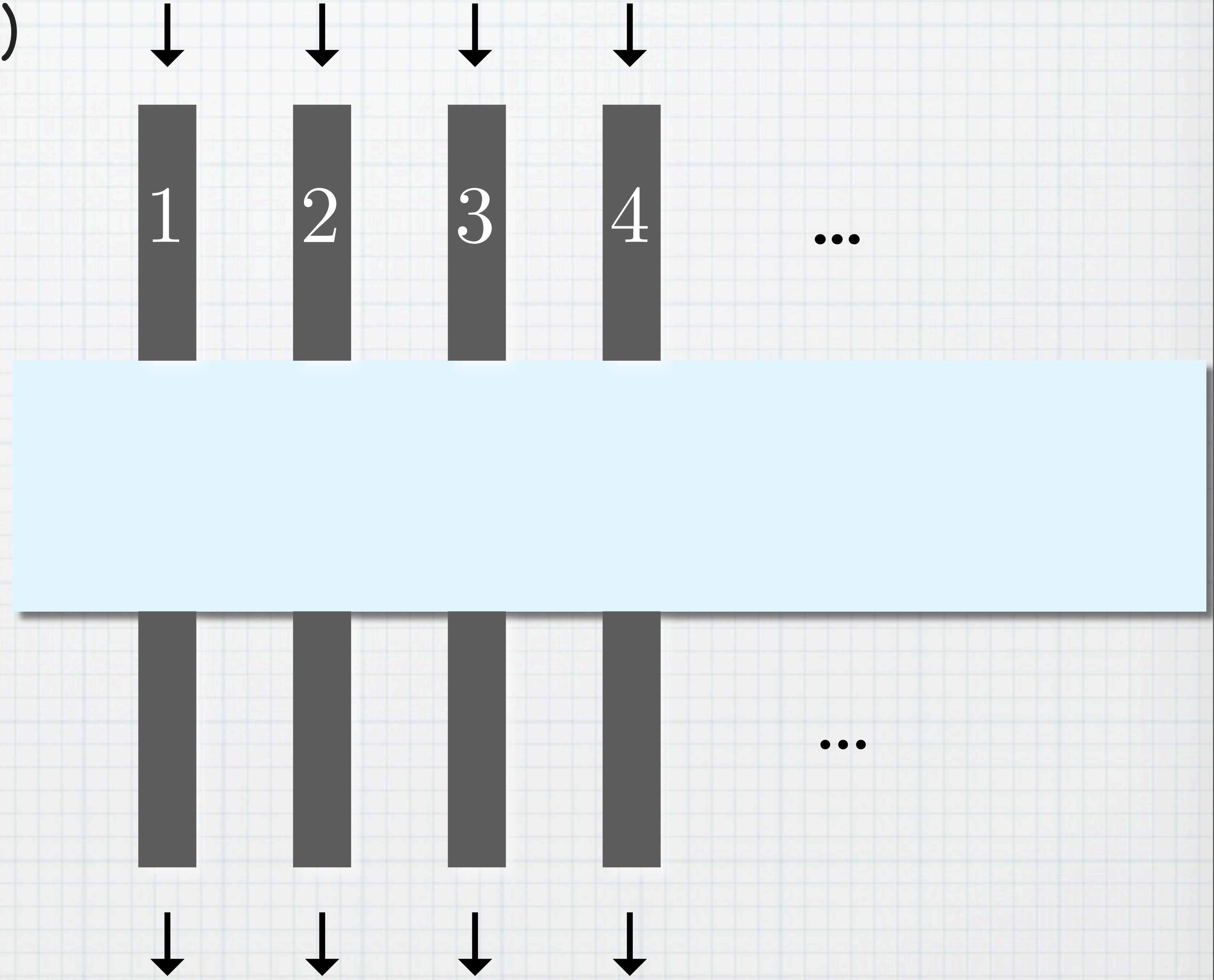

Hasuo (Tokyo)

Why Categorical Generalization?: Examples Other Than Pfn [AHS02]

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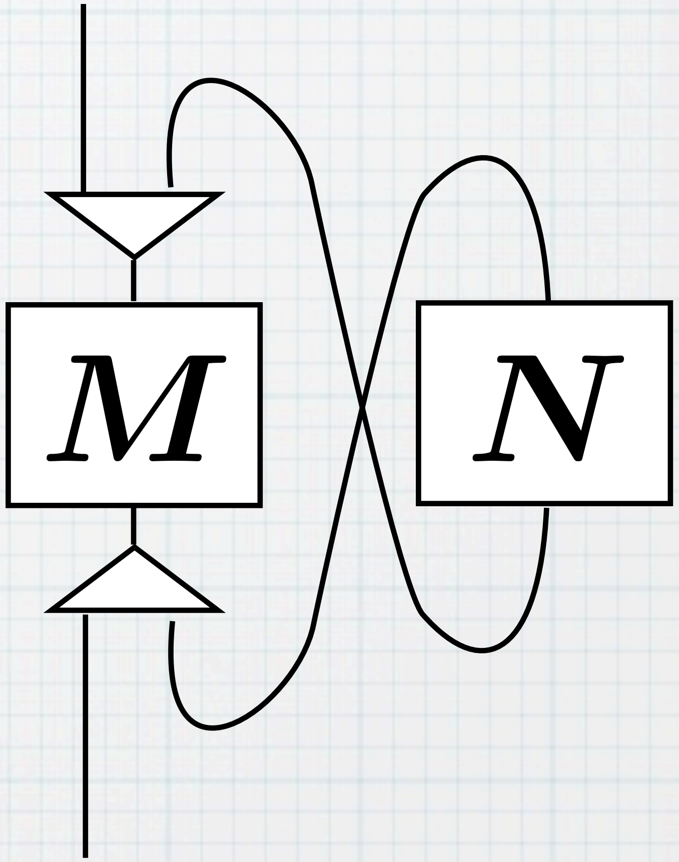

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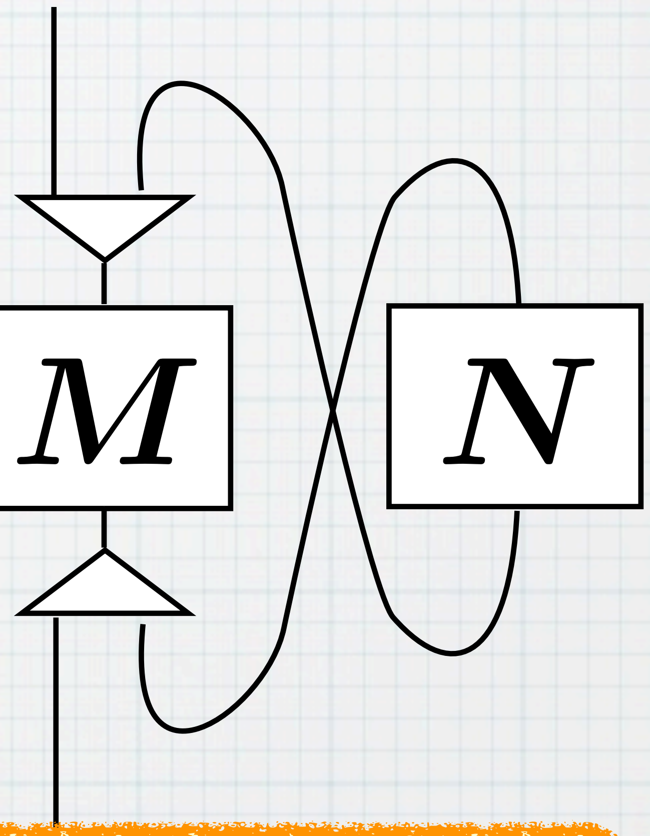

Essential to have subdistribution, for infinite loops

yo)

The Coauthor

- Naohiko Hoshino
  - DSc (Kyoto, 2011)
    - Supervisor: Masahito "Hassei" Hasegawa
  - Currently at RIMS, Kyoto U.
    - http://www.kurims.kyoto-u.ac.jp/~naophiko/

A Coalgebraic View

- Theory of coalgebra =
  Categorical theory of state-based dynamic systems (LTS, automaton, Markov chain, ...)

- In [Hasuo, Jacobs, Sokolova '07]:
  - Coalgebras in a Kleisli category $Kl(B)$

$$\begin{array}{c} \boldsymbol{X} \to \boldsymbol{Y} \ \text{ in } \mathcal{K}\ell(B) \\ \hline\hline \boldsymbol{X} \to \boldsymbol{BY} \ \text{ in } \mathbf{Sets} \end{array}$$

  - ➜ Generic theory of "trace semantics"

Why Catego
Examples Other Than Fun

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Probabilistic branching

Why Catego
Examples

$Kl(B)$ for different branching monads B

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Probabilistic branching

Branching Monad: Source of Particle-Style GoI Situations

Thm. ([Jacobs,CMCS10])
Given a "branching monad" $\boldsymbol{B}$ on **Sets**, the monoidal category

$$(\boldsymbol{\mathcal{K}\ell}(\boldsymbol{B}), +, \mathbf{0})$$

is

- a *unique decomposition category* [Haghverdi,PhD00], hence is
- a traced symmetric monoidal category.

Cor.
$\big((\boldsymbol{\mathcal{K}\ell}(\boldsymbol{B}), +, \mathbf{0}),\ \mathbb{N}\cdot\_\ ,\ \mathbb{N} \big)$ is a GoI situation.

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(Roughly) monads in
[Hasuo, Jacobs, Sokolova '07]

- ＊ Kl(B) is $\mathrm{Cpo}_{\perp}$-enriched
- ＊ like $\mathcal{L}$, $\mathcal{P}$, $\mathcal{D}$

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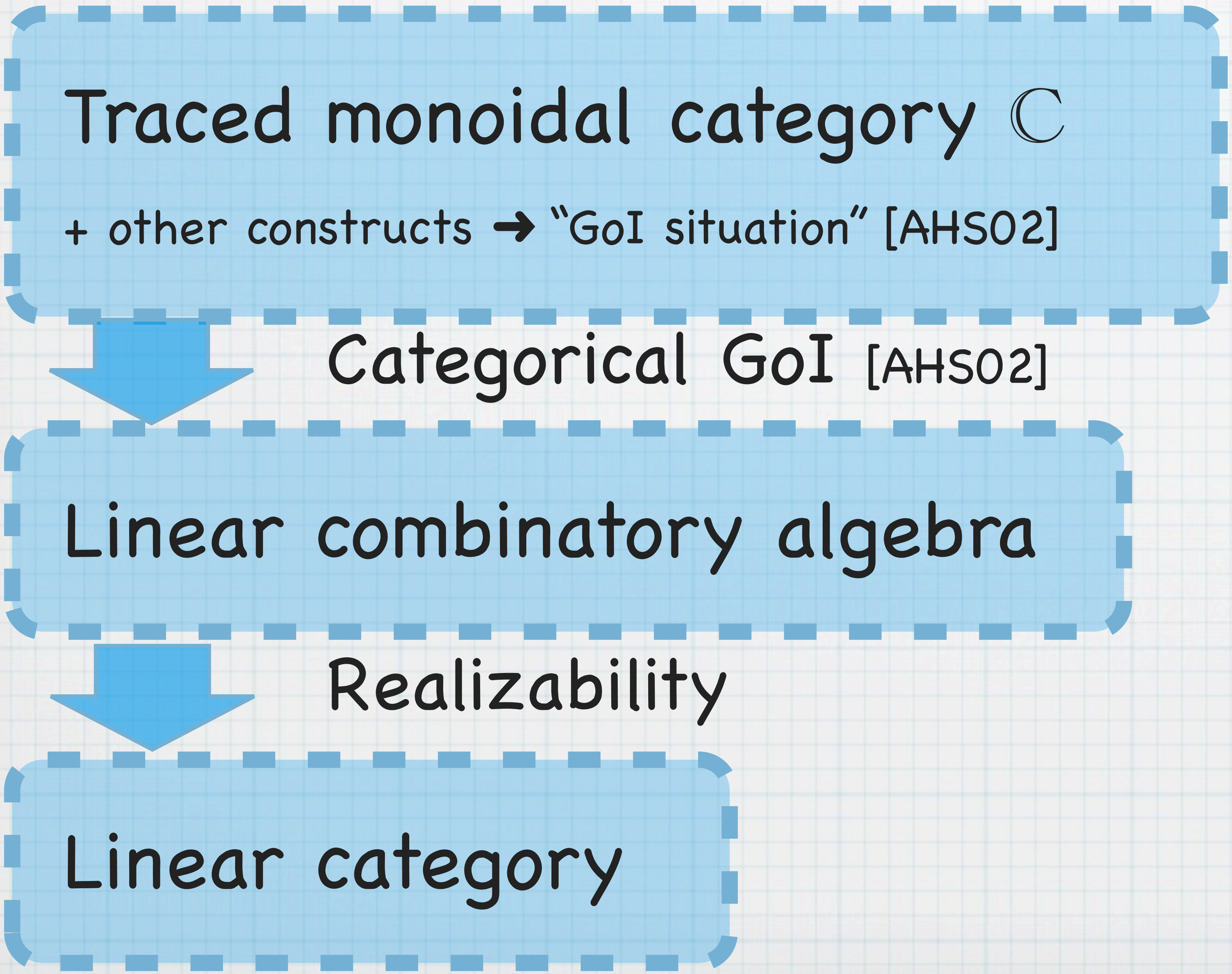
Particle-style: trace via the execution formula

$$\mathbf{tr}(f) = f_{XY} \sqcup \left(\coprod_{n\in\mathbb{N}} f_{ZY} \circ (f_{ZZ})^n \circ f_{XZ} \right)$$

The Categorical GoI Workflow



Traced monoidal category $\mathbb{C}$

+ other constructs ➔ “GoI situation” [AHS02]

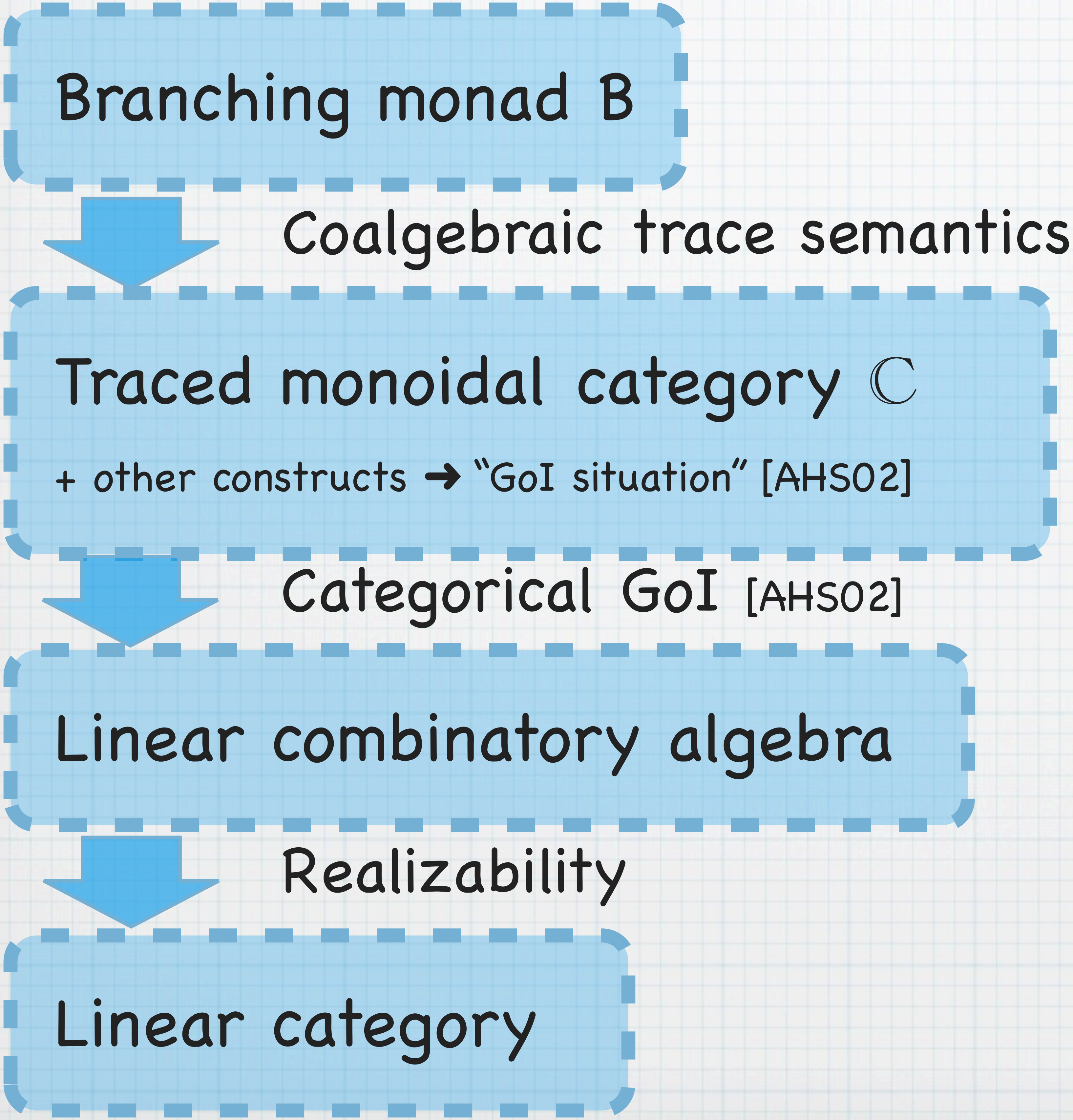
Categorical GoI [AHS02]

Linear combinatory algebra

Realizability

Linear category

The Categorical GoI Workflow



Branching monad B

Coalgebraic trace semantics

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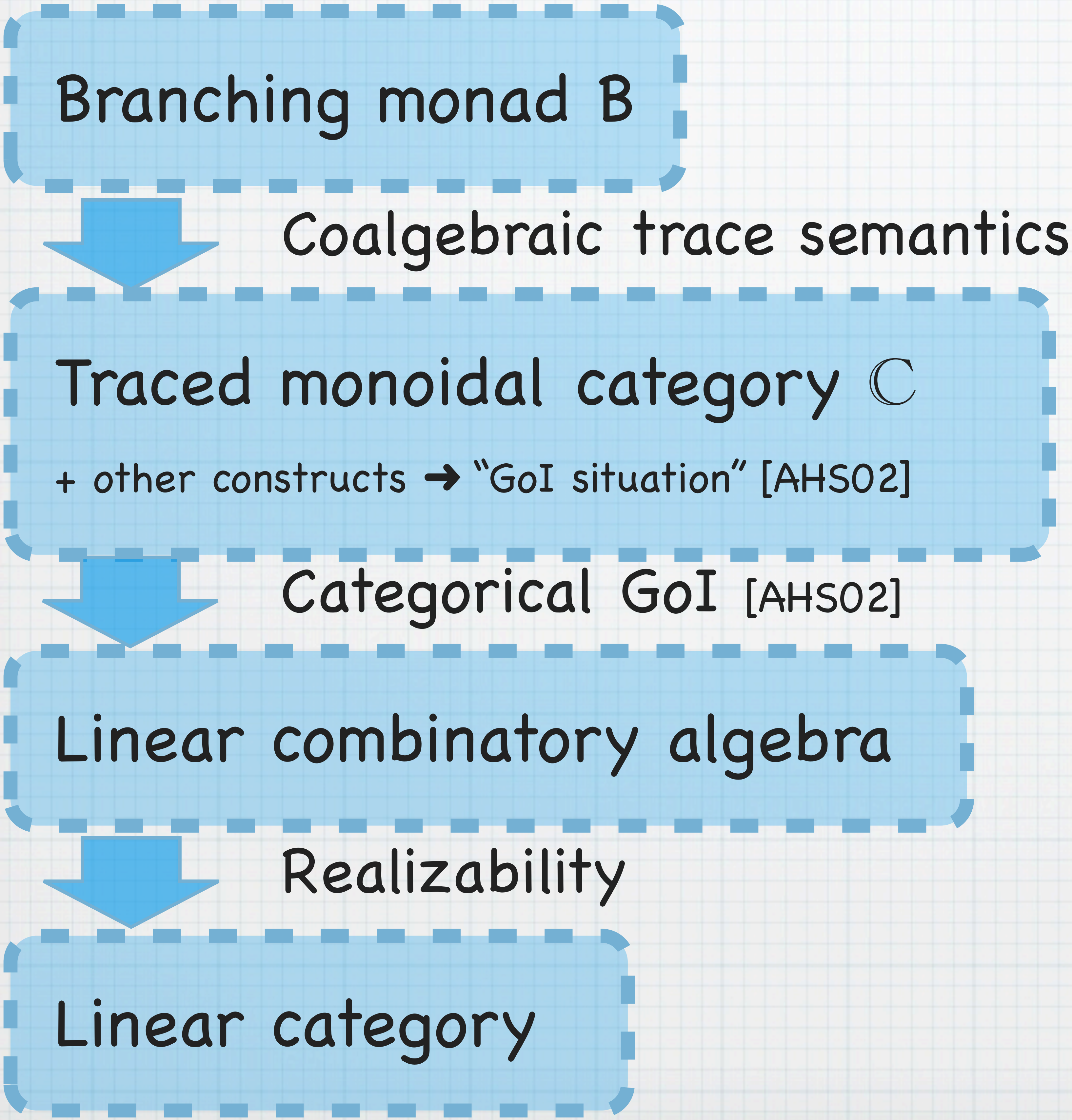

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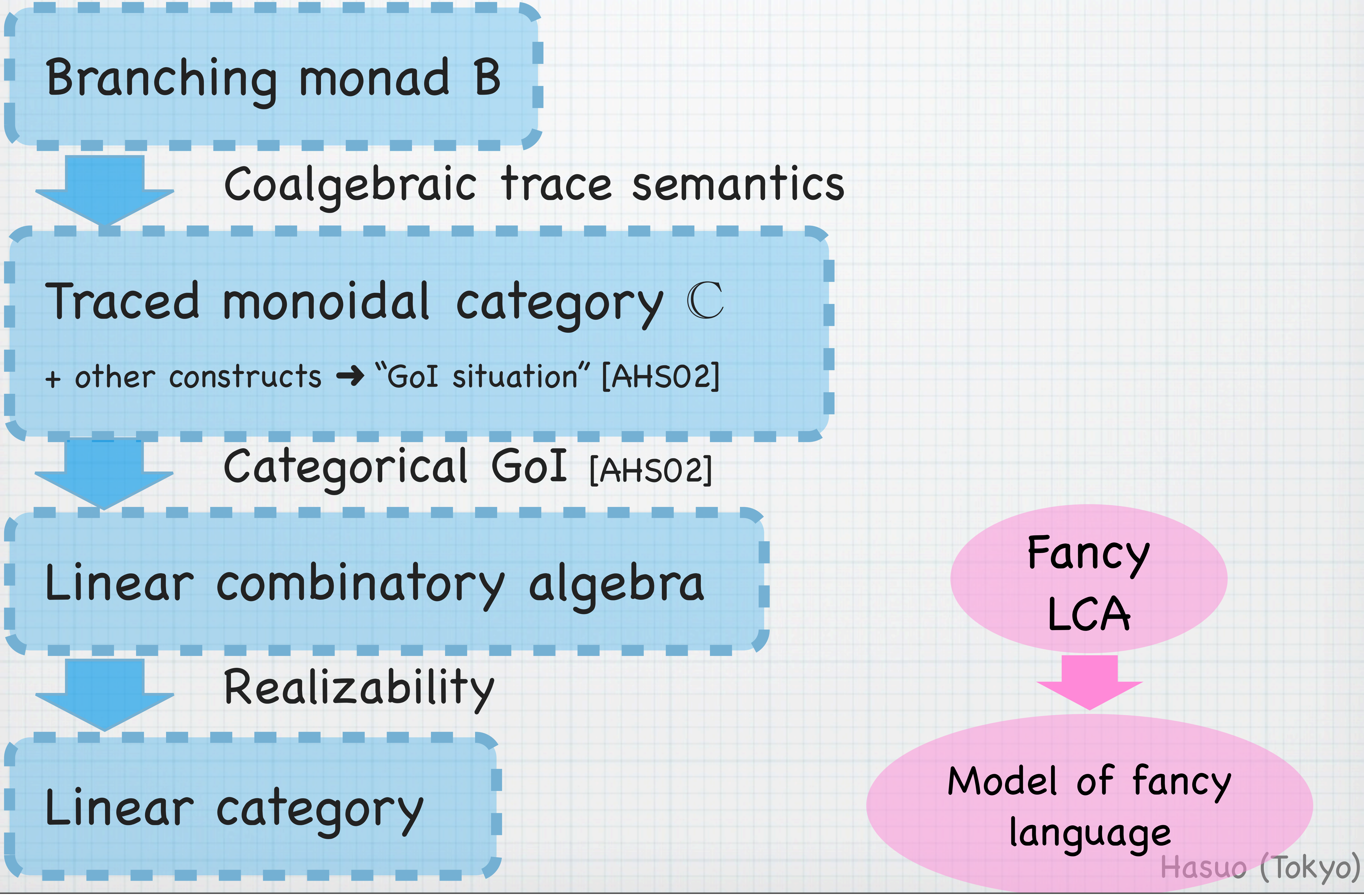

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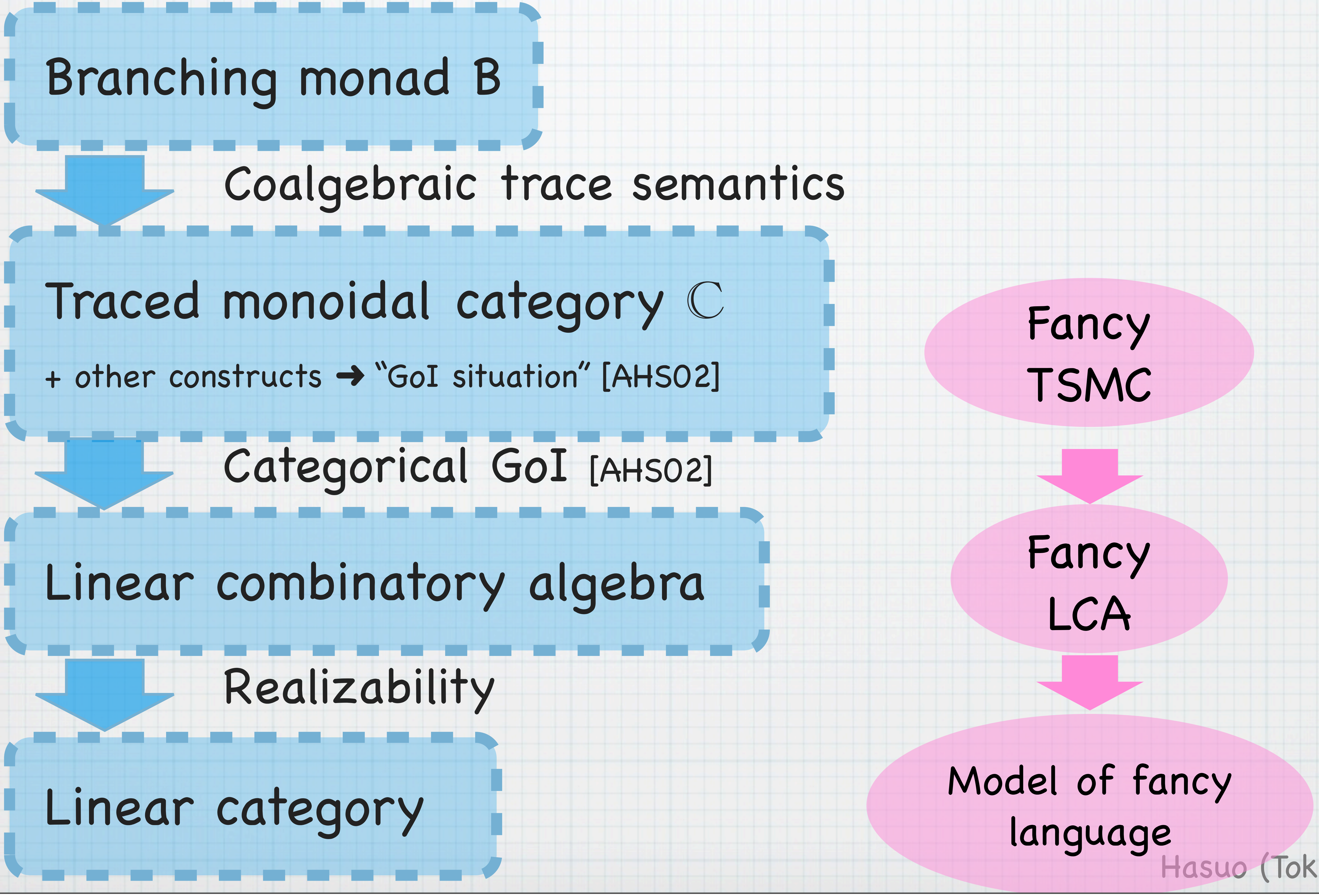

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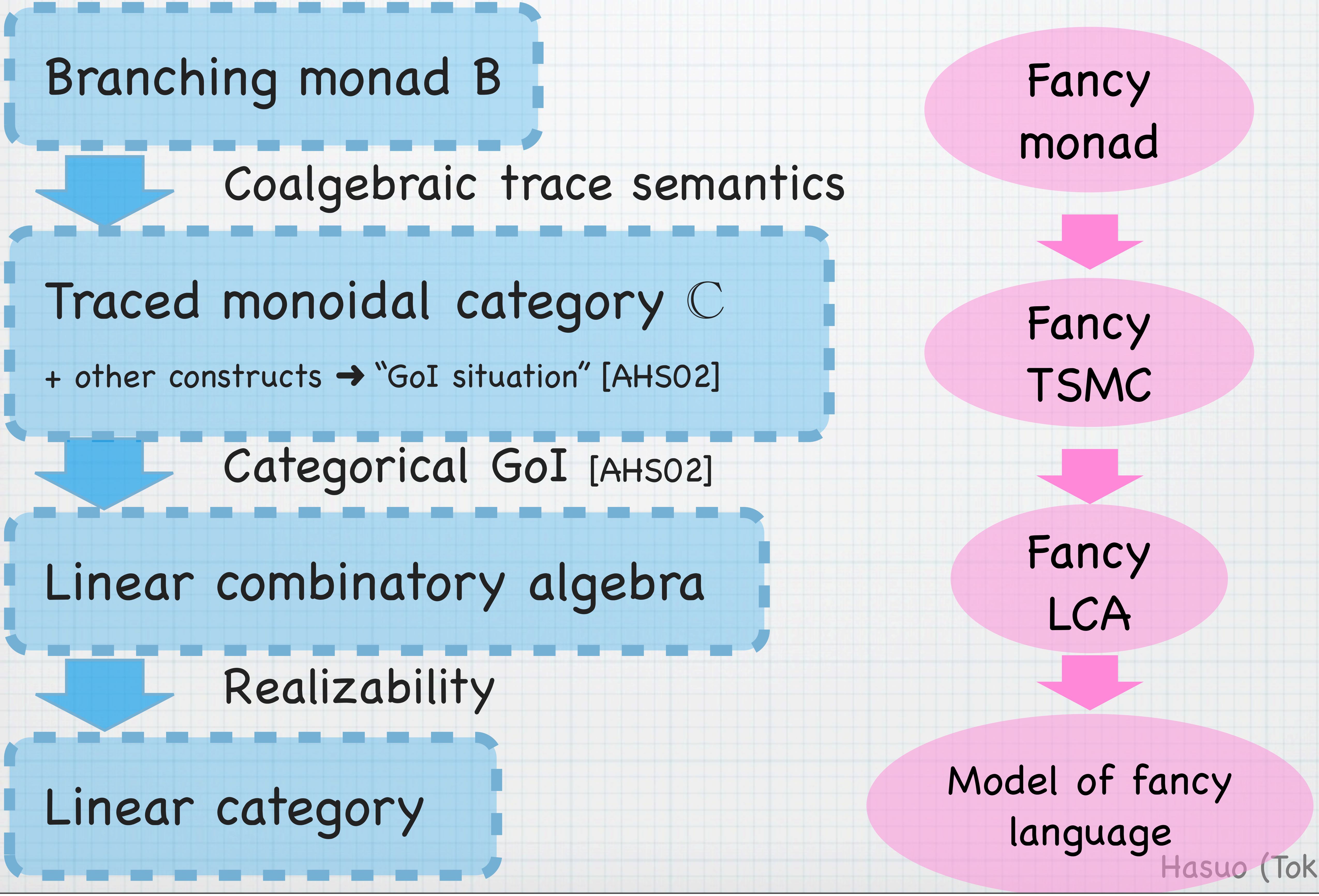

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Model of fancy language

What is Fancy, Nowadays?

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- Biology?

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- Hybrid systems?
  - Both discrete and continuous data, typically in **cyber-physical systems** (CPS)
  - ➜ Our approach via **non-standard analysis** [Suenaga, Hasuo ICALP'11]

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- Quantum?
  - Yes this worked!

Part 3

Phil Scott.
Tutorial on Geometry of Interaction, FMCS 2004.
Page 47/47

Future Directions

- GoI 2 : Non-converging algms (untyped λ-calc / PCF)
  - uses more topological info on operatn algs
- GoI 3: uses additives & additive proof nets —
- GoI 4 (last month) : von Neumann algebras : $Ex(f, \tau)$ for f arb (not necessarily coming from proof)
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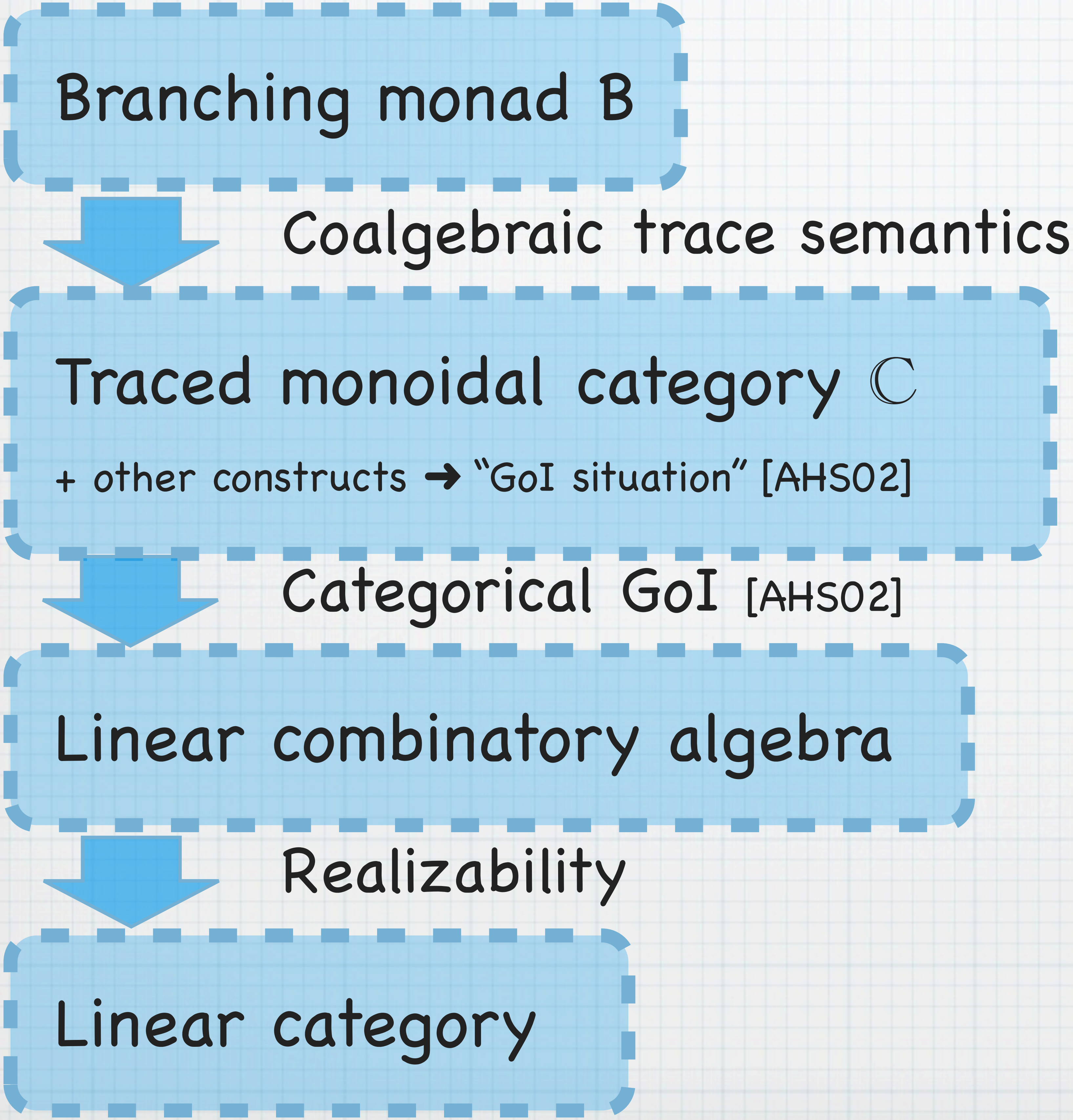
- $\boxed{\text{Quantum GoI}}$?

The Categorical GoI Workflow



Branching monad B

Coalgebraic trace semantics

Traced monoidal category $\mathbb{C}$

+ other constructs ➜ “GoI situation” [AHS02]

Categorical GoI [AHS02]

Linear combinatory algebra

Realizability

Linear category

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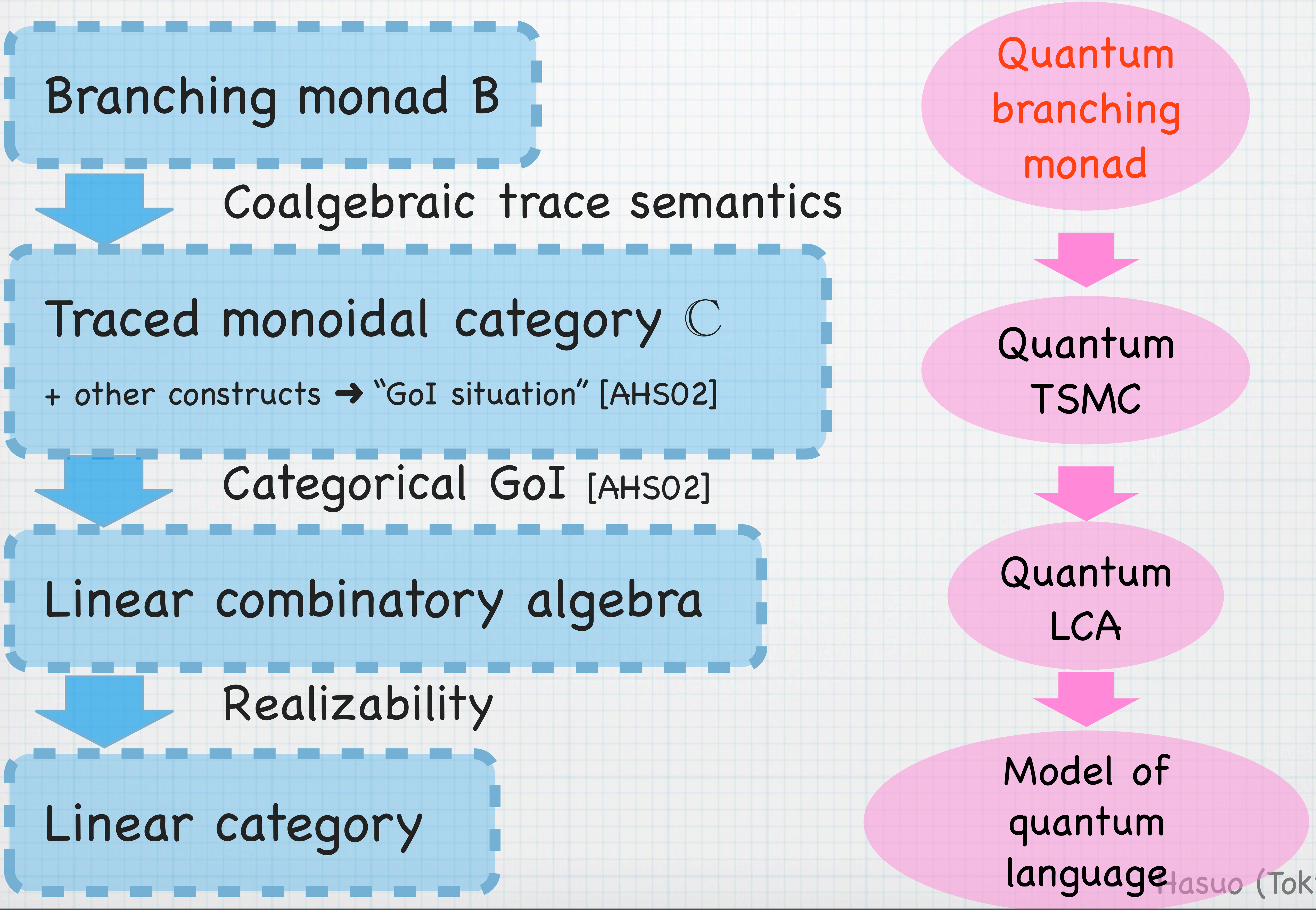

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Quantum branching monad

Quantum TSMC

Quantum LCA

Model of quantum language

The Quantum Branching Monad

$$\mathcal{Q}Y = \left\{ c : Y \to \prod_{m,n\in\mathbb{N}} \mathbf{QO}_{m,n} \;\middle|\; \text{the trace condition} \right\}$$

The Quantum Branching M

$$\mathbf{QO}_{\boldsymbol{m},\boldsymbol{n}} := \left\{ \begin{array}{l} \text{quantum operations,} \\ \text{from dim. } \boldsymbol{m} \text{ to dim. } \boldsymbol{n} \end{array} \right\}$$

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$$QY = \left\{ c : Y \to \prod_{m,n\in\mathbb{N}} \underline{\mathbf{QO}_{m,n}} \;\middle|\; \underline{\text{the trace condition}} \right\}$$

$$\sum_{y\in Y} \sum_{n\in\mathbb{N}} \mathsf{tr}\left[\left(c(y)\right)_{m,n}(\rho)\right] \leq 1 \ ,$$

$$\forall m \in \mathbb{N},\ \forall \rho \in D_m .$$

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- Compare with

$$\mathcal{P}Y = \Big\{ c : Y \to 2 \Big\}$$

$$\mathcal{D}Y = \Big\{ c : Y \to [0,1] \;\Big|\; \sum_{y\in Y} c(y) \le 1 \Big\}$$

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$$\frac{X \xrightarrow{f} Y \text{ in } \mathcal{K}\ell(\mathcal{Q})}{\overline{X \xrightarrow{f} \mathcal{Q}Y \text{ in } \mathbf{Sets}}}$$

- Given $x \in X$, $y \in Y$, $m \in \mathbb{N}$, $n \in \mathbb{N}$ determines a quantum operation

$$\Big(f(x)(y) \Big)_{m,n} \;:\; D_m \to D_n$$

- Subject to the **trace condition**

Hasuo (Tokyo)

The Quantum Branching Monad

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Any opr. on quantum data:

combination of

- ● preparation
- ● unitary transf.
- ● measurement

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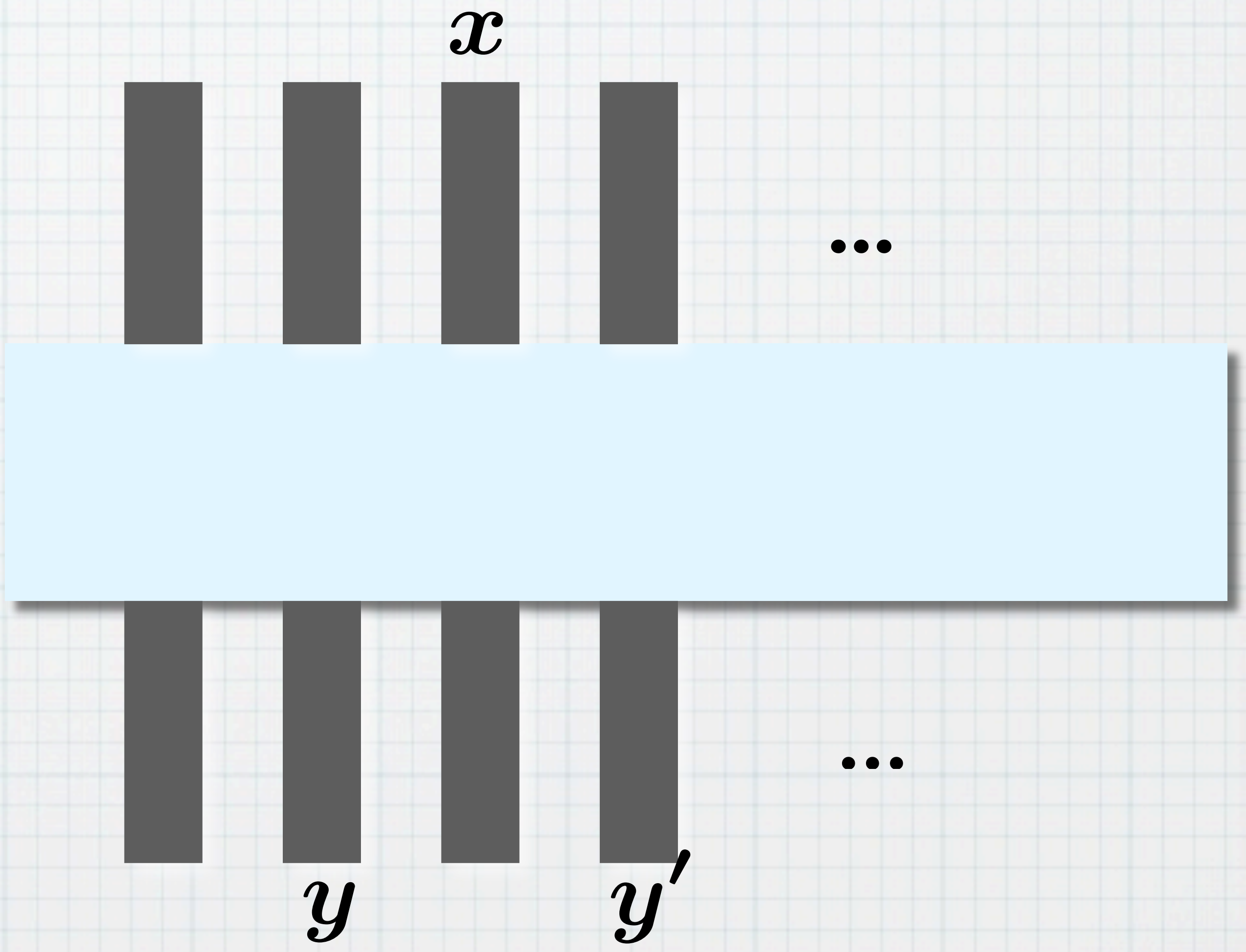

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- Given $\overset{\text{entrance}}{x \in X}$, $\overset{\text{exit}}{y \in Y}$, $\overset{\text{in dim.}}{m \in \mathbb{N}}$, $\overset{\text{out dim.}}{n \in \mathbb{N}}$ determines a quantum operation $\Big(f(x)(y)\Big)_{m,n}$
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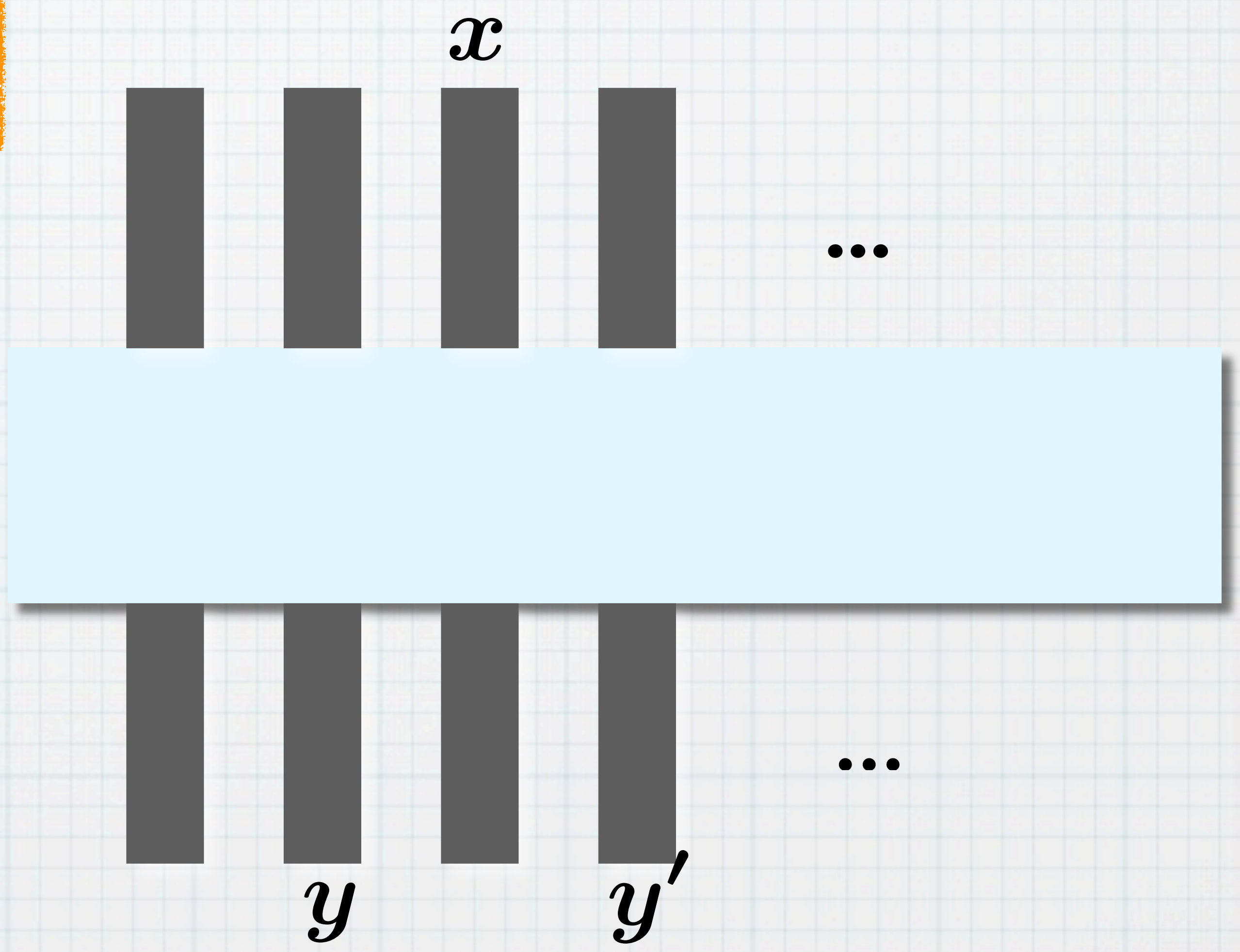

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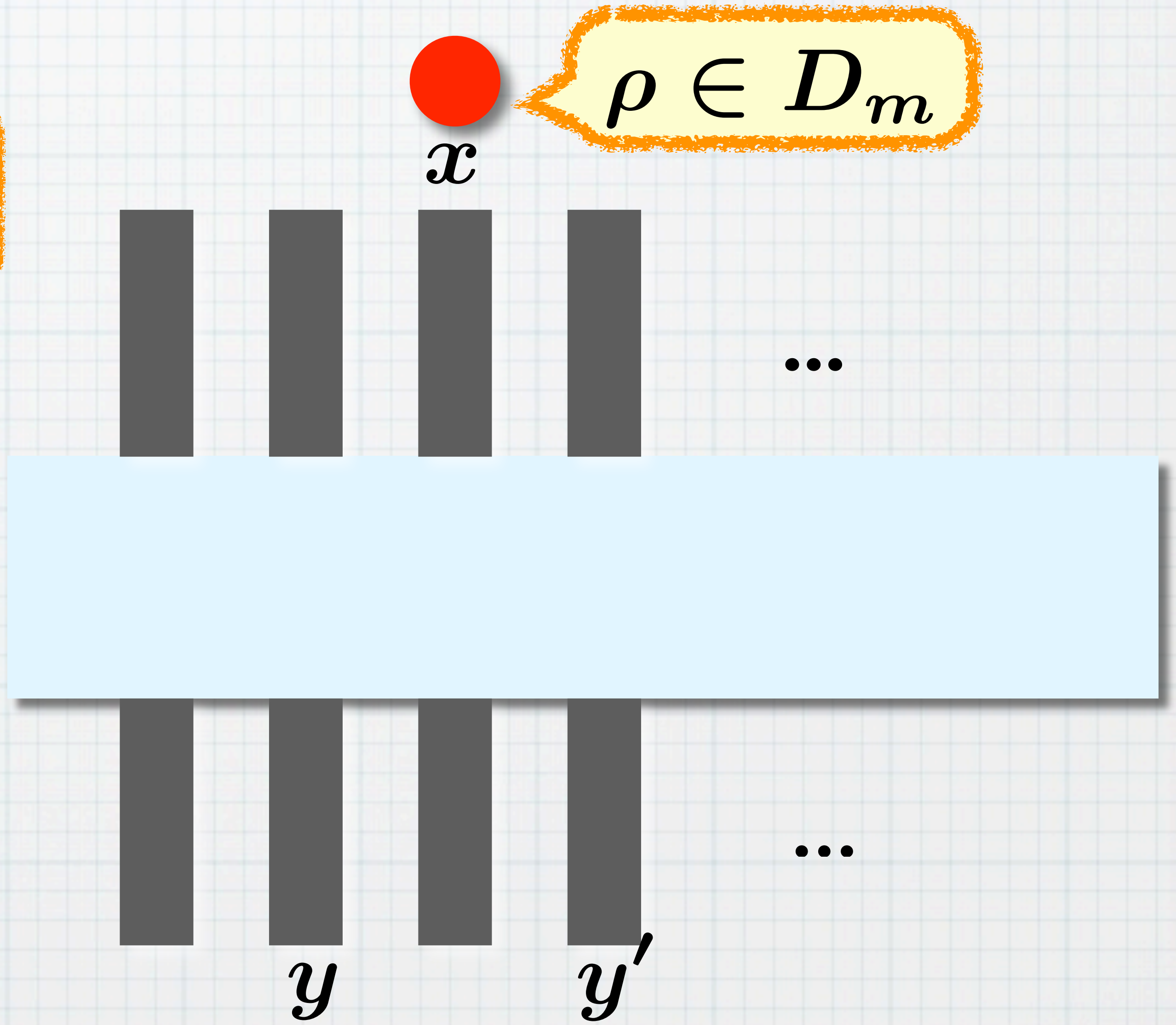
entrance

exit

in dim.

out dim.

- * Given $x \in X$, $y \in Y$, $m \in \mathbb{N}$, $n \in \mathbb{N}$ determines a quantum operation $\Big(f(x)(y)\Big)_{m,n}$
- * **trace cond.:**


$\rho \in D_m$

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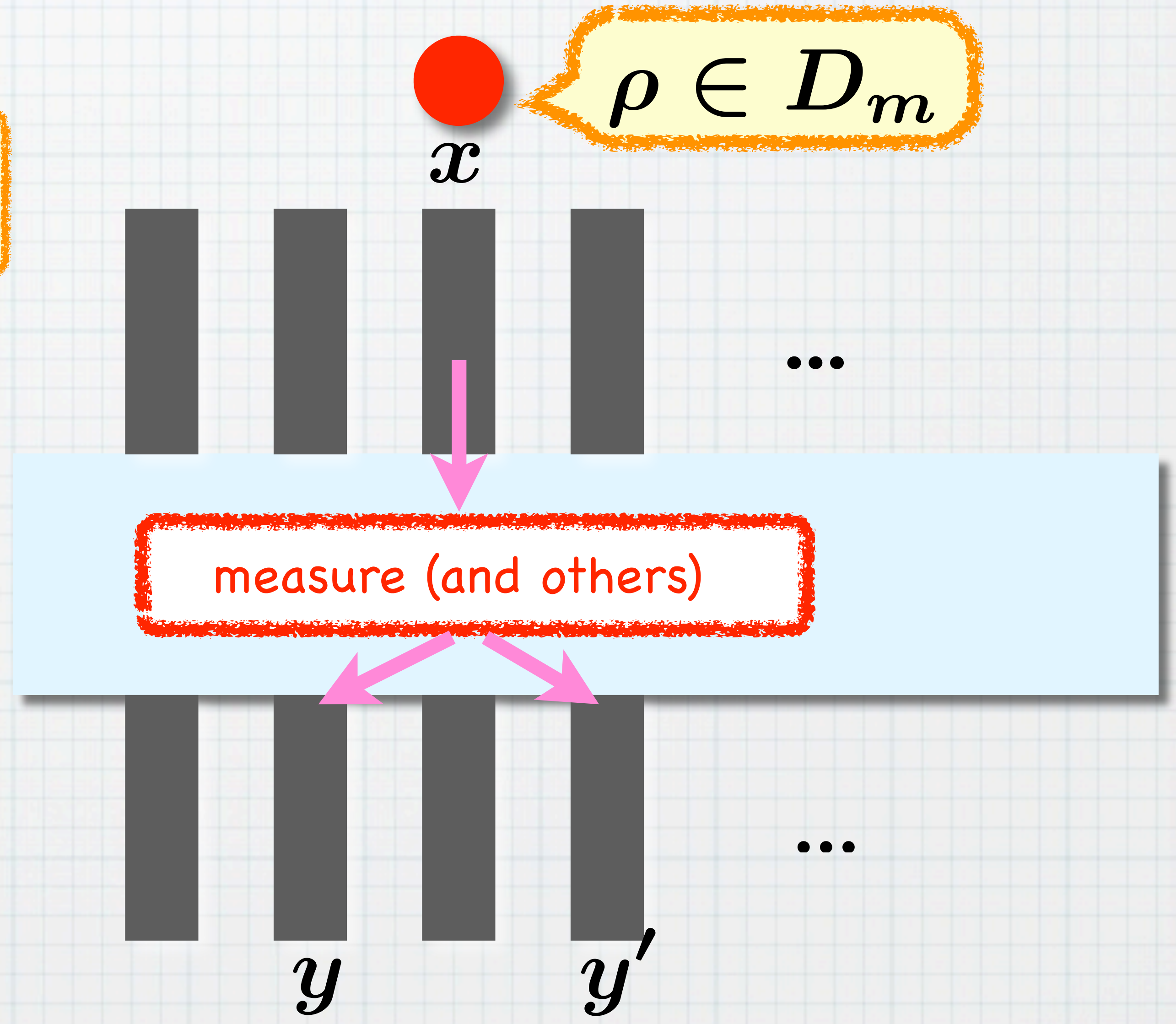

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entrance exit in dim. out dim.

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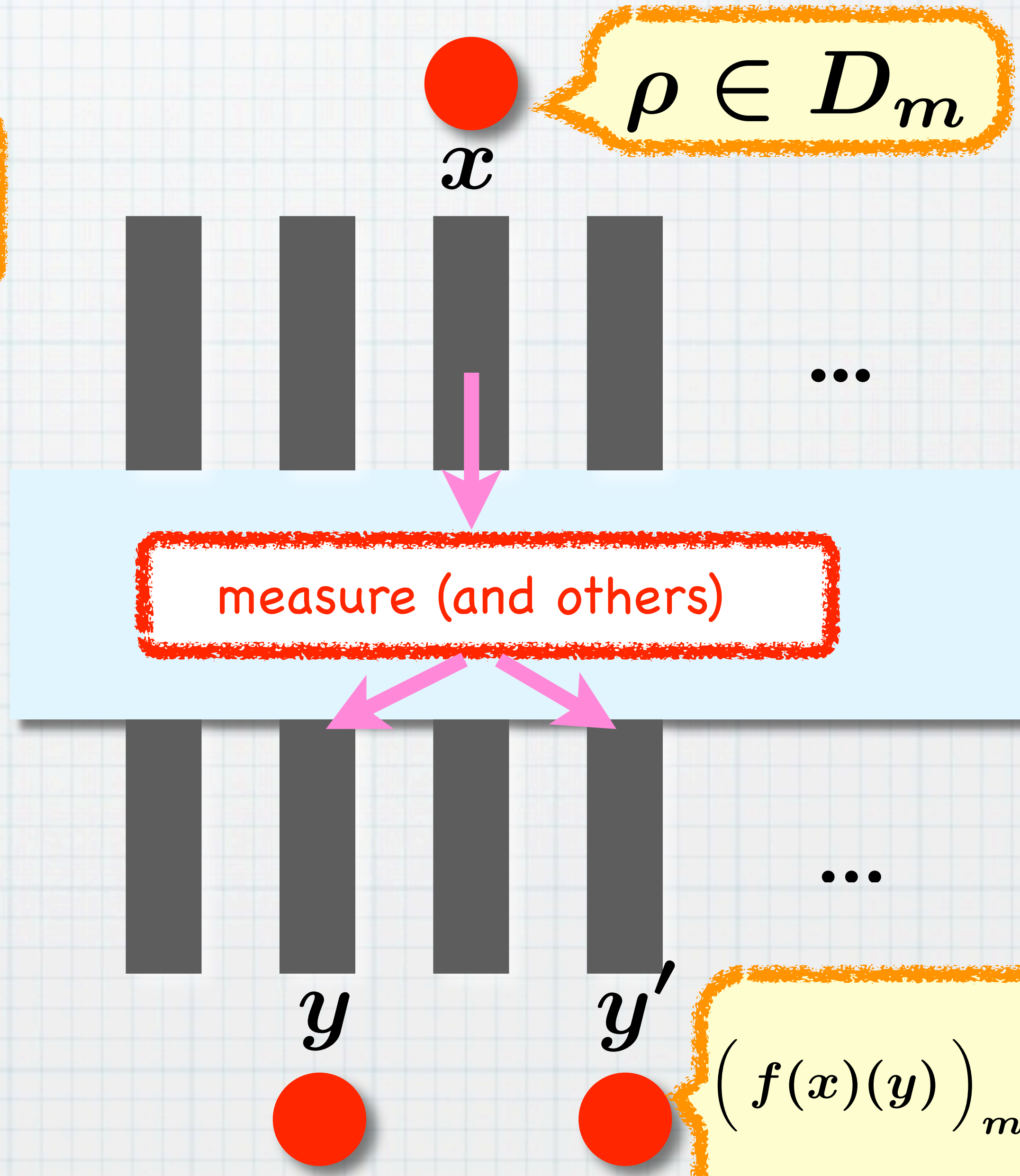

$\rho \in D_m$

$\Big(f(x)(y) \Big)_{m,n}(\rho) \in D_n$

for each n

The Quantum Branching Monad

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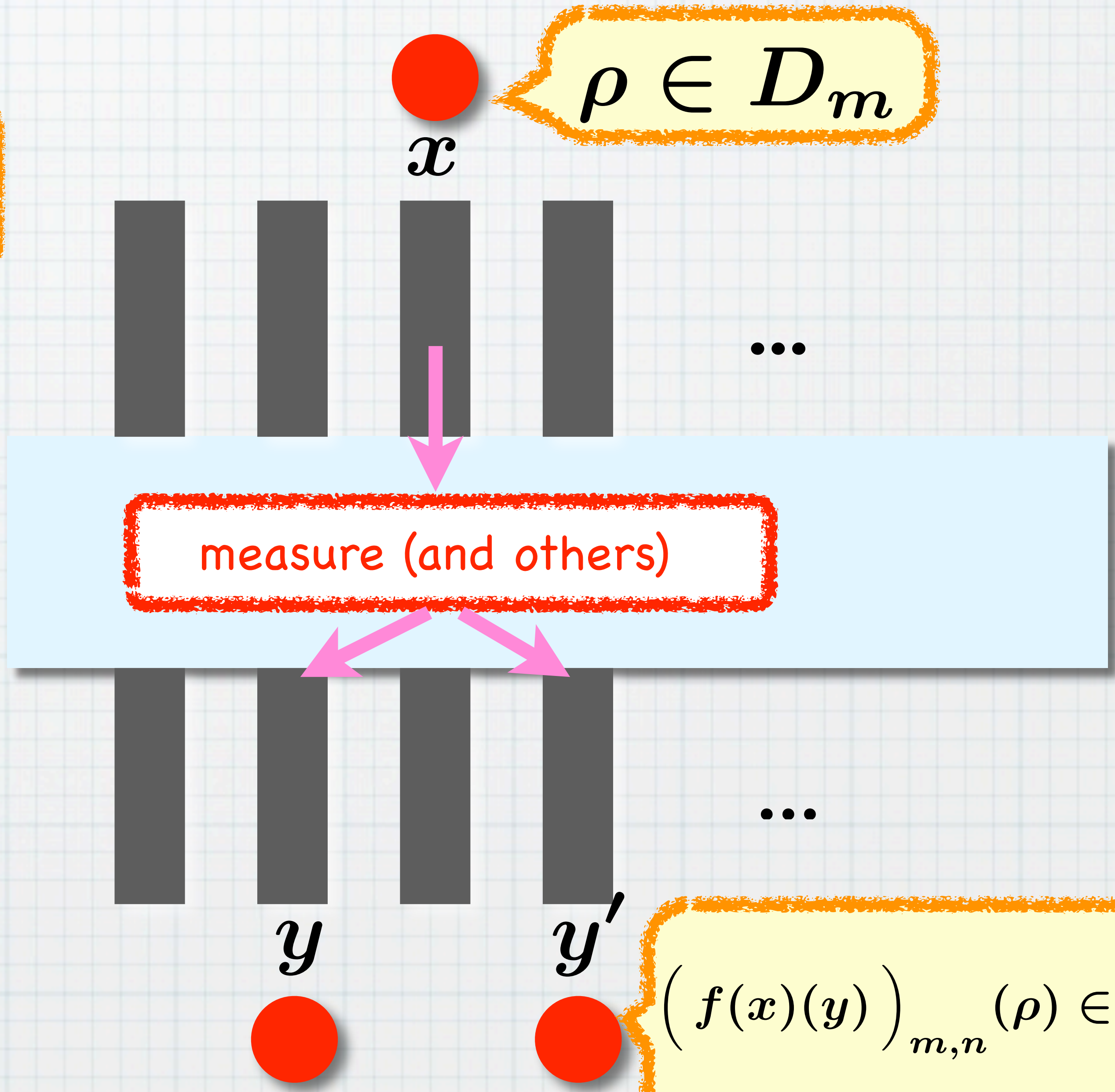
- **trace cond.:**

$$\sum_{y,n} \Pr\left(\begin{array}{c}\text{Token led} \\ \text{to } y \\ \text{with dim. } n\end{array} \right) \leq 1$$

$\rho \in D_m$

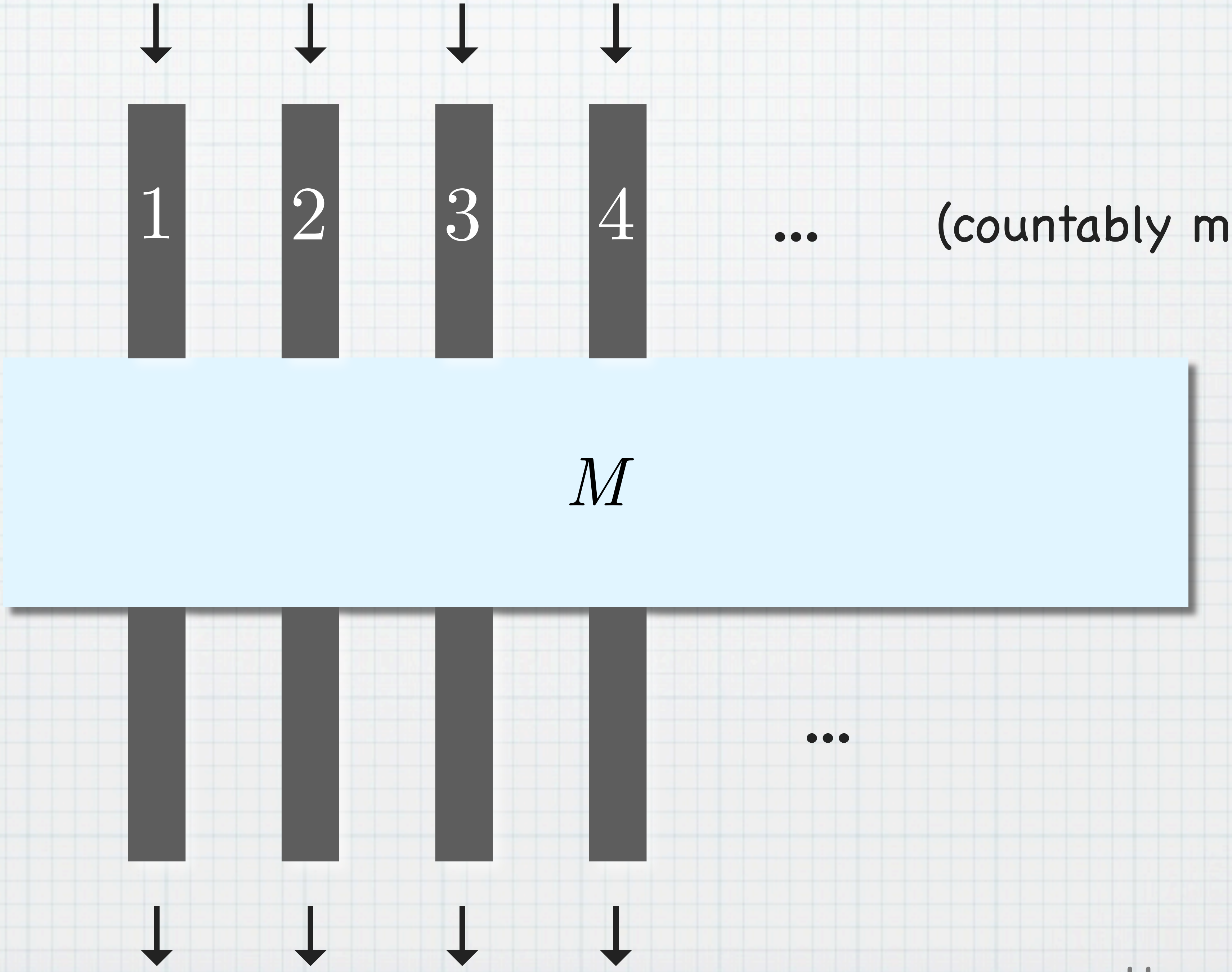


$\big(f(x)(y)\big)_{m,n}(\rho) \in D_n$

for each n

Quantum
Geometry of Interaction

$[\![M]\!] =$

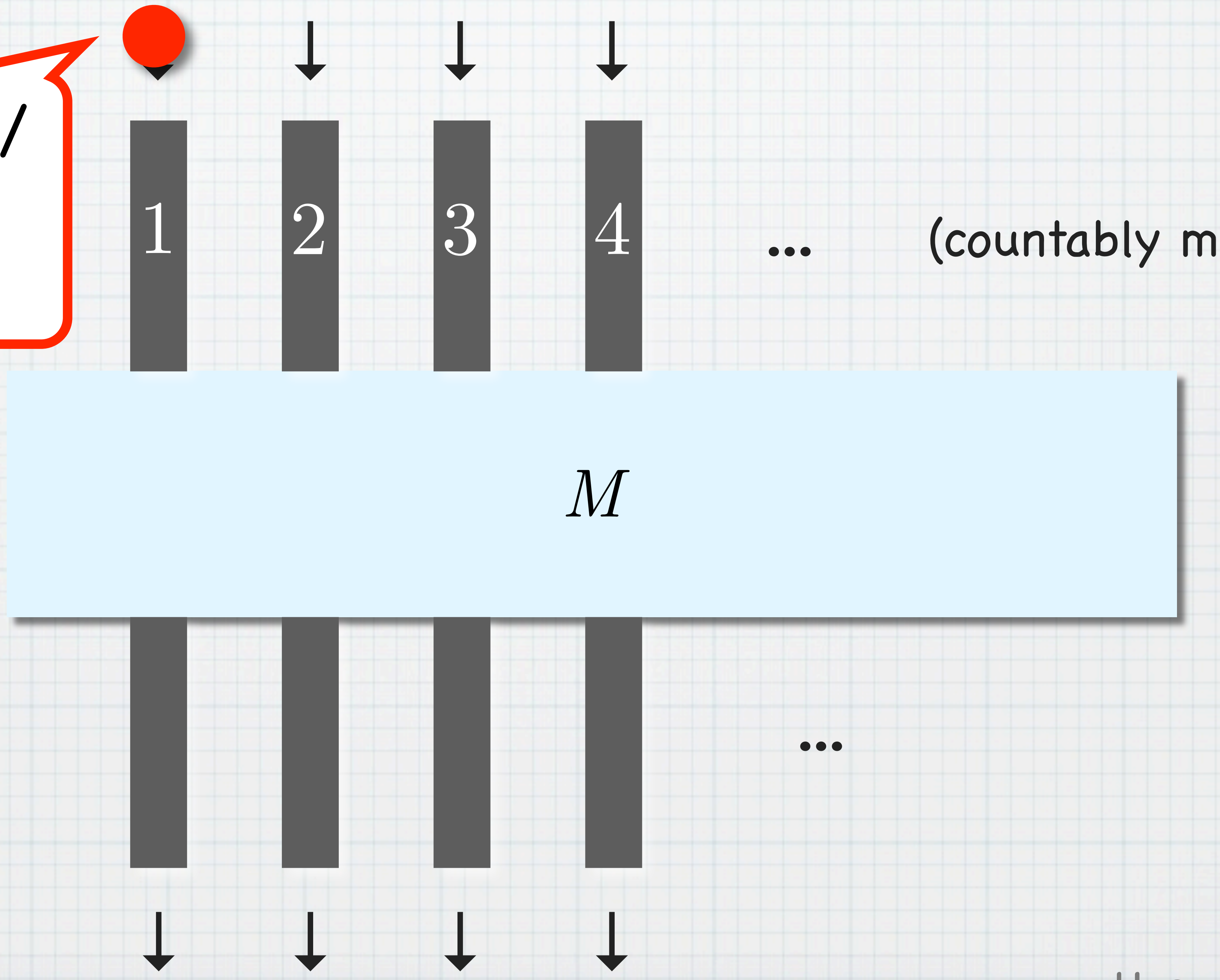


(countably many)

Quantum
Geometry of Interaction

Not just a token/ particle, but quantum state!

$$[\![M]\!] =$$



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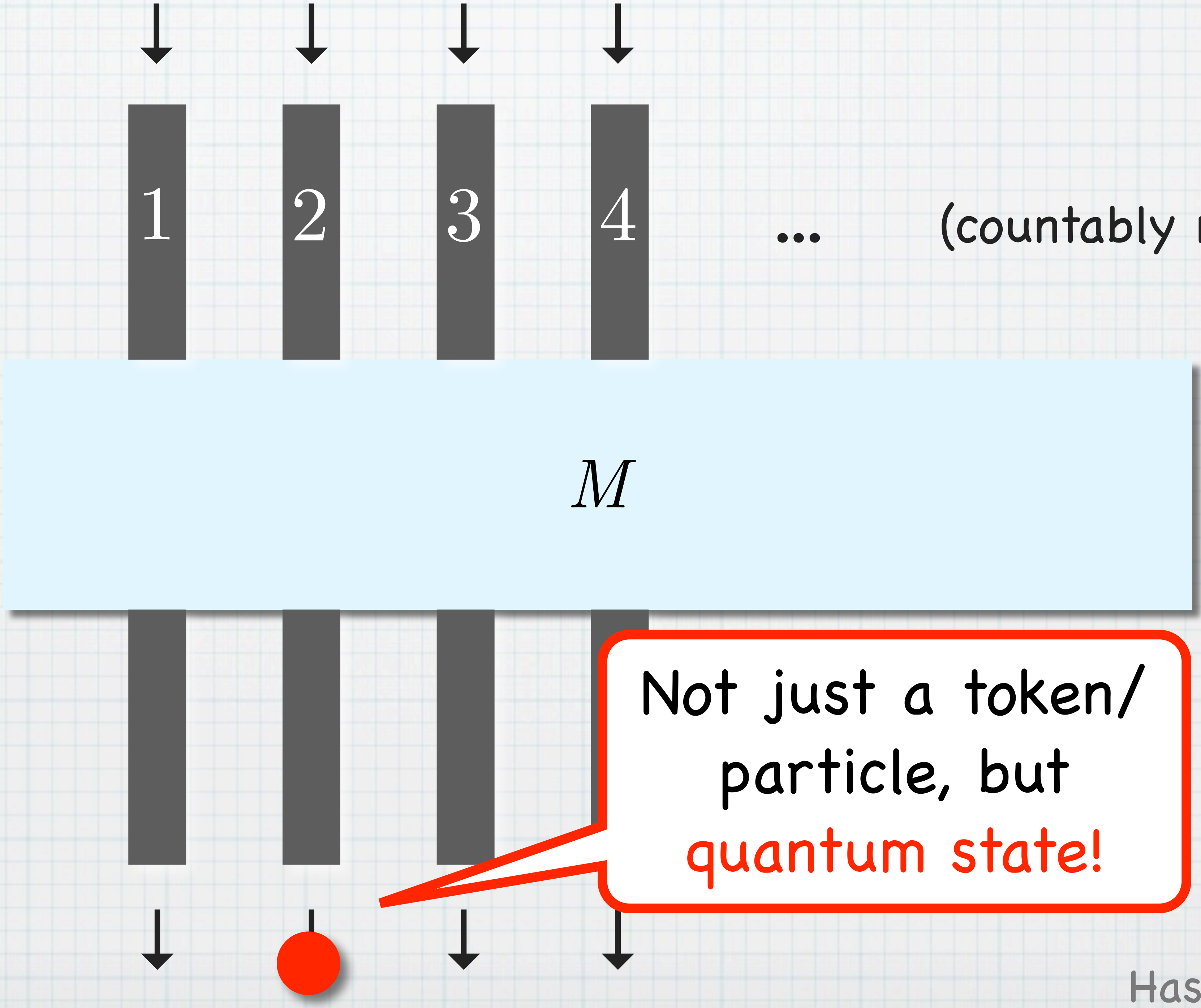

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Hasuo (Tokyo)

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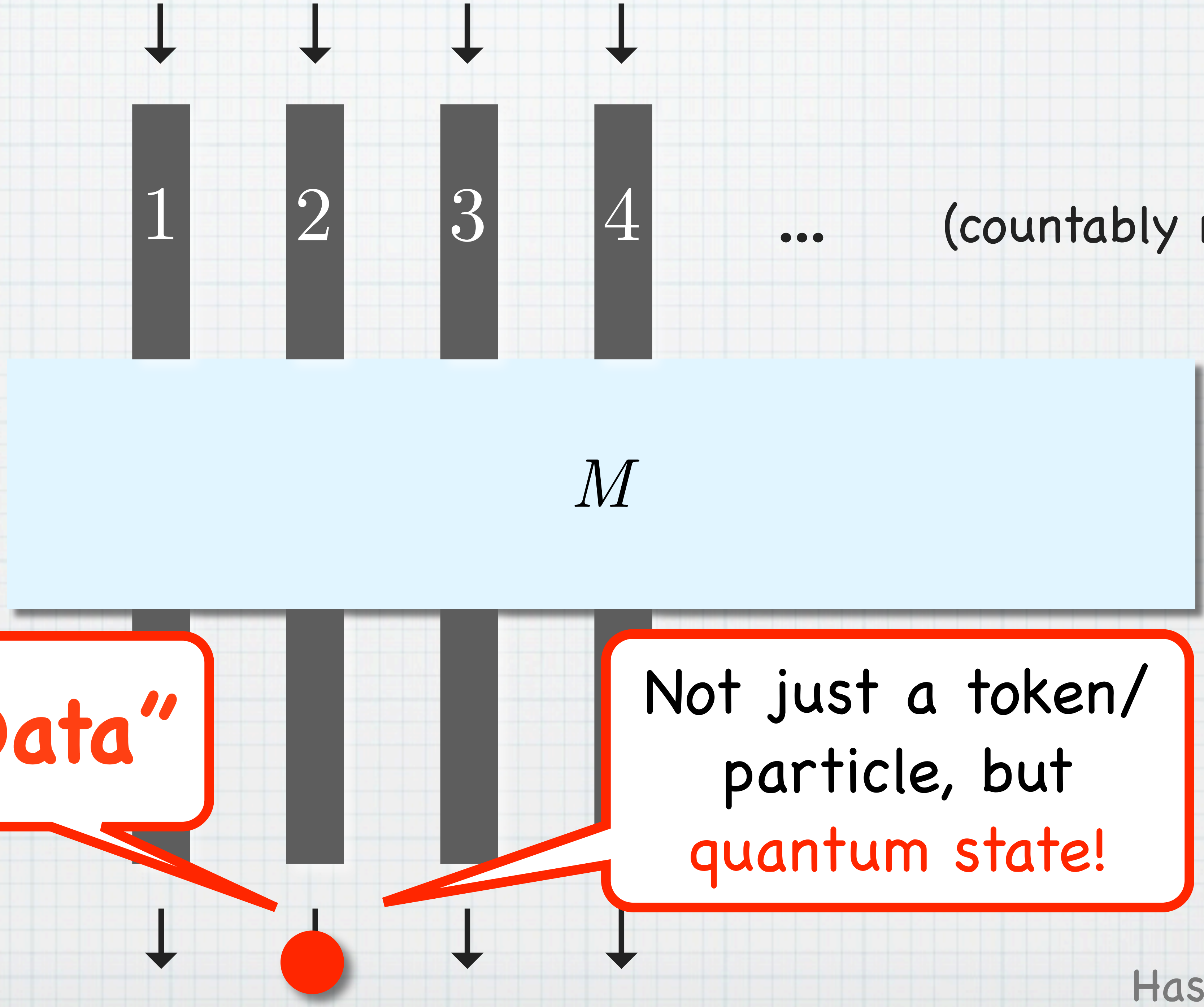

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"Quantum Data"

Not just a token/ particle, but quantum state!

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Quantum
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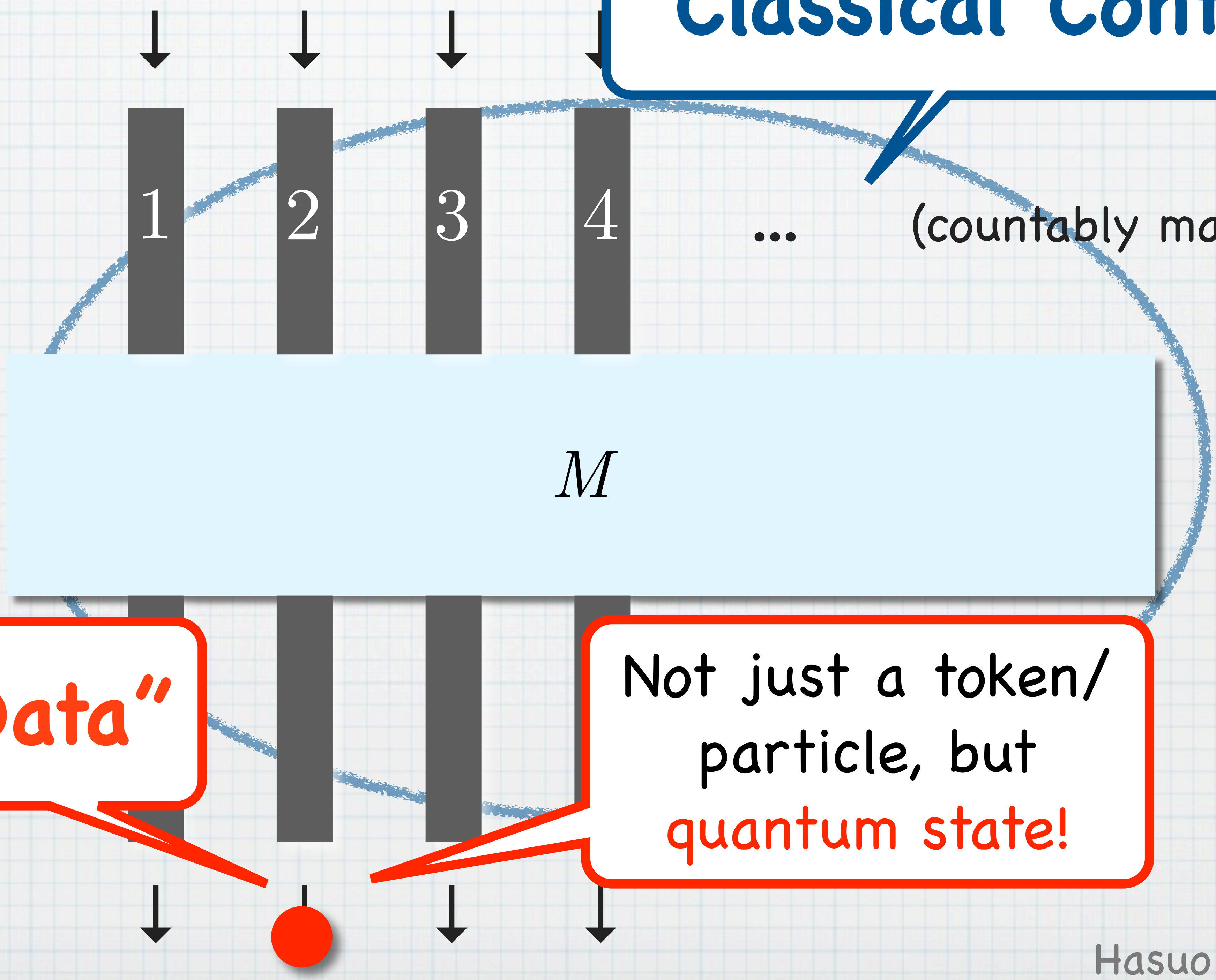


(countably many)

“Classical Control”

“Quantum Data”

Not just a token/ particle, but quantum state!

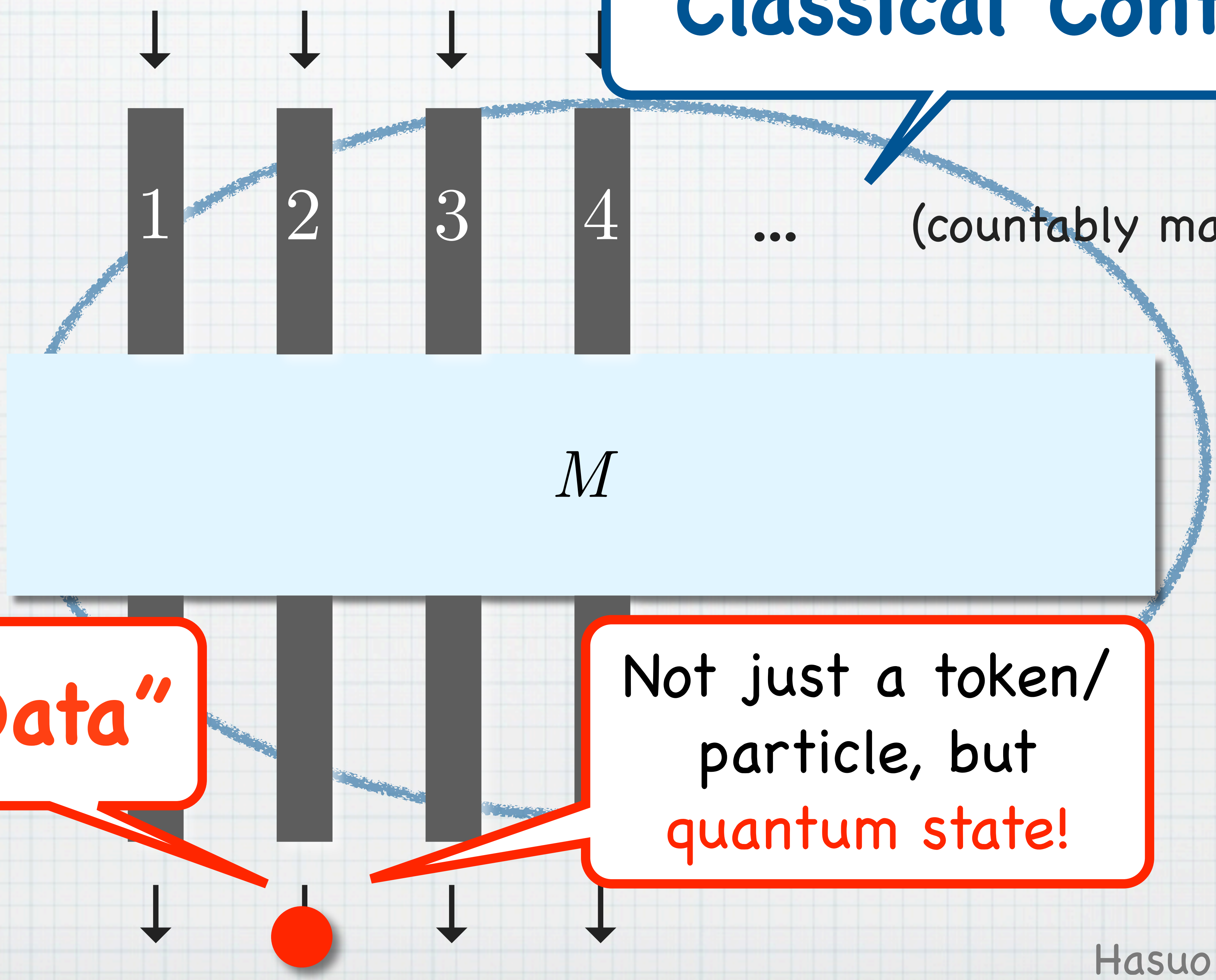
Hasuo (Tokyo)

Quan
Geometry o

- "in which pipe"
- (measurement → case-distinction) leads a token to different pipes

"Classical Control"

$[\![M]\!] =$



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"Quantum Data"

Not just a token/ particle, but quantum state!

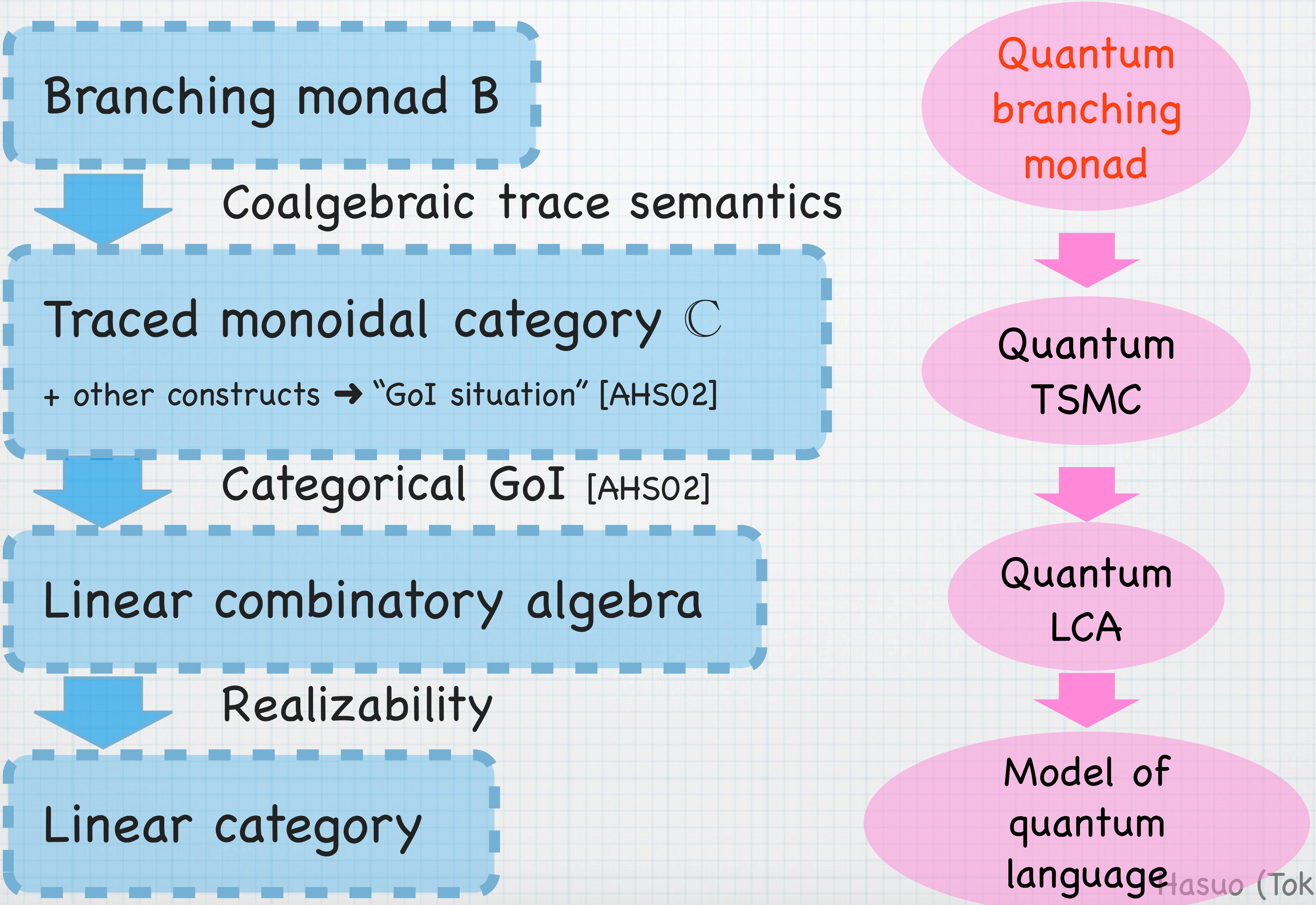
Indeed...

- The monad $\mathcal{Q}$ qualifies as a "branching monad"
- The quantum GoI workflow leads to a linear category $\mathbf{PER}_{\mathcal{Q}}$
- From which we construct an adequate denotational model

End of the Story?

- No! All the technicalities are yet to come:
  - CPS-style interpretation (for partial measurement)
  - Result type: a final coalgebra in $\mathbf{PER}_Q$
  - **Admissible PERs** for recursion
  - ...
- On the next occasion :-)

Conclusion: the Categorical GoI Workflow



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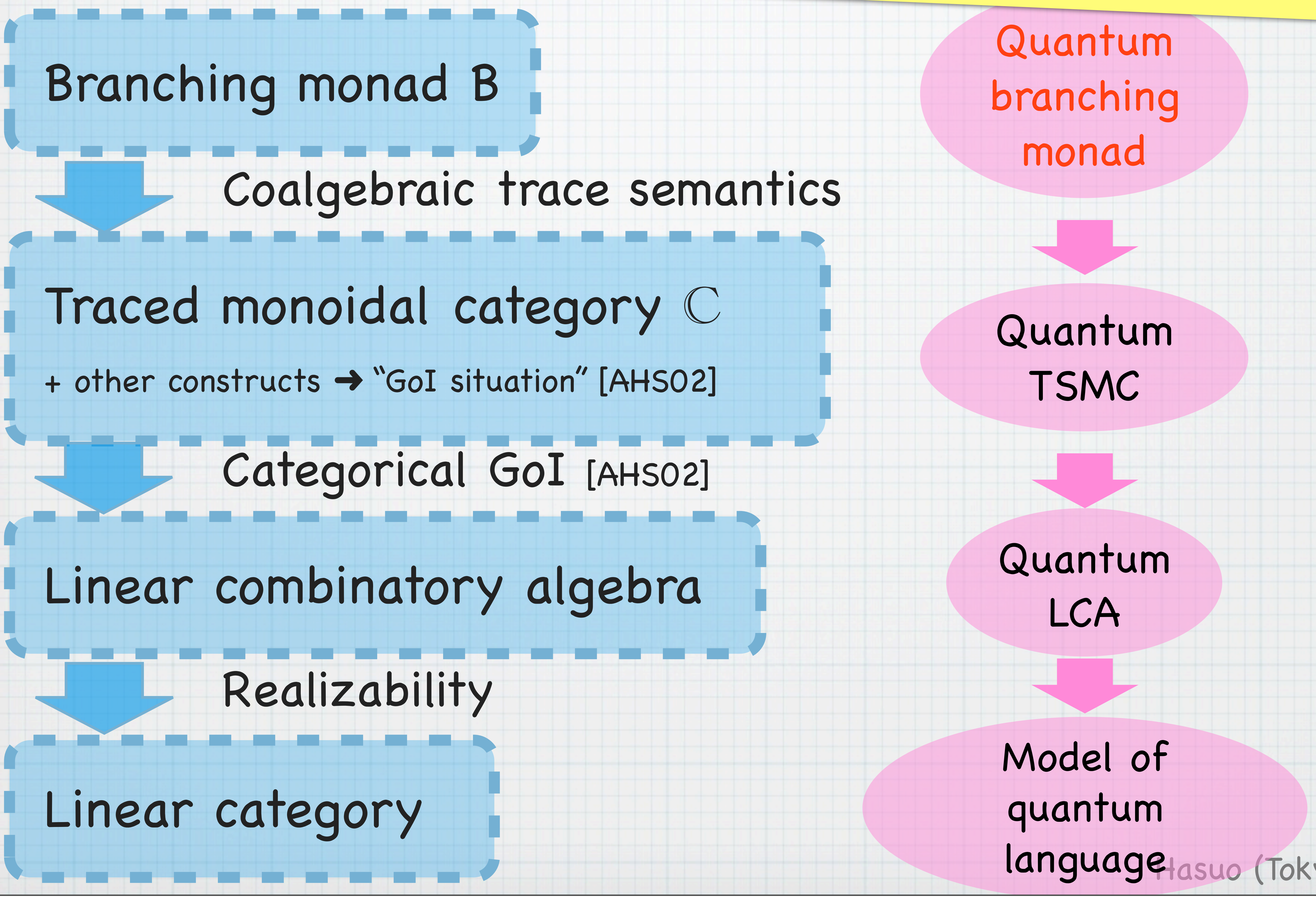
Model of quantum language

Thank you for your attention!

Ichiro Hasuo (Dept. CS, U Tokyo)

http://www-mmm.is.s.u-tokyo.ac.jp/~ichiro/

Conclusion: the Cat



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Model of quantum language

The Language $q\lambda_\ell$

- Roughly: linear λ + quantum primitives
- "Quantum data, classical control"
  - No superposed threads
- Based on [Selinger&Valiron'09]
  - With slight modifications
  - Notably: quantum $\otimes$ vs. linear logic ☒
    - The same in [Selinger&Valiron'09]
      ➜ clean type system, aids programming
    - But... problem with GoI-style semantics

The Language $q\lambda_\ell$

The *types* of $\mathbf{q}\boldsymbol{\lambda}_\ell$ are:

$$\boldsymbol{A}, \boldsymbol{B} ::= \boldsymbol{n}\text{-}\mathtt{qbit} \mid \boldsymbol{!}\,\boldsymbol{A} \mid \boldsymbol{A} \multimap \boldsymbol{B} \mid \top \mid \boldsymbol{A} \boxtimes \boldsymbol{B} \mid \boldsymbol{A} + \boldsymbol{B} \;,$$

with conventions $\mathtt{qbit} := \mathbf{1}\text{-}\mathtt{qbit}$ and $\mathtt{bit} := \top + \top$.

The *terms* of $\mathbf{q}\boldsymbol{\lambda}_\ell$ are:

$$\begin{aligned}
&\boldsymbol{M}, \boldsymbol{N}, \boldsymbol{P} ::= \\
&\quad \boldsymbol{x} \mid \boldsymbol{\lambda x^A . M} \mid \boldsymbol{M\,N} \mid \langle \boldsymbol{M}, \boldsymbol{N} \rangle \mid * \mid \\
&\quad \mathtt{let}\,\langle \boldsymbol{x^A}, \boldsymbol{y^B} \rangle = \boldsymbol{M}\;\mathtt{in}\;\boldsymbol{N} \mid \mathtt{let}\,* = \boldsymbol{M}\;\mathtt{in}\;\boldsymbol{N} \mid \\
&\quad \mathtt{inj}_\ell^{\boldsymbol{B}}\;\boldsymbol{M} \mid \mathtt{inj}_r^{\boldsymbol{A}}\;\boldsymbol{M} \mid \\
&\quad \mathtt{match}\;\boldsymbol{P}\;\mathtt{with}\;(\boldsymbol{x^A} \mapsto \boldsymbol{M} \mid \boldsymbol{y^B} \mapsto \boldsymbol{N}) \mid \\
&\quad \mathtt{letrec}\;\boldsymbol{f^A x} = \boldsymbol{M}\;\mathtt{in}\;\boldsymbol{N} \mid \\
&\quad \mathtt{new}\,|\mathbf{0}\rangle \mid \mathtt{meas}_i^{n+1} \mid \boldsymbol{U} \mid \mathtt{cmp}_{\boldsymbol{m},\boldsymbol{n}} \;,
\end{aligned}$$

with conventions $\mathtt{tt} := \mathtt{inj}_\ell^{\top}(*)$ and $\mathtt{ff} := \mathtt{inj}_r^{\top}(*)$.

The Langua

Different from quantum ⊗
(Unlike [Selinger-Valiron'09]);
same as the one in **PER**

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Different from quantum ⊗ (Unlike [Selinger-Valiron'09]); same as the one in **PER**

2-qbit ≅ qbit ⊗ qbit

$$A, B ::= n\text{-}\mathtt{qbit} \mid\, !A \mid A \multimap B \mid \top \mid A \boxtimes B \mid A + B\ ,$$

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Hasuo (Tokyo)

Monday, November 7, 2011

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\end{aligned}$$

Recursion

with conventions $\mathtt{tt} := \mathtt{inj}_\ell^\top(*)$ and $\mathtt{ff} := \mathtt{inj}_r^\top(*)$.

The Langua

Different from quantum ⊗
(Unlike [Selinger-Valiron'09]);
same as the one in **PER**

2-qbit ≅ qbit ⊗ qbit

$$A, B ::= n\text{-}\mathtt{qbit} \mid {!A} \mid A \multimap B \mid \top \mid A \boxtimes B \mid A + B\ ,$$

with conventions $\mathtt{qbit} := \mathbf{1}\text{-}\mathtt{qbit}$ and $\mathtt{bit} := \top + \top$.

The *terms* of $\mathbf{q}\boldsymbol{\lambda}_\ell$ are:

$$\begin{aligned} M, N, P ::= \\ & x \mid \lambda x^A.M \mid MN \mid \langle M, N\rangle \mid * \mid \\ & \mathtt{let}\ \langle x^A, y^B\rangle = M\ \mathtt{in}\ N \mid \mathtt{let} * = M\ \mathtt{in}\ N \mid \\ & \mathtt{inj}_\ell^B\ M \mid \mathtt{inj}_r^A\ M \mid \\ & \mathtt{match}\ P\ \mathtt{with}\ (x^A \mapsto M \mid y^B \mapsto N) \mid \\ & \mathtt{letrec}\ f^A x = M\ \mathtt{in}\ N \mid \\ & \mathtt{new}\ |\mathbf{0}\rangle \mid \mathtt{meas}_i^{n+1} \mid U \mid \mathtt{cmp}_{m,n}\ , \end{aligned}$$

Recursion

Quantum primitives

with conventions $\mathtt{tt} := \mathtt{inj}_\ell^\top(*)$ and $\mathtt{ff} := \mathtt{inj}_r^\top(*)$.

Hasuo (Tokyo)

Monday, November 7, 2011

Implicit linearity tracking via subtyping <:

e.g. !A <: A, !A <: !!A

(following [Selinger-Valiron'09])

$$\frac{n = 0 \Rightarrow m = 0 \ (*)}{!^n\, k\text{-}\mathtt{qbit} <: !^m\, k\text{-}\mathtt{qbit}}\ (k\text{-}\mathtt{qbit}) \qquad \frac{n = 0 \Rightarrow m = 0}{!^n \top <: !^m \top}\ (\top)$$

$$\frac{A_1 <: B_1 \quad A_2 <: B_2 \quad (*)}{!^n(A_1 \boxdot A_2) <: !^m(B_1 \boxdot B_2)}\ (\boxdot) \text{ with } \boxdot \in \{\boxtimes, +\}$$

$$\frac{B_1 <: A_1 \quad A_2 <: B_2 \quad (*)}{!^n(A_1 \multimap A_2) <: !^m(B_1 \multimap B_2)}\ (\multimap)$$

Measurements

$$\begin{array}{rcl}
A_{\mathrm{new}|0\rangle} & := & \mathtt{qbit} \\
A_{\mathrm{meas}_i^{n+1}} & := & (n+1)\text{-}\mathtt{qbit} \multimap (\mathtt{bit} \boxtimes n\text{-}\mathtt{qbit}) \text{ for } n \geq 1 \\
A_{\mathrm{meas}_1^1} & := & \mathtt{qbit} \multimap \mathtt{bit} \\
A_U & := & n\text{-}\mathtt{qbit} \multimap n\text{-}\mathtt{qbit} \quad \text{for a } 2^n \times 2^n \text{ matrix } U \\
A_{\mathrm{cmp}_{m,n}} & := & (m\text{-}\mathtt{qbit} \boxtimes n\text{-}\mathtt{qbit}) \multimap (m+n)\text{-}\mathtt{qbit}
\end{array}$$

Bookkeeping

(due to ⊗ vs. ⊠)

$$\frac{A <: A'}{!\,\Delta, x : A \vdash x : A'}\ (\mathrm{Ax.1}) \qquad \frac{!\,A_c <: A}{!\,\Delta \vdash c : A}\ (\mathrm{Ax.2})$$

$$\frac{\Delta \vdash M : !^n A}{\Delta \vdash \mathtt{inj}_\ell^B M : !^n(A+B)}\ (+.\mathrm{I}_1)$$

$$\frac{\Delta \vdash N : !^n B}{\Delta \vdash \mathtt{inj}_r^A N : !^n(A+B)}\ (+.\mathrm{I}_2)$$

$$\frac{!\,\Delta, \Gamma_1 \vdash P : !^n(A+B) \quad \begin{array}{c} !\,\Delta, \Gamma_2, x : !^n A \vdash M : C \\ !\,\Delta, \Gamma_2, y : !^n B \vdash N : C \end{array}}{\begin{array}{l} !\,\Delta, \Gamma_1, \Gamma_2 \\ \qquad \vdash \mathtt{match}\, P\, \mathtt{with}\, (x^{!^n A} \mapsto M \mid y^{!^n B} \mapsto N) : C \end{array}}\ (+.\mathrm{E}), (\dagger)$$

$$\frac{x : A, \Delta \vdash M : B}{\Delta \vdash \lambda x^A.M : A \multimap B}\ (\multimap.\mathrm{I}_1)$$

$$\frac{x : A, !\,\Delta \vdash M : B}{!\,\Delta \vdash \lambda x^A.M : !^n(A \multimap B)}\ (\multimap.\mathrm{I}_2)$$

$$\frac{!\,\Delta, \Gamma_1 \vdash M : A \multimap B \quad !\,\Delta, \Gamma_2 \vdash N : A}{!\,\Delta, \Gamma_1, \Gamma_2 \vdash MN : B}\ (\multimap.\mathrm{E}), (\dagger)$$

$$\frac{!\,\Delta, \Gamma_1 \vdash M_1 : !^n A_1 \quad !\,\Delta, \Gamma_2 \vdash M_2 : !^n A_2}{!\,\Delta, \Gamma_1, \Gamma_2 \vdash \langle M_1, M_2 \rangle : !^n(A_1 \boxtimes A_2)}\ (\boxtimes.\mathrm{I}), (\dagger)$$

$$\frac{}{!\,\Delta \vdash * : !^n \top}\ (\top.\mathrm{I})$$

$$\frac{\begin{array}{c} !\,\Delta, \Gamma_2, x_1 : !^n A_1, x_2 : !^n A_2 \vdash N : A \\ !\,\Delta, \Gamma_1 \vdash M : !^n(A_1 \boxtimes A_2) \end{array}}{!\,\Delta, \Gamma_1, \Gamma_2 \vdash \mathtt{let}\, \langle x_1^{!^n A_1}, x_2^{!^n A_2} \rangle = M\, \mathtt{in}\, N : A}\ (\boxtimes.\mathrm{E}), (\dagger)$$

$$\frac{!\,\Delta, \Gamma_1 \vdash M : \top \quad !\,\Delta, \Gamma_2 \vdash N : A}{!\,\Delta, \Gamma_1, \Gamma_2 \vdash \mathtt{let}\, * = M\, \mathtt{in}\, N : A}\ (\top.\mathrm{E}), (\dagger)$$

$$\frac{\begin{array}{c} !\,\Delta, \Gamma, f : !(A \multimap B) \vdash N : C \\ !\,\Delta, f : !(A \multimap B), x : A \vdash M : B \end{array}}{!\,\Delta, \Gamma \vdash \mathtt{letrec}\, f^{A \multimap B} x = M\, \mathtt{in}\, N : C}\ (\mathrm{rec}), (\dagger)$$

Operational Semantics

$$\begin{aligned}
&E[\,(\lambda x^{A}.M)V\,] \to_1 E[\,M[V/x]\,]\\
&E[\,\mathtt{let}\,\langle x^{A}, y^{B}\rangle = \langle V, W\rangle\,\mathtt{in}\,M\,] \to_1 E[\,M[V/x, W/y]\,]\\
&E[\,\mathtt{let}\,* = *\,\mathtt{in}\,M\,] \to_1 E[\,M\,]\\
&E[\,\mathtt{match}\,(\mathtt{inj}^{B}_{\ell}V)\,\mathtt{with}\,(x^{!^n A} \mapsto M \mid y^{!^n B} \mapsto N)\,]\\
&\qquad\qquad\qquad\qquad\qquad\qquad \to_1 E[\,M[V/x]\,]\\
&E[\,\mathtt{match}\,(\mathtt{inj}^{A}_{r}V)\,\mathtt{with}\,(x^{!^n A} \mapsto M \mid y^{!^n B} \mapsto N)\,]\\
&\qquad\qquad\qquad\qquad\qquad\qquad \to_1 E[\,N[V/y]\,]\\
&E[\,\mathtt{letrec}\,f^{A\multimap B}x = M\,\mathtt{in}\,N\,]\\
&\qquad\qquad \to_1 E[\,N[\lambda x^{A}.\mathtt{letrec}\,f^{A\multimap B}x = M\,\mathtt{in}\,M/f]\,]\\
&E[\,\mathtt{meas}^{n+1}_{i}(\mathtt{new}\,\rho)\,] \to_1 E[\,\langle\,\mathtt{tt},\,\mathtt{new}\,\langle 0_i|\rho|0_i\rangle\,\rangle\,]\\
&E[\,\mathtt{meas}^{n+1}_{i}(\mathtt{new}\,\rho)\,] \to_1 E[\,\langle\,\mathtt{ff},\,\mathtt{new}\,\langle 1_i|\rho|1_i\rangle\,\rangle\,]\\
&E[\,\mathtt{meas}^{1}_{1}(\mathtt{new}\,\rho)\,] \to_{\langle 0|\rho|0\rangle} E[\,\mathtt{tt}\,]\\
&E[\,\mathtt{meas}^{1}_{1}(\mathtt{new}\,\rho)\,] \to_{\langle 1|\rho|1\rangle} E[\,\mathtt{ff}\,]\\
&E[\,U(\mathtt{new}\,\rho)\,] \to_1 E[\,\mathtt{new}\,(U\rho)\,]\\
&E[\,\mathtt{cmp}_{m,n}\langle\mathtt{new}\,\rho, \mathtt{new}\,\sigma\rangle\,] \to_1 E[\,\mathtt{new}\,(\rho\otimes\sigma)\,]
\end{aligned}$$

- Standard small-step one, CBV, but with probabilistic branching (measurement)