Semantics of Higher-Order Quantum Computation via Geometry of Interaction

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Ichiro Hasuo
University of Tokyo (JP)

Naohiko Hoshino
RIMS, Kyoto University (JP)

東京大学 THE UNIVERSITY OF TOKYO

京都大学 KYOTO UNIVERSITY

Contribution

One of the first to cover the full features!

- !-modality for duplicable data
- recursion

Adequacy

Denotational semantics of a **functional quantum programming language**

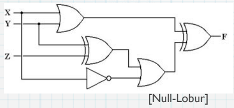
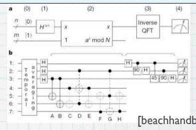
Linear λ-calculus + quantum primitives

... via **GoI** (Geometry of Interaction)

Part 1

Functional QPL: Some Contexts

Quantum Programming Language

Classical		Quantum	
(Boolean) circuit	 [Null-Lobur]	Quantum circuit	 [beachhandball.es]
Programming language		Quantum programming language	$\textbf{telep} = \quad \mathit{let}\ \langle x, y\rangle = \mathbf{EPR} * \mathit{in}\ \mathit{let}\ f = \mathbf{BellMeasure}\ x\ \mathit{in}\ \mathit{let}\ g = \mathbf{U}\ y\ \mathit{in}\ \langle f, g\rangle.$ [Selinger-Valiron]

```
int i,j;
int factorial(int k)
{
    j=1;
    for (i=1; i<=k; i++)
        j=j*i;
    return j;
}
```

- For discovery of **algorithms**
- For **reasoning, verification**

Functional Quantum Programming Language

- A **real man's** programming style
- Heavily used in the **financial sector**
- …

ICFP'11 Sponsers (Tokyo, Sep 2011)

- **Mathematically nice and clean**
  - Aids semantical study
  - **Transfer** from classical to quantum

Functional QPL: Syntax

- **Linear λ-calculus**
  + quantum primitives

Preparation/Unitary transformation/Measurement

[van Tonder, Selinger, Valiron, ...]

- Linearity for **no-cloning**
  - "Input can be used only once"
  - Not allowed/typable: $\lambda x.\ \langle \mathtt{meas}\, x,\ \mathtt{meas}\, x \rangle$
  - Duplicable (classical) data: by the **!-modality**

$\vdash \mathtt{tt} : \mathbf{!}\,\mathtt{bit}$

"arbitrary many copies"

Functional QPL: Semantics

- Semantics = **mathematical model**
- Operational semantics: [Selinger & Valiron, '09]
  - "Quantum closure," reduction with probabilistic branching

$$[\alpha|Q_0\rangle + \beta|Q_1\rangle, |x_1 \dots x_n\rangle, meas\ \ x_i\,] \to_{|\alpha|^2} [\,|Q_0\rangle, |x_1 \dots x_n\rangle, 0\,]$$
$$[\alpha|Q_0\rangle + \beta|Q_1\rangle, |x_1 \dots x_n\rangle, meas\ \ x_i\,] \to_{|\beta|^2} [\,|Q_1\rangle, |x_1 \dots x_n\rangle, 1\,]$$

- Allows to identify **linear logic** $\otimes$ and **quantum** $\otimes$
  (feature of the Selinger-Valiron language; not in ours)

Functional QPL: Semantics

- Denotational semantics
  - $[\![M]\!]$: a **function**, or an **arrow** of a category
  - **Compositionality:** $[\![MN]\!] = [\![M]\!] \circ [\![N]\!]$
  - **Linear category:** [Benton & Wadler, Bierman]
    (axioms for) a categorical model of linear λ-calculus

Defn.
A *linear category* $(\mathbb{C}, \otimes, \mathbf{I}, \multimap, !)$ is a sym. monoidal closed cat. with a *linear exponential comonad* $!$.

  - For functional QPL? Is **Hilb** (or alike) a linear cat.?

Functional QPL: Semantics

- **Hilb** (or alike) is **not** a linear category

monoidal closed str. $(\mathbb{C}, \otimes, \mathbf{I}, \multimap)$

duality $V \cong V^{\perp}$

finite dim.

! (for duplicable data)

infinite dim.

- **Challenge**: coexistence of **quantum** and **classical** data
- Only partial results
  - [Selinger & Valiron, '08]: for strictly linear fragmant (w/o **!**)

"Quantum Data, Classical Control"

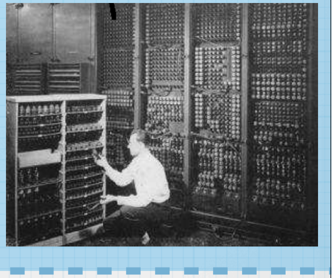
Illustration by N. Hoshino

Quantum data

$\frac{1}{\sqrt{2}}$

$+\frac{1}{\sqrt{2}}$

Classical control

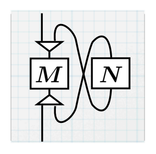


What We Do

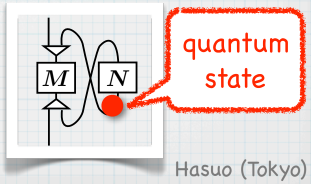
- **GoI (Geometry of Interaction)** [Girard '89]
  An "implementation" of **classical control**

$$\mathbf{tr}(f) = f_{XY} \sqcup \left(\coprod_{n \in \mathbb{N}} f_{ZY} \circ (f_{ZZ})^{n} \circ f_{XZ} \right)$$

- **Categorical GoI** [Abramsky, Haghverdi, Scott '02]
  Its categorical axiomatics



- We add a **quantum layer** to GoI
  - ➜ "Quantum data, classical control"
  - Used: theory of coalgebra
    [Hasuo, Jacobs, Sokolova '07] [Jacobs '10]



Part 2

The Categorical GoI Workflow

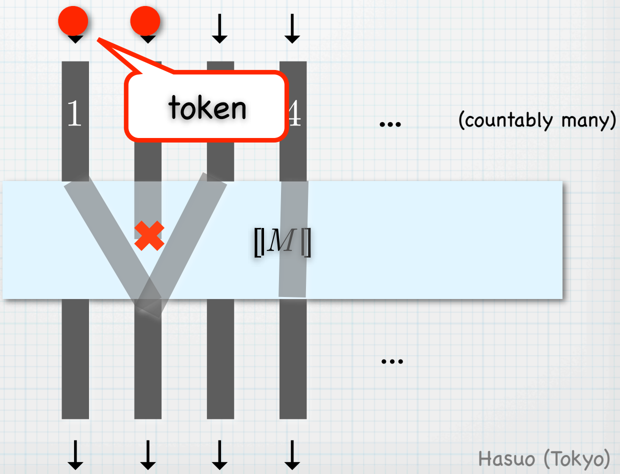
GoI:
Geometry of Interaction

- J.-Y. Girard, at Logic Colloquium '88
- **But** I'm no linear logician!
- In this talk:
  - Its categorical formulation
    [Abramsky, Haghverdi, Scott '02]
  - "The GoI Animation"

The GoI Animation

$[\![M]\!] \;=\; (\,\mathbb{N} \rightharpoonup \mathbb{N},\ \text{a partial function}\,)$

$=$ "piping"

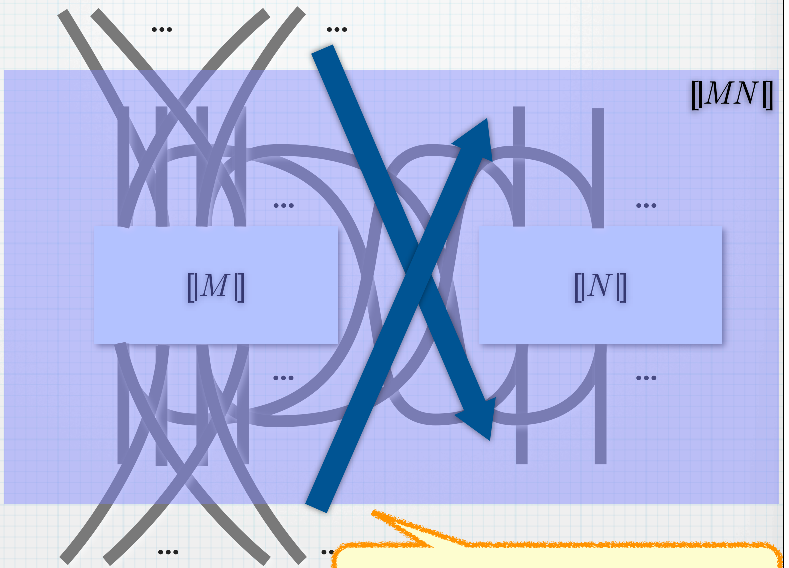


The GoI Animation

- Function application $[\![MN]\!]$
  - by "parallel composition + hiding"

$[\![MN]\!]$

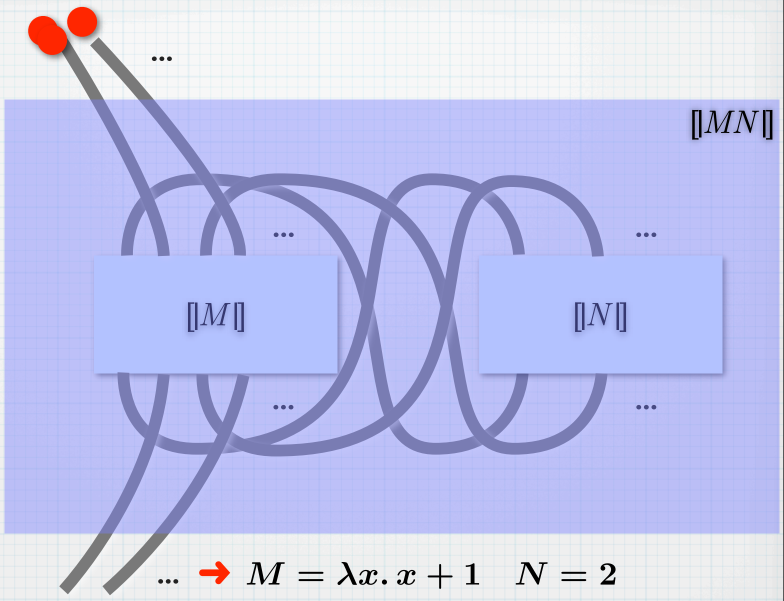
$=$



"parallel composition + hiding"
(cf. games)

$[\![MN]\!]$

$=$



... ➜ $M = \lambda x.\, x+1$ $N = 2$

➜ $M = \lambda x.\, 1$ $N = 2$

➜ $M = \lambda f.\, f1$ $N = \lambda x.\,(x+1)$

Categorical GoI

- Axiomatics of GoI in the categorical language
- Our main reference:
  - [AHS02] S. Abramsky, E. Haghverdi, and P. Scott, "Geometry of interaction and linear combinatory algebras," MSCS 2002
  - Especially its technical report version (Oxford CL), since it's a bit more detailed

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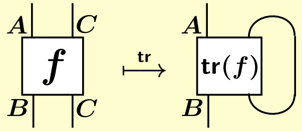
The Categorical GoI Workflow

Traced monoidal category $\mathbb{C}$

+ other constructs ➜ "GoI situation" [AHS02]



Categorical GoI [AHS02]

- Applicative str. + combinators
- Model of untyped calculus

Linear combinatory algebra

Realizability

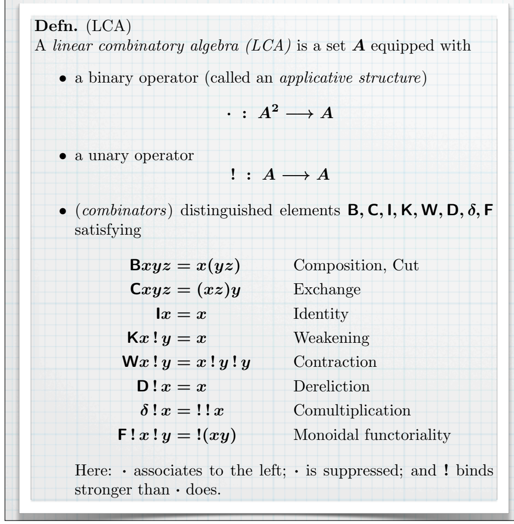
- PER, ω-set, assembly, ...
- "Programming in untyped λ"

Linear category

Model of typed calculus

Linear Combinatory Algebra (LCA)

What we want (outcome)



Defn. (LCA)
A *linear combinatory algebra (LCA)* is a set $\boldsymbol{A}$ equipped with

- a binary operator (called an *applicative structure*)

$$\cdot \;:\; \boldsymbol{A}^2 \longrightarrow \boldsymbol{A}$$

- a unary operator

$$! \;:\; \boldsymbol{A} \longrightarrow \boldsymbol{A}$$

- (*combinators*) distinguished elements $\mathsf{B}, \mathsf{C}, \mathsf{I}, \mathsf{K}, \mathsf{W}, \mathsf{D}, \delta, \mathsf{F}$ satisfying

$\mathsf{B}xyz = x(yz)$	Composition, Cut
$\mathsf{C}xyz = (xz)y$	Exchange
$\mathsf{I}x = x$	Identity
$\mathsf{K}x\,!y = x$	Weakening
$\mathsf{W}x\,!y = x\,!y\,!y$	Contraction
$\mathsf{D}\,!x = x$	Dereliction
$\delta\,!x = !\,!x$	Comultiplication
$\mathsf{F}\,!x\,!y = !(xy)$	Monoidal functoriality

Here: $\cdot$ associates to the left; $\cdot$ is suppressed; and $!$ binds stronger than $\cdot$ does.

- Model of untyped linear λ
- a ∈ A ≈ closed linear λ-term
- No **S** or **K** (linear!)
- Combinatory completeness: e.g.

$$\lambda xyz.\, zxy$$

designates an elem. of A

What we use (ingredient)

GoI situation

Defn. (GoI situation [AHS02])
A *GoI situation* is a triple $(\mathbb{C}, \boldsymbol{F}, \boldsymbol{U})$ where

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- $\boldsymbol{F} : \mathbb{C} \to \mathbb{C}$ is a traced symmetric monoidal functor, equipped with the following retractions (which are monoidal natural transformations).

$e : FF \lhd F : e'$	Comultiplication
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$w : K_I \lhd F : w'$	Weakening

Here K_I is the constant functor into the monoidal unit I;

- $U \in \mathbb{C}$ is an object (called *reflexive object*), equipped with the following retractions.

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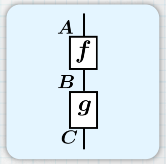
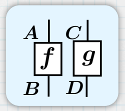
- **Monoidal category** $(\mathbb{C}, \otimes, I)$
  - **String diagrams**

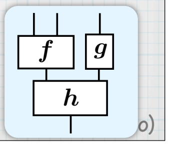
$$\frac{A \xrightarrow{f} B \quad B \xrightarrow{g} C}{A \xrightarrow{g \circ f} C}$$



$$\frac{A \xrightarrow{f} B \quad C \xrightarrow{g} D}{A \otimes C \xrightarrow{f \otimes g} B \otimes D}$$



$$h \circ (f \otimes g)$$



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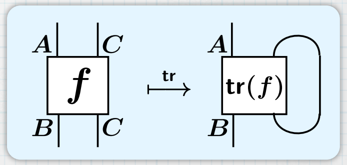
- **Traced monoidal category**
  - **"feedback"**

$$\frac{A \otimes C \xrightarrow{f} B \otimes C}{A \xrightarrow{\mathrm{tr}(f)} B}$$

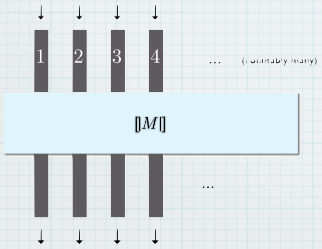
that is



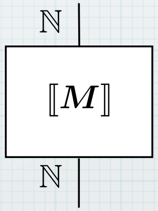
String Diagram vs. "Pipe Diagram"

- I use two ways of depicting partial functions $\mathbb{N} \rightharpoonup \mathbb{N}$

In the monoidal category $(\mathbf{Pfn}, +, 0)$



Pipe diagram

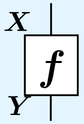


String diagram

Traced Sym. Monoidal Category
$(\mathbf{Pfn}, +, 0)$

- Category **Pfn** of partial functions
  - **Obj.** A set X
  - **Arr.** A partial function

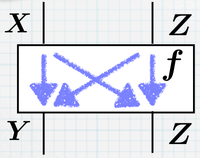
$$\frac{X \to Y \text{ in } \mathbf{Pfn}}{X \rightharpoonup Y, \text{ partial function}}$$



- is traced symmetric monoidal

Traced Sym. Monoidal Category
$(\mathbf{Pfn}, +, 0)$

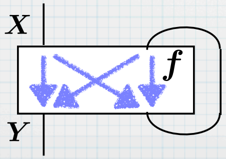
- $$\frac{X + Z \xrightarrow{f} Y + Z \quad \text{in } \mathbf{Pfn}}{X \xrightarrow{\mathsf{tr}(f)} Y \quad \text{in } \mathbf{Pfn}}$$ How?
-



$$f_{XY}(x) := \begin{cases} f(x) & \text{if } f(x) \in Y \\ \bot & \text{o.w.} \end{cases}$$

Similar for f_{XZ}, f_{ZY}, f_{ZZ}

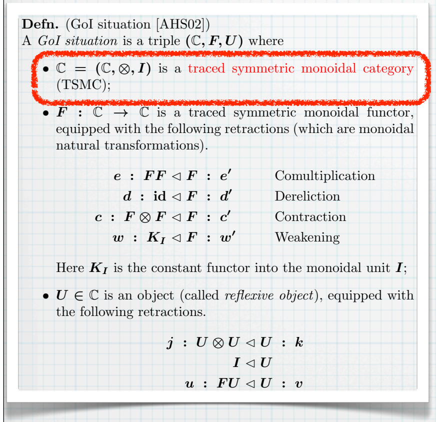
- Trace operator:



- **Execution formula** (Girard)
- Partiality is essential (infinite loop)

$$\mathsf{tr}(f) = f_{XY} \sqcup \left(\coprod_{n \in \mathbb{N}} f_{ZY} \circ (f_{ZZ})^n \circ f_{XZ} \right)$$

GoI situation



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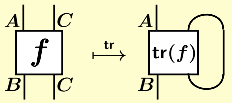
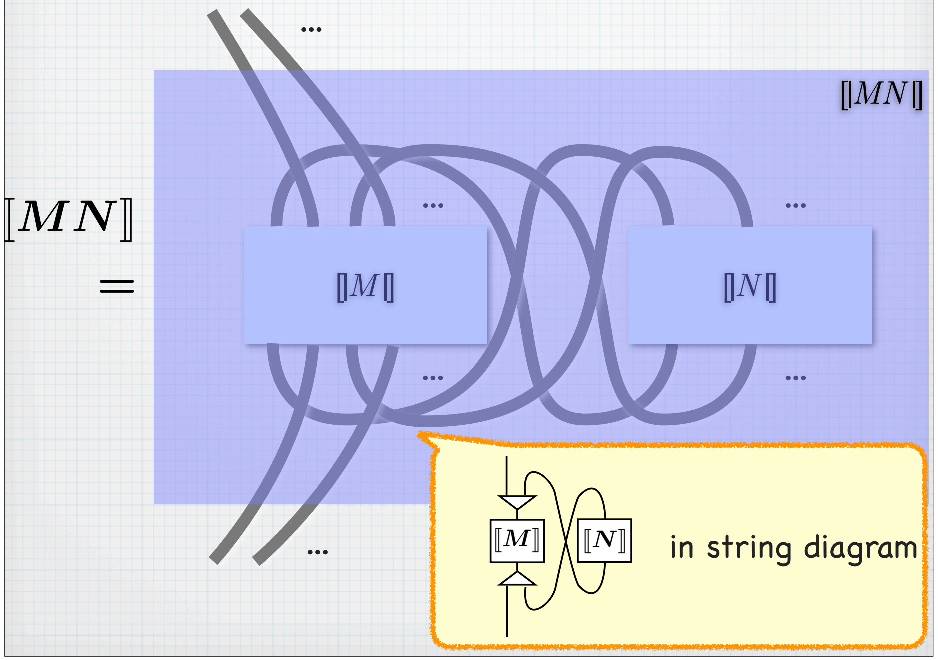
- Traced sym. monoidal cat.
  - Where one can "feedback"



  - Why for GoI?



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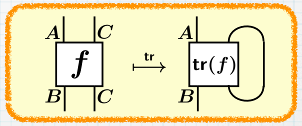

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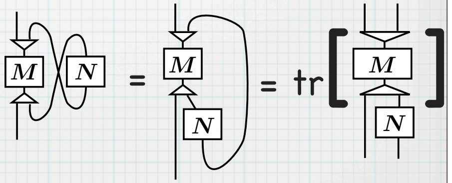

- Leading example: **Pfn**

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Defn. (Retraction)
A *retraction* from X to Y,

$$f : X \lhd Y : g\ ,$$

is a pair of arrows

$$\begin{tikzcd} X \arrow[loop left, "\mathrm{id}"] \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd}$$

"embedding"

"projection"

such that $g \circ f = \mathrm{id}_X$.

- Functor F
  - For obtaining $! : A \to A$

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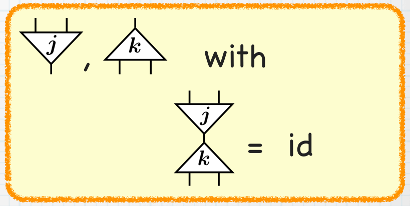
- The reflexive object U
  - Retr. $\begin{tikzcd} U \otimes U \arrow[r, bend left, "j"] & U \arrow[l, bend left, "k"] \end{tikzcd}$



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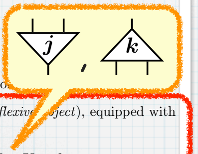
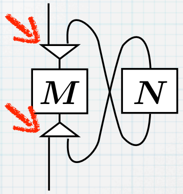
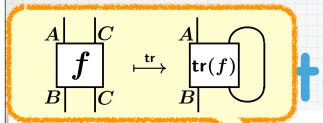


- The reflexive object U
  - Why for GoI?



- Example in **Pfn**:

$\mathbb{N} \in \mathbf{Pfn}$, with
$\mathbb{N} + \mathbb{N} \cong \mathbb{N}$,
$\mathbb{N} \cdot \mathbb{N} \cong \mathbb{N}$



tuation: Summary

Defn. (GoI situation [AHS02])
A *GoI situation* is a triple $(\mathbb{C}, F, U)$ wh e

- $\mathbb{C} = (\mathbb{C}, \otimes, I)$ is a traced symmetric monoidal category (TSMC);
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$$e : FF \lhd F : e' \qquad d : \mathrm{id} \lhd F : d' \qquad c : F \otimes F \lhd F : c' \qquad w : K_I \lhd F : w'$$

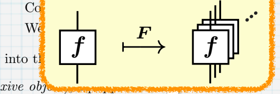
De
Co
We

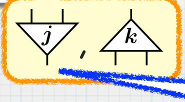
Here K_I is the constant functor into t

- $U \in \mathbb{C}$ is an object (called *reflexive obj* the following retractions.

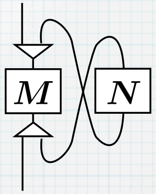
$$j : U \otimes U \lhd U : k \qquad I \lhd U \qquad u : FU \lhd U : v$$

For ! , via



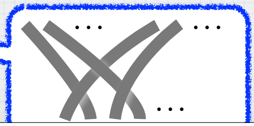


- Categorical axiomatics of the "GoI animation"



- Example:

$(\mathbf{Pfn}, \mathbb{N} \cdot \_ , \mathbb{N})$

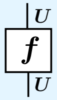


Categorical GoI: Constr. of an LCA

Thm. ([AHS02])
Given a GoI situation $(\mathbb{C}, F, U)$, the homset

$$\mathbb{C}(U, U)$$

carries a canonical LCA structure.

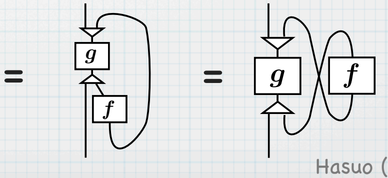


$\in \mathbb{C}(U, U)$

- Applicative str. ·
- ! operator
- Combinators B, C, I, ...

- $g \cdot f$

$$:= \mathbf{tr}(\,(U \otimes f) \circ k \circ g \circ j\,)$$



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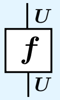
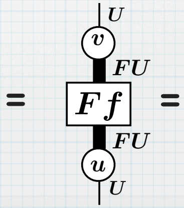
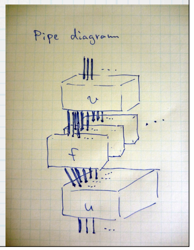


$\in \mathbb{C}(U, U)$

- Applicative str. ·
- ! operator
- Combinators B, C, I, ...

- $!f \;:=\; u \circ Ff \circ v$





Categorical GoI: Constr. of an LCA

- Combinator $Bxyz = x(yz)$

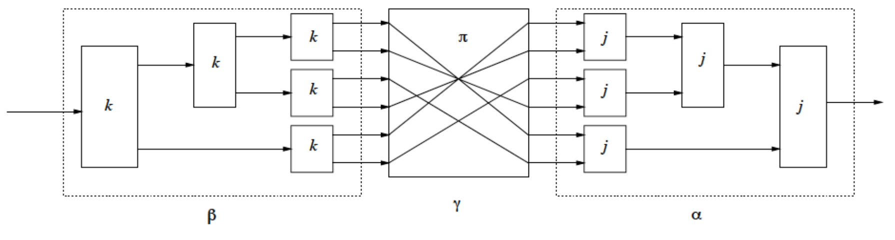


Figure 7: Composition Combinator B

from [AHS02]

Categorical GoI: Constr. of an LCA

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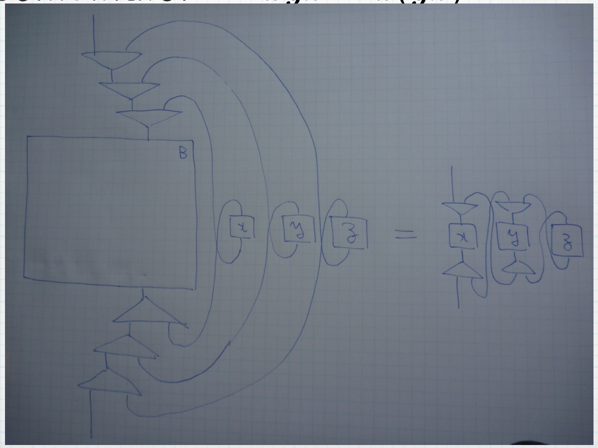

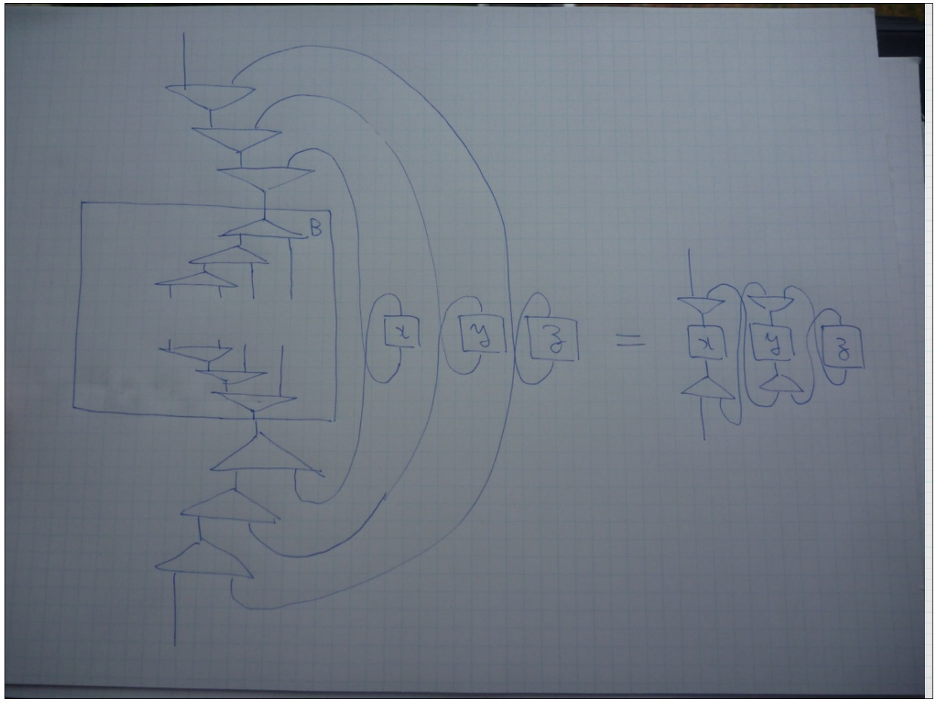

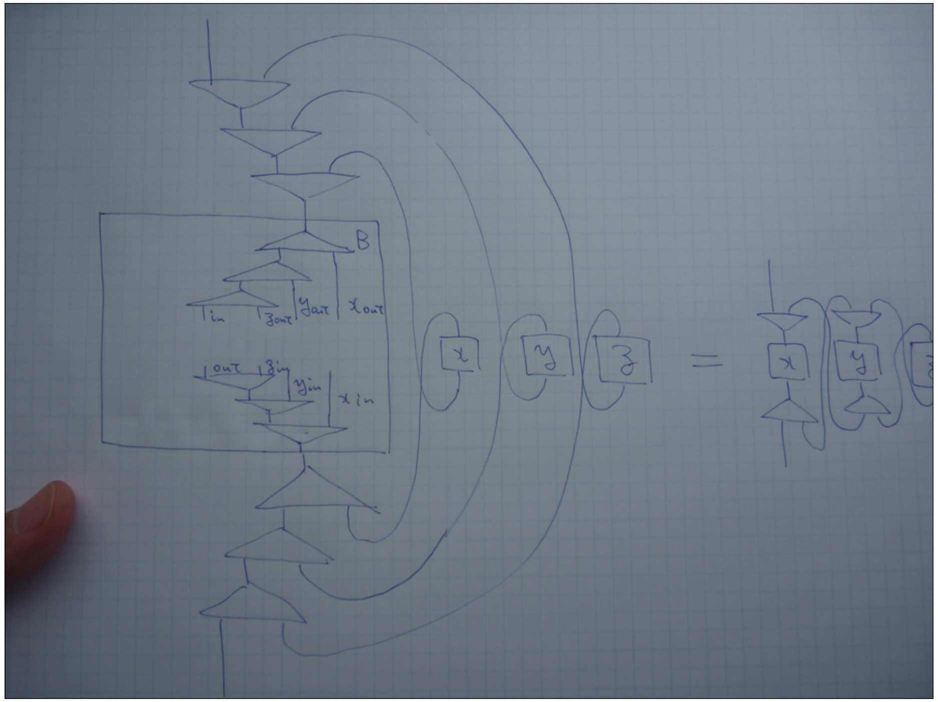

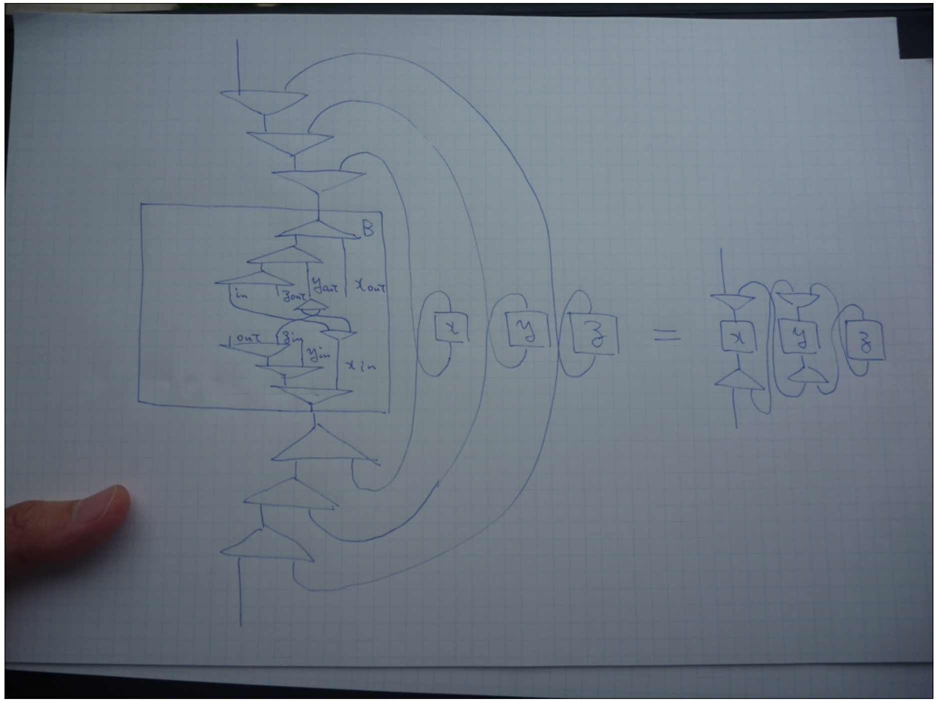

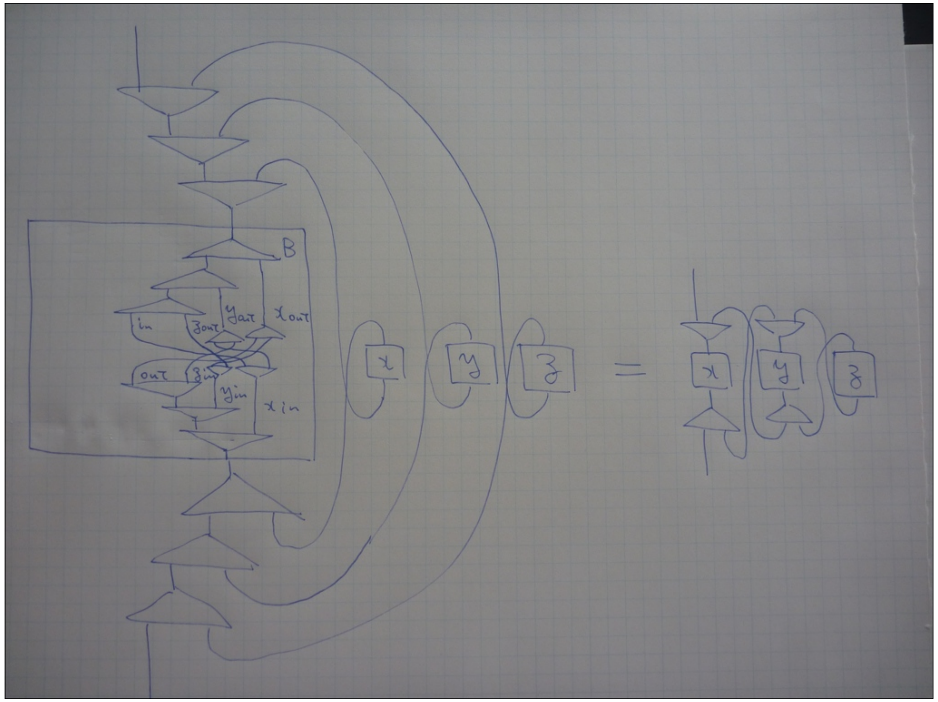

Categorical GoI: Constr. of an LCA

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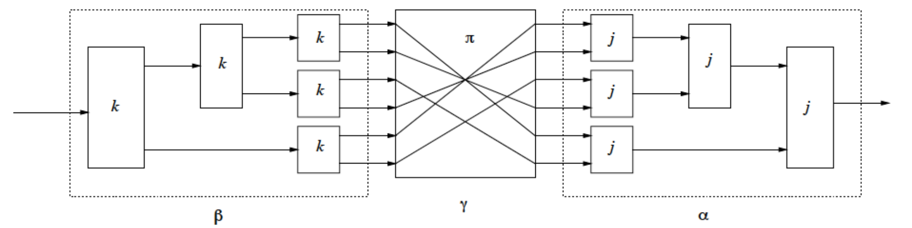

Figure 7: Composition Combinator B

from [AHS02]

Nice dynamic interpretation of (linear) computation!!

Summary: Categorical GoI

Defn. (GoI situation [AHS02])
A *GoI situation* is a triple $(\mathbb{C}, F, U)$ where

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$e : FF \lhd F : e'$	Comultiplication
$d : \mathrm{id} \lhd F : d'$	Dereliction
$c : F \otimes F \lhd F : c'$	Contraction
$w : K_I \lhd F : w'$	Weakening

Here K_I is the constant functor into the monoidal unit I;

- $U \in \mathbb{C}$ is an object (called *reflexive object*), equipped with the following retractions.

$$j : U \otimes U \lhd U : k$$
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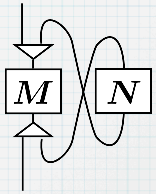
Why Categorical Generalization?: Examples Other Than $\mathbf{Pfn}$ [AHS02]

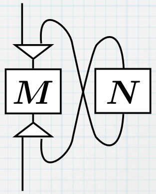


- Strategy: find a TSMC!
- "Wave-style" examples
  - ⊗ is Cartesian product(-like)
    - in which case,

      trace ≈ fixed point operator [Hasegawa/Hyland]
  - An example: $\bigl((\omega\text{-}\mathbf{Cpo}, \times, \mathbf{1}), (\_)^{\mathbb{N}}, A^{\mathbb{N}}\bigr)$
  - (... less of a dynamic flavor)

Why Categorical Generalization?: Examples Other Than **Pfn** [AHS02]

- "Particle-style" examples
  - Obj. X∈C is set-like; ⊗ is coproduct-like
    - The GoI animation is valid
  - Examples:
    - Partial functions $((\mathbf{Pfn},+,0),\ \mathbb{N}\cdot\_\,,\ \mathbb{N})$
    - Binary relations $((\mathbf{Rel},+,0),\ \mathbb{N}\cdot\_\,,\ \mathbb{N})$
    - "Discrete stochastic relations" $((\mathbf{DSRel},+,0),\ \mathbb{N}\cdot\_\,,\ \mathbb{N})$



Why Categorical Generalization?: Examples Other Than **Pfn** [AHS02]

Categories of sets and (functions with different branching/partiality)

- **Pfn** (partial functions)

$$\begin{array}{c} X \to Y \text{ in } \mathbf{Pfn} \\ \hline\hline X \rightharpoonup Y, \text{ partial function} \\ \hline\hline X \to \mathcal{L}Y \text{ in } \mathbf{Sets} \end{array} \quad \text{where } \mathcal{L}Y = \{\bot\} + Y$$

(Potential) non-termination

- **Rel** (relations)

$$\begin{array}{c} X \to Y \text{ in } \mathbf{Rel} \\ \hline\hline R \subseteq X \times Y, \text{ relation} \\ \hline\hline X \to \mathcal{P}Y \text{ in } \mathbf{Sets} \end{array} \quad \text{where } \mathcal{P} \text{ is the powerset monad}$$

Non-determinism

- **DSRel**

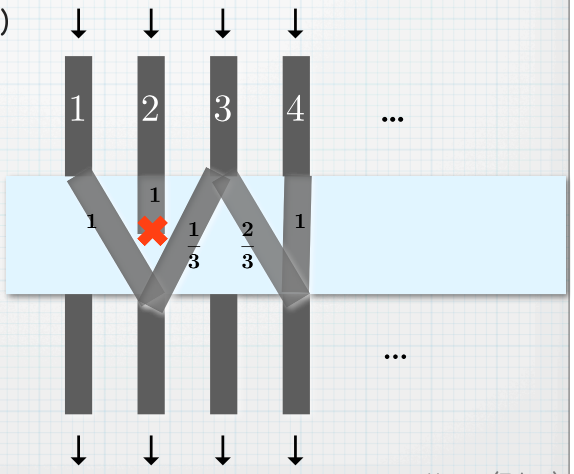
$$\begin{array}{l} X \to Y \text{ in } \mathbf{DSRel} \\ \hline\hline X \to \mathcal{D}Y \text{ in } \mathbf{Sets} \end{array}$$

$$\text{where } \mathcal{D}Y = \{d : Y \to [0,1] \mid \sum_y d(y) \le 1\}$$

Probabilistic branching

Different Branching in The GoI Animation

- **Pfn** (partial functions)
  - Pipes can be stuck
- **Rel** (relations)
  - Pipes can branch
- **DSRel**
  - Pipes can branch probabilistically



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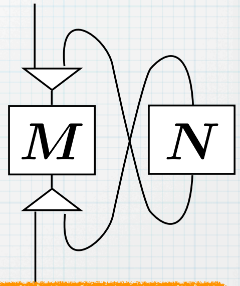

Essential to have subdistribution, for infinite loops

The Coauthor

- Naohiko Hoshino
  - DSc (Kyoto, 2011)
    - Supervisor: Masahito "Hassei" Hasegawa
  - Currently at RIMS, Kyoto U.
    - `http://www.kurims.kyoto-u.ac.jp/~naophiko/`

A Coalgebraic View

- Theory of coalgebra = Categorical theory of state-based dynamic systems (LTS, automaton, Markov chain, ...)
- In [Hasuo, Jacobs, Sokolova '07]:
  - Coalgebras in a Kleisli category $Kl(B)$

$$\frac{\dfrac{X \to Y \text{ in } \mathcal{K}\ell(B)}{}}{X \to BY \text{ in } \mathbf{Sets}}$$

  - ➜ Generic theory of "trace semantics"

Why Catego

Examples

$Kl(B)$ for different branching monads B

- **Pfn** (partial functions)

$$\begin{array}{c} X \to Y \text{ in } \mathbf{Pfn} \\ \hline\hline X \rightharpoonup Y, \text{ partial function} \\ \hline\hline X \to \mathcal{L}Y \text{ in } \mathbf{Sets} \end{array} \quad \text{where } \mathcal{L}Y = \{\perp\} + Y$$

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where $\mathcal{D}Y = \{d : Y \to [0,1] \mid \sum_{y} d(y) \leq 1\}$

Probabilistic branching

Branching Monad: Source of Particle-Style GoI Situations

Thm. ([Jacobs,CMCS10])
Given a "branching monad" B on **Sets**, the monoidal category

$$(\mathcal{K}\ell(B), +, 0)$$

is

- a *unique decomposition category* [Haghverdi,PhD00], hence is
- a traced symmetric monoidal category.

Cor.
$\big((\mathcal{K}\ell(B), +, 0),\ \mathbb{N}\cdot\_\,,\ \mathbb{N} \big)$ is a GoI situation.

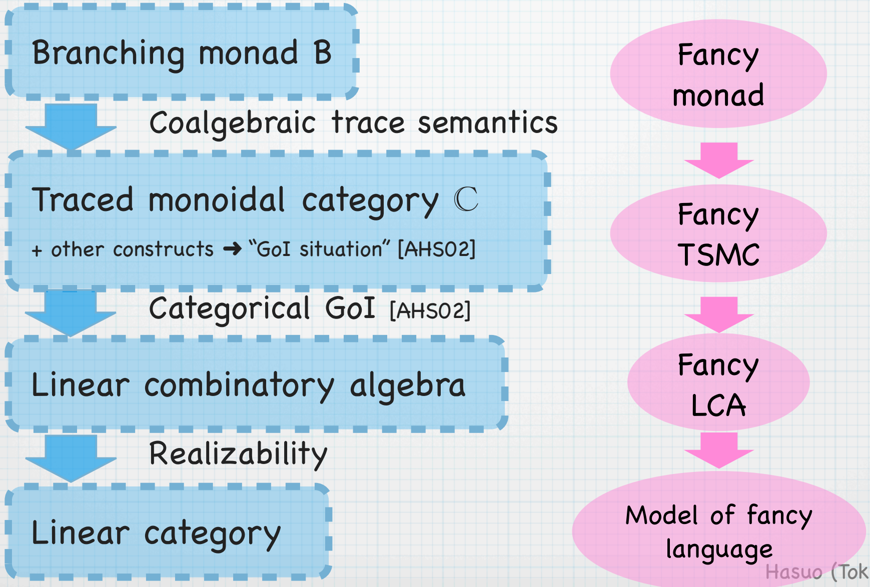
(Roughly) monads in [Hasuo, Jacobs, Sokolova '07]

- Kl(B) is $\mathbf{Cpo}_{\perp}$-enriched
- like $\mathcal{L}$, $\mathcal{P}$, $\mathcal{D}$

Particle-style: trace via the execution formula

$$\mathrm{tr}(f) = f_{XY} \sqcup \Big(\coprod_{n\in\mathbb{N}} f_{ZY} \circ (f_{ZZ})^n \circ f_{XZ} \Big)$$

The Categorical GoI Workflow



Branching monad B

Coalgebraic trace semantics

Traced monoidal category $\mathbb{C}$

+ other constructs ➔ "GoI situation" [AHS02]

Categorical GoI [AHS02]

Linear combinatory algebra

Realizability

Linear category

Fancy monad

Fancy TSMC

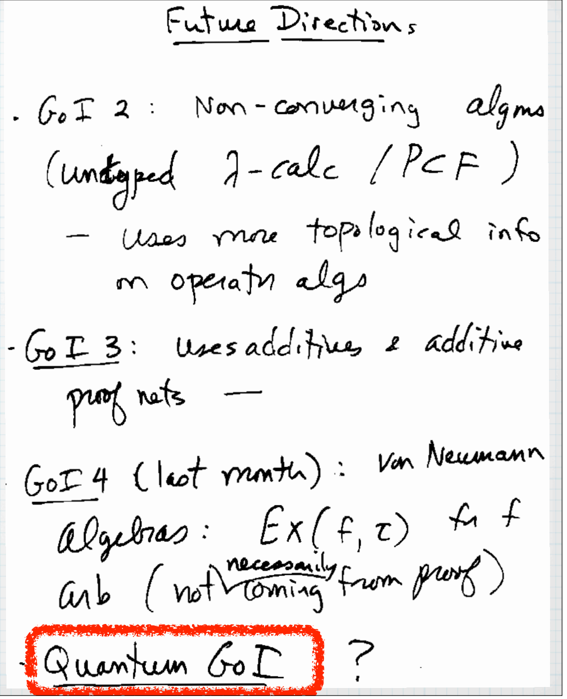
Fancy LCA

Model of fancy language

What is Fancy, Nowadays?

- Biology?
- Hybrid systems?
  - Both discrete and continuous data, typically in **cyber-physical systems** (CPS)
  - ➔ Our approach via **non-standard analysis** [Suenaga, Hasuo ICALP'11]
- Quantum?
  - Yes this worked!

Part 3



Future Directions

. GoI 2 : Non-converging algms (untyped λ-calc / PCF)

— uses more topological info on operatn algs

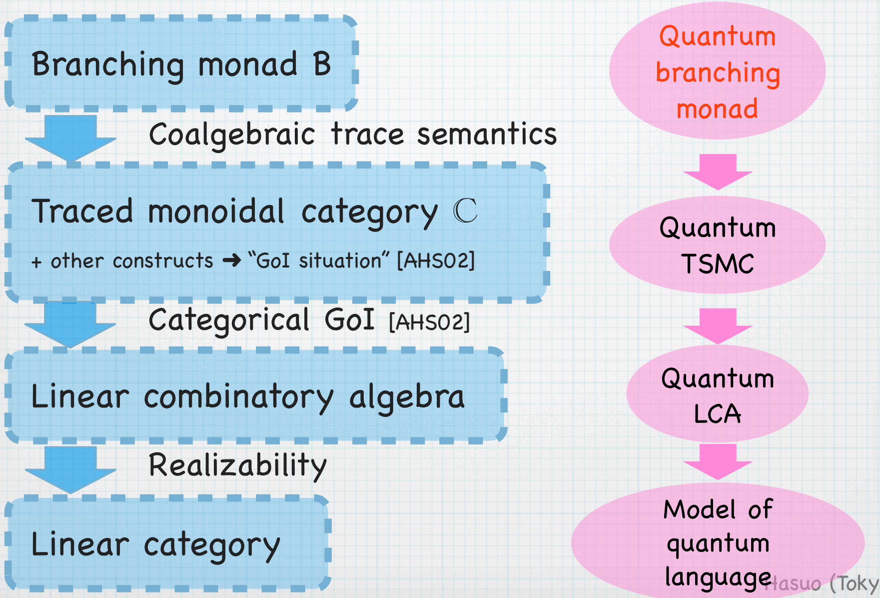
. GoI 3: uses additives & additive proof nets —

. GoI 4 (last month) : von Neumann algebras : Ex(f, τ) for f arb (not necessarily coming from proof)

. Quantum GoI ?

Phil Scott.
Tutorial on Geometry of Interaction, FMCS 2004.
Page 47/47

The Categorical GoI Workflow



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Quantum branching monad

Quantum TSMC

Quantum LCA

Model of quantum language

The Quantum Branching M

$\mathrm{QO}_{m,n} := \left\{ \begin{array}{l} \text{quantum operations,} \\ \text{from dim. } m \text{ to dim. } n \end{array} \right\}$

$$\mathcal{Q}Y = \left\{ c : Y \to \prod_{m,n\in\mathbb{N}} \mathrm{QO}_{m,n} \;\middle|\; \text{the trace condition} \right\}$$

$$\sum_{y\in Y}\sum_{n\in\mathbb{N}} \mathbf{tr}[(c(y))_{m,n}(\rho)] \le 1 \;,\quad \forall m\in\mathbb{N},\ \forall \rho\in D_m.$$

- Compare with

$$\mathcal{P}Y = \{c : Y \to 2\}$$

$$\mathcal{D}Y = \left\{c : Y \to [0,1] \;\middle|\; \sum_{y\in Y} c(y) \le 1\right\}$$

The Quantum Branching Monad

$$\mathcal{Q}Y = \left\{ c : Y \to \prod_{m,n\in\mathbb{N}} \mathrm{QO}_{m,n} \;\middle|\; \text{the trace condition} \right\}$$

$$\sum_{y\in Y}\sum_{n\in\mathbb{N}} \mathbf{tr}[(c(y))_{m,n}(\rho)] \le 1 \;,\quad \forall m\in\mathbb{N},\ \forall \rho\in D_m.$$

$$\frac{X \xrightarrow{f} Y \text{ in } \mathcal{K}\ell(\mathcal{Q})}{X \xrightarrow{f} \mathcal{Q}Y \text{ in } \mathbf{Sets}}$$

- Given $x\in X,\ y\in Y,\ m\in\mathbb{N},\ n\in\mathbb{N}$ determines a quantum operation

$$\big(f(x)(y)\big)_{m,n} \;:\; D_m \to D_n$$

- Subject to the **trace condition**

Any opr. on quantum data:

combination of

- preparation
- unitary transf.
- measurement

The Quantum Branching Monad

$$\mathcal{Q}Y = \left\{ c : Y \to \prod_{m,n\in\mathbb{N}} \mathrm{QO}_{m,n} \;\middle|\; \text{the trace condition} \right\}$$

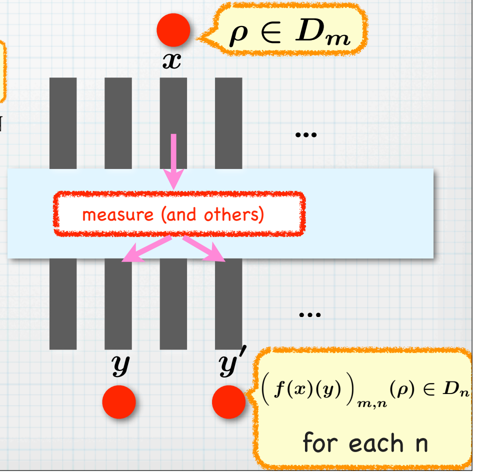
$$\sum_{y\in Y}\sum_{n\in\mathbb{N}} \mathbf{tr}[(c(y))_{m,n}(\rho)] \le 1 \;,\quad \forall m\in\mathbb{N},\ \forall \rho\in D_m.$$

$$\frac{X \xrightarrow{f} Y \text{ in } \mathcal{K}\ell(\mathcal{Q})}{X \xrightarrow{f} \mathcal{Q}Y \text{ in } \mathbf{Sets}}$$

entrance exit in dim. out dim.

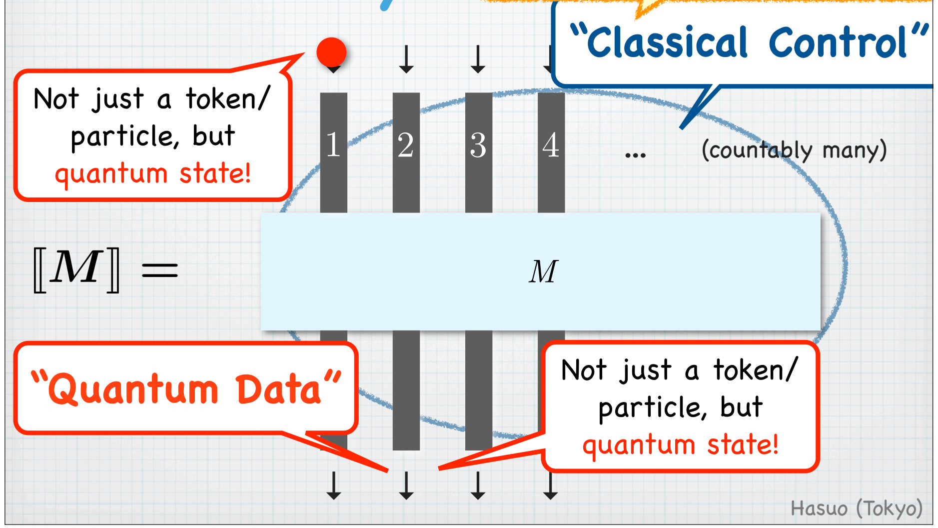
- Given $x\in X,\ y\in Y,\ m\in\mathbb{N},\ n\in\mathbb{N}$ determines a quantum operation $\big(f(x)(y)\big)_{m,n}$
- **trace cond.:**

$$\sum_{y,n} \Pr\left(\begin{array}{c}\text{Token led} \\ \text{to } y \\ \text{with dim. } n\end{array}\right) \le 1$$



Quan
Geometry o

- "in which pipe"
- (measurement ➜ case-distinction) leads a token to different pipes



Not just a token/ particle, but quantum state!

"Quantum Data"

Not just a token/ particle, but quantum state!

Indeed...

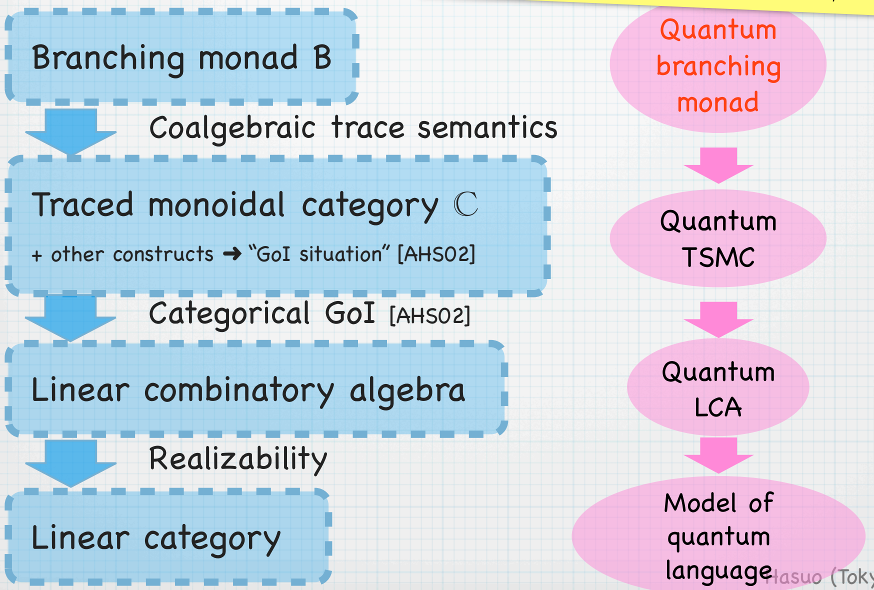
- The monad $\mathcal{Q}$ qualifies as a "branching monad"
- The quantum GoI workflow leads to a linear category $\mathbf{PER}_{\mathcal{Q}}$
- From which we construct an adequate denotational model

End of the Story?

- No! All the technicalities are yet to come:
    - CPS-style interpretation (for partial measurement)
    - Result type: a final coalgebra in $\mathbf{PER}_{\mathcal{Q}}$
    - **Admissible PERs** for recursion
    - ...
- On the next occasion :-)

Conclusion: the Cat

Thank you for your attention!
Ichiro Hasuo (Dept. CS, U Tokyo)
http://www-mmm.is.s.u-tokyo.ac.jp/~ichiro/



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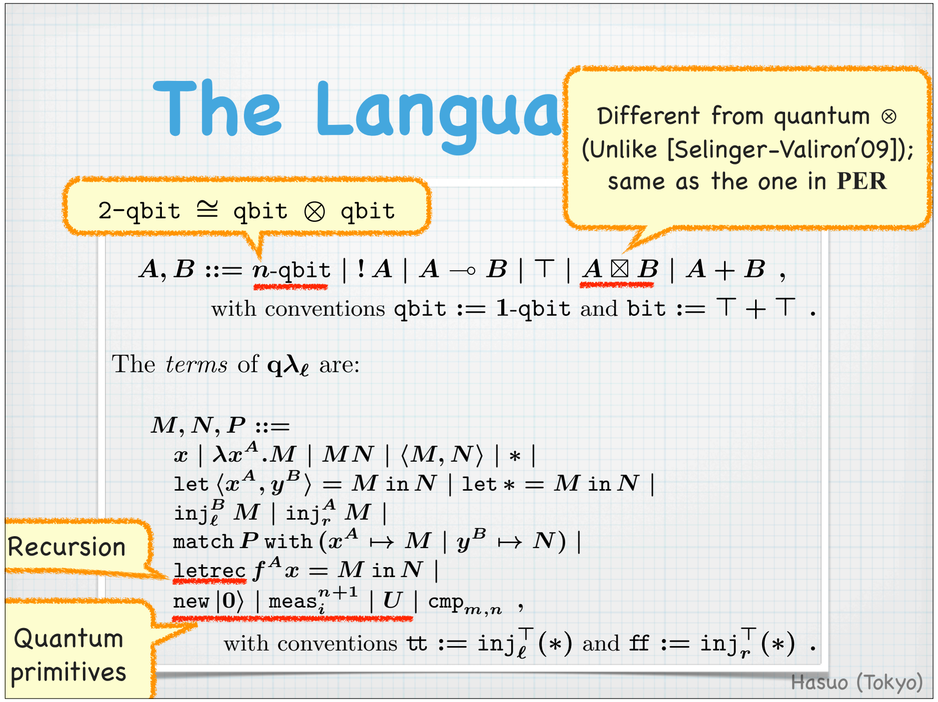
Quantum LCA

Model of quantum language

The Language q$\lambda\ell$

- Roughly: linear λ + quantum primitives
- "Quantum data, classical control"
    - No superposed threads
- Based on [Selinger&Valiron'09]
    - With slight modifications
    - Notably: quantum $\otimes$ vs. linear logic $\boxtimes$
        - The same in [Selinger&Valiron'09] ➜ clean type system, aids programming
        - But... problem with GoI-style semantics



The Langua

Different from quantum ⊗
(Unlike [Selinger-Valiron'09]);
same as the one in **PER**

2-qbit ≅ qbit ⊗ qbit

$$\boldsymbol{A, B} ::= \underline{\boldsymbol{n}\text{-}\mathtt{qbit}} \mid \boldsymbol{!\,A} \mid \boldsymbol{A \multimap B} \mid \top \mid \underline{\boldsymbol{A \boxtimes B}} \mid \boldsymbol{A + B} \ ,$$

with conventions $\mathtt{qbit} := \mathbf{1}\text{-}\mathtt{qbit}$ and $\mathtt{bit} := \top + \top$.

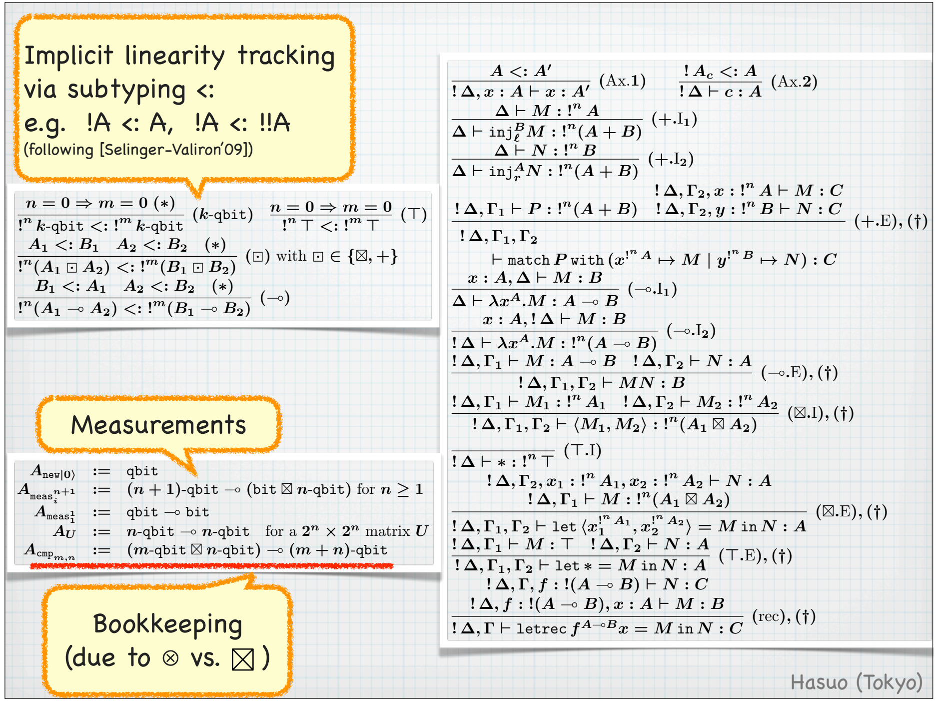
The *terms* of $\mathbf{q}\boldsymbol{\lambda}_{\boldsymbol{\ell}}$ are:

$$
\begin{array}{l}
\boldsymbol{M, N, P} ::= \\
\quad x \mid \lambda x^{A}.M \mid MN \mid \langle M, N \rangle \mid * \mid \\
\quad \mathtt{let}\, \langle x^{A}, y^{B} \rangle = M \,\mathtt{in}\, N \mid \mathtt{let}\, * = M \,\mathtt{in}\, N \mid \\
\quad \mathtt{inj}_{\ell}^{B}\, M \mid \mathtt{inj}_{r}^{A}\, M \mid \\
\quad \mathtt{match}\, P \,\mathtt{with}\, (x^{A} \mapsto M \mid y^{B} \mapsto N) \mid \\
\quad \underline{\mathtt{letrec}}\, f^{A} x = M \,\mathtt{in}\, N \mid \\
\quad \underline{\mathtt{new}\, |0\rangle \mid \mathtt{meas}_{i}^{n+1} \mid U} \mid \mathtt{cmp}_{m,n} \ ,
\end{array}
$$

with conventions $\mathtt{tt} := \mathtt{inj}_{\ell}^{\top}(*)$ and $\mathtt{ff} := \mathtt{inj}_{r}^{\top}(*)$.

Recursion

Quantum
primitives



Implicit linearity tracking
via subtyping <:
e.g. !A <: A, !A <: !!A
(following [Selinger-Valiron'09])

$$\frac{n = 0 \Rightarrow m = 0 \ (*)}{!^{n}\, k\text{-}\mathtt{qbit} <: \,!^{m}\, k\text{-}\mathtt{qbit}}\ (k\text{-}\mathtt{qbit}) \qquad \frac{n = 0 \Rightarrow m = 0}{!^{n} \top <: \,!^{m} \top}\ (\top)$$

$$\frac{A_1 <: B_1 \quad A_2 <: B_2 \quad (*)}{!^{n}(A_1 \boxdot A_2) <: \,!^{m}(B_1 \boxdot B_2)}\ (\boxdot) \text{ with } \boxdot \in \{\boxtimes, +\}$$

$$\frac{B_1 <: A_1 \quad A_2 <: B_2 \quad (*)}{!^{n}(A_1 \multimap A_2) <: \,!^{m}(B_1 \multimap B_2)}\ (\multimap)$$

Measurements

$$
\begin{array}{rcl}
A_{\mathtt{new}|0\rangle} & := & \mathtt{qbit} \\
A_{\mathtt{meas}_{i}^{n+1}} & := & (n+1)\text{-}\mathtt{qbit} \multimap (\mathtt{bit} \boxtimes n\text{-}\mathtt{qbit}) \text{ for } n \geq 1 \\
A_{\mathtt{meas}_{1}^{1}} & := & \mathtt{qbit} \multimap \mathtt{bit} \\
A_{U} & := & n\text{-}\mathtt{qbit} \multimap n\text{-}\mathtt{qbit} \quad \text{for a } 2^{n} \times 2^{n} \text{ matrix } U \\
A_{\mathtt{cmp}_{m,n}} & := & (m\text{-}\mathtt{qbit} \boxtimes n\text{-}\mathtt{qbit}) \multimap (m+n)\text{-}\mathtt{qbit}
\end{array}
$$

Bookkeeping
(due to ⊗ vs. ⊠)

$$\frac{A <: A'}{!\,\Delta, x : A \vdash x : A'}\ (\mathrm{Ax}.1) \qquad \frac{!\,A_c <: A}{!\,\Delta \vdash c : A}\ (\mathrm{Ax}.2)$$

$$\frac{\Delta \vdash M : !^{n}\, A}{\Delta \vdash \mathtt{inj}_{\ell}^{B} M : !^{n}(A + B)}\ (+.\mathrm{I}_1)$$

$$\frac{\Delta \vdash N : !^{n}\, B}{\Delta \vdash \mathtt{inj}_{r}^{A} N : !^{n}(A + B)}\ (+.\mathrm{I}_2)$$

$$\frac{!\,\Delta, \Gamma_1 \vdash P : !^{n}(A + B) \quad \begin{array}{c} !\,\Delta, \Gamma_2, x : !^{n}\, A \vdash M : C \\ !\,\Delta, \Gamma_2, y : !^{n}\, B \vdash N : C \end{array}}{\begin{array}{l} !\,\Delta, \Gamma_1, \Gamma_2 \\ \quad \vdash \mathtt{match}\, P \,\mathtt{with}\, (x^{!^{n} A} \mapsto M \mid y^{!^{n} B} \mapsto N) : C \end{array}}\ (+.\mathrm{E}), (\dagger)$$

$$\frac{x : A, \Delta \vdash M : B}{\Delta \vdash \lambda x^{A}.M : A \multimap B}\ (\multimap.\mathrm{I}_1)$$

$$\frac{x : A, !\,\Delta \vdash M : B}{!\,\Delta \vdash \lambda x^{A}.M : !^{n}(A \multimap B)}\ (\multimap.\mathrm{I}_2)$$

$$\frac{!\,\Delta, \Gamma_1 \vdash M : A \multimap B \quad !\,\Delta, \Gamma_2 \vdash N : A}{!\,\Delta, \Gamma_1, \Gamma_2 \vdash MN : B}\ (\multimap.\mathrm{E}), (\dagger)$$

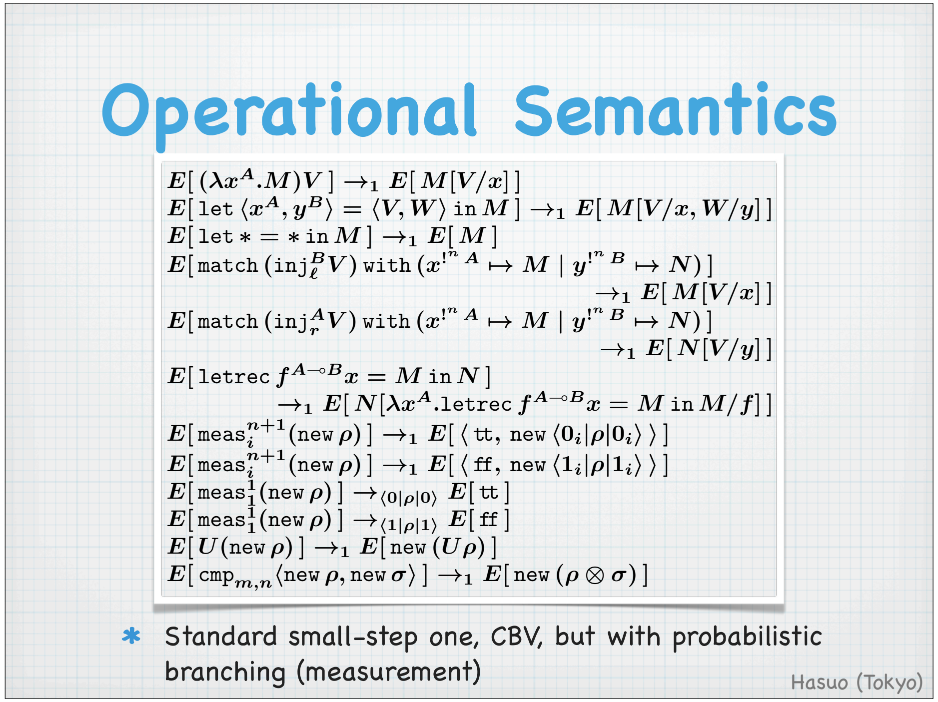
$$\frac{!\,\Delta, \Gamma_1 \vdash M_1 : !^{n}\, A_1 \quad !\,\Delta, \Gamma_2 \vdash M_2 : !^{n}\, A_2}{!\,\Delta, \Gamma_1, \Gamma_2 \vdash \langle M_1, M_2 \rangle : !^{n}(A_1 \boxtimes A_2)}\ (\boxtimes.\mathrm{I}), (\dagger)$$

$$\frac{}{!\,\Delta \vdash * : !^{n} \top}\ (\top.\mathrm{I})$$

$$\frac{\begin{array}{c} !\,\Delta, \Gamma_2, x_1 : !^{n}\, A_1, x_2 : !^{n}\, A_2 \vdash N : A \\ !\,\Delta, \Gamma_1 \vdash M : !^{n}(A_1 \boxtimes A_2) \end{array}}{!\,\Delta, \Gamma_1, \Gamma_2 \vdash \mathtt{let}\, \langle x_1^{!^{n} A_1}, x_2^{!^{n} A_2} \rangle = M \,\mathtt{in}\, N : A}\ (\boxtimes.\mathrm{E}), (\dagger)$$

$$\frac{!\,\Delta, \Gamma_1 \vdash M : \top \quad !\,\Delta, \Gamma_2 \vdash N : A}{!\,\Delta, \Gamma_1, \Gamma_2 \vdash \mathtt{let}\, * = M \,\mathtt{in}\, N : A}\ (\top.\mathrm{E}), (\dagger)$$

$$\frac{\begin{array}{c} !\,\Delta, \Gamma, f : !(A \multimap B) \vdash N : C \\ !\,\Delta, f : !(A \multimap B), x : A \vdash M : B \end{array}}{!\,\Delta, \Gamma \vdash \mathtt{letrec}\, f^{A \multimap B} x = M \,\mathtt{in}\, N : C}\ (\mathrm{rec}), (\dagger)$$



Operational Semantics

$$
\begin{array}{l}
E[\,(\lambda x^{A}.M)V\,] \to_1 E[\,M[V/x]\,] \\
E[\,\mathtt{let}\, \langle x^{A}, y^{B} \rangle = \langle V, W \rangle \,\mathtt{in}\, M\,] \to_1 E[\,M[V/x, W/y]\,] \\
E[\,\mathtt{let}\, * = * \,\mathtt{in}\, M\,] \to_1 E[\,M\,] \\
E[\,\mathtt{match}\,(\mathtt{inj}_{\ell}^{B} V)\,\mathtt{with}\,(x^{!^{n} A} \mapsto M \mid y^{!^{n} B} \mapsto N)\,] \\
\hfill \to_1 E[\,M[V/x]\,] \\
E[\,\mathtt{match}\,(\mathtt{inj}_{r}^{A} V)\,\mathtt{with}\,(x^{!^{n} A} \mapsto M \mid y^{!^{n} B} \mapsto N)\,] \\
\hfill \to_1 E[\,N[V/y]\,] \\
E[\,\mathtt{letrec}\, f^{A \multimap B} x = M \,\mathtt{in}\, N\,] \\
\qquad \to_1 E[\,N[\lambda x^{A}.\mathtt{letrec}\, f^{A \multimap B} x = M \,\mathtt{in}\, M/f]\,] \\
E[\,\mathtt{meas}_{i}^{n+1}(\mathtt{new}\,\rho)\,] \to_1 E[\,\langle\, \mathtt{tt},\ \mathtt{new}\, \langle 0_i | \rho | 0_i \rangle\, \rangle\,] \\
E[\,\mathtt{meas}_{i}^{n+1}(\mathtt{new}\,\rho)\,] \to_1 E[\,\langle\, \mathtt{ff},\ \mathtt{new}\, \langle 1_i | \rho | 1_i \rangle\, \rangle\,] \\
E[\,\mathtt{meas}_{1}^{1}(\mathtt{new}\,\rho)\,] \to_{\langle 0|\rho|0\rangle} E[\,\mathtt{tt}\,] \\
E[\,\mathtt{meas}_{1}^{1}(\mathtt{new}\,\rho)\,] \to_{\langle 1|\rho|1\rangle} E[\,\mathtt{ff}\,] \\
E[\,U(\mathtt{new}\,\rho)\,] \to_1 E[\,\mathtt{new}\,(U\rho)\,] \\
E[\,\mathtt{cmp}_{m,n}\langle \mathtt{new}\,\rho, \mathtt{new}\,\sigma \rangle\,] \to_1 E[\,\mathtt{new}\,(\rho \otimes \sigma)\,]
\end{array}
$$

- Standard small-step one, CBV, but with probabilistic branching (measurement)